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ON SOME RESULTS ON THETA CONSTANTS (I).

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Dedicated to Professor Katuzi Ono on his 60th birtbday

D. Mumford has shown an excelent algebralization of theory of theta constants and theta functions in his papers: On the equations defining abelian varieties I, II, III (Invent. Math. 1. 237–354 (1966), 3. 75–135 (1967), 3. 215–244) (1967). Our starting point and idea, however, are something different from those of Mumford; we begin our study at characterizing abelian addition formulae among all the possible addition formulae, and we want to give expressions to everything in words of matric notations.

§ 1. Commutative composition and 2-division points.

We mean by K the universal domain and by $ch(K)$ the characteristic of K. For each finite additive group G we associate a system of indeterminates X_a $(a \in G)$ and the projective space P_G with the homogeneous coordinate ring $K[(X_a)_{a\in G}]$.

In the following we shall assume that the order $|G|$ of G is always odd and shall use the following notation for brevity;

Point in P_G	Homogeneous coordinates	The a-component
x	$(x_a)_{a\in G}$	x_a
x^{-1}	$(x_{-a})_{a\in G}$	x_{-a}
$x(b)$	$(x_{a+b})_{a\in G}$	x_{a+b}
Matrix	**The (a, b)-component**	
$(x_{-a+b}y_{a+b})_{a\in G,\, b\in G}$	$x_{-a+b}y_{a+b}$	
${}^t(x_{-a+b}y_{a+b})_{a\in G,\, b\in G}$	$x_{-b+a}y_{b+a}$	

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1. Commutative composition.

We choose a point $e=(e_a)_{a\in G}$ in P_G such that $e_{-a}=e_a$ $(a\in G)$ and put

$$n = \operatorname{rank}(e_{-a+b}e_{a+b})_{a\in G,\, b\in G}\,.$$

We shall define a commutative composition $\circ$ relating with the point e. Since the $|G|\times|G|$-matrix $(e_{-a+b}e_{a+b})_{a\in G,\, b\in G}$ is symmetric, we can find a $|G|\times n$-matrix S of rank n such that.

$$(e_{-a+b}e_{a+b})_{a\in G,\, b\in G} = S{}^tS\,.$$

The matrix S is uniquely determined up to the right multiplication $S\longrightarrow SM$ by orthogonal $n\times n$-matrices M. We shall fix the pair (e,S) in the first half of the present paragraph.

DEFINITION (1. 1. 1). Let $x=(x_a)_{a\in G}$ and $y=(y_a)_{a\in G}$ be two points in P_G. We say that the composition $x\circ y$ is *well-defined with respect to* e, if there exist non-zero vectors $(u_a)_{a\in G}$ and $(v_a)_{a\in G}$ such that

$$\operatorname{rank}\begin{bmatrix}(e_{-a+b}e_{a+b})_{a\in G,\, b\in G} & (y_{-a+b}y_{a+b})_{a\in G,\, b\in G}\\ {}^t(x_{-a+b}x_{a+b})_{a\in G,\, b\in G} & (u_{-a+b}v_{a+b})_{a\in G,\, b\in G}\end{bmatrix}$$
$$= \operatorname{rank}(e_{-a+b}e_{a+b})_{a\in G,\, b\in G}\,.$$

This definition does not depend on the choice of homogeneous coordinates. If non-zero vectors $(u_a)_{a\in G}$ and $(v_a)_{a\in G}$ satisfy the above relation, then

$$\begin{aligned}&\operatorname{rank}((e_{-a+b}e_{a+b})_{a\in G,\, b\in G},\ (x_{-a+b}x_{a+b})_{a\in G,\, b\in G})\\ &= \operatorname{rank}((e_{-a+b}e_{a+b})_{a\in G,\, b\in G},\ (y_{-a+b}y_{a+a})_{a\in G,\, b\in G})\\ &= \operatorname{rank}(e_{-a+b}e_{a+b})_{a\in G,\, b\in G} = \operatorname{rank} S = n\,.\end{aligned}$$

Therefore we obtain two $|G|\times n$-matrices $T^{(x)}$ and $T^{(y)}$ such that

$$\begin{aligned}(x_{-a+b}x_{a+b}) &= S{}^tT^{(x)}\\ (y_{-a+b}y_{a+b}) &= S{}^tT^{(y)}\,,\end{aligned}$$

were $T^{(x)}$ and $T^{(y)}$ are uniquely determined by the matrix S and points x, y up to the multiplication by non-zero scalars. Since

$$\operatorname{rank}\begin{pmatrix}(e_{-a+b}e_{a+b})_{a\in G,\, b\in G} & (y_{-a+b}y_{a+b})_{a\in G,\, b\in G}\\ {}^t(x_{-a+b}x_{a+b})_{a\in G,\, b\in G} & (u_{-a+b}v_{a+b})_{a\in G,\, b\in G}\end{pmatrix}$$
$$= \operatorname{rank}\begin{pmatrix}S & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I & {}^tT^{(y)}\\ T^{(x)} & (u_{-a+b}v_{a+b})_{a\in G,\, b\in G}\end{pmatrix}\begin{pmatrix}S & 0\\ 0 & I\end{pmatrix}$$

the matrix $(u_{-a+b}v_{a+b})_{a\in G,\, b\in G}$ equals to $T^{(x)t}T^{(y)}$. By virtue of the oddenss of $|G|$ the pair $(-a+b,\ a+b)$ runs over all the elements in $G\times G$; this means that the points $u=(u_a)_{a\in G}$ and $v=(v_a)_{a\in G}$ in P_G are uniquely determined by the given points x and y.

If we denote by $x\circ y$ the point v, the point u is nothing else than $x^{-1}\circ y$, i.e. the composition of x^{-1} with y.

PROPOSITION (1. 1. 2). *$x\circ y$ is well-defined with respect to e, if and only if there exist two $|G|\times n$-matrices $T^{(x)}$ and $T^{(y)}$ such that*

$$(x_{-a+b}x_{a+b})_{a\in G,\, b\in G}=S^tT^{(x)}$$
$$(y_{-a+b}y_{a+b})_{a\in G,\, b\in G}=S^tT^{(y)},$$

and

$$((x^{-1}\circ y)_{-a+b}(x\circ y)_{a+b})_{a\in G,\, b\in G}=\lambda T^{(x)t}T^{(y)}$$

with a non-zero scalar λ, where the scalar λ depends on the choice of homogeneous coordinates of the points.

This is the summation of the above results. It is also remarked that $x^{-1}\circ y$ is well-defined if and only if $x\circ y$ is well-defined.

PROPOSITION (1. 1. 3). *If $x\circ y$ is well-defined, then $y\circ x$ and $x^{-1}\circ y^{-1}$ are well-defined and*

(1. 1. 3. 1) $\qquad x\circ e=x,\quad y\circ e=y,$

(1. 1. 3. 2) $\qquad x\circ y=y\circ x \qquad$ (*commutativity*),

(1. 1. 3. 3) $\qquad (x\circ y)^{-1}=x^{-1}\circ y^{-1}.$

Proof. From the relations

$$T^{(y)t}T^{(x)}={}^t(T^{(x)t}T^{(y)})=\lambda^{-1t}((x^{-1}\circ y)_{-a+b}(x\circ y)_{a+b}),$$
$$\operatorname{rank}\begin{pmatrix}S^tS & S^tT^{(x)}\\ T^{(y)t}S & T^{(y)t}T^{(x)}\end{pmatrix}=\operatorname{rank}\begin{pmatrix}S^tS & S^tT^{(y)}\\ T^{(x)t}S & T^{(x)t}T^{(y)}\end{pmatrix},$$

we can conclude that

$$((y^{-1}\circ x)_{-a+b}(y\circ x)_{a+b})_{a\in G,\, b\in G}=\lambda' T^{(y)t}T^{(x)}$$

with a non-zero scalar λ', i.e. $y\circ x$ and $y^{-1}\circ x$ are well-defined. Replacing x by x^{-1}, we know that $y^{-1}\circ x^{-1}$ is well-defined. The commutativity comes from the symmetricity of the matrix $(e_{-a+b}e_{a+b})_{a\in G,\, b\in G}$ and, combining the above result with the commutativity, we have $x^{-1}\circ y^{-1}=(x\circ y)^{-1}$. Finally $x\circ e=x$ is a direct consequence of the matric equation.

$$(x_{-a+b}x_{a+b})_{a\in G,\, b\in G}=S^tT_x.$$

2. Orthogonal matrices associated with 2-division points.

A point $e(f)$ in P_G is called a 2-division point of e if $e(f)\circ e(f)$ is well-defined and

$$e(f)^{-1}\circ e(f)=e(f)\circ e(f)=e\,.$$

In other words

$$S^tS=\lambda T^{(e(f))t}T^{(e(f))}$$

with a non-zero scalar λ. When $\lambda=1$, the homogeneous coordinates $(e_a(f))_{a\in G}$ $e(f)$ is said to be normalized. We can always choose exactly four normalized homogeneous coordinates:

$$(e_a(f))_{a\in G},\ \ (-e_a(f))_{a\in G},\ \ (\sqrt{-1}e_a(f))_{a\in G},\ \ (-\sqrt{-1}e_a(f))_{a\in G},$$

where, if $e(f)\neq e$, $e_a(f)$ is replaced by $\lambda^{-\frac{1}{4}}e_a(f)$ $\quad(a\in G)$.

LEMMA (1. 2. 1). *If $e(f)$ is a 2-division point, then $e(f)^{-1}=e(f)$.*

Proof. If we choose $(e_{-a}(f))_{a\in G}$ as a homogeneous coordinates of $e(f)$, then

$$T^{(e(f)^{-1})}=(t^{(e(f))}_{-a,i})_{a\in G,1\leqslant i\leqslant n}$$

and

$$\lambda\sum_{i=1}^{n}t^{(e(f)^{-1})}_{a,i}t^{(e(f))}_{b,i}=\lambda\sum_{i=1}^{n}t^{(e(f))}_{-a,i}t^{(e(f))}_{b,i}$$

$$=e_{-(-a)+b}e_{-a+b}=\lambda\sum_{i=1}^{n}t^{(e(f))}_{a,i}t^{(e(f))}_{b,i}\qquad(a,b\in G)$$

with a non-zero scalar λ. Since rank $T^{(e(f))}=n$, we can conclude $T^{(e(f)^{-1})}-T^{(e(f))}=0$, i.e. $e(f)^{-1}=e(f)$.

From $e(f)^{-1}=e(f)$ we obtain a scalar $\varepsilon_{e(f)}=1$ or -1 such that

$$e_{-a}(f)=\varepsilon_{e(f)}e_a(f)\qquad(a\in G)\,.$$

We call the scalar $\varepsilon_{e(f)}$ the signature of the 2-division point $e(f)$.

PROPOSITION (1. 2. 2). *A vector* $(x_a)_{a\in G}$ *is a normalized homogeneous coordinates of a 2-division point, if and only if there exists an orthogonal* $n\times n$*-matrix* M_x *such that*

$$(x_{-a+b}x_{a+b})_{a\in G,\ b\in G} = S^tM_x{}^tS,$$

i.e.

$$T^{(x)} = SM_x\,.$$

Proof. If $(x_a)_{a\in G}$ is a normalized homogeneous coordinates of a 2-division point x, then

$$(x_{-a+b}x_{a+b})_{a\in G,\ b\in G} = S^tT^{(x)}$$

and

$$S^tS = (e_{-a+b}e_{a+b})_{a\in G,\ b\in G} = T^{(e(f))t}T^{(e(f))}.$$

Hence we can choose the unique orthogonal matrix M_x such that $T^{(x)} = SM_x$. Conversely, if an orthogonal $n\times n$-matrix M and a non-zero vector $(x_a)_{a\in G}$ satisfy the relation

$$(x_{-a+b}x_{a+b})_{a\in G,\ b\in G} = S^tM^tS,$$

then it follows

$$\begin{aligned}
&\operatorname{rank}\begin{pmatrix} S^tS & S^tM^tS \\ SM^tS & S^tS \end{pmatrix} = \operatorname{rank}\begin{pmatrix} S^tS & S^tM^tS \\ SM^tS & SM^tM^tS \end{pmatrix} \\
&= \operatorname{rank}\left(\begin{pmatrix} S & 0 \\ 0 & SM \end{pmatrix}\begin{pmatrix} I & I \\ I & I \end{pmatrix}\begin{pmatrix} {}^tS & 0 \\ 0 & {}^tM^tS \end{pmatrix}\right) \\
&= n = \operatorname{rank} S^tS.
\end{aligned}$$

This means that

$$(e_{-a+b}e_{a+b})_{a\in G,\ b\in G} = ((x^{-1}\circ x)_{-a+b}(x\circ x)_{a+b})_{a\in G,\ b\in G}\,,$$

i.e. $(x_a)_{a\in G}$ is a normalized homogeneous coordinates of a 2-division point x.

The orthogonal matrix $M_{e(f)}$ is uniquely determined up to the multiplication by ± 1. We call both $\pm M_{e(f)}$ the orthogonal matrix associated with a 2-division point $e(f)$.

LEMMA (1. 2. 3). $M_{e(f)}M_{e(f)} = \varepsilon_{e(f)}I$.

Proof. We choose the normalized homogeneous coordinates $(e_a(f))_{a\in G}$ such that

$$(e_{-a+b}(f)e_{a+b}(f))_{a\in G,\, b\in G} = SM_{e(f)}{}^tS.$$

Since $e_{-a}(f) = \varepsilon_{e(f)}e_a(f)$ $(a\in G)$ and $\varepsilon^2_{e(f)} = 1$, we have

$$\begin{aligned} SM_{e(f)}{}^tS &= {}^t(e_{-a+b}(f)e_{a+b}(f))_{a\in G,\, b\in G} \\ &= \varepsilon_{e(f)}(e_{-a+b}(f)e_{a+b}(f))_{a\in G,\, b\in G} = \varepsilon_{e(f)}S^tM_{e(f)}{}^tS. \end{aligned}$$

This implies $M^{-1}_{e(f)} = {}^tM_{e(f)} = \varepsilon_{e(f)}M_{e(f)}$.

