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Nagoya Math. J.
Vol. 38 (1970), 103–111

# ON A NECESSARY CONDITION FOR THE SAMPLE PATH CONTINUITY OF WEAKLY STATIONARY PROCESSES

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***To Professor Katuzi Ono on the occasion of his 60th birthday***

## 1. Introduction

We shall discuss the sample path continuity of a stationary process assuming that the spectral distribution function $F(\lambda)$ is given. Many kinds of sufficient conditions have been given in terms of the covariance function or the asymptotic behavior of the spectral distribution function. We also shall give a sufficient condition (Theorem 1) expressed in the form

$$\sum\sqrt{F(n+1)-F(n)} < \infty .$$

Such a formula has also inspired our investigation for necessary conditions observing the case where $dF(\lambda)$ is discrete.

We are mainly interested in necessary conditions for the sample path continuity. For any distribution function $F(\lambda)$, there always exists a stationary process with the spectral distribution function $F(\lambda)$ which has continuous sample paths; there might, however, exist a stationary process, with the same $F(\lambda)$ such that any version of the process has no continuous sample paths (c.f. Theorem 2). In view of this, we shall be concerned with the problem to determine the class of distribution functions $F(\lambda)$ satisfying the following condition:

CONDITION $(C)$ *Any weakly stationary process with the spectral distribution function $F(\lambda)$ has a version, sample paths of which are continuous.*

The last section will be devoted to clarify relation between the sufficient condition and the necessary condition which are stated in Theorem 1 and

Received March 19, 1969

Theorem 2. In fact, the one is close enough to the other. Propositions and examples will serve in illustrating this situation.

The author wishes to express his hearty thanks to Professor T. Kawata who sent the preprint [2] and letters which gave private communications encouraging the author in the course of this note.

## 2. Sufficient condition

In this section we shall state a sufficient condition for sample path continuity of a weakly stationary process. The result is a slight generalization of the results given by T. Kawata [2].

Let $\{X(t)\}$ be a weakly stationary process with mean zero and spectral distribution function $F(\lambda)$. Suppose that $\{X(t)\}$ has the spectral representation

$$X(t) = \int e^{i\lambda t} dZ(\lambda) \tag{2.1}$$

where the $\{Z(\lambda)\}$ process has orthogonal increments, and

$$E|Z(\lambda)|^2 = F(\lambda). \tag{2.2}$$

We now define approximate weakly stationary processes $\{X_T(t)\}$, $0 < T < \infty$, in the following way according to the idea of T. Kawata. Set

$$X_T(t) = \sum_{n=-\infty}^{\infty} e^{2\pi int/T} \left\{ Z\left(\frac{2\pi(n+1)}{T}\right) - Z\left(\frac{2\pi n}{T}\right) \right\}. \tag{2.3}$$

This Fourier series converges in $L^2$-sense for each $t$. In order to discuss the absolute convergence of the Fourier series (2.3), we prepare the following simple lemma.

Lemma 1. *Let $F(\lambda)$ be a distribution function (not necessary $F(\infty) = 1$). Then for $a \geq 1$, the following inequalities hold:*

$$\frac{1}{\sqrt{[a]+1}} \sum_{n=-\infty}^{\infty} \sqrt{F(n+1)-F(n)} \leq \sum_{n=-\infty}^{\infty} \sqrt{F(a(n+1))-F(an)} \leq 2 \sum_{n=-\infty}^{\infty} \sqrt{F(n+1)-F(n)}^{*)}.$$

Lemma 2. *Let $F(\lambda)$ be the spectral distribution function of a weakly stationary process $\{X(t)\}$ with mean zero. If it holds that*

$$\sum_{n=-\infty}^{\infty} \sqrt{F(n+1)-F(n)} < \infty, \tag{2.4}$$

*) [.] denotes the Gauss symbol.

*then the Fourier series* (2. 3) *converges absolutely with probability* 1. *Moreover, the sample paths of the weakly stationary process* $\{X_T(t)\}$ *defined by* (2. 3) *are continuous with probability* 1.

