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A REMARK ON THE MOYAL'S CONSTRUCTION OF MARKOV PROCESSES

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To Professor Katuji Ono on the occasion of his 60th birthday.

§ 1. **Result.** In the author's previous paper [3], we used Theorem 1 of the present paper to assure the existence of a signed branching Markov process with age satisfying given conditions in [3]. The purpose of this paper is to give a proof of Theorem 1.

Let $X = \{X_t, \zeta, \mathscr{B}_t, P_X;\ x \in E\}$ be a right continuous Markov process[1] on a locally compact Hausdorff space E satisfying the second axiom of countability, and Ω be the sample space of X. A non-negative function $\sigma(\omega)$ $(\omega \in \Omega)$ is called a $\mathscr{B}_t$-Markov time if it holds that for each $t \geqq 0$

$$\{\omega \in \Omega;\ \sigma(\omega) \leqq t < \zeta(\omega)\} \in \mathscr{B}_t.$$

For any Markov time $\sigma, \mathscr{B}_\sigma$ is defined as the collection of the sets A such that for any $t \geqq 0$

$$A \in \bigvee_{t \geqq 0} \mathscr{B}_t \text{ and } A \cap \{\omega;\ \sigma(\omega) \leqq t < \zeta(\omega)\} \in \mathscr{B}_t,$$

where $\bigvee_{t \geqq 0} \mathscr{B}_t$ denotes the σ-algebra generated by the sets of $\mathscr{B}_t$, $t \geqq 0$. Let $C(E)$ be the space of all bounded continuous functions on E. A right continuous Markov process X is said to be strong Markov if it holds that for any Markov time σ, $t \geqq 0$, $x \in E$, $f \in C(E)$, and $A \in \mathscr{B}_\sigma$,

$$E_x[f(X_{t+\sigma});\ A \cap \{\sigma < \zeta\}] = E_x[E_{X_\sigma}[f(X_t)];\ A \cap \{\sigma < \zeta\}],$$

where $E_x[\,\cdot\,; A]$ expresses the integral over A by P_x.

Let $\chi_0(t, x, \cdot)$ and $\Psi(x;\ t, \cdot)$ be substochastic measures on the σ-algebra $\mathscr{B}(E)$[2], and suppose that $\chi_0(\cdot, \cdot, B)$ and $\Psi(\cdot\,; \cdot, B)$ are Borel measurable

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1) A Markov process is said to be right continuous if their almost all sample paths are right continuous in $t \geqq 0$.

2) $\mathscr{B}(\mathscr{X})$ denotes the class of Borel set on the topological space $\mathscr{X}$.

functions of $(t, x) \in [0, \infty) \times E$ for any fixed $B \in \mathscr{B}(E)$. A pair of χ_0 and Ψ is said to be satisfied Moyal's $\chi_0\Psi$-condition if they satisfy the following conditions[3]:

$$(1) \quad \chi_0(t+s, x, B) = \int_E \chi_0(t, x, dy)\chi_0(s, y, B), \quad \chi_0(0, x, E) = 1,$$

$$(2) \quad \lim_{t\to\infty} \Psi(x; t, E) = 1 - \lim_{t\to\infty} \chi_0(t, x, E)$$

$$(3) \quad \Psi(x; t+s, B) = \Psi(x; t, B) + \int_E \chi_0(t, x, dy)\Psi(y; s, B)$$

$$(4) \quad \Psi(x; t, E) \text{ is continuous in } t \quad t \geqq 0, \quad x \in E, \quad B \in \mathscr{B}(E).$$

Now, suppose that the $\chi_0\Psi$-condition is satisfied for a given pair of χ_0 and Ψ_0. By virtue of (3), $\Psi(x; t, B)$ is monotone nondecreasing in t, and hence it determines a measure $\Psi(x; dt, dy)$ on $\mathscr{B}([0, \infty)\times E)$. Using this measure, we shall define measures $\Psi_r(x; \cdot, \cdot)$ and $\chi_r(t, x, \cdot)$ as follows:

$$\Psi_1(x; dt, dy) = \Psi(x; dt, dy),$$

$$(5) \quad \Psi_{r+1}(x; dt, dy) = \int_0^t\int_E \Psi_r(x; ds, dz)\Psi(z; d(t-s), dy),$$