LEMMA (1. 2. 4). *Let $e(f)$ and $e(g)$ be two 2-division points of e such that $e(f)\circ e(g)$ is well-defined and it is also a 2-division point of e. Let $M_{e(f)}$, $M_{e(g)}$ and $M_{e(f)\circ e(g)}$ be the orthogonal matrices associated with $e(f)$, $e(g)$ and $e(f)\circ e(g)$, respectively. Then there exist scalars $\mu_{e(f),\, e(g)}$ and $\mu_{e(g),\, e(f)}$ such that*

(1. 2. 4. 1) $\quad M_{e(f)}M_{e(g)} = \mu_{e(f),\, e(g)}M_{e(f)\circ e(g)},\quad M_{e(g)}M_{e(f)} = \mu_{e(g),\, e(f)}M_{e(f)\circ e(g)}$

(1. 2. 4. 2) $\quad \mu^2_{e(f),\, e(g)} = \mu^2_{e(g),\, e(f)} = 1,$

(1. 2. 4. 3) $\quad \mu_{e(f),\, e(g)}\mu_{e(g),\, e(f)} = \varepsilon_{e(f)}\varepsilon_{e(g)}\varepsilon_{e(f)\circ e(g)}$

(1. 2. 4. 4) $\quad M_{e(f)}M_{e(g)} = \varepsilon_{e(f)}\varepsilon_{e(g)}\varepsilon_{e(f)\circ e(f)}M_{e(f)}M_{e(g)}$ *(commutator relation).*

Proof. We choose the normalized homogeneous coordinates $(e_a(f))_{a\in G}$, $(e_a(g))_{a\in G}$, $(e(f\circ g))_{a\in G}$ such that

$$\begin{aligned} &(e_{-a+b}(f)e_{a+b}(f))_{a\in G,\, b\in G} = S^tM_{e(f)}{}^tS, \\ &(e_{-a+b}(g)e_{a+b}(g))_{a\in G,\, b\in G} = S^tM_{e(g)}{}^tS, \\ &((e(f)\circ e(g))_{-a+b}(e(f)\circ e(g))_{a+b})_{a\in G,\, b\in G} = S^tM_{e(f)\circ e(g)}{}^tS. \end{aligned}$$

Since $T^{(e(f))} = SM_{e(f)}$, $T^{(e(g))} = SM_{e(g)}$
and $((e(f)\circ e(g))_{-a+b}(e(f)\circ e(g))_{a+b})_{a\in G,\, b\in G} = \lambda^{-1}_{e(f),\, e(g)}T^{(e(f))t}T^{(e(g))}$
with non-zero scalars $\lambda_{e(f),\, e(g)}$, hence we have

$$\begin{aligned} \lambda_{e(f),\, e(g)}S^tM_{e(f)\circ e(g)}{}^tS &= SM_{e(f)}{}^tM_{e(g)}{}^tS, \\ \lambda_{e(g),\, e(f)}S^tM_{e(g)\circ e(f)}{}^tS &= SM_{e(g)}{}^tM_{e(f)}{}^tS. \end{aligned}$$

From rank $S = n$ we can conclude

$$\begin{aligned} \lambda_{e(f),\, e(g)}{}^tM_{e(f)\circ e(g)} &= M_{e(f)}{}^tM_{e(g)}, \\ \lambda_{e(g),\, e(f)}{}^tM_{e(f)\circ e(g)} &= M_{e(g)}{}^tM_{e(f)}, \\ \lambda_{e(f),\, e(g)}{}^tM_{e(f)\circ e(g)} &= \lambda_{e(g),\, e(f)}M_{e(f)\circ e(g)}. \end{aligned}$$

By virtue of (1. 2. 3) we have

$$\lambda_{e(f),\,e(g)}\varepsilon_{e(f)\circ e(g)}\varepsilon_{e(g)}M_{e(f)\circ e(g)} = M_{e(f)}M_{e(g)},$$
$$\lambda_{e(g),\,e(f)}\varepsilon_{e(f)\circ e(g)}\varepsilon_{e(f)}M_{e(f)\circ e(g)} = M_{e(g)}M_{e(f)},$$
$$\lambda_{e(f),\,e(g)}\varepsilon_{e(f)\circ e(g)} = \lambda_{e(g),\,e(f)},$$
$$\lambda^2_{e(f),\,e(g)}I = (\lambda_{e(f),\,e(g)}{}^tM_{e(f)\circ e(g)})(\lambda_{e(f),\,e(g)}M_{e(f)\circ e(g)})$$
$$= (M_{e(f)}{}^tM_{e(g)})^t(M_{e(f)}{}^tM_{e(g)}) = I.$$

Hence, putting

$$\mu_{e(f),\,e(g)} = \varepsilon_{e(f)}\varepsilon_{e(f)\circ e(g)}\lambda_{e(f),\,e(g)},$$
$$\mu_{e(g),\,e(f)} = \varepsilon_{e(g)}\varepsilon_{e(f)\circ e(g)}\lambda_{e(g),\,e(f)}$$

we can conclude that

$$\mu^2_{e(f),\,e(g)} = \mu^2_{e(g),\,e(f)} = 1,$$
$$\mu_{e(f),\,e(g)}\mu_{e(g),\,e(f)} = \varepsilon_{e(f)}\varepsilon_{e(g)}\lambda_{e(f),\,e(g)}\lambda_{e(g),\,e(f)}$$
$$= \varepsilon_{e(f)}\varepsilon_{e(g)}\varepsilon_{e(f)\circ e(g)}\lambda^2_{e(f),\,e(g)} = \varepsilon_{e(f)}\varepsilon_{e(g)}\varepsilon_{e(f)\circ e(g)},$$
$$M_{e(f)}M_{e(g)} = \mu_{e(f),\,e(g)}M_{e(f)\circ e(g)},$$
$$M_{e(g)}M_{e(f)} = \mu_{e(g),\,e(f)}M_{e(f)\circ e(g)}.$$

The commutator relation is the direct consequence form (1. 2. 4. 1), (1. 2. 4. 2), (1. 2. 4. 3).

PROPOSITION (1. 2. 5). *Let Δ be an additive group of exponent two and $\{e(f)\mid f\in\Delta\}$ be the group of 2-division point of e such that $f\longrightarrow e(f)$ is an isomorphism. Let $\{M_{e(f)}\mid f\in\Delta\}$ be orthogonal matrices associated with $e(f)$ $(f\in\Delta)$ and $\mu_{e(f),\,e(g)}$ be the sign such that $M_{e(f)}M_{e(g)}=\mu_{e(f),\,e(g)}M_{e(f)\circ e(g)}$. We denote by $\Gamma(\Delta)$ the set $\{(\alpha,f)\mid\alpha=\pm1, f\in\Delta\}$ with the composition*

$$(\alpha,f)(\beta,g) = (\alpha\beta\mu_{e(f),\,e(g)}, f+g),$$

and put

$$M((\alpha,f)) = \alpha M_{e(f)}.$$

Then $\Gamma(\Delta)$ is two step nilpotent group such that

1) *the exponent of $\Gamma(\Delta)$ is two or four*
2) $|\Delta|\leqslant 2^{2n-1}n!$, *where* $n=\mathrm{rank}\ (e_{-a+b}e_{a+b})_{a\in G,\,b\in G}$,
3) $(\alpha,f)\longrightarrow M((\alpha,f))$ *is a faithful matric representation of $\Gamma(\Delta)$.*

Proof. From the definition it follows

$$\begin{aligned} M((\alpha, f))M((\beta, g)) &= \alpha\beta M_{e(f)}M_{e(g)} = \alpha\beta\mu_{e(f),\, e(g)}M_{e(f)\circ e(g)} \\ &= \alpha\beta\mu_{e(f),\, e(g)}M_{e(f+g)} = M(\alpha\beta\mu_{e(f),\, e(g)}, f+g)) \\ &= M((\alpha, f)(\beta, g)). \end{aligned}$$

This shows that $(\alpha, f) \longrightarrow M((\alpha, f))$ is a matric representation of $\Gamma(\Delta)$ and $\Gamma(\Delta)$ is associative. By virtue of (1. 2. 3) $M^2_{e(f)} = \varepsilon_{e(f)}\ I$, hence the inverse of (α, f) is given by $(\alpha\varepsilon_{e(f)}, f)$. Therefore $\Gamma(\Delta)$ is a two step nilpotent group whose exponent is two or four. For each finite subgroup $\sum$ of Δ the subset $\Gamma(\sum) = \{(\alpha, f) \mid \alpha = \pm 1, f \in \sum\}$ is a subgroup of $\Gamma(\Delta)$. Since $\Gamma(\sum)$ is a finite two step nilpotent group whose exponent is two or four and the characteristic of K is not two, by virtue of theory of representation of finite nilpotent group, the representation $(\alpha, f) \longrightarrow M((\alpha, f))$ is equivalent to a monoidal representation whose matric components are contained in $\{\pm 1, \pm\sqrt{-1}\}$. This means $|\Gamma(\sum)| \leqslant 4^n n!$, i.e. $|\sum| \leqslant 2^{2n-1} n!$. Since a finite set in Δ generates a finite subgroup of Δ, we may conclude that $|\Delta| \leqslant 2^{2^{n-1}}\ n!$, where $n = \operatorname{rank}(e_{-a+b}e_{a+b})_{a\in G,\, b\in G} = \deg M_{e(f)}$.

We call the group $\Gamma(\Delta)$ the two-step nilpotent group associated with group $\{e(f) \mid f \in \Delta\}$ of 2-division points.

DEFINITION (1. 2. 6). Let Δ_r be an additive group of type $(\overbrace{2, \dots, 2}^{r})$ and $\hat{\Delta}_r$ be the dual group of Δ_r, i.e. the group of homomorphisms $f \longrightarrow <\hat{f}, f>$ of Δ_r into the roots of unity in K. Then the *Heisenberg group of dimension r* is defined as a group isomorphic to the group $H_r = \{(\alpha, f + \hat{f}) \mid \alpha = \pm 1, f + \hat{f} \in \Delta_r \oplus \hat{\Delta}_r\}$ with the composition

$$(\alpha, f + \hat{f})(\beta, g + \hat{g}) = (\alpha\beta <\hat{f}, g>, (f+g) + (\hat{f} + \hat{g})).$$

LEMMA (1. 2. 7). *Let Δ be an additive group of type $(\overbrace{2, \cdots, 2}^{2r})$ and $\mu_{f,g}$ $(f, g \in \Delta)$ be numbers such that*

$$\mu_{f,0} = \mu_{0,f} = 1, \quad \mu^2_{f,g} = 1, \quad \mu_{f,g}\mu_{f+g,h} = \mu_{f,g+h}\mu_{g,h} \qquad (f, g, h \in \Delta)$$

Then the group $\Gamma = \{(\alpha, f) \mid \alpha = \pm 1, f \in \Delta\}$ with the composition $(\alpha, f)(\beta, g) = (\alpha\beta\mu_{f,g}, f+g)$ is the Heisenberg group H_r if and only if the center of Γ is $\{(1, 0), (-1, 0)\}$.

Proof. We shall prove by induction on r. When $r=1, \Gamma$ is isomorphic to H_1, Assume Lemma for $r-1$. Let f_1 and f_2 be fixed elements in Δ such that $(1,f_1)(1,f_2) \neq (1,f_2)(1,f_1)$ N be the subgroup consisting of all elements commutative with $(1,f_1)$ and $(1,f_2)$. Then $(1,f_1)$, $(1,f_2)(1,f_1+f_2)$ are not contained in N, more over

$$\Gamma = N + (1,f_1)N + (1,f_2)N + (1,f_1+f_2)N$$

is a left coset decomposition. Therefore $|\Gamma : N| = 4$. We shall show that the center of N is again $\{(1,0),(-1,0)\}$. Let (α, g) be an element of the center of N. Then (α, g) commutes with $(1,f_1)$ and $(1,f_2)$, therefore (α, g) commutes with every element in Γ, i.e. $(\alpha, g) = (\pm 1, 0)$. Since $|N| = 2^{2(r-1)+1}$ this means that N is isomorphic to H_{r-1}. We denote by $\sum$ the subgroup $\{g \mid (1,g) \in N\}$ and by ρ an isomorphism of Δ onto $\Delta_1 \oplus \Delta_{r-1} \oplus \Delta_1 \oplus \Delta_{r-1}$ such that

1) $\rho(f_1) \in \Delta_1$, $\Delta(f_2) \in \Delta_1$,
2) ρ induces an isomorphism of N onto H_{r-1}.

This isomorphism induces an isomorphism of Γ onto H_r.

We denote by $\{\pm U_{f+\hat{f}} \mid f + \hat{f} \in \Delta_r \oplus \hat{\Delta}_r\}$ the irreducible representation of the Heisenberg group H_r defined as follows.

$$U_{f+\hat{f}} = (u_{g,h}(f+\hat{f}))_{g \in \Delta_r, h \in \Delta_r}$$
$$u_{g,h}(f+\hat{f}) = \langle \hat{f}, g \rangle \delta_{g+f,h} \quad (f,g,h \in \Delta_r, \hat{f} \in \hat{\Delta}_r).$$

This is the only irreducible representation of H_r whose degree is greater than one, because

$$H = \{(1,0)\} + \{(-1,0)\} \sum_{f+\hat{f} \neq 0} \{(1,f+\hat{f}),(-1,f+\hat{f})\}$$

is the conjugate class decomposition.

THEOREM (1. 2. 8). *Let G be a finite additive group of odd order $|G|$ such that $(ch(K), 2|G|) = 1$, and $e = (e_a)_{a \in G}$ be a point in P_G having the following properties*

1) $e_{-a} = e_a \quad (a \in G)$,
2) $rank\,(e_{-a+b}e_{a+b})_{a \in G,\, b \in G} = 2^r$

3) *there exists a group of 2-division points of e of order* 4^r *whose associated 2-step nilpotent group has the center of order* 2.