*Proof.* Since inequalities

$$(2.\ 5)\qquad E\Big\{\sum_{n=-\infty}^{\infty}\Big|Z\Big(\frac{2\pi(n+1)}{T}\Big)-Z\Big(\frac{2\pi n}{T}\Big)\Big|\Big\}\leq\sum_{n=-\infty}^{\infty}\sqrt{E\Big|Z\Big(\frac{2\pi(n+1)}{T}\Big)-Z\Big(\frac{2\pi n}{T}\Big)\Big|^2}$$

$$=\sum_{n=-\infty}^{\infty}\sqrt{F\Big(\frac{2\pi(n+1)}{T}\Big)-F\Big(\frac{2\pi n}{T}\Big)}\leq\sum_{n=-\infty}^{\infty}\sqrt{\Big[\frac{T}{2\pi}\Big]+1}\sqrt{F(n+1)-F(n)}<\infty$$

hold by (2. 4) and Lemma 1, the series $\sum|Z(2\pi(n+1)/T)-Z(2\pi n/T)|$ converges with probability 1. Hence we have proved that the series (2. 3) converges absolutely with probability 1.

THEOREM 1. *Let* $F(\lambda)$ *be a spectral distribution function. If* (2. 4) *holds,* $F(\lambda)$ *satisfies Condition* (*C*).

*Proof.* Let $\{X(t)\}$ be a weakly stationary process with the spectral distribution function $F(\lambda)$ and have the spectral representation (2. 1) with $\{Z(\lambda)\}$, and let $\{X_T(t)\}$ be the process defined by (2. 3). Then we have

$$P\Big(\max_{|t|\leq A}|X_{2^{k+1}}(t)-X_{2^k}(t)|>\varepsilon_k\Big)\leq P\Big(\frac{\pi A}{2^k}\sum_{n=-\infty}^{\infty}\Big|Z\Big(\frac{\pi(n+1)}{2^k}\Big)-Z\Big(\frac{\pi n}{2^k}\Big)\Big|>\varepsilon_k\Big)$$

$$\leq\frac{\pi A}{2^k\varepsilon_k}E\Big\{\sum_{n=-\infty}^{\infty}\Big|Z\Big(\frac{\pi(n+1)}{2^k}\Big)-Z\Big(\frac{\pi n}{2^k}\Big)\Big|\Big\}\leq\frac{\pi A}{2^k\varepsilon_k}\sqrt{\frac{2^k}{\pi}+1}\sum_{n=-\infty}^{\infty}\sqrt{F(n+1)-F(n)}$$

by (2. 5). Hence if we put $\varepsilon_k=2^{-k/4}$, the series

$$\sum_k P\Big(\max_{|t|\leq A}|X_{2^{k+1}}(t)-X_{2^k}(t)|>\varepsilon_k\Big)$$

converges. Using Borel-Cantelli lemma, by the same method in Theorem 9 in [2], we can prove that $X_{2^k}(t)$ converges uniformly in $[-A, A]$ as $k\to\infty$ with probability 1. Hence the asserion is proved by Lemma 2.

COROLLARY 1. (T. Kawata) *If there exists a function* $g(u)>0$ *which satisfies the following conditions*:

(a) $g(u)\leq g(v)$ *and* $g(-u)\leq g(-v)$ *hold for* $0\leq u<v$;

(b) *it holds that*

$$(2.\ 6)\qquad \int\frac{1}{g(u)}du<\infty$$

*and*

$$\int g(\lambda)dF(\lambda) < \infty, \tag{2.7}$$

*then* $F(\lambda)$ *satisfies* (2. 4) *and hence Condition* (*C*).

*Proof.* By (2. 6) and (2. 7), we have inequalities

$$\Big(\sum_{|n|\geq 2}\sqrt{F(n+1)-F(n)}\,\Big)^2 \leq \sum_{n=1}^{\infty} g(n+1)\,(F(n+1)-F(n))\cdot \sum_{n=1}^{\infty}\frac{1}{g(n+1)}$$
$$+\sum_{n=-2}^{-\infty} g(n)\,(F(n+1)-F(n))\cdot \sum_{n=-2}^{-\infty}\frac{1}{g(n)}$$
$$\leq \int g(\lambda)dF(\lambda)\cdot \int \frac{1}{g(u)}\,du < \infty.$$

Hence the assertion follows from Theorem 1.