$$\chi_r(t, x, dy) = \int_0^t\int_E \Psi_r(x; ds, dz)\chi_0(t-s, z, dy),$$

$$r \geqq 1, \quad t \geqq 0, \quad B \in \mathscr{B}(E).$$

Further we set

$$(6) \quad \Psi_r(x; t, dy) = \int_0^t \Psi_r(x; ds, dy), \quad r \geqq 1.$$

Then we have

THEOREM. *(J.E. Moyal) If the $\chi_0\Psi$-condition is satisfied, then it holds that for any* $t, s \geqq 0$, $x \in E$, and $B \in \mathscr{B}(E)$,

$$(7) \quad \Psi_{r+r'}(x; dt, B) = \int_0^t\int_E \Psi_r(x; ds, dy)\Psi_{r'}(y; d(t-s), B), \quad r, r' \geqq 1,$$

$$(8) \quad \chi_{r+r'}(t, x, B) = \int_0^t\int_E \Psi_r(x; ds, dy)\chi_{r'}(t-s, y, B), \quad r \geqq 1, \quad r' \geqq 0,$$

$$(9) \quad \chi_r(t+s, x, B) = \sum_{r'=0}^{r} \int_E \chi_{r'}(t, x, dy)\chi_{r-r'}(s, y, B), \quad r \geqq 0,$$

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3) J.E. Moyal [2] defined the $\chi_0\Psi$-condition for non-stationary Markov processes. The condition stated here is the one for stationary case with an additional condition (4).

(10) $\sum_{r=0}^{\infty} \chi_r(t, x, E) = 1 - \lim_{r \to \infty} \Psi_r(x, t, E).$

Moreover, if we set

(11) $\chi(t, x, B) = \sum_{r=0}^{\infty} \chi_r(t, x, B), \quad t \geqq 0, \quad x \in E, \quad B \in \mathscr{B}(E),$

then χ *satisfies so-called Chapman-Kolmogorov's equation, i.e.,*

(12) $\chi(t + s, x, B) = \int_E \chi(t, x, dy)\chi(s, y, B),$

and further χ *is the minimal non-negative solution of the equation:*

(13) $\chi(t, x, B) = \chi_0(t, x, B) + \int_0^t \int_E \Psi(x; ds, dy)\chi(t - s, y, B).$

In addition, χ *is the unique solution of* (13) *if it holds that for each* $t \geqq 0$

(14) $\lim_{r \to \infty} \Psi_r(x; t, E) = 0.$

According to Kolmogorov's extension theorem, (1) and (12) imply that there exist two Markov process X and X^0 whose transition functions are given by χ and χ_0 respectively. We shall consider the relation between X and X^0.

Let $E \cup \{\Delta\}$ be the one-point compactification of E and set

$$C_0(E) = \{f; f \in C(E) \text{ and } \lim_{x \to \Delta} f(x) = 0\},$$

$$\| f \| = \sup \{ |f(x)| ; x \in E\},$$

$$T_t^{(r)} f(x) = \int_E \chi_r(t, x, dy) f(y), \quad r \geqq 0, \quad f \in C_0(E),$$

and

$$T_t f(x) = \int_E \chi(t, x, dy) f(y), \qquad f \in C_0(E).$$

Then (1) and (12) imply $T_{t+s}^{(0)} = T_t^{(0)} T_s^{(0)}$ and $T_{t+s} = T_t T_s$ if they act on $C_0(E)$. Now we can state

THEOREM 1. *Let the semi-group* $T_t^{(0)}$, $t \geqq 0$, *be strongly continuous on* $C_0(E)$ *with respect to the norm* $\| \;\; \|$, *and assume that for any* $r \geqq 1$, $T_t^{(r)}$ *maps* $C_0(E)$ *into itself and it holds that*