Let Δ_r *be the additive group of type* $(\overbrace{2, \cdots, 2}^{r})$ *and* $\hat{\Delta}_r$ *be its dual. Then there exist an isomorphism of* $\Delta_r \oplus \hat{\Delta}_r$ *onto the group of 2-division points* $f + \hat{f} \longrightarrow e(f + \hat{f})$, *normalized homogeneous coordinates* $(e_a(f + \hat{f}))_{a \in G} (f + \hat{f} \in \Delta_r \oplus \hat{\Delta}_r)$ *and a* $|G| \times 2^r$*-matrix* S_0 *such that*

$$(1.2.8.1) \qquad (e_{-a+b} e_{a+b})_{a \in G,\, b \in G} = S_0{}^t S_0$$

$$(1.2.8.2) \qquad (e_{-a+b}(f + \hat{f}) e_{-a+b}(f + \hat{f}))_{a \in G,\, b \in G} = S_0{}^t U_{f+\hat{f}}{}^t S_0$$

$$(1.2.8.3) \qquad \varepsilon_{e(f+\hat{f})} = \langle \hat{f}, f \rangle$$

Proof. Let Δ be an additive group of exponent two such that $\{e(g) | g \in \Delta\}$ is the given group of 2-division points and $g \longrightarrow e(g)$ is an isomorphism. By the assumption the two step nilpotent group $\Gamma(\Delta)$ has the center $\{(1, 0), (-1, 0)\}$. Hence, by virtue of Lemma (1. 2. 7) $\Gamma(\Delta)$ is isomorphic to the Heisenberg group H_r. Namely we may choose an isomorphism of $\Delta_r \oplus \hat{\Delta}_r$ onto the given group of 2-division points

$$f + \hat{f} \longrightarrow e(f + \hat{f})$$

and normalized homogeneous coordinates $(e_a(f + \hat{f}))$ such that the map $(\alpha, f + \hat{f}) \longrightarrow \alpha M_{e(f+\hat{f})}$ is a faithful representation of the Heisenberg group H_r. Hence there exists an orthogonal matrix P such that $M_{e(f+\hat{f})} = P U_{f+\hat{f}} P^{-1}$ $(f + \hat{f} \in \Delta_r \oplus \hat{\Delta}_r)$, because $\{\pm M_{e(f+f)}\}, \{\pm U_{f+\hat{f}}\}$ are equivalent orthogonal representations of H_r. Putting $S_0 = SP$, we have

$$(e_{-a+b} e_{a+b})_{a \in G,\, b \in G} = S^t S = S_0{}^t S_0 ,$$
$$(e_{-a+b}(f + \hat{f}) e_{a+b}(f + \hat{f}))_{a \in G,\, b \in G} = S M_{e(f+\hat{f})}{}^t S = S_0{}^t U_{f+\hat{f}}{}^t S_0 .$$

Since $(1, f + \hat{f})(1, f + \hat{f}) = (\langle \hat{f}, f \rangle, 0)$ and $M_{e(f+\hat{f})} = \varepsilon_{e(f+f)} I$, we have $\varepsilon_{e(f+f)} = \langle \hat{f}, f \rangle$.

3. Specialization of group of 2-division points.

Since $(\alpha, f + \hat{f}) \longrightarrow \alpha U_{f+\hat{f}}$ is an irreducible representation of the Heisenberg group H_r, the matrices $U_{f+\hat{f}}$ $(f + \hat{f} \in \Delta_r \oplus \hat{\Delta}_r)$ are linearly independent and they generate the $m \times m$-full matric algebra over the prime field k_0.

LEMMA (1. 3. 1). *If* $S=(s_{a,g})_{a\in G,\, g\in\Delta_r}$ *satisfies* $(e_{-a+b}(f+\hat{f})e_{a+b}(f+\hat{f}))_{a\in G,\, b\in G} = S^t U_{f+\hat{f}}{}^t S$. $(f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r)$, *then specializations* $S\longrightarrow S'$ *of* S *correspond one-to-one to specializations of* $e(f+\hat{f})\longrightarrow e'(f+\hat{f})$ $(f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r)$ *where specializations* $S\longrightarrow S'$ *mean specializations as points of a projective space.*

Proof. By the above consideration, quadratic monomials

$$\{s_{a,f}s_{a,g}\,|\,a,b\in G;\ g,h\in\Delta_r\}$$

and

$$\{e_a(f+\hat{f})e_b(f+\hat{f})\,|\,a,b\in G;\ f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r\}$$

are mutually linear combinations of the others over the prime field k_0. This proves Lemma.

THEOREM (1. 3. 2). *Let* G *be an additive group of odd order* $|G|$ *such that* $(ch(K), 2|G|)=1$. *Let* $e=(e_a)_{a\in G}$ *be a point in* P_G *such that* $e_{-a}=e_a$. *Assume that rank* $(e_{-a+b}e_{a+b})_{a\in G,\, b\in G}=2^r$ *and there exists a group* $\{e(f+\hat{f})\,|\,f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r\}$ *of* 2*-division points of* e *of order* 4^r *whose associated* 2*-step nilpotent group is the Heisenberg group of dimension* r. *Let* $\{e'(f+\hat{f})\,|\,f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r\}$ *be a specialization of* $\{e(f+\hat{f})\,|\,f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r\}$ *such that* $\{e'(f+\hat{f})\,|\,f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r\}$ *is a group of* 2*-division points of the new origin* $e'=e'(0)$. *Then rank* $(e'_{-a+b}e'_{a+b})=2^s$ *with non-zero* s *and the* 2*-step nilpotent group associated with* $\{e'(f+\hat{f})\,|\,f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r\}$ *is the Heisenberg group of dimensions* s.

Proof. Denote $m=\operatorname{rank}(e'_{-a+b}e'_{a+b})_{a\in G,\, b\in G}$. Then, from the assumption there exist a $|G|\times m$-matrix S'' and orthogonal $m\times m$-matrices $M_{e'(f+\hat{f})}$, $(f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r)$ such that

$$(e'_{-a+b}(f+\hat{f})e'_{a+b}(f+\hat{f}))_{a\in G,\, b\in G}=S''{}^tM_{e'(f+\hat{f})}{}^tS''$$

$$(e'_{-a+b}(f+\hat{f}+g+\hat{g})e_{a+b}(f+\hat{f}+g+\hat{g}))_{a\in G,\, b\in G}$$

$$=S''M_{e'(f+\hat{f})}{}^tM_{e'(g+\hat{g})}{}^tS''.$$

On the other hand, if we denote by S the $|G|\times 2^r$-matrix satisfying

$$(e_{-a+b}(f+\hat{f})e_{a+b}(f+\hat{f}))_{a\in G,\, b\in G}=S^tU_{f+\hat{f}}{}^tS,\ (f+\hat{f}\in\Delta_r\oplus\hat{\Delta}_r),$$

the specialization $e(f+\hat{f})\longrightarrow e'(f+\hat{f})$ is induced by a specialization $S\longrightarrow S'$ as follows:

$$(e'_{-a+b}(f+\hat{f})e'_{a+b}(f+\hat{f}))_{a\in G,\, b\in G} = S'^{t}U_{f+\hat{f}}{}^{t}S'.$$

We denote by S''_0 a non-singular minor $m \times m$-matrix in the $|G| \times m$-matrix S'' and by S'_0 the minor $m \times 2^r$-matrix in the $|G| \times 2^r$-matrix S' corresponding to S''_0. Then

$${}^{t}M_{e'(f+\hat{f})} = (S''^{-1}_0 S'_0)^{t} U_{f+\hat{f}}{}^{t}(S''^{-1}_0 S'_0).$$

Since $U_{f+\hat{f}}$ $(f+\hat{f} \in \varDelta_r \oplus \hat{\varDelta}_r)$ generate the full $2^r \times 2^r$-matric algebra, the orthogonal matrix $M_{e'(f+\hat{f})}(f+\hat{f} \in \varDelta_r \oplus \hat{\varDelta}_r)$ generate the full $m \times m$-matric algebra. Hence the 2-step nilpotent group associated with the group $F = \{e'(f+\hat{f}) | f+\hat{f} \in \varDelta_r \oplus \hat{\varDelta}_r\}$ has the center of order two. So, by virtue of (1. 2. 7), it is sufficient to show the order $|F|$ equals to 2^{2s} with a non-negative s. The relation

$$e_{-a}(f+\hat{f}) = <\hat{f}, f> e_a(f+\hat{f})$$

implies

$$e'_{-a}(f+\hat{f}) = <\hat{f}, f>\, e'_a(f+\hat{f}),$$

hence by virtue of (1. 2. 2) we have

$$\begin{aligned} M^{-1}_{e'(f+\hat{f})} M_{e'(g+\hat{g})} M_{e'(f+\hat{f})} &= <\hat{f}, f> <\hat{g}, g> <\hat{f}+\hat{g}, f+g> M_{e'(g+\hat{g})} \\ &= <\hat{f}, g> <\hat{g}, f> M_{e'(g+\hat{g})}. \end{aligned}$$

The matric group $\{\pm M_{e'(f+\hat{f})} | f+\hat{f} \in \varDelta_r \oplus \hat{\varDelta}_r\}$ is an irreducible faithful representation of $\varGamma$ and the quotient of $\varGamma$ by its center is isomorphic to F. Hence

$$\begin{aligned} &\{\pm M_{e'(f+\hat{f})} | f+\hat{f} \in \varDelta_r \oplus \hat{\varDelta}_r\} \\ &\qquad = \{I\} + \{-I\} + \sum_{e'(f+\hat{f}) \neq e'} \{\pm M_{e'(f+\hat{f})}\} \end{aligned}$$

is the conjugate class decomposition of $\varGamma$. Therefore the number of conjugate classes of $\varGamma$ equals to $|F|+1$. On the other hand there exist $|F|$ irreducible characters of degree one of $\varGamma$, hence there exists only one higher degree irreducible character that is the trace of $\{\pm M_{e'(f+\hat{f})} | f+\hat{f} \in \varDelta_r \oplus \hat{\varDelta}_r\}$.

This shows the relation

$$2|F| = |\Gamma| = F + m^2.$$

Namely the order $|F|$ of $\{e'(f+\hat{f}) | f+\hat{f} \in \Delta_r \oplus \hat{\Delta}_r\}$ is a square number $= 2^{2s}$.

4. Problems.

Denote by A_e the projective variety in P_G consisting of all the points x such that for every 2-division point $e(f)$ of e the composition $e(f) \circ x$ is well-defined.

Then the following are, in some sense, very big problems:

PROBLEM. Under what condition on e is A_e an abelian variety of dimension r with the composition given by e ?

PROBLEM. Is the condition $\operatorname{rank}(e_{-a+b}e_{a+b})_{a\in G,\, b\in G} = 2^r$ necessary for A_e to be an abelian variety of dimension r ?

PROBLEM. Under what condition on e has A_e a group of 2-division points of e whose associated 2-step milpotent group is the Heisenberg group of dimension r ?.

§ 2. Symmetric theta structures on abelian varieties.

1. Symmetric theta structures.

Let A be an abelian variety and X be a divisor on A. We mean by g_X the subgroup of A consisting of all the points t such that $X_t \sim X$, i.e. there exists a function f such that $(f) = X_t - X$. When g_X is a finite group, the divisor X is called to be non-degenerate.

Let X be a non-degenerate positive divisor such that $l(X)$ is coprime to the characteristic $ch(K)$, where $l(X) = \dim|X| + 1$. Then the order of g_X is exactly $l(X)^2$ and there exists a non-degenerate skew symmetric bicharacter $e_X(s,t)$ on g_X, i.e. $e_X(s,t)$ is a function on $g_X \times g_X$ with values roots of unity in the universal domain such that

$$e_X(s+s',t) = e_X(s,t)e_X(s',t),$$
$$e_X(s,t+t') = e_X(s,t)e_X(s,t'),$$
$$e_X(s,t)e_X(t,s) = 1, \quad e_X(s,s) = 1 \qquad (s,s',t,t' \in g_X)$$

and moreover $e_X(s,*)\equiv 1$ if and only if $s=0$.

A divisor X is called to be symmetric if $(-\delta_A)^{-1}(X)=X$, where $-\delta_A$ is the involution $u \longrightarrow -u$. We shall give a definition of a symmetric theta structure which is more concrete as compared with the Mumford's original definition in its expression.

DEFINITION (2. 1. 1). Let G be a finite additive group of which order $|G|$ is coprime to the characteristic $ch(K)$. *A symmetric G-theta structure* means a pair (X, ρ) of a symmetric positive divisor X on an abelian variety A and an isomorphism ρ of $G\oplus\hat{G}$ onto the subgroup $g_X=\{t\in A|X_t\sim X\}$ such that

(2. 1. 1. 1) $\qquad X_{\hat{a}\rho}=X,$

(2. 1. 1. 2) $\qquad \langle\!\langle a,\hat{a}\rangle\!\rangle = e_X(a\rho,\hat{a}\rho)=e_X(-\hat{a}\rho,a\rho),$

(2. 1. 1. 3) $\qquad e_X(a\rho,b\rho)=e_X(\hat{a}\rho,\hat{b}\rho)=1$

$$(a,b\in G;\ \hat{a},\hat{b}\in\hat{G}),$$

where $\hat{G}$ means the dual of G and $\langle\!\langle a,\hat{a}\rangle\!\rangle$ means the pairi which defines $\hat{G}$.

THEOREM (2. 1. 2). *Let (X,ρ) be a symmetric G-theta structure on an abelian variety A such that $l(X)$ and the order $|G|$ are coprime to $2ch(K)$. Then there exists a unique system of functions $\varphi_a(u)$ $(a\in G)$ on A such that*

(2. 1. 2. 1) $\qquad (\varphi_a)+X\succ 0,$

(2. 1. 2. 2) $\qquad \varphi_0(u)\equiv 1,\quad \varphi_{-a}(u)=\varphi_a(-u),$

(2. 1. 2. 3) $\qquad \varphi_{a+b}(u)=\varphi_a(u+b\rho)\varphi_b(u),$

(2. 1. 2. 4) $\qquad \varphi_a(u+\hat{a}\rho)=\langle\!\langle a,\hat{a}\rangle\!\rangle\varphi_a(u)$

$$(a,b\in G;\ \hat{a},\hat{b}\in\hat{G}).$$

Proof. From the definition of g_X there exists a system of functions $f(u)$ $(a\in G)$ such that $(f_a)=X_{-a\rho}-X$. The zero divisor of $f_a(u+b\rho)$ as a function in u is given by

$$X_{-(a+b)\rho}-X_{-b\rho}=(X_{-(a+b)\rho}-X)-(X_{-b\rho}-X)$$
$$=(f_{(a+b)\rho})-(f_{b\rho})=(f_{(a+b)\rho}f_{b\rho}^{-1}),\quad (a,b\in G).$$

This implies that

$$f_{a+b}(u)=\gamma_{a,b}f_a(u+b\rho)f_b(u)\qquad(a,b\in G)$$

with non-zero constants $\gamma_{a,b}$; this γ can be regarded as a 2-cocycle of G with coefficients in the multiplicative group i.e.,

$$\partial\gamma(a,b,c)=\gamma_{b,c}\gamma_{a+b,c}^{-1}\gamma_{a,b+c}\gamma_{a,b}^{-1}=1,\quad(a,b,c\in G).$$

Since the 2-cohomology of a finite group with coefficients in the multiplicative group of an algebraically closed field is always trivial, hence there exist non-zero constants β_a $(a\in G)$ such that

$$\gamma_{a,b}=\partial\beta(a,b)=\beta_{a+b}^{-1}\beta_a\beta_b\qquad(a,b\in G).$$