## 3. Necessary condition

We now discuss a necessary condition for any weakly stationary process with a given spectral distribution function $F(\lambda)$ to have a version with continuous sample paths. For this purpose, we shall constract a strictly stationary process with the given spectral distribution function $F(\lambda)$.

Corresponding to the given $F(\lambda)$, there exist finite measures $\sigma(1,\lambda)$ and $\sigma(-1,\lambda)$ on $\left[0, \frac{2\pi}{T}\right)$ and a non-negative function $\rho(\lambda)$ on $(-\infty,\infty)$ such that

$$dF\left(\lambda+\frac{2\pi n}{T}\right) = \rho\left(\lambda+\frac{2\pi n}{T}\right)d\sigma(\theta(n),\lambda) \quad \lambda\in\left[0, \frac{2}{T}\right), \tag{3.1}$$

where $\theta(n)=1$ if $n\geq 0$ and $=-1$ if $n<0$. Such measures $\sigma(1, \lambda)$ and $\sigma(-1,\lambda)$ and such a function $\rho(\lambda)$ are not unique. They can be given, for example, in the following forms:

$$\begin{aligned} d\sigma(1,\lambda) &= \sum_{n=0}^{\infty} dF\left(\lambda+\frac{2\pi n}{T}\right) \\ d\sigma(-1,\lambda) &= \sum_{n=-1}^{-\infty} dF\left(\lambda+\frac{2\pi n}{T}\right). \end{aligned} \tag{3.2}$$

Now we define a probability space $(\Omega, P)$. Set

$$\Omega = \left\{\omega=(\mu,\nu,x,y,\lambda)\,;\,\mu,\nu=\pm 1, \nu=\pm 1, 0\leq x<1,\ 0\leq y<2\pi,\ 0\leq\lambda<\frac{2\pi}{T}\right\}.$$

Define a probability measure $P(\mu,\nu,x,y,\lambda)$ on $\Omega$ by

$$dP(\mu, \nu, x, y, \lambda) = \frac{1}{4\pi\gamma} dx\ dy\ d\sigma(\nu, \lambda) \tag{3. 3}$$

with $\gamma = \sigma(1, [0, 2\pi/T)) + \sigma(-1, [0, 2\pi/T))$. We define a flow $\{S_t\}$ on $\Omega$ by

$$S_t(\mu, \nu, x, y, \lambda) = \left(\mu,\ \nu,\ x + \frac{t}{T} - \left[x + \frac{t}{T}\right],\ y + \lambda t - 2\pi\left[\frac{y + \lambda t}{2\pi}\right],\ \lambda\right).$$

It is obvious that the system $\{S_t\}$ of point transformations of $\Omega$ forms a group and each $S_t$ is measure preserving. Define a random variable $X(\omega)$, $\omega = (\mu, \nu, x, y, \lambda)$, by

$$X(\mu, \nu, x, y, \lambda) = \sqrt{\gamma}\mu e^{iy} \sum_{n=-\infty}^{\infty} \frac{1 + \nu\theta(n)}{2} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) e^{2\pi i n x},$$

(the convergence of this series is $L^2$-sence). With this $X(\omega)$, a strictly stationary process $\{X(t)\}$ is given by

$$X(t, \omega) = X(S_t\omega). \tag{3. 4}$$

Then it holds that

$$X(t, \omega) = \sqrt{\gamma}\ \mu e^{i(y+\lambda t)} \sum_{n=-\infty}^{\infty} \frac{1 + \nu\theta(n)}{3} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) e^{2\pi n i n(x+t/T)} \quad \text{a.e. } (dP). \tag{3. 5}$$

LEMMA 3. *The process $\{X(t)\}$ defined by (3. 4) is a weakly stationary process with mean zero and with the spectral distribution function $F(\lambda)$.*