(15) $\lim_{t\to 0} \|T_t^{(r)} f\| = 0, \qquad r \geqq 1, \quad f \in C_0(E).$

Then it holds that (i) *there exists a right and quasi-left continuous*[4] *strong Markov process* $X = \{X_t, \zeta, \mathscr{B}_t, P_x;\ x \in E\}$ *corresponding to the semi-group* T_t, (ii) *there exists a Markov time* τ *of* X_t *such that the killed process* $X^0 = \{X_t^0, \zeta^0, \mathscr{B}_t^0, P_x^0;\ x \in E\}$ *of* X *at time* τ[5] *corresponds to the semi-group* $T_t^{(0)}$, (iii) *setting*

$$\tau_0 = 0, \quad \tau_1 = \tau, \quad \tau_{r+1} = \tau_r + \theta_{\tau_r}\tau^{6)}, \quad r \geqq 1,$$

we have

(16) $P_x(X_t \in B, \tau_r \leqq t < \tau_{r+1}) = \chi_r(t, x, B),$

(17) $P_x(X_{\tau_r} \in B, \tau_r \in dt) = \Psi_r(x; dt, B),$

$$x \in E, \quad B \in \mathscr{B}(E), \quad t \geqq 0, \quad r \geqq 0.$$

§ 2. **Proof.** Let $N = \{0, 1, 2, \cdots\}$ and S be the product space $E \times N$ where the topology of S is introduced in a natural way. Then S is a locally compact Hausdorff space satisfying the second axiom of countability. We define a measure $P(t, (x, p), \cdot)$[7] on $\mathscr{B}(S)$ by

(18) $$P(t, (x, p), (B, q)) = \begin{cases} \chi_{q-p}(t, x, B), & \text{if } q \geqq p, \\ 0, & \text{otherwise,} \end{cases}$$

$$(x, p) \in S, \quad t \geqq 0, \quad B \in \mathscr{B}(E), \quad p, q \in N.$$

Then we have

LEMMA 1. *For* $t, s \geqq 0$, $(x, p) \in S$, $A \in \mathscr{B}(S)$, *it holds that*

$$P(t + s, (x, p), A) = \int_S P(t, (x, p), d(y, r)) P(s, (y, r), A).$$

Proof. It suffices to prove the above equality for $A = (B, q)$ where $q \geqq p$. By the definitions of $P(t, (x, p), \cdot)$ and (9), we have

4) A Markov process $X = \{X_t, \zeta, \mathscr{B}_t, P_x;\ x \in E\}$ is said to be quasi-left continuous if it holds that for any increasing sequence τ_r of Markov times,

$$P_x(\lim_{r\to\infty} X_{\tau_r} = X_\tau, \tau < \zeta) = P_x(\tau < \zeta),$$

where

$$\tau(\omega) = \lim_{r\to\infty} \tau_r(\omega).$$

5) The killed process X^0 of X at time τ means that

$$X_t^0(\omega) = \begin{cases} X_t(\omega), & \text{if } t < \tau, \\ \Delta, & \text{if } t \geqq \tau. \end{cases}$$

6) θ_t denotes the shift operator.

7) $P(\cdot, \cdot, (B, q))$ is $\mathscr{B}([0, \infty) \times S)$-measurable.

$$
\begin{aligned}
P(t+s,(x,p),(B,q)) &= \chi_{q-p}(t+s,x,B) \\
&= \sum_{r=0}^{q-p}\int_E \chi_r(t,x,dy)\chi_{q-p-r}(s,y,B) \\
&= \sum_{r=0}^{q-p}\int_E P(t,(x,p);(dy,p+r))P(s,(y,p+r);(B,q)) \\
&= \int_S P(t,(x,p),d(y,r))P(s,(y,r);(B,q)),
\end{aligned}
$$

as was to be proved. Q.E.D.

According to Lemma 1, there exists a Markov process $Y=\{Y_t=(X_t,N_t), \zeta, \mathscr{B}_t, P_{(x,p)}; (x,p)\in S\}$ with transition function $P(t,(x,p),\cdot)$ where $\mathscr{B}_t$ is the σ-algebra generated by sets of the form $\{Y_s\in A; s\leqq t, A\in\mathscr{B}(S)\}$. Since it follows from (18), (11), and (13) that for any $t,h\geqq 0$

