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ON ε-ENTROPY OF EQUIVALENT GAUSSIAN PROCESSES

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To Professor Katuzi Ono on the occasion of his 60th birthday

§1. Introduction.

Let $\xi = \{\xi(t);\ t \in T\}$ be a stochastic process, where T is a finite interval. The ε-entropy $H_\varepsilon(\xi)$ of ξ is defined as the following quantity:

$$H_\varepsilon(\xi) = \inf_\eta I(\xi, \eta), \tag{1}$$

where $I(\xi, \eta)$ is the amount of information about ξ contained in an another stochastic process $\eta = \{\eta(t);\ t \in T\}$ and the infimum is taken for all stochastic processes η satisfying the condition:

$$\int_T E|\xi(t) - \eta(t)|^2 dt \leqq \varepsilon^2. \tag{2}$$

Concerning the ε-entropy of Gaussian processes, M.S. Pinsker has got an explicit expression of it in terms of the spectral measure. More precisely, let $\xi = \{\xi(t);\ t \in T\}$ be a real valued mean continuous Gaussian process. We denote by $r(s, t)$ the covariance function of ξ, i.e. $r(s, t) = E\{(\xi(s) - E\xi(s))(\xi(t) - E\xi(t))\}$, and we define an integral operator on the space $L^2(T)$ by the following:

$$K\varphi(t) = \int_T r(s, t)\varphi(s)ds, \quad \varphi \in L^2(T), \quad t \in T. \tag{3}$$

Then K is a symmetric Hilbert-Schmidt operator with countable nonnegative eigenvalues $\{\lambda_n\}_{n=1}^\infty$. Using these eigenvalues, the ε-entropy of ξ is expressed in the form:

$$H_\varepsilon(\xi) = \frac{1}{2} \sum_{n=1}^\infty \log\left[\max\left(\frac{\lambda_n}{\theta^2},\ 1\right)\right], \tag{4}$$

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where θ is determined by the equation:

$$\sum_{n=1}^{\infty} \min(\lambda_n, \theta^2) = \varepsilon^2 . \tag{5}$$

The formula (4) enables us to estimate the asymptotic behavior of $H_\varepsilon(\xi)$ as ε tends to 0 (cf. [1], [2]).

On the other hand, there are many investigations concerning the equivalence of Gaussian processes. In a usual manner, every Gaussian process $\xi = \{\xi(t);\ t \in T\}$ determines a measure on the space R^T. Two Gaussian processes are said to be equivalent, if their corresponding measures are equivalent, i.e. they are mutually absolutely continuous.

The purpose of this paper is to investigate the relations between the equivalence of Gaussian processes and the asymptotic properties of their ε-entropy.

Let $\xi = \{\xi(t);\ t \in T\}$ and $\tilde{\xi} = \{\tilde{\xi}(t);\ t \in T\}$ be the real valued mean continuous Gaussian processes and let K and $\tilde{K}$ be the integral operators defined by the formula (3) for ξ and $\tilde{\xi}$, respectively. Denote by $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$ the eigenvalues of K and $\tilde{K}$, respectively. In §2 we shall give a necessary condition for the equivalence of ξ and $\tilde{\xi}$ in terms of $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$. Consider an operator S on l^2 defined by

$$S = \begin{pmatrix} \sqrt{\lambda_1} & & O \\ & \sqrt{\lambda_2} & \ddots \\ O & & \ddots \end{pmatrix}.$$

If $\tilde{\xi}$ is equivalent to ξ, then it will be shown that the eigenvalues $\{\tilde{\lambda}_n\}$ coincide with the eigenvalues of the operator SAS, where A is a self-adjoint operator such that $I - A$ (I = identity operator) is of Hilbert-Schmidt type (Theorem 1). Moreover, using the fact that $I - A$ is completely continuous, we shall prove that the asymptotic behavior of $\tilde{\lambda}_n$ is approximately similar to that of λ_n as $n \to \infty$ (Theorem 2).