*Proof.* It is easily seen that the mean of $X(t)$ is zero. We have

$$\begin{aligned} E\left\{X(t)\overline{X(s)}\right\} &= \sum_{\mu, \nu=\pm 1} \gamma\mu^2 \int_0^{\frac{2\pi}{T}} \int_0^{2\pi} \int_0^1 e^{i\lambda(t-s)} \sum_{n, m=-\infty}^{\infty} \left\{\frac{(1 + \nu\theta(n))(1 + \nu\theta(m))}{4}\right. \\ &\quad \left. \times e^{2\pi i(n-m)} e^{2\pi i(nt-ms)/T} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) \rho^{1/2}\left(\lambda + \frac{2\pi m}{T}\right)\right\} \\ &\quad \times \frac{1}{4\pi\gamma} dx\ dy\ d\sigma(\nu, \lambda) \\ &= \sum_{n=-\infty}^{\infty} \int_0^{\frac{2\pi}{T}} e^{i(\lambda+2\pi n/T)(t-s)} \rho\left(\lambda + \frac{2\pi n}{T}\right) d\sigma(\theta(n), \lambda) \\ &= \int e^{i\lambda(t-s)} dF(\lambda). \end{aligned}$$

Hence $\{X(t)\}$ is a weakly stationary process with the given spectral distribution function $F(\lambda)$.

LEMMA 4. *The stationary process* $\{X(t)\}$ *defined by* (3. 4) *has a version whose sample paths are continuous with probability* 1, *if and only if*

$$\sum_{n=0}^{\infty} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) < \infty \quad \text{a.e. } (d\sigma(1, \lambda)),$$

(3. 6)

$$\sum_{n=-1}^{-\infty} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) < \infty \quad \text{a.e. } (d\sigma(-1, \lambda))$$

*hold.*

*Proof.* Suppose that (3. 6) holds. Then the series (3. 4) converges absolutely for almost every $(\nu, \lambda)$ $(d\sigma(\nu, \lambda))$. Hence, for almost every $\omega$ $(dP)$, $X(t, \omega)$ is continuous in $t$. Conversely, if $X(t, \omega)$ has a version with continuous paths, then two processes (see (3. 5))

$$\sum_{n=0}^{\infty} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) e^{2\pi i n(x+t)/T} \text{ and } \sum_{n=-1}^{-\infty} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) e^{2\pi i n(x+t)/T}$$

have versions with continuous sample paths. Hence these Fourier series are regarded as those of some continuous functions of $t$ for almost every $(x, \lambda)$ $(dx\ d\sigma(1, \lambda))$ or $(dx\ d\sigma(-1, \lambda))$, respectively. Therefore the Fourier series

$$\sum_{n=0}^{\infty} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) e^{2\pi i n t/T} \left(\text{resp. } \sum_{n=-1}^{-\infty} \rho^{1/2}\left(\lambda + \frac{2\pi n}{T}\right) e^{2\pi i n t/T}\right) \tag{3. 7}$$

is a Fourier series of a continuous function for almost every $\lambda$ $(d\sigma(1, \lambda))$ (resp. $(d\sigma(-1, \lambda))$). By Fejér's theorem, the series (3. 7) is summable $(C. 1)$ at any point $t$. Since the Fourier coefficients are non-negative, the summability $(C. 1)$ at $t = 0$ implies the convergence of the series (3. 6).

By these lemmas, we can state the following theorem.

THEOREM 2. *If a distribution function* $F(\lambda)$ *satisfies Condition* $(C)$, *then* (3. 6) *holds for any* $\sigma(1, \lambda)$, $\sigma(-1, \lambda)$ *and* $\rho(\lambda)$ *which satisfy* (3. 1).

The following proposition is obvious but helpful in applications of our theorem.

PROPOSITION 1. *The convergence or the divergence of the series* (3. 6) *is independent of the choice of measures* $\sigma(1, \lambda)$, $\sigma(-1, \lambda)$ *and a function* $\rho(\lambda)$ *which satisfy* (3. 1).

## 4. Discussions

In this section we shall show that the conditions (2. 4), (3. 6) and $(C)$ are equivalent to each other under suitable assumptions.