$$P_{(x,p)}(N(t)>N(t+h))=0,$$

and

$$
\begin{aligned}
&P_{(x,p)}(N(t)<N(t+h)) \\
&= \sum_{r=0,s=1}^{\infty}\int_E \chi_r(t,x,dy)\chi_s(h,y,E) \\
&= \sum_{s=1}^{\infty}\int_E \chi(t,x,dy)\chi_s(h,y,E) \\
&= \int_E \chi(t,x,dy)\{\chi(h,y,E)-\chi_0(h,y,E)\} \\
&= \int_E \chi(t,x,dy)\int_0^h\int_E \Psi(y;du,dz)\chi(h-u,z,E) \\
&\longrightarrow 0 \quad \text{as} \quad h\longrightarrow 0,
\end{aligned}
$$

there exists a version of Y in which N_t is right continuous in t. So we take this version as Y.

Now let us consider $\chi_0(t,x,dy)$. As was stated already, χ_0 defines a Markov process $X^0=\{X_t^0,\zeta^0,\mathscr{B}_t^0,P_x^0; x\in E\}$ on E. Let us denote its sample space by $\Omega^0=\{\omega^0=\omega^0(t); \omega^0(t)$ is a mapping of $[0,\zeta^0)$ to $E\}$. Next we consider a function space $\hat{\Omega}_r$ which is a kind of copy of shifted Ω_0. This means that

$\hat{\Omega}_r=\{\hat{\omega}=(\hat{\omega}_1(t),\hat{\omega}_2(t)); \hat{\omega}$ is a mapping of $[\alpha_r,\beta_r)$ to $E\times\{r\}$ where $0\leqq\alpha_r(\hat{\omega})\leqq\beta_r(\hat{\omega})\leqq\infty$ and they may vary with $\hat{\omega}\}$, and, for each $\hat{\omega} \in \hat{\Omega}_r$, there corresponds one and only one $\omega^0 \in \Omega^0$, such that the graph $\{(t, \omega^0(t)); 0 \leqq t < \zeta^0(\omega^0)\}$ is identical to $\{(t, \hat{\omega}(t+\alpha_r));$ $0 \leqq t < \beta_r(\hat{\omega}) - \alpha_r(\hat{\omega})\}$. Let $\hat{\mathscr{F}}_r$ be the algebra generated by cylinder sets of the following type

$$
\begin{aligned}
&\hat{B} = \{\hat{\omega} \in \hat{\Omega}_r;\ t_0 \leqq \alpha_r(\hat{\omega}) < t_1, \hat{\omega}_1(\alpha_r(\hat{\omega})) \in B_0, \hat{\omega}_1(t_i) \in B_i,\ \ i = 1, 2, \cdots, n\} \\
&\qquad\qquad 0 \leqq t_0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n, \\
&\qquad\qquad B_i \in \mathscr{B}(E),\ \ i = 0, 1, 2, \cdots, n,\ \ n = 0, 1, 2, \cdots,
\end{aligned}
\tag{19}
$$

and define a finitely additive measure $\nu_x(\cdot)$ on $\hat{\mathscr{F}}_r$ by

$$
\nu_x(\hat{B}) = \int_{t_0}^{t_1}\int_{B_0} \Psi_r(x; dt, dy) P_y^0(X_{t_i - t}^0 \in B_i,\ \ i = 1, 2, \cdots, n). \tag{20}
$$

Then we have

LEMMA 2. *$\nu_x(\cdot)$ can be extended to a measure on the σ-algebra $\mathscr{B}_r$ generated by $\hat{\mathscr{F}}_r$.*

Remark. Consider a Markov time τ_r defined by

$$
\tau_r(\omega) = \inf\{t;\ N_t(\omega) = N_0(\omega) + r\},
$$

where N_t is the right continuous second coordinate of $Y_t = (X_t, N_t)$. If the distribution of the joint variable (τ_r, X_{τ_r}) is given by $\Psi_r(x, dt, dy)$, then $\nu_x(\cdot)$ is supposed to be the restricted measure of $P_{(x,0)}$ on $E \times \{r\}$. So intuitively, Lemma 2 is clear.