Putting $\phi_a(u)=\beta_a^{-1}f_a(u)$ $(a\in G)$, we obtain a system of functions $\phi_a(u)$ $(a\in G)$, such that

$$\begin{aligned}&(\phi_a)+X\succ 0,\quad \phi_0(u)\equiv 1,\\ &\phi_{a+b}(u)=\phi_a(u+b\rho)\phi_b(u),\quad(a,b\in G).\end{aligned}$$

Let π be the natural isogeny of A onto the quotient abelian variety $B=A/\hat{G}\rho$ of A by $\hat{G}\rho$. Then, since $X_{\hat{a}\rho}=X$ $(\hat{a}\in\hat{G})$, there exists a positive divisor U on B such that $\pi^{-1}(U)=X$. From the symmetricity it follows the symmetricity of U and the equality $X_{\hat{a}\rho}=X$ $(\hat{a}\in\hat{G})$ implies that there exist non-zero constants $\chi(a,\hat{a})$ such that

$$\begin{aligned}&\phi_a(u+\hat{a}\rho)=\chi(a,\hat{a})\phi_a(u),\\ &\chi(a,\hat{b}+\hat{c})=\chi(a,\hat{b})\chi(a,\hat{c}),\\ &\chi(a+b,\hat{c})=\chi(a,\hat{c})\chi(b,\hat{c}),\quad(a,b\in G;\ \hat{a},\hat{b},\hat{c}\in\hat{G}).\end{aligned}$$

Let n be the degree of the isogeny π and π' be the isogeny of B onto A such that $\pi\pi'=n\delta_B$, and let $F_s(u)$ $(ns=0)$ be the functions on B such that

$$(F_s)=(n\delta_B)^{-1}(U_{-s})-(n\delta_B)^{-1}(U).$$

Let s and t be points in B such that $n^2s=n^2t=0$, $\pi's\in G\rho$, $\pi't\in\hat{G}\rho$. Then by virtue of the definition of $e_{nU}(s,t)=e_{U,n}(s,t)$ and the equation $X=\pi^{-1}(U)$ it follows

$$\phi_{\pi' s}(\pi' u) = \gamma_s F_{ns}(u),$$
$$\phi_{\pi' s}(\pi' u + \pi' t) = \gamma_s F_{ns}(u + t) = \gamma_s E e_{U,n}(ns, t) F_{ns}(u)$$
$$= e_{U,n}(ns, t)\phi_{ns}(\pi' u)$$

with non-zero constants γ_s. Since $\pi'^{-1}(X) = (n\delta_B)^{-1}(U) \equiv n^2 U$, we obtain

$$e_X(\pi' s, \pi' t) = e_{n^2 U}(s, t) = e_{nU}(ns, t) = e_{U,n}(ns, t),$$
$$(n^2 s = 0, \quad \pi' s \in G\rho, \quad \pi' t \in \hat{G}\rho).$$

This means that

$$\phi_a(u + \hat{a}\rho) = e_X(a\rho, \hat{a}\rho)\phi_a(u) = \langle\!\langle a, \hat{a} \rangle\!\rangle \phi_a(u),$$

i.e.,

$$\chi(a, \hat{a}) = \langle\!\langle a, \hat{a} \rangle\!\rangle, \quad (a \in G, \hat{a} \in \hat{G}).$$

From the symmetricity $X = (-\delta_A)^{-1}(X)$ we have $(-\delta_A)^{-1}(X_{-a\rho}) = X_{a\rho}$ $(a \in G)$, hence there exist non-zero constants ρ_a $(a \in G)$ such that

$$\phi_a(-u) = \rho_a \phi_{-a}(u) \qquad (a \in G).$$

Since $\phi_{a+b}(u) = \phi_a(u + b\rho)\phi_b(u)$ $(a \in G)$, we have

$$\rho_{-a-b}\phi_{a-b}(u) = \phi_{a+b}(-u) = \phi_a(-u + b\rho)\phi_b(-u)$$
$$= \rho_{-a}\rho_{-b}\phi_{-a}(u - b\rho)\phi_b(u) = \rho_{-a}\rho_{-b}\phi_{-a-b}(u).$$

This means $\rho_a \rho_b = \rho_{a+b}$ $(a, b \in G)$. Hence, putting

$$\varphi_a(u) = \rho_{\frac{1}{2}a}\varphi_a(u) \qquad (a \in G),$$

we obtain the system of functions in Theorem. Let us prove the uniqueness. Let $\psi_a(u)$ $(a \in G)$ be another system of functionss satisfying the condition in Theorem. Then the quotients $\zeta_a = \psi_a(u)/\varphi_a(u)$ $(a \in G)$ are constants such that $\zeta_{a+b} = \zeta_a \zeta_b$ and $\zeta_a^2 = \zeta_{2a} = 1$ $(a, b \in G)$. The oddness of $|G|$ implies $\zeta_a = 1$ $(a \in G)$. This completes the proof of Theorem.

DEFINITION (2. 1. 3). Let (X, ρ) be a symmetric G-theta structure on an abelian variety A such that the order $|G|$ is coprime to $2ch(K)$. The system of functions $\varphi_a(u)$ $(a \in G)$ in Theorem (2. 1. 2) is called *the canonical system of functions on A associated with a symmetric G-theta structure* (X, ρ). If X is very ample the canonical system of functions $\varphi_a(u)$ $(a \in G)$ defines a projective embedding of A into P_G. We call the embedding *the projective embedding of A associated with a symmetric G-theta structure* (X, ρ) and we call the image of the origin *the system of symmetric theta null-values.* We use the same notation $u \longrightarrow \varphi(u)$ for the projective embedding of A into P_G.

PROPOSITION (2. 1. 4). *Let (X, ρ) be a symmetric G-theta structure on A such that $|G|$ is coprime to $2ch(K)$ and X is a very ample divisor satisfying that the* map $\alpha : \{f|(f)+X \succ 0\} \otimes \{f|(f)+X \succ 0\} \longrightarrow \{g|(g)+2X \succ 0\}$ *is surjective. Let $\varphi_a(u)$ $(a \in G)$ be the canonical system of functions on A associated with (X, ρ). Then the dimension of the linear space spaned by $\varphi_{-a}(u)\varphi_a(u)$ $(a \in G)$ is given by* $2^{\dim A}$.

Proof. The linear space of all the quadratic forms in $\varphi_a(u)$ $(a \in G)$ is the linear space corresponding to the complete linear system $|2X|$. Hence the dimension of the linear space is given by

$$l(2X) = 2^{\dim A} l(X) = 2^{\dim A}|G|.$$

From the oddenss of $|G|$ it follows that the functions $\varphi_{-a+b}(u)\varphi_{a+b}(u)$ $(a, b \in G)$ span the whole linear space. On the other hand

$$\begin{aligned}
&\varphi_{-a+b}(u)\varphi_{a+b}(u) = \varphi_{-a}(u+b\rho)\varphi_a(u+b\rho)\varphi_{-b}(u)^2, \\
&\varphi_{-a+b}(u+\hat{c}\rho)\varphi_{a+b}(u+\hat{c}\rho) = \langle\!\langle 2b, \hat{c} \rangle\!\rangle\, \varphi_{-a+b}(u)\varphi_{a+b}(u) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a, b \in G;\ \hat{c} \in \hat{G}),
\end{aligned}$$

hence, if we denote by r the dimension of the linear space spaned by $\varphi_{-a}(u)\varphi_a(u)$ $(a \in G)$, then we have

$$2^{\dim A}|G| = l(2X) = r|G|.$$

This proves $r = 2^{\dim A}$.

2. Addition formula.

We shall show that, if $\varphi(0)$ is the system of G-theta nullvalues associated with a symmetric G-theta structure on an abelian variety A, then

$$\operatorname{rank}(\varphi_{-a+b}(0)\varphi_{a+b}(0))_{a\in G,\, b\in G} = 2^{\dim A}$$

and

$$\varphi(u) \circ \varphi(v) = \varphi(u+v),$$

where $\circ$ is the commutative composition with respect to $\varphi(0)$.

LEMMA (2. 2. 1). *Let $(u, v) \longrightarrow \sigma(u, v) = (-u+v, u+v)$ be the endomorphism of $A \times A$, and let X a symmetric divisor on A. Then*

$$\sigma^{-1}(X \times A + A \times X) \sim 2(X \times A + A \times X).$$

Proof. Denote by k a field over which A is defined and X is rational. Let u and v be independent generic points over k. Then $X_v + X_{-v} - 2X \sim 0$ by Theorem 30 Corollary 2, § 8 [III], i.e., there exists a function $f(u)$ on A defined over $k(v)$ such that $(f) = X_v + X_{-v} - 2X$. Putting $F(u, v) = f(u)$, we have a function in u and v defined over k such that

$$(F)(A \times v) = (f) \times v$$

by Theorem 1 Corollary 3, VIII [IV]. Since $(-\delta_A)^{-1}(X_v) = X_{-v}$, it follows that

$$\begin{aligned} &\sigma^{-1}(X \times A + A \times X)(A \times v) = ((-\delta_A)^{-1}(X_v) + X_v) \times v \\ &= (X_{-v} + X_v) \times v. \end{aligned}$$

The divisor $\sigma^{-1}(X \times A + A \times X)$ has no component of the type $A \times Y_1$, hence we can conclude that

$$(F) = \sigma^{-1}(X \times A + A \times X) - 2X \times A - A \times Y$$

with a divisor Y on A. Let $\tau = (-\delta_A, \delta_A)\sigma$ be the endomorphism of $A \times A$ such that $\tau(u, v) = (u - v, u + v)$. Then by virtue of the symmetricity of X we have

$$\tau^{-1}(X \times A + A \times X) = \sigma^{-1}(X \times A + A \times X).$$

Therefore, exchanging (u, v) and σ by (v, u) and τ, we obtain similarly a function $F_1(u, v)$ such that

$$\begin{aligned}(F_1) &= \tau^{-1}(X \times A + A \times X) - Y_1 \times A - 2A \times X \\ &= \sigma^{-1}(X \times A + A \times X) - Y_1 \times A - 2A \times X\end{aligned}$$

with a divisor Y_1. This means

$$Y_1 \times A + 2A \times X \sim 2X \times A + A \times Y$$

and

$$Y_1 \sim 2X \sim Y.$$

Therefore finally we get

$$\sigma^{-1}(X \times A + A \times X) \sim 2(X \times A + A \times X).$$

THEOREM (2. 2. 2). (*Addition formula*). Let (X, ρ) *be a symmetric G-theta structure on an abelian variety A such that* $|G|$ *is coprime to* $2ch(K)$ *and X is a very ample divisor satisfying that the* map $\alpha : \{f|(f) + X \succ 0\} \otimes \{f|(f) + X \succ 0\} \longrightarrow \{g|(g) + 2X \succ 0\}$ *is surjective. Let* $u \longrightarrow \varphi(u)$ *be the projective embedding of A into* P_G *associated with* (X, ρ). *Then*

$$\varphi(u) \circ \varphi(v) = \varphi(u + v)$$

and

$$rank\, (\varphi_{-a+b}(0)\varphi_{a+b}(0))_{a \in G,\, b \in G} = 2^{\dim A},$$

where $\circ$ *is the commutative compositon with respect to* $\varphi(0)$.

Proof. For the sake of exactness we shall make the distinction between the components $\varphi_a(u)$ $(a \in G)$ of the projective embedding $u \longrightarrow \varphi(u)$ and the elements $\bar{\varphi}_a(u)$ $(a \in G)$ of the canonical system of functions associated with (X, φ) i.e., the fuctions such that

$$\begin{aligned}&(\bar{\varphi}_a) + X \succ 0, \quad \bar{\varphi}_0(u) \equiv 1, \quad \bar{\varphi}_{-a}(u) = \bar{\varphi}_a(-u), \\ &\bar{\varphi}_{a+b}(u) = \bar{\varphi}_a(u + b\rho)\bar{\varphi}_b(u), \quad \bar{\varphi}_a(u + \hat{c}\rho) = \langle\!\langle a, \hat{c} \rangle\!\rangle\, \bar{\varphi}_a(u), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (a, b \in G, \hat{c} \in \hat{G}).\end{aligned}$$

By virtue of (2. 2. 1) there exists a function $F(u,v)$ such that

$$(F) = \sigma^{-1}(X \times A + A \times X) - 2(X \times A + A \times X),$$

where σ is the endomorphism $(u,v) \longrightarrow (-u+v, u+v)$. First of all, putting

$$\Phi_{-a,b}(u,v) = \bar{\varphi}_{-a+b}(-u+v)\bar{\varphi}_{a+b}(u+v)F(u,v)$$

and

$$\Psi_{a,b}(u,v) = F(u+a\rho, v+b\rho)\bar{\varphi}_a(u)^2\bar{\varphi}_b(v)^2,$$

we shall show

$$\Phi_{a,b}(u,v) = \Psi_{a,b}(u,v), \qquad (a,b \in G).$$

By virtue of the symmetricity of X it follows

$$\begin{aligned}&(-\delta_{A\times A})^{-1}(\sigma^{-1}(X \times A + A \times X) - 2(X \times A + A \times X))\\&= \sigma^{-1}(X \times A + A \times X) - 2(X \times A + A \times X),\end{aligned}$$

hence we can conclude that $F(-u,-v) = \varepsilon F(u,v)$ with a non-zero constant ε. From the definitions we can caluculate the divisors of $\Phi_{a,b}$ and $\Psi_{a,b}$:

$$\begin{aligned}(\Phi_{a,b}) &= \sigma^{-1}(X_{-(a,b)} \times A + A \times X_{-(a+b)}) - 2(X \times A + A \times X)\\(\Psi_{a,b}) &= (\sigma^{-1}(X \times A + A \times X))_{(-a,-b)} - 2(X \times A + A \times X)\\&= \sigma^{-1}(X_{a-b} \times A + A \times X_{-a-b}) - 2(X \times A + A + X)\\&= (\Phi_{a,b}).\end{aligned}$$

