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ON A CLASS OF METRICAL AUTOMORPHISMS ON GAUSSIAN MEASURE SPACE

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To Professor KATUZI ONO on the occasion of his 60th birthday

1. Introduction. Let E be an infinite dimensional real nuclear space and H be its completion by a continuous Hilbertian norm $\| \ \|$ of E. Then we have the relation

$$E \subset H \subset E^*$$

where E^* is the conjugate space of E. Consider a function $C(\xi)$ on E defined by the formula

$$C(\xi) = e^{-\|\xi\|^2/2}, \quad \xi \in E. \tag{1}$$

Then $C(\xi)$ is a positive definite and continuous function with $C(0)=1$. Therefore, by Bochner-Minlos' theorem, there exists a unique probability measure μ on E^* such that

$$\int_{E^*} e^{i\langle x, \xi\rangle}\, d\mu(x) = e^{-\|\xi\|^2/2}, \quad \xi \in E, \tag{2}$$

where $\langle x, \xi\rangle$ being the canonical bilinear form. The measure μ is defined on the σ-algebra $\mathscr{L}$ generated by all cylinder sets of E^* ([1]). We call μ a *Gaussian measure.*

Let $O(H)$ be the group formed by all linear and orthogonal operators acting on H. After [3], we consider a subgroup $O(E)$ of the group $O(H)$ which is defined as the collection of all $g's$ of $O(H)$ having the property that each of g is a linear homeomorphism from E onto E. An operator g of $O(E)$ is called a rotation of E. As is seen from the formula (2), the conjugate operator g^* of a rotation g is a metrical automorphism on the space (E^*, μ). The purpose of this paper is to generalize this fact and we shall prove that

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(i) *for each* g *of* $O(H)$ *there exists an automorphism* T_g *on the space* (E^*, μ), *with the group property*

(ii) $T_{g_1}T_{g_2}=T_{g_1g_2}$ (*mod* 0), *for each* g_1 *and* g_2.

In the following section 2, we shall prepare lemmas used to prove the above assertions. In particular, we consider a unitary representation $(U_g, L_2(E^*, \mu))$ of the group $O(H)$ and we make use it in section 3 for the proof of Theorem.

2. Preliminaries. In this section we shall give some preparatory lemmas used in the following. For details we refer to [2] and [3].

Let μ be the Gaussian measure on the space $(E^*, \mathscr{L})$. We denote by $L_2=L_2(E^*, \mu)$ the Hilbert space of all square integrable complex-valued functions with the inner product $(\varphi, \psi)=\int_{E^*}\varphi(x)\overline{\psi(x)}d\mu(x)$. Then we have the following lemmas.

LEMMA 1. *The mapping* γ *from* E *into* L_2 *defined by*

$$\gamma : \xi \longrightarrow \langle x, \xi\rangle$$

can be extended to a linear isometric mapping from H *into* L_2. *Moreover, for each* f *of* H, $\gamma(f)$ (*we shall also denote it by* $\langle x, f\rangle$) *is a Gaussian random variable with mean* 0 *and variance* $\|f\|^2$ ([3], *Proposition* 1).

LEMMA 2. *The linear subspace* M *of* L_2 *spanned by* $\{e^{i<x,f>};\ f\in H\}$ *is dense in* L_2 ([2], *Lemma* 2. 1).

Let g be an orthogonal operator of H. We shall define a unitary operator U_g on L_2 by the following manner. First, we define U_g as an operator on M by the formula:

$$U_g\Big(\sum_{k=1}^{n}a_ke^{i<x,f_k>}\Big)=\sum_{k=1}^{n}a_ke^{i<x,gf_k>},\quad f_k\in H,\quad k=1,2,\cdots,n. \tag{3}$$