Proof. The proof is given by the same way as the construction of product measure. It suffices to prove that if a decreasing sequence $\{\hat{B}_n\} \subset \hat{\mathscr{F}}_r$, satisfies

$$
\nu_x(\hat{B}_n) \geqq c > 0, \qquad n = 1, 2, 3, \cdots,
$$

where c is a constant, then we have

$$
\bigcap_{n=1}^{\infty} \hat{B}_n \neq \phi.
$$

Since $\Psi_r(x; \cdot, E)$ is a finite measure on $[0, \infty)$,

$$
\nu_x(\{\hat{\omega};\ \alpha_r(\hat{\omega}) \geqq t\}) = \int_t^{\infty} \Psi(x; dt, E)
$$

tends to zero as t tends to infinity. Therefore, without loss of generality, we may assume that there exists $T > 0$ such that

$$\hat{B}_n \subset \{\hat{\omega};\, 0 \leqq \alpha_r(\hat{\omega}) < T\}, \qquad n = 1, 2, 3, \cdots.$$

Now let us express $\hat{B}_n$ in a form

$$(21) \quad \hat{B}_n = \sum_{j=1}^{k_n} \{\hat{\omega};\, t_{j0}^{(n)} \leqq \alpha_r(\hat{\omega}) < t_{j1}^{(n)},\ \hat{\omega}_1(\alpha_r(\hat{\omega})) \in B_{j0}^{(n)},$$
$$\hat{\omega}_1(t_{ji}^{(n)}) \in B_{ji}^{(n)},\quad i = 1, 2, \cdots, n_j\}^{8)}, \quad n = 1, 2, 3, \cdots,$$

where the following are assumed to be satisfied.

$$t_{j1}^{(n)} \leqq T,\ \ j = 1, 2, \cdots, k_n,\ \ n \geqq 1,$$
$$t_{ji}^{(n)} \leqq t_{ji+1}^{(n)},\ \ i = 0, 1, 2, \cdots, n_j - 1,\ \ n \geqq 1,$$
$$[t_{j0}^{(n)}, t_{j1}^{(n)}] \times B_{j0}^{(n)} \cap [t_{k0}^{(n)}, t_{k1}^{(n)}] \times B_{k0}^{(n)} = \phi \ \text{ if } \ j \neq k, n \geqq 1,$$

and for any n and j there exists j_0 such that

$$[t_{j0}^{(n+1)}, t_{j1}^{(n+1)}) \times B_{j0}^{(n+1)} \subset [t_{j_00}^{(n)}, t_{j_01}^{(n)}) \times B_{j_00}^{(n)}.$$

Set

$$C_j^{(n)} = \Big\{(t, y);\, t_{j0}^{(n)} \leqq t < t_{j1}^{(n)}, y \in B_{j0}^{(n)} \text{ and}$$
$$P_y^0(X^0_{t_{ji}^{(n)}-t} \in B_{ji}^{(n)},\ \ i = 1, 2, \cdots, n_j) > \frac{c}{2}\Big\}^{9)},$$
$$D_j^{(n)} = [t_{j0}^{(n)}, t_{j1}^{(n)}) \times B_{j0}^{(n)} - C_j^{(n)}.$$

Then we can see

$$\sum_{j=1}^{k_n} C_j^{(n)} \downarrow$$

and

$$\Psi_r(x; \sum_{j=1}^{k_n} C_j^{(n)}) > \frac{c}{2} > 0.$$

Accordingly there exist (t_0, y_0) and j_n such that

$$(22) \quad (t_0, y_0) \in C_{j_n}^{(n)},\ \ n = 1, 2, 3, \cdots,$$

which means

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8) For the set $\{\hat{\omega};\, \beta_r(\hat{\omega}) \leqq t\}$, we used the notation $\{\hat{\omega};\, 0 \leqq \alpha_r(\hat{\omega}) < t,\ \hat{\omega}_1(\alpha_r(\hat{\omega})) \in E,\ \hat{\omega}_1(t) \in \phi\}$. The last funny expression $\hat{\omega}_1(t) \in \phi$ means $\hat{\omega}_1(t)$ is not defined at t.