PROPOSITION 2. *For some period $T$ and for some $\sigma(1,\lambda)$, $\sigma(-1,\lambda)$ and $\rho(\lambda)$ which satisfy* (3. 1), *set*

$$a_n = \operatorname*{ess\ sup}_{0\leq\lambda<2\pi/T} \rho\Big(\lambda+\frac{2\pi n}{T}\Big) \qquad (d\sigma(\theta(n),\lambda))$$

*and*

$$b_n = \operatorname*{ess\ inf}_{0\leq\lambda<2\pi/T} \rho\Big(\lambda+\frac{2\pi n}{T}\Big) \qquad (d\sigma(\theta(n),\lambda)).$$

*If*

$$\sum_{n=-\infty}^{\infty} (a_n^{1/2}-b_n^{1/2}) < \infty \tag{4. 1}$$

*holds, the following three conditions are equivalent*:

(i) $F(\lambda)$ *satisfies Condition* $(C)$;

(ii) (2. 4) *holds*;

(iii) (3. 6) *holds*.

*Proof.* By Theorem 1 and Theorem 2, we can see that (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii). Since

$$F\Big(\frac{2\pi(n+1)}{T}\Big)-F\Big(\frac{2\pi n}{T}\Big)=\int_0^{\frac{2\pi}{T}} \rho\Big(\lambda+\frac{2\pi n}{T}\Big)\,d\sigma(\theta(n),\lambda)$$

holds, we have

$$a_n\sigma(\theta(n),\,[0,2\pi))\geq F\Big(\frac{2\pi(n+1)}{T}\Big)-F\Big(\frac{2\pi n}{T}\Big)\geq b_n\sigma(\theta(n),\,[0,2\pi)).$$

Hence we have

$$\sum_{n=-\infty}^{\infty}\sqrt{F\Big(\frac{2\pi(n+1)}{T}\Big)-F\Big(\frac{2\pi n}{T}\Big)}\leq\sum_{n=-\infty}^{\infty}a_n^{1/2}=\sum_{n=-\infty}^{\infty}(a_n^{1/2}-b_n^{1/2})+\sum_{n=0}^{\infty}b_n^{1/2}+\sum_{n=-1}^{-\infty}b_n^{1/2}\,.$$

Since

$$\sum_{n=0}^{\infty}b_n^{1/2}\leq\sum_{n=0}^{\infty}\rho^{1/2}\Big(\lambda+\frac{2\pi n}{T}\Big)<\infty \quad \text{a.e.} \quad (d\sigma(1,\lambda))$$

and

$$\sum_{n=-1}^{-\infty}b_n^{1/2}\leq\sum_{n=-1}^{-\infty}\rho^{1/2}\Big(\lambda+\frac{2\pi n}{T}\Big)<\infty \quad \text{a.e.} \quad (d\sigma(-1,\lambda))$$

hold under the condition (3. 6), (2. 4) follows imediately.

PROPOSITION 3. *Let* $\sigma(1,\lambda)$, $\sigma(-1,\lambda)$ *and* $\rho(\lambda)$ *satisfy* (3. 1) *for some period* $T$. *If there exists a non-negative function* $h(\lambda)$ *such that*

$$c_1 < \varliminf_{|\lambda|\to\infty} \frac{\rho(\lambda)}{h(\lambda)} \leq \varlimsup_{|\lambda|\to\infty} \frac{\rho(\lambda)}{h(\lambda)} < c_2,\ 0 < c_1 < c_2, \tag{4. 2}$$

*where* $h(\lambda)$ *is monotone non--increasing in* $[0,\infty)$ *and non-decreasing in* $(-\infty,0]$. *Then the following conditions are equivalent*:

(i) $F(\lambda)$ *satisfies Condition* $(C)$;

(ii) (2. 4) *holds*;

(iii) (3. 6) *holds*;

(iv) $\int \rho^{-1/2}(\lambda)\, dF(\lambda) < \infty$;

(v) $\sigma(\nu,[0,\pi)) \int_{\lambda>A} \rho^{1/2}(\nu\lambda)\, d\lambda < \infty$, $\nu=\pm 1$, *for sufficiently large* $A$;

(vi) *there exists a function* $g(u)$ *which satisfies the conditions* (a) *and* (b) *in Corollary* 1.