Then, by lemma 1, we obtain the following relation:

$$\begin{aligned}&\Big(U_{g_1}\Big(\sum_{k=1}^{n}a_ke^{i<x,f_k>}\Big),\ U_{g_2}\Big(\sum_{l=1}^{m}b_le^{i<x,h_l>}\Big)\Big)\\&=\sum_{k=1}^{n}\sum_{l=1}^{m}a_k\bar{b}_l\exp\Big\{-\frac{1}{2}\|g_1f_k-g_2h_l\|^2\Big\},\end{aligned} \tag{4}$$

where g_1 and g_2 being elements of $O(H)$. In particular, putting $g_1=g_2=g$, we know that U_g preserves the inner product in M. Hence, by lemma 2,

U_g can be extended to a unitary operator on L_2. Then we have the following lemma.

LEMMA 3. *The system* $\{U_g, g \in O(H); L_2\}$ *is a unitary representation of the group* $O(H)$:

(i) $$U_{g_1} U_{g_2} = U_{g_1 g_2},$$

and

(ii) *the mapping* $g \longrightarrow U_g$ *is continuous, that is,*
if $g_\nu f \longrightarrow gf$ $(\nu \to \infty)$ *for any* f *of* H, *then* $U_{g_\nu}\ \varphi \longrightarrow U_g \varphi (\nu \to \infty)$ *for any* φ *of* L_2.

Proof. Since each operator U_g is unitary, (i) is obvious by definition (3) of U_g and the lemma 2. To prove (ii), it is enough to show that (iii) holds for φ with the form $\sum_{k=1}^{n} a_k e^{i\langle x, f_k \rangle}$. By the relation (4), we have

$$\|U_g\varphi - Ug_\nu\varphi\|^2 = 2\|\varphi\|^2 - 2\sum_{k,l=1}^{n} a_k \bar{a}_l \exp\left\{-\frac{1}{2}\|gf_k - g_\nu f_l\|^2\right\}$$

$$\underset{(\nu\to\infty)}{\longrightarrow} 2\|\varphi\|^2 - 2\sum_{k,l=1}^{n} a_k \bar{a}_l \exp\left\{-\frac{1}{2}\|gf_k - gf_l\|^2\right\} = 0$$

This proves the lemma.

3. The theorem. The purpose of this section is to prove the following theorem.

THEOREM. *For any* g *of* $O(H)$ *there exists a unique (mod* 0) *metrical automorphism* T_g *of the space* $(E^*, \mathscr{L}, \mu)$ *with the following properties:*

(i) $$U_g \varphi(x) = \varphi(T_g^{-1} x) \text{ (a.e.)}, \quad \varphi \in L_2,$$

and

(ii) $$T_{g_1} T_{g_2} = T_{g_1 g_2} \ (\text{mod } 0), \textit{ for each } g_1 \textit{ and } g_2.$$

Proof. 1°. We put $U=U_g$ and prove that U is multiplicative: it holds that

(*) $$U(\varphi\psi) = U\varphi \cdot U\psi$$

for any bounded measurable functions. By the definition (3) of U the relation (*) is obvious if both φ and ψ are functions in M. Suppose φ is bounded and ψ belongs to M. By lemma 2, there exists such a sequence $\{\varphi_n\}$ of functions in M that

$$\varphi_n \longrightarrow \varphi \quad (n \to \infty), \text{ in } L_2.$$

Then the following inequality (which holds almost everywhere)

$$\begin{aligned} |U(\varphi\psi)(x) - U\varphi(x)\cdot U\psi(x)| &\leq |U(\varphi\psi)(x) - U(\varphi_n\psi)(x)| \\ &+ |U(\varphi_n\psi)(x) - U\varphi_n(x)\cdot U\psi(x)| + |U\varphi_n(x)\cdot U\psi(x) - U\varphi(x)\cdot U\psi(x)| \end{aligned}$$