9) If $B = \phi$, $P_x^0(X_t \in B)$ is regarded as $1 - P_x^0(X_t \in E)$.

$$P_{y_0}^{(0)}(X^0_{t_{j_n i}-t_0} \in B^{(n)}_{j_n i},\ i=1,2,\cdots,n_{j_n}) > \frac{c}{2} > 0.$$

By the monotonicity of $\hat{B}_n$, the events in the above parentheses are monotone non-increasing. So we can take ω^0 such that for all $n \geqq 1$

$$(23)\quad X^0_{t_{j_n i}-t_0}(\omega^0) \in B^{(n)}_{j_n i},\ i=1,2,3,\cdots,n_{j_n}.$$

If we put

$$\alpha_r(\hat{\omega}) = t_0,\ \beta_r(\hat{\omega}) = t_0 + \zeta^0(\omega^0),\ \hat{\omega}_1(t_0) = y_0$$

and

$$\hat{\omega}(t+t_0) = (\omega^0(t), r),\ 0 \leqq t < \zeta^0(\omega^0),$$

then (21), (22) and (23) show

$$\bigcap_{n=1}^{\infty} \hat{B}_n \ni \hat{\omega},$$

as was to be proved. Q.E.D.

Now we return to the process $Y = \{Y_t = (X_t, N_t), \zeta, \mathscr{B}_t, P_{(x,p)}; (x,p) \in S\}$. Since N_t is right continuous, τ_r defined by

$$\tau_r(\omega) = \inf\{t; N_t(\omega) = N_0(\omega) + r\},$$

are $\mathscr{B}_t$-Markov times. Then we have

LEMMA 3. *Let X^0 be a Markov process on E corresponding to the transition function $\chi_0(t,x,\cdot)$. If X^0 is right continuous, Y has a right continuous version and, for this version, we have*

$$(24)\quad P_{(x,p)}(Y_t \in (B, p+r)) = \chi_r(t,x,B),$$

$$(25)\quad P_{(x,p)}(Y_{\tau_{r+1}} \in (B, p+r+1), \tau_{r+1} \in dt) = \Psi_{r+1}(x; dt, B)$$
$$B \in \mathscr{B}(E),\ r \geqq 0.$$

Proof. By (5), (18) and (20), we can see that for $r \geqq 1$,

$$(26)\quad P_{(x,p)}(Y_{t_i} \in (B_i, p+r), i=1,2,\cdots,n) = \nu_x(\{\hat{\omega}; 0 \leqq \alpha_r(\hat{\omega}) < t_1, \hat{\omega}_1(t_i) \in B_i,$$
$$i=1,2,\cdots,n\}).$$

Hence $P_{(x,p)}$ defines a measure on the space of sub-trajectories of Y_t in the time interval $[\tau_r, \tau_{r+1})$ which is equivalent to ν_x. On the other hand,

$\nu_x(\cdot)$ is a measure on $\hat{\mathscr{B}}_r$ which is obtained from the sample space of X^0 by shift of starting time point. So we may consider that on the time interval $[\tau_r, \tau_{r+1})$, Y_t has the same continuity property with X^0. Since $r \geqq 1$ is arbitrary, we may regard that the right continuity of X^0 implies the right continuity of Y_t on $[\tau_1, \zeta)$. Evidently Y_t restricted on $[0, \tau_1)$ is equivalent to X^0, and hence we can have a right continuous version of Y_t. Furthermore, the event in parentheses of left hand side of (25) is measurable if Y_t is right continuous. Then the definition of ν_x and (26) implies

$$P_{(x,p)}(\tau_r(\omega) \in dt, X_{\tau_r}(\omega) \in (B, p+r)) = \nu_x(\{\hat{\omega}; \alpha_r(\hat{\omega}) \in dt, \hat{\omega}(\alpha_r) \in B\})$$
$$= \Psi_r(z; dt, B),$$

which proves (25). Since (24) is obtained from (18) we have proved the lemma. Q.E.D.

Now Theorem 1 is proved easily as follows.

Proof of Theorem 1. Since $T_t^{(0)}$ is strongly continuous on $C_0(E)$, by the general theory of Markov processes[10], a Markov process $X^0 = \{X_t^0, \zeta^0, \mathscr{B}_t^0, P_x^0; x \in E\}$ corresponding to $T_t^{(0)}$ can be considered to be right continuous. Accordingly, by Lemma 3, we may regard Y_t is right continuous.