*Proof.* From (4. 2), it follows that

$$c_1 h(\lambda) < \rho(\lambda) < c_2 h(\lambda),\quad |\lambda| \geq \frac{2\pi N}{T} \tag{4. 3}$$

for sufficiently large number $N$. Set $I_1(\nu) = \sum_{n\geq N+1} h^{1/2}\left(\frac{2\pi\nu n}{T}\right)$ and $I_2(\nu) = \sum_{n\geq N} h^{1/2}\left(\frac{2\pi\nu n}{T}\right)$, $\nu=\pm 1$. Then it follows, from (3. 1) and (4. 3), that

$$c_1 I_1(\nu) \leq \sum_{n\geq N} \rho^{1/2}\left(\lambda + \frac{2\pi\nu n}{T}\right) \leq c_2 I_2(\nu),\quad \nu=\pm 1,$$

$$\frac{2\pi}{T} c_1 I_1(\nu) \leq c_1 \int_N^\infty h^{1/2}(\nu\lambda)\, d\lambda \leq \int_N^\infty \rho^{1/2}(\nu\lambda)\, d\lambda \leq$$

$$\leq c_2 \int_N^\infty h^{1/2}(\nu\lambda)\, d\lambda \leq \frac{2\pi}{T} c_2 I_2(\nu),\quad \nu = \pm 1,$$

and that

$$\sigma\left(\nu, \frac{2\pi}{T}\right)\right) c_1 I_1(\nu) \leq c_1 \int_N^\infty h^{-1/2}(\nu\lambda)\, dF(\nu\lambda) \leq \int_N^\infty \rho^{-1/2}(\nu\lambda)\, dF(\nu\lambda) \leq$$

$$\leq c_2 \int_N^\infty h^{-1/2}(\nu\lambda)\, dF(\nu\lambda) \leq \sigma\left(\nu, \left[0, \frac{2\pi}{T}\right)\right) c_2 I_2(\nu),\quad \nu = \pm 1.$$

Hence the equivalence of (iii), (iv) and (v) is obvious. Moreover, this fact implies that the function $g(u)$ defined by

$$g(u) = \begin{cases} h^{-1/2}(u) & \text{for } \nu u > 0 \quad \text{if } \sigma\left(\nu, \left[0, \frac{2\pi}{T}\right)\right) > 0, \\ 1 + u^2 & \text{for } \nu u > 0 \quad \text{if } \sigma\left(\nu, \left[0, \frac{2\pi}{T}\right)\right) = 0, \end{cases}$$

$\nu = \pm 1$, satisfies the conditions (a) and (b) in Corollary 1, if (v) holds. Since the proof of (vi) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) follows from Corollary 1 and Theorem 2. Thus our assertion has been proved.

Now let us give some examples.

EXAMPLE 1. Let $F(\lambda)$ be a spectral distribution function of some weakly stationary process with a period $T$ (i.e. the measure $dF(\lambda)$ is supported by the discrete set $\{2\pi n/T;\ n = 0, \pm 1, \pm 2, \cdots\}$). Then $F(\lambda)$ satisfies Condition $(C)$ if and only if (2. 4) holds. (For the proof we refer Proposition 2.)

EXAMPLE 2. Suppose that $F(\lambda)$ has the density $f(\lambda)$ with respect to the Lebesgue measure. Then we may set $d\sigma(1, \lambda) = d\sigma(-1, \lambda) = d\lambda$ and $\rho(\lambda) = f(\lambda)$.

If, in particular, $\rho(\lambda) = f(\lambda)$ satisfies (3. 7) with some $h(\lambda)$, then $F(\lambda)$ satisfies Condition $(C)$ if and only if

$$\int f^{1/2}(\lambda)\, d\lambda < \infty.$$

EXAMPLE 3. Let $F(\lambda)$ satisfy (4. 2) with some $\sigma(1, \lambda)$, $\sigma(-1, \lambda)$, $\rho(\lambda)$ and $h(\lambda)$. Suppose that $h(\lambda)$ is a function of the form

$$h(\lambda) = \{|\lambda| \cdot \log|\lambda| \cdot \log_{(2)}|\lambda| \cdot \cdots \cdot \log_{(n)}|\lambda| \cdot (\log_{(n+1)}|\lambda|)^{1+\varepsilon}\}^{-1}$$

for large $|\lambda|$, where $\log_{(n+1)}\lambda = \log(\log_{(n)}\lambda)$ and $\log_{(1)}\lambda = \log\lambda$. Then $F(\lambda)$ satisfies Condition $(C)$ if $\varepsilon > 0$ and dose not if $\varepsilon \leq 0$

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