implies that

$$\begin{aligned} \|U(\varphi\psi) - U\varphi\cdot U\psi\|_{L_1} &\leq \|U(\varphi\psi) - U(\varphi_n\psi)\|_{L_1} + \|(U\varphi_n - U\varphi)U\psi\|_{L_1} \\ &\leq \|U(\varphi\psi) - U(\varphi_n\psi)\| + \|(U\varphi_n - U\varphi)U\psi\| \\ &\leq \left\{\sup_x |\psi(x)| + \sup_x |U\psi(x)|\right\} \|\varphi_n - \varphi\| \longrightarrow 0 \ (n \to \infty), \end{aligned}$$

where $\| \ \|_{L_1}$ being the L_1-norm. Hence we have $U(\varphi\psi) = U\varphi\cdot U\psi$. Furthermore, let ψ be bounded and find $\psi_n \in M$ such that

$$\psi_n \longrightarrow \psi \quad (n \to \infty), \text{ in } L_2.$$

Then using the above result, we obtain

$$\begin{aligned} \|U(\varphi\psi) - U\varphi\cdot U\psi\|_{L_1} &\leq \|U(\varphi\psi) - U(\varphi\psi_n)\|_{L_1} + \|U(\varphi\psi_n) - U\varphi\cdot U\psi_n\|_{L_1} \\ &\quad + \|(U\psi_n - U\psi)U\varphi\|_{L_1} \\ &\leq \|U(\varphi\psi) - U(\varphi\psi_n)\| + \|U\varphi\|\cdot\|U\psi_n - U\psi\| \\ &\leq \left\{\sup_x |\varphi(x)| + \|\varphi\|\right\} \|\psi_n - \psi\| \longrightarrow 0 \ (n \to \infty), \end{aligned}$$

so that we have $U(\varphi\psi) = U\varphi\cdot U\psi$. This proves the assertion.

2^0 Since E is a nuclear space, the conjugate space E^* of E is a separable complete metric space and the class of all Borel sets of this space coincides with the σ-algebra $\mathscr{L}$ generated by all cylinder sets (see [6]). Moreover, the Gaussian measure μ is regular. Therefore, on account of the results of von Neumann [4, 5], we know that there exists a unique (mod 0) automorphism T_g satisfying the relation of (i). Finally, applying (i) of lemma 3, we get

$$\varphi(T^{-1}_{g_1g_2}x) = U_{g_1g_2}\varphi(x) = U_{g_1}U_{g_2}\varphi(x) = \varphi(T^{-1}_{g_2}T^{-1}_{g_1}x) \text{ (a.e.)}.$$

Thus we obtain $T_{g_1g_2} = T_{g_1}T_{g_2}$ (mod 0). This concludes the proof.

COROLLARY. *Let* (g_t), $-\infty < t < \infty$, *be a strongly continuous one-parameter subgroup of* $O(H)$. *Then setting*

(5) $$T_t = T_{g_t}, \text{ and } U_t = U_{g_t}, \quad -\infty < t < \infty,$$

we have a flow (T_t):

(6) $$T_t T_s = T_{t+s} \text{ (mod 0), flor each } t \text{ and } s,$$

and a continuous unitary group (U_t) *such that*

$$U_t \varphi(x) = \varphi(T_t^{-1} x), \text{ (a.e.)}, \quad \varphi \in L_2.$$

For, the mapping $t \longrightarrow U_t$ is continuous by (ii) of lemma 3.

EXAMPLE. Let $X = X(t)$: $-\infty < t < \infty$ be a real stationary Gaussian process defined on a certain probability space, which is mean continuous with expectation 0. We define h_t by $h_t X(s) = X(s+t)$ and extend h_t to an orthogonal operator on the real Hibert space H_X spanned by all linear combinations of $X(t)$, $-\infty < t < \infty$. Then we get a strongly continuous one-parameter subgroup (h_t), $-\infty < t < \infty$, of $O(H_X)$. Now, assume an additional condition that H_X is infinite dimensional and let σ be an isometric mapping from H onto H_X. Then it can be shown that the flow $(T_{\sigma^{-1} h_t \sigma})$, $-\infty < t < \infty$, on E^* is spectrally equivalent to the flow of the process X.

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