Now let V_t be the semi-group on $C_0(S)$ induced by Y_t and $g \in C_0(S)$[11]. Then we have

$$\begin{aligned} V_t g(x,p) - g(x,p) &= \sum_{r=0}^{\infty} \int_E P(t,(x,p),(dy,p+r)) g(y,p+r) - g(x,p) \\ &= \int_E \chi_0(t,x,dy) g(y,p) - g(x,p) \\ &\quad + \sum_{r=1}^{\infty} \int_E \chi_r(t,x,dy) g(y,p+r). \end{aligned} \tag{27}$$

Since $g(x,p)$ belongs to $C_0(S), g(x,p)$ tends to zero uniformly in x as p tends to infinity. Furthermore the assumption on $T_t^{(r)}$ implies

$$\Big\| \sum_{r=1}^{\infty} \int_E \chi_r(t,x,dy) g(y,p+r) \Big\| \longrightarrow 0 \text{ as } t \longrightarrow 0.$$

Then we can see from (27) and the assumption on $T_t^{(0)}$ that V_t is strongly

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10) cf. [1] Theorem 3.14, p. 104.

11) $g(x,n)$ belongs to $C_0(S)$ if it holds that $g(\cdot,n) \in C_0(E)$ for any fixed $n \in N$ and $g(x,n)$ tends to zero, uniformly in x, when n tends to infinity.

continuous on $C_0(S)$. Therefore we may consider that Y is a right con tinuous and quasi-left continuous[12] strong Markov process.

Now let Ω^0 be a sample space of the process X_t^0, and Ω^i $(i=1,2,3,\cdots)$ be infinitely many copies of Ω^0. Let us set

$$\tilde{\Omega}=\prod_{i=0}^{\infty}\Omega^i,$$

and, for any $\tilde{\omega}=(\omega^0,\omega^1,\cdots,\omega^i,\cdots)\in\tilde{\Omega}$, set

$$\sigma_0(\tilde{\omega})=0,\quad \sigma_r(\tilde{\omega})=\sum_{i=0}^{r-1}\zeta^0(\omega^i),\qquad r\geqq 1,$$

$$\tilde{X}_t(\tilde{\omega})=\tilde{\omega}(t)=\omega^r(t-\sum_{i=0}^{r-1}\zeta^0(\omega^i))\ \text{ if }\ \sigma_r(\tilde{\omega})\leqq t<\sigma_{r+1}(\tilde{\omega})^{13)},$$

$$\tilde{\zeta}(\tilde{\omega})=\lim_{r\to\infty}\sigma_r(\tilde{\omega}).$$

Further set

$$\theta_t\tilde{\omega}=(\theta_{t-\sigma_r(\tilde{\omega})}\omega^r,\omega^{r+1},\cdots)\ \text{ if }\ \sigma_r(\tilde{\omega})\leqq t<\sigma_{r+1}(\tilde{\omega}).$$

Then we consider the σ-algebra $\tilde{\mathscr{B}}_t$ generated by the cylinder sets of the form of

$$\{\tilde{\omega}\in\tilde{\Omega};\ \tilde{\omega}(t)\in B,\sigma_r(\tilde{\omega})\leqq t\},\quad B\in\mathscr{B}(E),\quad r\geqq 0,$$

and set

$$\tilde{\mathscr{B}}=\bigvee_{t\geqq 0}\tilde{\mathscr{B}}_t.$$

If we consider the correspondence of

$$\{\tilde{\omega}\in\tilde{\Omega};\ \tilde{\omega}(t)\in B,\sigma_r(\tilde{\omega})\leqq t<\sigma_{r+1}(\tilde{\omega})\}$$

and

$$\{\omega\in\Omega;\ Y_t(\omega)=(\omega_1(t),\omega_2(t))\in(B,r),\omega_2(0)=0\},$$

then it induces the correspondence between $\tilde{\mathscr{B}}_t$ and $\mathscr{F}_t$ defined by

$$\mathscr{F}_t=\mathscr{B}_t\cap\{\omega\in\Omega;\ N_0(\omega)=0\}.$$

So, $\tilde{P}_x(\cdot)$ defined by

$$\tilde{P}_x(\tilde{A})=P_{(x,0)}(A),$$

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12) cf. [1] Theorem 3.14, p. 104.