In §3 it will be shown that the ε-entropy of equivalent Gaussian processes are "asymptotically equal" (Theorem 3). This means the following: Given a Gaussian process whose ε-entropy is known. Then we can estimate the ε-entropy of any Gaussian process which is equivalent to the given process.

§2. A necessary condition for the equivalence of Gaussian processes.

Let $\xi = \{\xi(t);\ t\in T\}$ and $\tilde{\xi} = \{\tilde{\xi}(t);\ t\in T\}$ be real valued mean continuous Gaussian processes with covariance function $r(s,t)$ and $\tilde{r}(s,t)$ $(s,\ t\in T)$, respectively. We may assume, without loss of generality, that the mean of $\xi(t)$ is zero $(t\in T)$. Define the operators K and $\tilde{K}$ by the formula (3), and arrange the eigenvalues of K and $\tilde{K}$ in a decreasing order: $\lambda_1 \geqq \lambda_2 \geqq \cdots$ and $\tilde{\lambda}_1 \geqq \tilde{\lambda}_2 \geqq \cdots$, respectively. Define operators S and $\tilde{S}$ in l^2 as follows:

$$S = \begin{pmatrix} \sqrt{\lambda_1} & & O \\ & \sqrt{\lambda_2} & \\ O & & \ddots \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} \sqrt{\tilde{\lambda}_1} & & O \\ & \sqrt{\tilde{\lambda}_2} & \\ O & & \ddots \end{pmatrix}. \tag{6}$$

We first derive a necessary condition for the equivalence of the processes ξ and $\tilde{\xi}$ in terms of the corresponding eigenvalues $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$.

THEOREM 1. *If a Gaussian process $\tilde{\xi}$ is equivalent to a Gaussian process ξ, then there exists an operator A in l^2 such that A is self-adjoint and $I-A$ is of Hilbert-Schmidt type and that the eigenvalues $\{\tilde{\lambda}_n\}$ of $\tilde{K}$ (or of $\tilde{S}^2$) coincide with those of SAS.*

Proof. Let $\{\varphi_n\}_{n=1}^{\infty}(\{\tilde{\varphi}_n\}_{n=1}^{\infty})$ be the eigenfunctions, which forming a *c. o. n. s.* of $L^2(T)$, of the operator K $(\tilde{K})$ corresponding to the eigenvalues $\{\lambda_n\}$ $(\{\tilde{\lambda}_n\})$. Define Gaussian random variable ξ_n by

$$\xi_n = \frac{1}{\sqrt{\lambda_n}} \int_T \xi(t)\varphi_n(t)dt, \quad (n = 1, 2, \cdots).$$

Then $\xi(t)$ can be expressed as follows:

$$\xi(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\ \varphi_n(t)\xi_n. \tag{7}$$

Since $\tilde{\xi}$ is equivalent to ξ, there is an invertible bounded self-adjoint positive definite operator F such that $I-F^2$ is of Hilbert-Schmidt type and that

$$E(F\xi(s)\cdot F\xi(t))^{*)} = \tilde{r}(s,t), \quad s,\ t\in T, \tag{8}$$

(Yu.A. Rozanov [3]). Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ be the eigenvalues and corresponding eigenvectors of F which form a *c. o. n. s.*. Set

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*) $E(X)$ denotes the expectation of a random variable X.

$$c_{nm} = E\xi_n\eta_m \text{ and } C = \left(c_{nm}\right)_{n,m=1}^{\infty}.$$

Then C is a unitary operator on l^2 and by (7) it follows that

$$\xi(t) = \sum_{n,m=1}^{\infty} \sqrt{\lambda_n}\, c_{nm}\varphi_n(t)\eta_m,$$

so

$$F\xi(t) = \sum_{n,m=1}^{\infty} \sqrt{\lambda_n}\, c_{nm}\varphi_n(t)\mu_m\eta_m.$$