Therefore there exist non-zero constants $\xi_{a,b}$ such that

$$\Phi_{a,b}(u,v) = \xi_{a,b}\Psi_{a,b}(u,v), \quad (a,b \in G).$$

Let us show the relation

$$\xi_{a,b}\xi_{a',b'} = \xi_{a+a',\, b+b'}, \quad (a,a',b,b' \in G).$$

From the definition we have

$$\begin{aligned}&\xi_{a,b}\Phi_{a+a',b+b'}(u,v)\\&= \xi_{a,b}F(u+(a+a')\rho, v+(b+b')\rho)\bar{\varphi}_{a+a'}(u)^2\bar{\varphi}_{b+b'}(v)^2\\&= \xi_{a,b}F(u+(a+a')\rho, v+(b+b')\rho)\bar{\varphi}_a(u+a\rho)^2\bar{\varphi}_b(v+b'\rho)^2\bar{\varphi}_a(u)^2\bar{\varphi}_b(v)^2\\&= \Psi_{a,b}(u+a'\rho, v+b'\rho)\bar{\varphi}_{a'}(u)^2\bar{\varphi}_{b'}(v)^2\end{aligned}$$

$$= \bar{\varphi}_{-a+b}(-u+v+(-a'+b')\rho)\bar{\varphi}_{a+b}(u+v+(a'+b')\rho)F(u+a'\rho, \\ v+b'\rho)\bar{\varphi}_{a'}(u)^2\bar{\varphi}_{b'}(v)^2$$

$$= \frac{\bar{\varphi}_{-(a+a')+(b+b')}(-u+v)\bar{\varphi}_{(a+a')+(b+b')}(u+v)F(u,v)\Psi_{a',b'}(u,v)}{\bar{\varphi}_{a'+b'}(-u+v)\bar{\varphi}_{a'+b'}(u+v)F(u,v)}$$

$$= \frac{\Phi_{a+a',\,b+b'}(u,v)}{\Phi_{a'b'}(u,v)}\Psi_{a',b'}(u,v) = \frac{\xi_{a+a',\,b+b'}}{\xi_{a',b'}}\Psi_{a+a',\,b+b'}(u,v).$$

This means

$$\xi_{a+a',\,b+b'} = \xi_{a,b}\xi_{a',b'}, \quad (a, a', b, b' \in G).$$

We shall next show

$$\xi_{-a,-b} = \xi_{a,b}, \quad (a, b \in G).$$

Since

$$F(-u, -v) = \varepsilon F(u,v)$$

and

$$\bar{\varphi}_{-a}(u) = \bar{\varphi}_a(-u) \qquad (a \in G),$$

it follows

$$\begin{aligned}
&\xi_{a,b}F(-u+a\rho, -v+b\rho)\bar{\varphi}_a(-u)^2\bar{\varphi}_b(-v)^2 \\
&= \xi_{a,b}\Psi_{a,b}(-u,-v) = \Phi_{a,a}(-u,-v) \\
&= \bar{\varphi}_{-a+b}(u-v)\bar{\varphi}_{a+b}(-u-v)F(-u,-v) \\
&= \varepsilon\bar{\varphi}_{a-b}(-u+v)\bar{\varphi}_{a+b}(u+v)F(u,v) \\
&= \varepsilon\Phi_{-a,-b}(u,v) = \varepsilon\xi_{-a,-a}\Psi_{-a,-b}(u,v) \\
&= \xi_{-a,-b}\varepsilon F(u-a\rho, -v+b\rho)\bar{\varphi}_{-a}(u)^2\bar{\varphi}_{-b}(v)^2 \\
&= \xi_{-a,-b}F(-u+a\rho, -v+b\rho)\bar{\varphi}_{-a}(-u)^2\bar{\varphi}_{-b}(-v)^2.
\end{aligned}$$

This implies

$$\xi_{-a,-b} = \xi_{a,b} \qquad (a, b \in G).$$

and

$$1 = \xi_{-a,-b}\xi_{b,b} = \xi_{a,b}\xi_{a,b} = \xi_{2a,\,2b} \qquad (a, b \in G).$$

Since the order $|G|$ is odd, it follows $\xi_{a,b} = 1$

and

$$\bar{\varphi}_{-a+b}(-u+v)\bar{\varphi}_{a+b}(u+v)F(u,v)=F(u+a\rho,v+b\rho)\bar{\varphi}_a(u)^2\bar{\varphi}_b(v)^2,$$
$$(a,b\in G).$$

By virtue of (2. 1. 4) there exists a subset H of G such that the cardinal $|H|$ is $2^{\dim A}$ and the functions $\bar{\varphi}_{-a^+}(u)\bar{\varphi}_{a^+}(u)$ $(a^+\ni H)$ are linearly independent. Let π be the natural isogeny of A onto the quotient $B=A/\hat{G}\rho$ and U be the positive divisor on B such that $\pi^{-1}(U)=X$. Denoting by the same symbol σ the endomorphism $(\bar{u},\bar{v})\longrightarrow(-\bar{u}+\bar{v},\bar{u}+\bar{v})$ of $B\times B$, by virtue of (2. 2. 1) we have a function on $B\times B$ such that

$$(f)=\sigma^{-1}(U\times B+B\times U)-2(U\times B+B\times U).$$

Since

$$\begin{aligned}(F)&=\sigma^{-1}(X\times A+A\times X)-2(X\times A+A\times X)\\&=(\pi,\pi)^{-1}(\sigma^{-1}(U\times B+B\times U)-2(U\times B+B\times U))\\&=(\pi,\pi)^{-1}((f)),\end{aligned}$$

we may assume that $F(u,v)=f(\pi u,\pi v)$.
From the relation

$$\bar{\varphi}_{-a^+}(u+\hat{a}\rho)\bar{\varphi}_{a^+}(u+\hat{a}\rho)=\bar{\varphi}_{-a^+}(u)\bar{\varphi}_{a^+}(u)\qquad(a^+\in H,\hat{a}\in\hat{G})$$

we can conclude that there exist linearly independent functins $g_{a^+}(u)$ $(a^+\in H)$ such that

$$\bar{\varphi}_{-a^+}(u)\bar{\varphi}_{a^+}(u)=g_{a^+}(\pi u),\quad(a^+\in H^+).$$

Since

$$l(2U)=2^{\dim A}l(U)$$

and

$$|G|=\sqrt{l(X)}=\deg(\pi)=|\hat{G}|\,l(U),$$

we can conclude that $l(2U)=2^{\dim A}$ and $g_{a^+}(u)$ $(a^+\in H)$ form a linear base of the space of functions on B corresponding to the linear system $|2U|$. The functions $g_{-a^+}(u)g_{a^+}(u)$ $(a^+,b^+\in H)$ form a linear base of the functions corresponding to the linear system $2(U\times B+B\times U)$ on $B\times B$, hence there exist constants $\alpha_{a^+,b^+}(a^+,b^+\in H)$ such that

$$f(\pi u, \pi v) = \sum_{c^+, d^+ \in H} \alpha_{c^+, d^+} g_{c^+}(\pi u) g_{d^+}(\pi v)$$

and

$$F(uv) = \sum_{c^+, d^+ \in H} \alpha_{c^+, d^+} \bar{\varphi}_{-c^+}(u) \bar{\varphi}_{c^+}(u) \varphi_{-d^+}(v) \varphi_{d^+}(v).$$

By virtue of the relation $(*)$, translating the variables u and v, we have

$$\begin{aligned} &\bar{\varphi}_{-a+b}(-u+v)\bar{\varphi}_{a+b}(u+v)F(u,v) \\ &= F(u+a\rho, v+b\rho)\bar{\varphi}_b(u)^2\varphi_b(v)^2 \\ &= \sum_{c^+, d^+ \in H} \alpha_{c^+d^+}\bar{\varphi}_{-c^+}(u+a\rho)\bar{\varphi}_{c^+}(u+a\rho)\bar{\varphi}_{-d^+}(v+b\rho)\bar{\varphi}_{d^+}(v+b\rho)\bar{\varphi}_d(u)^2\bar{\varphi}_b(v)^2 \\ &= \sum_{c^+, d^+ \in H} \alpha_{c^+, d^+}\bar{\varphi}_{-c^++a}(u)\bar{\varphi}_{c^++a}(u)\bar{\varphi}_{-d^++b}(v)\bar{\varphi}_{d^++b}(v). \end{aligned}$$

In the words of the homogeneous coordinates $(\varphi_a(u))_{a\in G}$ of $\varphi(u)$ this relation can be expressed as follows

$$(**) \qquad \begin{aligned} &\varphi_{-a+b}(-u+v)\varphi_{a+b}(u+v) \\ &= \gamma(u,v) \sum_{c^+, d^+ \in H} \alpha_{c^+, d^+}\varphi_{-c^++a}(u)\varphi_{c^++a}(u)\varphi_{-d^++b}(v)\varphi_{d^++d}(v) \qquad (a, b \in G). \end{aligned}$$

with a non-zero $\gamma(u,v)$, where $\gamma(u,v)$ depends on the homogeneous coordinates $(\varphi_a(u))_{a\in G}$, $(\varphi_a(v))_{a\in G}$, $(\varphi_a(-u+v))_{a\in G}$, $(\varphi_a(u+v))_{a\in G}$. Replacing $F(u,v)$ and α_{c^+,d^+} by $\gamma(0,0)^{-1}F(u,v)$ and $\gamma(0,0)^{-1}\alpha_{c^+,d^+}$, we may assume that

$$(***) \qquad \varphi_{-a+b}(0)\varphi_{a+b}(0) = \sum_{c^+, d^+ \in H} \alpha_{c^+, d^+}\varphi_{-c^++a}(0)\varphi_{c^++a}(0)\varphi_{-d^++b}(0)\varphi_{d^++b}(0), \qquad (a, b \in G).$$

We shall next show the relation

$$(****) \qquad (\alpha_{a^+b^+})_{a^+\in H,\, b^+\in H} = (\varphi_{-a^++b^+}(0)\varphi_{a^++b^+}(0))^{-1}_{a^+\in H,\, b^+\in H}\,.$$

We mean by l the exponent of G and O_l the subring of the rational number field Q consisting of all the elements of which denominators are not divided by l. We denote by Ω the 2r-times direct sum $Q/O_l \oplus Q/O_l \oplus \ldots \oplus Q/O_l$ of the quotient additive group Q/O_l and by Ω_m the subgroup of Ω consisting of all the elements of which order are at most l^m, were $r = \dim A$. The additive group $G \oplus \hat{G}$ can be regarded as a subgroup of Ω by a fixed monomorphism and the isomorphism ρ of $G \oplus \hat{G}$ onto g_X can be extended to an isomorphism of Ω onto the group of the l-power division points on A as abstruct groups. Since $\rho(\Omega)$ is dense on A in Zariskis sense and the functions $\varphi_{-a^+}(u)\varphi_{b^+}(u)$ $(a^+ \in H)$ are linearly independent, there exists a positive integer m such that $\Omega_m \supset G \oplus \hat{G}$ and

$$\operatorname{rank}(\varphi_{-a^+}(u+b\rho)\varphi_{a^+}(u+b\rho))_{a^+\in H,\ b\in\Omega_m} = 2^{\dim A}.$$

Putting $u=0$, $b=0$ and $v=u+b\rho$ in (**), we have

$$\begin{aligned}&(\varphi_{-a^+}(u+b\rho)\varphi_{a^+}(u+b\rho))_{a^+\in H,\ b\in\Omega_m}\\ &= \gamma(0,0)^t(\varphi_{-a^++b^+}(0)\varphi_{a^++b^+}(0))_{a^+\in H,\ b^+\in H}(\alpha_{a^+,b^+})_{a^+\in H,\ b^+\in H}(\varphi_{-a^+}(u+b\rho)\varphi_{a^+}(u+b\rho))_{a^+\in H,\ b\Omega_m}.\end{aligned}$$

Since

$$|H| = 2^{\dim A},\quad \gamma(0,0)=1$$

and

$$\varphi_{-a^+}(0) = \varphi_{a^+}(0),$$

we can finally conclude that

$$\gamma(0,0)^t(\varphi_{-a^++b^+}(0)\varphi_{a^++b^+}(0))_{a^+\in H,\ b^+\in H}(\alpha_{a^+,b^+}) = I$$

and

$$(\alpha_{a^+,b^+})_{a^+\in H,\ b^+\in H} = (\varphi_{-a^++b^+}(0)\varphi_{a_++b^+}(0))^{-1}_{a^+\in H,\ b^+\in H}.$$

Therefore

$$\begin{aligned}&\operatorname{rank}\begin{pmatrix}(\varphi_{-a+b}(0)\varphi_{a+b}(0))_{a\in G,\ b\in G} & (\varphi_{-+b}(v)\varphi_{a+b}(v))_{a\in G,\ b\in G}\\ {}^t(\varphi_{-a+b}(u)\varphi_{a+b}(u))_{a\in G,\ b\in G} & (\gamma(u,v)\varphi_{-a+b}(-u+v)\varphi_{a+b}(u+v))_{a\in G,\ b\in G}\end{pmatrix}\\ &= \operatorname{rank}(\varphi_{-a+b}(0)\varphi_{a+b}(0))_{a\in G,\ b\in G}.\end{aligned}$$

This proves the relation.

$$\varphi(u)\circ\varphi(v) = \varphi(u+v).$$

§ 3. Symplectic group.

In the present paragraph we shall assmue that the order of G is always coprime to $2ch(K)$. Canonically identifying G with the bidual $\hat{\hat{G}}$, we may put

$$\langle\!\langle \hat{a}, a\rangle\!\rangle = \langle\!\langle a, \hat{a}\rangle\!\rangle \qquad (a\in G,\ \hat{a}\in\hat{G}).$$

If $a \longrightarrow a\sigma$ is a homomorphism of G into another finite additive group H, the dual $\hat{\sigma}$ of σ is the homomorphism of $\hat{H}$ into $\hat{G}$ such that

$$\langle\!\langle a\sigma, \hat{b}\rangle\!\rangle = \langle\!\langle a, \hat{b}\hat{\sigma}\rangle\!\rangle, \qquad (a \in G,\ \hat{b} \in \hat{H}).$$

Each endomorphism $(a, \hat{a}) \longrightarrow (a, \hat{a})\sigma$ of $G \oplus \hat{G}$ has the unique matric representation

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (a, \hat{a})\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (a\alpha + \hat{a}\gamma,\ a\beta + \hat{a}\delta)$$

such that $\alpha, \beta, \gamma, \delta,$ are homomorphisms of G into G, G into $\hat{G}, \hat{G}$ into $G, \hat{G}$ into $\hat{G},$ respectively. The dual $\hat{\sigma}$ of σ is given by the endomorphism of $\hat{G} \oplus G$

$$\hat{\sigma} = \begin{pmatrix} \hat{\alpha} & \hat{\gamma} \\ \hat{\beta} & \hat{\delta} \end{pmatrix}, \quad (\hat{a}, a)\begin{pmatrix} \hat{\alpha} & \hat{\gamma} \\ \hat{\beta} & \hat{\delta} \end{pmatrix} = (\hat{a}\hat{\alpha} + a\hat{\beta},\ \hat{a}\hat{\gamma} + a\hat{\delta}),$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ are the duals of $\alpha, \beta, \gamma, \delta,$ respectively.