13) To define θ_t completely, we have to consider an extra point Δ as a grave of $\tilde{X}$ and an $\tilde{\omega}$ such that $\tilde{\omega}(t)=\Delta$, $t\geqq 0$.

where $A \in \mathscr{F}_t$ corresponds to $\tilde{A} \in \tilde{\mathscr{B}}_t$, defines a measure on $\tilde{\mathscr{B}}$. Further, setting $f(x, p) = \tilde{f}(x)$ for any bounded continuous function $\tilde{f}$ on E, we can see that

$$\begin{aligned} \tilde{E}_x[\tilde{f}(\tilde{X}_t); t < \tilde{\zeta}] &= \int_{\Omega} \tilde{f}(\tilde{X}_t(\tilde{\omega})) d\tilde{P}_x \\ &= E_{(x,0)}[f(Y_t); t < \zeta] \\ &= E_{(x,p)}[f(Y_t); t < \zeta]. \end{aligned}$$

Since, for fixed $B \in \mathscr{B}(E)$, $r \geqq 0$, $P_{(x,p)}((X_t, N_t) \in (B, p + r))$ is independent of p, we can see from the above equalities that

$$\begin{aligned} &\tilde{P}_x(\{\tilde{\omega} \in \tilde{\Omega}; \tilde{\omega}(t_i) \in B_i \text{ and } \sigma_{r_i}(\tilde{\omega}) \leqq t_i < \sigma_{r_i+1}(\tilde{\omega}); i = 1, 2\}) \\ &= P_{(x,0)}(\{\omega \in \Omega; \omega(t_i) \in (B_i, r_i), i = 1, 2\}) \\ &= E_{(x,0)}[P_{(X_{t_1}, N_{t_1})}((X_{t_2-t_1}, N_{t_2-t_1}) \in (B_2, r_2)); (X_{t_1}, N_{t_1}) \in (B_1, r_1)] \\ &= E_{(x,0)}[P_{(X_{t_1}, N_{t_1})}((X_{t_2-t_1}, N_{t_2-t_1}) \in (B_2, r_2 - r_1 + N_{t_1})); (X_{t_1}, N_{t_1}) \in (B_1, r_1)] \\ &= \tilde{E}_x[P_{\tilde{X}_{t_1}}(\tilde{X}_{t_2-t_1} \in B_2, \sigma_{r_2-r_1}(\tilde{\omega}) \leqq t_2 - t_1 < \sigma_{r_2-r_1+1}(\tilde{\omega})); \\ &\qquad\qquad \tilde{X}_{t_1} \in B_1, \sigma_{r_1}(\tilde{\omega}) \leqq t_1 < \sigma_{r_1+1}(\tilde{\omega})], \end{aligned}$$

which proves the Markov property of $\tilde{P}_x$. So we have a right continuous Markov Process $\tilde{X} = \{\tilde{X}_t, \tilde{\zeta}, \tilde{\mathscr{B}}_t, \tilde{P}_x; x \in E\}$ on E. Similarly, for a $\tilde{\mathscr{B}}_t$-Markov time ρ, if we consider a $\mathscr{B}_t$-Markov time σ of Y defined by

$$\sigma(\omega) = \begin{cases} t, & \text{if } \omega \in A \text{ where } \tilde{A} = \{\tilde{\omega} \in \tilde{\Omega}; \rho(\tilde{\omega}) = t\}, \quad t \geqq 0 \\ \infty, & \text{if } \omega \notin \{\omega \in \Omega; N_0(\omega) = 0\}, \end{cases}$$

then we can see that $\tilde{X}$ is strong Markov and quasi-left continuous since Y is. Furthermore, by the definition of $\tilde{\mathscr{B}}_t, \sigma_r$ is a $\tilde{\mathscr{B}}_t$-Markov time of $\tilde{X}$ and (16), (17) are obtained from Lemma 3 and the definition of $\tilde{P}_x$. Thus taking $\tilde{X}$ as X, we complete the proof. Q.E.D.

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