Therefore (8) implies that

$$\tilde{r}(s,t) = \sum_{l,m,n=1}^{\infty} \sqrt{\lambda_l}\,\sqrt{\lambda_m}\, c_{ln}c_{mn}\mu_n^2\varphi_l(s)\varphi_m(t). \tag{9}$$

On the other hand, applying Mercer's expansion theorem we have

$$\tilde{r}(s,t) = \sum_{n=1}^{\infty} \tilde{\lambda}_n\tilde{\varphi}_n(s)\tilde{\varphi}_n(t). \tag{10}$$

Set

$$b_{nm} = \int_T \varphi_n(s)\tilde{\varphi}_m(s)ds \text{ and } B = \left(b_{nm}\right)_{n,m=1}^{\infty}.$$

Then B is a unitary operator on l^2. Multiplying the both sides of (9) and (10) by $\tilde{\varphi}_p(s)\tilde{\varphi}_q(t)$ and integrating them, it follows that

$$\delta_{pq}\tilde{\lambda}_p = \sum_{l,m,n=1}^{\infty} \sqrt{\lambda_l}\,\sqrt{\lambda_m}\, c_{ln}c_{mn}\mu_n^2 b_{lp}b_{mq}, \quad p, q = 1, 2, \cdots.$$

Observing this relation, we get

$$\tilde{S}^2 = B^* SASB,$$

where $A = CD^2C$ and $D = \begin{pmatrix} \mu_1 & & O \\ & \mu_2 & \\ O & & \ddots \end{pmatrix}$.

The eigenvalues $\{\tilde{\lambda}_n\}$ of $\tilde{S}^2 = B^*SASB$ coincide with those of SAS since B is unitary. The fact that $I - F^2$ is an operator of Hilbert-Schmidt type implies that $I - A$ is of Hilbert-Schmidt type. Thus the theorem is proved.

From Theorem 1 we can obtain the following theorem concerning the asymptotic behavior of the eigenvalues $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$.

THEOREM 2. *If $\tilde{\xi}$ is equivalent to ξ then for any $\delta > 0$ there exists an integer n_0 such that*

$$(1-\delta)\lambda_{n+n_0} \leqq \tilde{\lambda}_n \leqq (1+\delta)\lambda_{n-n_0}, \quad n > n_0. \tag{11}$$

Proof. For the proof it is sufficient to prove the next lemma.

LEMMA. *Let S be a self-adjoint completely continuous operator on a Hilbert space, and A be a self-adjoint operator such that $I - A$ is completely continuous and that SAS is positive definite. Let λ_n and μ_n $(n = 1, 2, \cdots)$ be the n-th largest eigenvalue of S^2 and SAS, respectively. Then for any $\delta > 0$ there exists an integer n_0 such that*

$$(1-\delta)\lambda_{n+n_0} \leqq \mu_n \leqq (1+\delta)\lambda_{n-n_0}, \quad n > n_0.$$

Proof. Let $\{\nu_n\}$ (arrange them so that $|\nu_1| \geqq |\nu_2| \geqq \cdots$) be the eigenvalues of $I - A$ and ζ_n be the eigenvector corresponding to ν_n. Let H_n be the closed linear subspace spanned by ζ_j, $j = 1, \cdots, n$. Since $I - A$ is completely continuous, for any $\delta > 0$ there exists an integer n_0 such that

$$|\nu_n| \leqq \delta, \qquad n > n_0.$$

Define operators A_1 and A_2 as follows:

$$A_1 = \begin{cases} I, & \text{on } H_{n_0}, \\ A, & \text{on } H_{n_0}^{\perp}, \end{cases}$$

$$A_2 = \begin{cases} A - I, & \text{on } H_{n_0}, \\ 0, & \text{on } H_{n_0}^{\perp}. \end{cases}$$