1. Symplectic group and its action on P_G.

Let J be the isomorphism of $G \oplus \hat{G}$ onto $\hat{G} \oplus G$ such that

$$(a, \hat{a})J = (-\hat{a}, a).$$

The symplectic group $Sp(G \oplus \hat{G})$ means the group consisting of all the automorphisms

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of $G \oplus \hat{G}$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} J \begin{pmatrix} \hat{\alpha} & \hat{\gamma} \\ \hat{\beta} & \hat{\delta} \end{pmatrix} = J,$$

namely

$$\alpha\hat{\beta} = \beta\hat{\alpha}, \quad \gamma\hat{\delta} = \delta\hat{\gamma},$$
$$\alpha\hat{\delta} - \beta\hat{\gamma} = id_G, \quad \delta\hat{\alpha} - \gamma\hat{\beta} = id_{\hat{G}}.$$

THEOREM (3. 1. 1). *Let G be a finite additive group of which order is coprime to $2ch(K)$. Then the symplectic group $Sp(G \oplus \hat{G})$ acts on the projective space P_G as follows*

$$\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}x\right)_a = |G|^{-1}\sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2}a, -\hat{a}\gg \ll \frac{1}{2}(a\alpha+\hat{a}\gamma),\ a\beta+\hat{a}\delta \gg x_{a\alpha+\hat{a}\gamma}$$

$$\left(a\in G, \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in Sp(G\oplus\hat{G})\right),$$

where $\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}x\right)_a$ *means the a-component of the homogeneous coordinates of the point* $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}x$ *in* P_G.

COROLLARY (3. 1. 2)

$$\left(\begin{pmatrix}\alpha & \beta\\ 0 & \hat{\alpha}^{-1}\end{pmatrix}x\right)_a = \ll \frac{1}{2}a\alpha, a\beta\gg x_{a\alpha},$$

$$\left(\begin{pmatrix}0 & -\hat{\gamma}^{-1}\\ \gamma & 0\end{pmatrix}x\right)_a = |G|^{-1}\sum_{\hat{a}\in\hat{G}} \ll a, -\hat{a}\gg x_{\hat{a}\gamma}.$$

Proof of Theorem (3. 1. 1).

It is sufficient to show

$$\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\right)x = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\left(\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}x\right),$$

$$\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\in SP\,(G\oplus\hat{G})\right).$$

Putting

$$\begin{pmatrix}\alpha'' & \beta''\\ \gamma'' & \delta''\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}$$

and

$$z_a = \left(\begin{pmatrix}\alpha'' & \beta''\\ \gamma'' & \delta''\end{pmatrix}x\right)_a,$$

$$u_a = \left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\left(\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}x\right)\right)_a,$$

we have

$$z_a = |G|^{-1} \sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2} a, -\hat{a} \gg \ll \frac{1}{2}(a\alpha'' + \hat{a}\gamma''), a\beta'' + \hat{a}\delta'' \gg x_{a\alpha''+\hat{a}\gamma}$$

and

$$u_a = |G|^{-1} \sum_{\hat{b},\hat{c}\in\hat{G}} \ll \frac{1}{2} a, -\hat{b} \gg \ll \frac{1}{2}(a\alpha + \hat{b}\gamma), a\beta + \hat{b}\delta \gg \ll \frac{1}{2}(a\alpha + \hat{b}\gamma), -\hat{c} \gg$$

$$\ll \frac{1}{2}(a\alpha + \hat{b}\gamma) + \hat{c}\gamma', (a\alpha + \hat{b}\gamma)\beta' + \hat{c}\delta' \gg x_{(a+\hat{b}\gamma)\alpha'+\hat{c}\gamma)}.$$

Replacing $\hat{c}$ by $\hat{c} + a\beta + \hat{b}\delta$, we can conclude that

$$u_a = |G|^{-1} \sum_{\hat{b},\hat{c}\in\hat{G}} \ll \frac{1}{2} a, -\hat{b} \gg \ll \frac{1}{2}(a\alpha + \hat{b}\gamma), -\hat{c} \gg$$

$$\ll \frac{1}{2}(a\alpha' 4\hat{b}\gamma'' + c\gamma'), a\beta'' + \hat{b}\delta'' + \hat{c}\delta' \gg x_{d\alpha'' f\hat{b}\gamma''+\hat{c}\gamma'}.$$

Since

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix} \begin{pmatrix} \hat{\delta}' & -\hat{\beta}' \\ -\hat{\gamma}' & \hat{\delta}' \end{pmatrix} = \begin{pmatrix} \alpha''\hat{\delta}' - \beta''\hat{\gamma}', & * \\ \gamma''\hat{\delta}' - \delta''\hat{\gamma}', & * \end{pmatrix},$$

it follows

$$\ll \frac{1}{2}(a\alpha + \hat{b}\gamma), -\hat{c} \gg = \ll \frac{1}{2} a(\alpha''\hat{\delta}' - \beta''\hat{\gamma}') + \frac{1}{2} b(\gamma''\hat{\delta}' - \delta''\hat{\gamma}'), -\hat{c} \gg$$

$$= \ll \frac{1}{2}(a\alpha'' + \hat{b}\gamma'') - \frac{1}{2}(a\beta'' + \hat{b}\delta''), -\hat{c} \gg$$

$$= \ll \frac{1}{2}(a\alpha'' + \hat{b}\gamma''), -\hat{c}\delta' \gg \ll \frac{1}{2}(a\beta'' + \hat{b}\delta''), \hat{c}\gamma' \gg.$$

Hence we obtain the relations

$$(*) \qquad u_a = |G|^{-1} \sum_{\hat{b},\hat{c}\in\hat{G}} \ll \frac{1}{2} a, -\hat{b} \gg \ll \frac{1}{2}(a\alpha'' + \hat{b}\gamma''), a\beta'' + \hat{b}\delta'' \gg$$

$$\ll \bar{a}\beta'' + \hat{b}\delta_{,}', \hat{c}\gamma' \gg \ll \frac{1}{2} \hat{c}\gamma', \hat{c}\delta' \gg x_{a\alpha''+\hat{b}\gamma''+\hat{c}\gamma}.$$

Now we shall divide the proof of Theorem into five steps.

STEP 1.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} x \right) = x.$$

Proof. If we put

$$\begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix} = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}^{-1} = \begin{pmatrix}\hat{\delta} & -\hat{\beta} \\ -\hat{\gamma} & \hat{\alpha}\end{pmatrix}$$

in $(*)$, then we get

$$u_a = |G|^{-1} \sum_{\hat{b}, \hat{c} \in \hat{G}} \ll \frac{1}{2} a, \hat{b} \gg \ll \frac{1}{2} a, \hat{b} \gg \ll \hat{b}, -\hat{c}\hat{\gamma} \gg \ll -\frac{1}{2} \hat{c}\hat{\gamma}, \hat{c}\hat{\alpha} \gg x_{a-\hat{c}\hat{\gamma}}.$$

From the orthogonal relation

$$|G|^{-1} \sum_{\hat{b} \in \hat{G}} \ll \hat{b}, -\hat{c}\hat{\gamma} \gg = \begin{cases} 1 & \text{for } \hat{c}\hat{\gamma} = 0 \\ 0 & \text{otherwise} \end{cases}$$

we obtain $u_a = x_a (a \in G)$.

STAP 2. If $\gamma' = 0$, *then*

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \left(\begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix} x \right) = \left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix} \right) x.$$

Proof. Putting $\gamma' = 0$ in $(*)$, we have

$$u_a = |G|^{-1} \sum_{\hat{b} \in \hat{G}} \ll \frac{1}{2} a, -\hat{b} \gg \ll \frac{1}{2} (a\alpha'' + b\beta''), a\beta'' + \hat{b}\delta'' \gg x_{a\alpha'' + \hat{b}\gamma''} = z_a \ (a \in G).$$

STEP 3. *If* $\alpha' = \delta' = 0$ *and* $ker(\delta) = \{0\}$, *then*

$$\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \begin{pmatrix}0 & -\hat{\gamma}'^{-1} \\ \gamma' & 0\end{pmatrix} \right) x = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \left(\begin{pmatrix}0 & -\hat{\gamma}'^{-1} \\ \gamma' & 0\end{pmatrix} x \right).$$

Proof. Putting $\alpha' = \delta' = 0$ in $(*)$, we get.

$$u_a = |G|^{-1} \sum_{\hat{b}, \hat{c} \in \hat{G}} \ll \frac{1}{2} a, -\hat{b} \gg \ll \frac{1}{2} (a\alpha'' + \hat{b}\gamma''), a\beta'' + \hat{b}\delta'' \gg$$
$$\ll a\beta'' + \hat{b}\delta'', \hat{c}\gamma' \gg x_{a\alpha'' + \hat{b}\gamma'' + \hat{c}\gamma'}.$$

Since $\ker(\delta) = \{0\}$, we may replace $\hat{c}$ by $\hat{c}\delta$. Therefore from $\gamma'' = \delta\gamma'$ we get

$$u_a = |G|^{-2} \sum_{\hat{b}, \hat{c} \in \hat{G}} \ll \frac{1}{2} a, -(\hat{b} + \hat{c}) \gg \ll \frac{1}{2} (a\alpha'' + (\hat{b} + \hat{c})\gamma'', a\beta'' + (\hat{b} + \hat{c})\delta'' \gg$$
$$\ll \frac{1}{2} \hat{c}\gamma'' - \hat{c}\delta'' \gg \ll \frac{1}{2} a, \hat{c} \gg \ll \frac{1}{2} (a\beta'' + \hat{b}\delta''), \hat{c}\gamma'' \gg$$
$$\ll \frac{1}{2} (a\alpha'' + \hat{b}\gamma''), -\hat{c}\delta'' \gg x_{a\alpha'' + (s + \hat{c})\gamma''}.$$

On the other hand

$$\ll\frac{1}{2}a,\hat{c}\gg\ll\frac{1}{2}(a\beta''+\hat{b}\delta''),\ \hat{c}\gamma''\gg\ll\frac{1}{2}(a\alpha''+\hat{b}\gamma''),\ -\hat{c}\delta''\gg=1,$$

hence

$$u_a=(|G|^{-1}\sum_{\hat{c}\in\hat{G}}\ll\frac{1}{2}\hat{c}\gamma'',-\hat{c}\delta''\gg)z_a\qquad(a\in G).$$

By virtue of the result in Step 1 we know that each element in $S_p(G\oplus\hat{G})$ induces an invertible projective transformation on P_G; this means that

$$|G|^{-1}\sum_{\hat{c}\in\hat{G}}\ll\frac{1}{2}\hat{c}\gamma'',\ -\hat{c}\delta''\gg\neq 0.$$

Therefore $u=z$ as poins in P_G.

STEP 4. If $\alpha'=\delta'=0$ *and* $\ker(\delta)\neq\{0\}$, *then*

$$\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}0 & -\hat{\gamma}'^{-1}\\ \gamma' & 0\end{pmatrix}\right)x=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\left(\begin{pmatrix}0 & -\hat{\gamma}'^{-1}\\ \gamma' & 0\end{pmatrix}x\right)$$

Proof. Putting $\alpha'=\delta'=0$ in (*), we have

$$u_a=|G|^{-2}\sum_{\hat{b},\hat{c}\in\hat{G}}\ll\frac{1}{2}a,-\hat{b}\gg\ll\frac{1}{2}(a\alpha''+\hat{b}\gamma''),\ a\beta''+\hat{b}\delta''\gg$$
$$\ll a\beta''+\hat{b}\delta'',\ \hat{c}\gamma'\gg x_{a\alpha''+\hat{b}\gamma''+\hat{c}\gamma'}\,.$$

From the relation

$$\begin{pmatrix}\alpha'' & \beta''\\ \gamma'' & \delta''\end{pmatrix}=\begin{pmatrix}\beta\gamma' & -\alpha\hat{\gamma}'^{-1}\\ \delta\gamma' & -\gamma\hat{\gamma}'^{-1}\end{pmatrix},$$

we get

$$u_a=|\ker(\delta)|^{-1}|G|^{-2}\sum_{\substack{\hat{d}\in\ker(\delta)\\ \hat{b},\hat{c}\in\hat{G}}}\ll\frac{1}{2}a,\ -(\hat{b}+\hat{d})\gg\ll\frac{1}{2}(a\alpha''+(\hat{b}+\hat{d})\gamma'',$$
$$a\beta''+(\hat{b}+\hat{d})\delta''\gg\ll a\beta''+(\hat{b}+\hat{d})\delta'',\hat{c}\gamma'\gg x_{a\alpha''+(\hat{b}+\hat{d})\gamma''+\hat{c}\gamma'}\,.$$

Since $\delta\hat{\alpha}-\gamma\hat{\beta}=id_{\hat{G}}$, we observe that

$$\ll\hat{d}\gamma,\hat{c}\gg=0\quad(\hat{d}\in\ker(\delta))$$

if and only if $\hat{c}=0$. Therefore we can conclude that

$$u_a = |G|^{-1} \sum_{\hat{b} \in \hat{G}} \ll \frac{1}{2} a, -\hat{b} \gg \ll \frac{1}{2} (a\alpha'' + \hat{b}\gamma''), a\beta'' + \hat{b}\delta'' \gg x_{a\alpha''+\hat{b}\delta''} = z_a \quad (a \in G).$$

STEP 5. Denote by M_1 the subset of $Sp(G \oplus \hat{G})$ consisting of all the elements such that $\gamma = 0$ or $\alpha = \delta = 0$, and denote by M_n $(n = 1, 2, 3, \cdots)$ the subsets of $Sp(G \oplus \hat{G})$ which are the natural images of the products $M_1 \times M_1 \times \cdots \times M_1$ into $Sp(G \oplus \hat{G})$. Since M_1 generates the whole group $Sp(G \oplus \hat{G})$, we have

$$Sp(G \oplus \hat{G}) = \bigcup_n M_n, \qquad M_1 \subset M_2 \subset M_3 \subset \cdots.$$

By virtue of the results in Step 2, 3, 4 it follows that

$$(\sigma\sigma')x = \sigma(\sigma' x), \qquad (\sigma \in Sp(G \oplus \hat{G}),\ \sigma' \in M_1).$$

We shall prove Theorem by the iduction on n. Assume that

$$(\sigma\sigma')x = \sigma(\sigma' x), \quad (\sigma \in Sp(G \oplus \hat{G}),\ \sigma' \in M_{n-1}).$$

If σ is an element in $Sp(G \oplus \hat{G})$ and $\sigma_1', \cdots, \sigma_n'$ be elements in M, then by virtue of the assumption we can conclude that

$$\begin{aligned}
&\sigma((\sigma_1' \cdots \sigma_n')x) = \sigma((\sigma_1' \cdots \sigma_{n-1}')(\sigma_n' x)) \\
&= (\sigma(\sigma_1' \cdots \sigma_{n-1}'))(\sigma_n' x) = (\sigma\sigma_1 \cdots \sigma_{n-1})(\sigma_n' x) \\
&= (\sigma\sigma_1' \cdots \sigma_n')x.
\end{aligned}$$

This completes the long proof of Theorem (3. 1. 1).

2. Action of $Sp(G \oplus \hat{G})$ on commutative compositions.

It will be shown that the action of $Sp(G \oplus \hat{G})$ on P_G carries commutative compositions to commutative compositions.

THEOREM (3. 2. 1). *Let G be a finite additive group of which order is coprime to $2ch(K)$. Let $e = (e_a)_{a \in G}$ be a point in P_G satisfying $e_{-a} = e_a$ $(a \in G)$ and σ be an element in $Sp(G \oplus \hat{G})$. Then the composition $x \circ y$ of two points x, y is well-defined if and only if the composition $\sigma x \circledcirc \sigma y$ is well-defined, where $\circ$ means the composition with respect to e and $\circledcirc$ means the composition with respect to σe. Moreover it follows*

$$\sigma x \circledcirc \sigma y = \sigma(x \circ y)$$

and

$$rank\,((\sigma e)_{-a+b}(\sigma e)_{a+b})_{a\in G,\, b\in G} = rank\,(e_{-a+b}\, e_{a+b})_{a\in G,\, b\in G}.$$

Proof. First of all we shall show that $(\sigma e)_{-a} = (\sigma e)_a$ $(a \in G)$:

$$\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} e\right)_{-a} = |G|^{-1} \sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2}((-a), -\hat{a}\gg \ll \frac{1}{2}((-a)\alpha + \hat{a}\gamma), (-a)\beta + \hat{a}\delta \gg e_{(-a)\alpha+\hat{a}\gamma}$$

$$= |G|^{-1} \sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2} a, -\hat{a}\gg \ll \frac{1}{2}(a\alpha + \hat{a}\gamma), a\beta + \hat{a}\delta \gg e_{a\alpha+\hat{a}\gamma}$$

$$= \left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} e\right)_a .$$

It is sufficient to prove Theorem for the special elements

$$\begin{pmatrix}\alpha & \beta\\ 0 & \hat{\alpha}^{-1}\end{pmatrix}, \quad \begin{pmatrix}0 & -\hat{\gamma}^{-1}\\ \gamma & 0\end{pmatrix}$$

in $Sp(G \oplus \hat{G})$.

Case 1.

$$\sigma = \begin{pmatrix}\alpha & \beta\\ 0 & \hat{\alpha}^{-1}\end{pmatrix}.$$

By virtue of (3. 1. 2) we have

$$\sigma(e)_{-a+b}\sigma(e)_{a+b} = \ll a\alpha, a\beta\gg \ll b\alpha, b\beta\gg e_{-(-a+b)\alpha} e_{(a+b)\alpha}, \quad (a, b \in G),$$

hence, denoting by D the $|G|\times|G|$-diagonal matrix of which (a, a)-component is $\ll a\alpha, a\beta\gg$, we have

$$\begin{aligned}
&\mathrm{rank}\,((\sigma e)_{-a+b}(\sigma e)_{a+b})_{a\in G,\, b\in G} = \mathrm{rank}\,(D(e_{(-a+b)\alpha} e_{(a+b)\alpha})_{a\in G,\, b\in G} D)\\
&= \mathrm{rank}\,(e_{-a+b} e_{a+b})_{a\in G,\, b\in G},\\
&= \mathrm{rank}\begin{pmatrix} (e_{-a+b} e_{a+b})_{a\in G,\, b\in G} & (y_{a+b} y_{a+b})_{a\in G,\, b\in G}\\ {}^t(x_{-a+b} x_{a+b})_{a\in G,\, b\in G} & (\lambda(x^{-1}\circ y)_{-a+b}(x\circ y)_{a+b})_{a\in G,\, b\in G}\end{pmatrix}\\
&= \mathrm{rank}\begin{pmatrix} D(e_{(-a+b)\alpha} e_{(a+b)\alpha})_{a\in G,\, b\in G} D & D(y_{(-a+b)\alpha} y_{(a+b)\alpha})_{a\in G,\, b\in G} D\\ D^t(x_{(-a+b)\alpha} x_{(a+b)\alpha})_{a\in G,\, b\in G} D & D(\lambda^t(x\circ y)_{(-a+b)\alpha}(x\ \ y)_{(a+b)\alpha})_{a\in G} D\end{pmatrix}\\
&= \mathrm{rank}\begin{pmatrix} ((\sigma e)_{-a+b}(\sigma e)_{a+b})_{a\in b})_{a\in G,\, b\in G} & ((\sigma y)_{-a+b}(\sigma y)_{a+b})_{a\in G,\, b\in G}\\ ((\sigma x)_{-a+b}(\sigma x)_{a+b})_{a\in G,\, b\in G} & (\lambda\sigma(x\circ y)_{-a+b}\sigma(x\circ y)_{a+b})_{a\in G,\, b\in G}\end{pmatrix}
\end{aligned}$$

with a non-zero scalar. This proves $\sigma(x\circ y) = \sigma x \circledcirc \sigma y$.

Case 2.

$$\sigma=\begin{pmatrix}0 & -\hat{\gamma}^{-1}\\ \gamma & 0\end{pmatrix}.$$

By virtue of (3. 1. 2) we have

$$\begin{aligned}(\sigma e)_{-a+b}(\sigma e)_{a+b} &= |G|^{-2}\sum_{\hat{a},\hat{b}\in\hat{G}}\langle\!\langle -a+b,\ -\hat{a}\rangle\!\rangle\langle\!\langle a+b,\ -\hat{b}\rangle\!\rangle e_{(\hat{a}+\hat{b})\gamma}e_{(\hat{a}+\hat{b})\gamma}\\ &= |G|^{-2}\sum_{\hat{a},\hat{b}\in G}\langle\!\langle -a,\ -\hat{a}+\hat{b}\rangle\!\rangle\langle\!\langle -b,\ \hat{a}+\hat{b}\rangle\!\rangle e_{(-\hat{a}+\hat{b})\gamma}\, e_{(\hat{a}+\hat{b})\gamma}\,.\end{aligned}$$

Denote by D the $|G|\times|G|$-matrix of which $(a,\hat{a})$-component is $|G|^{-1}\langle\!\langle -a,\hat{a}\rangle\!\rangle$. Then, since γ is an isomorphism of $\hat{G}$ onto G, we have

$$\begin{aligned}&\operatorname{rank}((\sigma e)_{-a+b}(\sigma e)_{a+b})_{a\in G,\,b\in G}=\operatorname{rank}(D(e_{(-\hat{a}+\hat{b})\gamma}e_{(\hat{a}+\hat{b})\gamma})_{\hat{a}\in\hat{G},\hat{b}\in\hat{G}}{}^tD)\\ &=\operatorname{rank}(e_{-a+b}e_{a+b})_{a\in G,\,b\in G}\\ &=\operatorname{rank}\begin{pmatrix}(e_{-a+b}e_{a+b})_{a\in G,\,b\in G} & (y_{-a+b}y_{a+b})_{a\in G}\\ {}^t(x_{-a+b}x_{a+b})_{a\in G,\,b\in G} & (\lambda(x^{-1}\circ y)_{-a+b}(x\circ y)_{a+b})_{a\in G,\,b\in G}\end{pmatrix}\\ &=\operatorname{rank}\begin{pmatrix}D(e_{(-\hat{a}+\hat{b})\gamma}\,e_{(\hat{a}+\hat{b})\gamma})_{\hat{a}\in\hat{G},\hat{b}\in\hat{G}}{}^tD) & D(y_{(-\hat{a}+\hat{b})\gamma}\,y_{(\hat{a}+\hat{b})\gamma})_{\hat{a}\in\hat{G},\hat{b}\in\hat{G}}{}^tD)\\ D^t(x_{(-\hat{a}+\hat{b})\gamma}\,x_{(\hat{a}+\hat{b})\gamma})_{\hat{a}\in\hat{G},\hat{b}\in\hat{G}}{}^tD & D(\lambda(x^{-1}\circ y)_{(-\hat{a}+\hat{b})\gamma}(x\circ y)_{(\hat{a}+\hat{b})\gamma})_{\hat{a}\in\hat{G},\hat{b}\in\hat{G}}{}^tD\end{pmatrix}\\ &=\operatorname{rank}\begin{pmatrix}((\sigma e)_{-a+b}(\sigma e)_{a+b})_{a\in G,\,c\in G} & ((\sigma y)_{-a+b}(\sigma y)_{a+b})_{a\in G,\,b\in G}\\ {}^t((\sigma x)_{-a+b}(\sigma x)_{a+b})_{a\in G,\,b\in G} & (\lambda\sigma(x^{-1}\circ y)_{-a+b}\sigma(x\ y))_{a+b})_{a\in G,b\in G}\end{pmatrix}\end{aligned}$$

with a non-zero scalar. This proves $\sigma(x\circ y)=\sigma x\circledcirc\sigma y$.

3. Action of $Sp(G\oplus\hat{G})$ on symmetric G-theta structures.

We shall show that the action of $Sp(G\oplus\hat{G})$ on P_G defined in the beginning of §3 is nothing else than the action on symmetric G-theta structures.

THEOREM (3. 3. 1). *Let (X,ρ) be a symmetric G-theta structure on an abelian variety A such that the order of G is coprime to $ch(K)$ and X is very ample. Let $\varphi_a(u)$ $(a\in G)$ be the canonical system of functions associated with (X,ρ) and let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ be an element in $Sp(G\oplus\hat{G})$. Let X' be the zero divisor of the function*

$$\sum_{\hat{a}\in\hat{G}}\langle\!\langle \frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta\rangle\!\rangle\varphi_{\hat{a}\gamma}(u)$$

and let ρ' *be the isomorphism of* $G \oplus \hat{G}$ *into* A *such that*

$$\rho' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ \rho ,$$

i.e., $(a, \hat{a})\rho' = ((a\alpha + \hat{a}\gamma)\rho, (a\beta + \hat{a}\delta)\rho)$. *Then* (X', ρ') *is a symmetric G-theta structure such that the regular map*

$$u \longrightarrow \psi(u) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \varphi(u)$$

is the canonical projective embedding of A *associated with* (X', ρ'), *where*

$$\psi_a(u) = |G|^{-1} \sum_{\hat{a} \in \hat{G}} \ll \frac{1}{2} a, -\hat{a} \gg \ll \frac{1}{2} (a\alpha + \hat{a}\gamma), a\beta + \hat{a}\delta \gg \varphi_{a\alpha + \hat{a}\gamma}(u).$$

Conversely let (X', ρ') *be a symmetric G-theta structure on* A *such that* X' *is linearly equivalent to* X. *Then there exists an element* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *in* $Sp(G \oplus \hat{G})$ *such that*

$$\rho' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rho$$

and X' *is the zero divisor of the function*

$$|G|^{-1} \sum_{\hat{a} \in \hat{G}} \ll \frac{1}{2} \hat{a}\gamma, \hat{a}\delta \gg \varphi_{\hat{a}\gamma}(u).$$

Proof. We shall prove the first part. We may put

$$\bar{\psi}_a(u) = \frac{\sum_{\hat{a} \in \hat{G}} \ll \frac{1}{2} a, -\hat{a} \gg \ll \frac{1}{2} (a\alpha + \hat{a}\gamma), a\beta + \hat{a}\delta \gg \varphi_{a\alpha + \hat{a}\gamma}(u)}{\sum_{\hat{a} \in \hat{G}} \ll \frac{1}{2} \hat{a}\gamma, \hat{a}\delta \gg \varphi_{\hat{a}\gamma}(u)}$$

because $\varphi_a(u)$ $(a \in G)$ are linearly independent and

$$\sum_{\hat{a} \in \hat{G}} \ll \frac{1}{2} \hat{a}\gamma, \hat{a}\delta \gg \varphi_{\hat{a}\gamma}(u) = |\ker(\gamma)| \varphi_0(u) + \cdots .$$

It is sufficient to show

$$\begin{aligned} &\bar{\psi}_0(u) \equiv 1, \qquad \bar{\psi}_a(u) = \bar{\psi}_a(-u), \\ &\bar{\psi}_{a+b}(u) = \bar{\psi}_a(u + b\rho')\bar{\psi}_b(u), \\ &\bar{\psi}_a(u + \hat{c}\rho') = \ll a, \hat{c} \gg \bar{\psi}_a(u), \qquad (a, b \in G, \ \hat{c} \in \hat{G}). \end{aligned}$$

The first two relations are the direct consequences of the definition of $\tilde{\psi}_a(u)$ $(a \in G)$. From the elementary properties of symplectic matrices and the bicharacter $\ll\ ,\ \gg$ we get the following relations:

$$\ll\frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta\gg\ll\hat{a}\gamma,\ b\beta\gg$$

$$= \ll\frac{1}{2}\hat{a}\gamma,\ b\beta+\hat{a}\delta\gg\ll\frac{1}{2}\hat{a}\gamma,\ b\beta\gg -a, b$$

$$= \ll\frac{1}{2}\hat{a}\gamma,\ b\beta+\hat{a}\delta\gg\ll\frac{1}{2}b,\ -\hat{a}\gg\ll\frac{1}{2}\hat{a}\delta,\ b\alpha\gg$$

$$= \ll\frac{1}{2}b\alpha,\ -b\beta\gg\ll\frac{1}{2}(b\alpha+\hat{a}\gamma),\ b\beta+\hat{a}\delta\gg,$$

$$\ll\frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta\gg\ll\hat{a}\Gamma,\ c\delta\gg$$

$$= \ll\frac{1}{2}\hat{a}\gamma,\ (\hat{a}+\hat{c})\delta\gg\ll\frac{1}{2}\hat{a}\gamma,\ \hat{c}\delta\gg$$

$$= \ll\frac{1}{2}\hat{a}\gamma,\ (\hat{a}+\hat{c})\delta\gg\ll\frac{1}{2}\hat{a}\delta,\ \hat{c}\gamma\gg$$

$$= \ll\frac{1}{2}\hat{c}\gamma,\ -\hat{c}\delta\gg\ll\frac{1}{2}(\hat{a}+\hat{c})\gamma,\ (\hat{a}+\hat{c})\delta\gg,$$

$$\ll\frac{1}{2}a,-\hat{a}\gg\ll\frac{1}{2}(a\alpha+\hat{a}\gamma),\ a\beta+\hat{a}\delta\gg\ll a\alpha+\hat{a}\gamma,\ b\beta\gg$$

$$= \ll\frac{1}{2}(a+b),\ -\hat{a}\gg\ll\frac{1}{2}(a\alpha+\hat{a}\gamma),\ (a+b)\beta+\hat{a}\delta\gg\ll\frac{1}{2}b,\hat{a}\gg$$

$$\ll\frac{1}{2}(a\alpha+\hat{a}\gamma),\ b\beta\gg$$

$$= \ll\frac{1}{2}b\alpha,-b\beta\gg\ll\frac{1}{2}(a+b),\ -\hat{a}\gg\ll\frac{1}{2}((a+b)\beta+\hat{a}\gamma),\ (a+b)\beta+\hat{a}\delta\gg$$

$$\ll\frac{1}{2}b\alpha,\ -a\beta-\hat{a}\delta\gg\ll\frac{1}{2}b,\hat{a}\gg\ll\frac{1}{2}a\alpha,b\beta\gg\ll\frac{1}{2}\hat{a}\gamma,b\beta\gg$$

$$= \ll\frac{1}{2}b\alpha,-b\beta\gg\ll\frac{1}{2}(a+b),\ -\hat{a}\gg\ll\frac{1}{2}((a+b)\beta+\hat{a}\gamma),\ (a+b)\beta+\hat{a}\delta\gg,$$

$$-\ \ll\frac{1}{2}a,-\hat{a}\gg\ll\frac{1}{2}(a\alpha+\hat{a}\gamma),\ a\beta+\hat{a}\delta\gg\ll a\alpha+\hat{a}\gamma,\ \hat{c}\delta\gg$$

$$= \ll\frac{1}{2}a,\ -(\hat{a}+\hat{c})\gg\ll\frac{1}{2}(a\alpha+\hat{a}\gamma),\ a\beta+(\hat{a}+\hat{c})\delta\gg\ll\frac{1}{2}a\alpha,\ \hat{c}\delta\gg$$

$$\ll\frac{1}{2}\hat{a}\Gamma,\hat{c}\delta\gg\ll\frac{1}{2}a,\hat{c}\gg$$

$$= \langle\!\langle \frac{1}{2}\hat{a}\gamma,\ -\hat{c}\delta \rangle\!\rangle \langle\!\langle a, \hat{c} \rangle\!\rangle \langle\!\langle \frac{1}{2}a,\ -(\hat{a}+\hat{c} \rangle\!\rangle \langle\!\langle \frac{1}{2}(a\alpha+(\hat{a}+\hat{c})\gamma), a\beta+(\hat{a}+\hat{c})\delta \rangle\!\rangle$$

$$\langle\!\langle \frac{1}{2}a, \hat{c} \rangle\!\rangle \langle\!\langle \frac{1}{2}a\alpha,\ -\hat{c}\delta \rangle\!\rangle \langle\!\langle \frac{1}{2}\hat{a}\gamma,\ -\hat{c}\gamma \rangle\!\rangle \langle\!\langle \frac{1}{2}a\alpha,\ \hat{c}\delta \rangle\!\rangle \langle\!\langle \frac{1}{2}\hat{a}\gamma, \hat{c}\delta \rangle\!\rangle$$

$$= \langle\!\langle \frac{1}{2}\hat{c}\gamma, \hat{c}\delta \rangle\!\rangle \langle\!\langle a,\ \hat{c} \rangle\!\rangle \langle\!\langle \frac{1}{2}a,\ -(\hat{a}+\hat{c}) \rangle\!\rangle \langle\!\langle \frac{1}{2}(a\alpha+(\hat{a}+\hat{c})\gamma, a\beta+(\hat{a}+\hat{c})\delta \rangle\!\rangle.$$

On the other hand from the definition of ρ' it follows

$$\varphi_{a\alpha+\hat{a}\gamma}(u+b\rho') = \varphi_{a\alpha+\hat{a}\gamma}(u+b\alpha\rho+b\beta\rho)$$

$$= \frac{\varphi_{(a+b)\alpha+\hat{a}\gamma}(u+b\beta\rho)}{\varphi_{b\alpha}(u+b\beta\rho)} = \frac{\langle\!\langle a\alpha+\hat{a}\gamma,\ b\beta \rangle\!\rangle \varphi_{(a+b)\alpha+\hat{a}\gamma}(u)}{\varphi_{q\alpha}(u)}$$

and

$$\varphi_{a\alpha+\hat{a}\gamma}(u+\hat{c}\rho') = \varphi_{a\alpha+\hat{a}\gamma}(u+\hat{c}\gamma\rho+\hat{c}\delta\rho)$$

$$= \frac{\varphi_{a\alpha+(\hat{a}+\hat{c})\gamma}(u+\hat{c}\delta\rho)}{\varphi_{\hat{c}\gamma}(u+\hat{c}\delta\rho)} = \frac{\langle\!\langle a\alpha+\hat{a}\gamma,\ \hat{c}\delta \rangle\!\rangle \varphi_{a\alpha+(\hat{a}+\hat{c})\gamma}(u)}{\varphi_{\hat{c}\gamma}(u)}.$$

Hence we can conclude

$$\psi_a(u+b\rho')\,\psi_b(u)$$

$$= \frac{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}a, -\hat{a} \rangle\!\rangle \langle\!\langle \frac{1}{2}(a\alpha+\hat{a}\gamma),\ a\beta+\hat{a}\delta \rangle\!\rangle \varphi_{a\alpha+\hat{a}\gamma}(u+b\rho')}{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta \rangle\!\rangle \varphi_{\hat{a}\gamma}(u+b\rho')}$$

$$\frac{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}b, -\hat{a} \rangle\!\rangle \langle\!\langle \frac{1}{2}(b\alpha+\hat{a}\gamma),\ b\beta+\hat{a}\delta \rangle\!\rangle \varphi_{a\alpha+\hat{a}\gamma}(u)}{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}\hat{a}\gamma, \hat{a}\delta \rangle\!\rangle \varphi_{\hat{a}\gamma}(u)}$$

$$= \frac{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}a, -\hat{a} \rangle\!\rangle \langle\!\langle \frac{1}{2}(a\alpha+\hat{a}\gamma),\ a\beta+\hat{a}\delta \rangle\!\rangle \langle\!\langle a\alpha+\hat{a}\gamma,\ b\beta \rangle\!\rangle \varphi_{(a+b)\alpha+\hat{a}\gamma}(u)}{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta \rangle\!\rangle \langle\!\langle \hat{a}\gamma,\ b\beta \rangle\!\rangle \varphi_{b\alpha+\hat{a}\gamma}(u)}$$

$$\frac{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}b, -\hat{a} \rangle\!\rangle \langle\!\langle \frac{1}{2}(b\alpha+\hat{a}\gamma),\ b\beta+\hat{a}\delta \rangle\!\rangle \varphi_{b\alpha+\hat{a}\gamma}(u)}{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta \rangle\!\rangle \varphi_{\hat{a}\gamma}(u)}$$

$$= \frac{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}(a+b),\ -\hat{a} \rangle\!\rangle \langle\!\langle \frac{1}{2}((a+b)\alpha+\hat{a}\gamma),\ (a+b)\beta+\hat{a}\delta \rangle\!\rangle \varphi_{(a+b)\alpha+\hat{a}\gamma}(u)}{\sum_{\hat{a}\in\hat{G}} \langle\!\langle \frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta \rangle\!\rangle \varphi_{\hat{a}\gamma}(u)}$$

$$= \bar{\psi}_{a+b}(u),$$

$$
\begin{aligned}
&\bar{\varphi}_a(u + \hat{c}\rho') \\
&= \frac{\sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2} a, -\hat{a} \gg \ll \frac{1}{2}(a\alpha + \hat{a}\gamma),\, a\beta + \hat{a}\delta \gg \varphi_{a\alpha + \hat{a}\gamma}(u + \hat{c}\rho')}{\sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2}\hat{a}\gamma,\, \hat{a}\delta \gg \varphi_{\hat{a}\gamma}(u + \hat{c}\rho')} \\
&= \frac{\sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2} a, -\hat{a} \gg \ll \frac{1}{2}(a\alpha + \hat{a}\gamma),\, a\beta + \hat{a}\delta \gg \ll a\alpha + \hat{a}\gamma,\, \hat{c}\delta \gg \varphi_{a\alpha + (\hat{a}+\hat{c})\gamma}(u)}{\sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2}\hat{c}\gamma,\, \hat{c}\delta \gg \ll \frac{1}{2}\hat{a}\gamma, \hat{a}\delta \gg \ll \hat{a}\gamma,\, \hat{c}\delta \gg \varphi_{a\alpha + (\hat{a}+\hat{c})\gamma}(u)} \\
&= \frac{\ll a, \hat{c} \sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2}(a, -(a + c) \gg \ll \frac{1}{2}(a\alpha + (\hat{a} + \hat{c})\gamma),\, a\beta + (\hat{a} + \hat{c})\delta \gg \varphi_{a\alpha + (\hat{a}+\hat{a})\gamma}(u)}{\sum_{\hat{a}\in\hat{G}} \ll \frac{1}{2}(\hat{a} + \hat{c})\gamma,\, (\hat{a} + \hat{c})\delta \gg \varphi_{(\hat{a}+\hat{c})\gamma}(u)} \\
&= \ll a, \hat{c} \gg \bar{\varphi}_a(u).
\end{aligned}
$$

This proves the first part of the theorem. We shall next show that if (X', ρ) is a symmetric G-theta structure on A such that X' is linearly equivalent to X, then $X' = X$. Let $\varphi'_a(u)$ $(a \in G)$ be the canonical system of functions associated with (X', ρ). Then the quotients $\varphi'_a(u)/\varphi_u(u)$ $(a \in G)$ satisfy the conditions

$$
\begin{gathered}
(\varphi'_a/\varphi_a) = X'_{-a} - X_{-a}, \quad \varphi'_0(u)/\varphi'_0(u) \equiv 1, \\
\varphi'_a(u + \hat{c}\rho)/\varphi_a(u + \hat{c}\rho) = \varphi'_a(u)/\varphi_a(u), \quad (a \in G,\ \ \hat{c} \in \hat{G}).
\end{gathered}
$$

If we denote by π the natural isogeny of A onto the quotient $B = A/\hat{G}$ and by U and V the divisors on B such that $\pi^{-1}(U) = X$ and $\pi^{-1}(V) = X'$, then there exist functions $h_a(u)$ $(a \in G)$ satisfying

$$(h_a) = V_{-\pi a} - U_{-\pi a}$$

and

$$\varphi'_a(u)/\varphi_a(u) = h_a(\pi u), \quad (a \in G).$$

On the other hand $|G| = \sqrt{l(X)} = \sqrt{l(X')} = \deg(\pi)\sqrt{l(U)} = \deg(\pi)\sqrt{l(V)}$ and $\deg(\pi) = |\hat{G}| = |G|$. Hence $l(U) = l(V) = 1$. This means that the functions $h_a(\bar{u})$ $(a{\in}G)$ are constants, i.e., $X = X'$. Finally we shall complete the proof of the second part of the theorem. Let α, β, γ, δ be the homomorphisms of G into G, G into $\hat{G}$, $\hat{G}$ into G, $\hat{G}$ into $\hat{G}$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \rho'\rho^{-1}.$$

Then

$$1 = e_X(a\rho', b\rho') = e_X((a\alpha + a\beta)\rho,\ (b\alpha + b\beta)\rho) = \langle\!\langle a, b(\beta\hat{\alpha} - \alpha\hat{\beta})\rangle\!\rangle.$$

and

$$\begin{aligned} 1 &= e_X(\hat{a}\rho', \hat{b}\rho') = e_X((\hat{a}\gamma + \hat{a}\delta)\rho,\ (\hat{b}\gamma + \hat{b}\delta)\rho) \\ &= \langle\!\langle \hat{a}, \hat{b}(\delta\hat{\gamma} - \gamma\hat{\delta})\rangle\!\rangle. \end{aligned}$$

This implies that $\beta\hat{\alpha} = \alpha\hat{\beta}$ and $\delta\hat{\gamma} = \gamma\hat{\delta}$. Moreover we have

$$\begin{aligned} \langle\!\langle a, \hat{a}\rangle\!\rangle &= e_X(a\rho', a\rho') = e((a\alpha + a\beta)\rho,\ (\hat{a}\gamma + \hat{a}\delta)\rho) \\ &= \langle\!\langle a\alpha, \hat{a}\delta\rangle\!\rangle \langle\!\langle -a\beta, \hat{a}\gamma\rangle\!\rangle = \langle\!\langle a, \hat{a}(\delta\hat{\alpha} - \gamma\hat{\beta})\rangle\!\rangle = \langle\!\langle a(\alpha\hat{\delta} - \beta\hat{\gamma}),\ \hat{a}\rangle\!\rangle \\ &\qquad\qquad (a \in G,\ \ \hat{a} \in G), \end{aligned}$$

hence

$$\delta\hat{\alpha} - \gamma\hat{\beta} = id_{\hat{G}},\ \ \alpha\hat{\gamma} - \beta\hat{\gamma} = id_{\hat{G}}.$$

This shows that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an element in $Sp(G \oplus \hat{G})$. Let X'' be the zero divisor of

$$\sum_{\hat{a} \in \hat{G}} \langle\!\langle \frac{1}{2}\hat{a}\gamma,\ \hat{a}\delta\rangle\!\rangle\, \varphi_{\hat{a}\gamma}(u).$$

Then (X'', ρ') is a symmetric G-theta structure on A such that X'' is linearly equivalent to X'. Hence by the above result we can conclude that $X'' = X'$. This completes the proof of Theorem (3. 3. 1).

References

[I] D. Mumford, On the equations defining abelian varieties I, II, III, Invent. Math., **1** (1966), 287–354. **3** (1967), 75–135, 215–244.

[II] A. Weil, Sur certaines groupes d'operateurs unitaire, Acta Math., **111** (1964) 145–211.

[III] ———, Variétés abeliennes et courbes algébriques, Hermann, Paris, (1948).

[IV] ———, Foundation of algebraic geometry, (1946).

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