Then A_1 satisfies

$$(1-\delta)I \leqq A_1 \leqq (1+\delta)I,$$

and so

$$(1-\delta)S^2 \leqq SA_1S \leqq (1+\delta)S^2.$$

Consequently we have

$$(1-\delta)\lambda_n \leqq \mu_{1,n} \leqq (1+\delta)\lambda_n, \qquad n = 1, 2, \cdots, \tag{12}$$

where $\mu_{1,n}$ denotes the n-th largest eigenvalue of SA_1S. On the other hand, noting that the dimension of the range of SA_2S is at most n_0, we get

$$\mu^+_{2,n} = \mu^-_{2,n} = 0, \qquad n > n_0, \tag{13}$$

where $\mu^+_{2,n}$ $(\mu^-_{2,n})$ denotes the n-th largest nonnegative (nonpositive) eigenvalue of SA_2S. From (12), (13) and the equality $SAS = SA_1S + SA_2S$, it follows that

$$\mu_n \leqq \mu_{1,n-n_0} + \mu^+_{2,\,n_0+1} = \mu_{1,n-n_0} \leqq (1+\delta)\lambda_{n-n_0}, \quad n > n_0,$$
$$\mu_n \geqq \mu_{1,n+n_0} + \mu^-_{2,\,n_0+1} = \mu_{1,n+n_0} \geqq (1-\delta)\lambda_{n+n_0}, \quad n = 1, 2, \cdots,$$

(cf. F. Riesz, B. Nagy [4] §95). These complete the proof.

§3 The ε-entropy of equivalent Gaussian processes.

Using Theorem 2 and the formula (4) for the ε-entropy of Gaussian processes, we have the following theorem.

THEOREM 3. *If a Gaussian process* $\tilde{\xi} = \{\tilde{\xi}(t);\ t \in T\}$ *is equivalent to a Gaussian process* $\xi = \{\xi(t);\ t \in T\}$, *then for any* $0 < \delta < 1$ *and* $\varepsilon > 0$ *there exists* $\delta_0 = \delta_0(\delta, \varepsilon)$ *such that, for any fixed* δ, $\delta_0(\delta, \varepsilon) \downarrow 0$ *as* $\varepsilon \downarrow 0$ *and that*

$$(1-\delta_0)H_{(1+\delta)(1+\delta_0)\varepsilon}(\xi) \leqq H_\varepsilon(\tilde{\xi}) \leqq (1+\delta_0)H_{(1-\delta)(1-\delta_0)\varepsilon}(\xi). \tag{14}$$

For the proof we prepare a lemma.

LEMMA. *Let* $\{\lambda_n\}$ *and* $\{\tilde{\lambda}_n\}$ *be the eigenvalues of the integral operator* K *and* $\tilde{K}$ *corresponding to the processes* ξ *and* $\tilde{\xi}$, *respectively, as before.*

(i) *If* $\lambda_n \leqq \tilde{\lambda}_n$ *for all* n, *then* $H_\varepsilon(\xi) \leqq H_\varepsilon(\tilde{\xi})$.

(ii) *If* $\lambda_n = a^2\tilde{\lambda}_n$ *for all* n *(*a *is a positive constant), then* $H_{a\varepsilon}(\xi) = H_\varepsilon(\tilde{\xi})$.

(iii) *If* $\lambda_n = \tilde{\lambda}_n$ *for all* $n > N$ *(*N *is a constant), then* $H_\varepsilon(\xi) = H_\varepsilon(\tilde{\xi}) + c$ *(*c *is a constant) for sufficiently small* ε.

(iv) *If* $\tilde{\lambda}_n = \lambda_{n+N}$ *for all* n *(*N *is a constant), then there exists* $\delta_1 = \delta_1(\varepsilon)$ *such that* $\delta_1(\varepsilon) \downarrow 0$ *as* $\varepsilon \downarrow 0$ *and* $H_{(1+\delta_1)\varepsilon}(\xi) \leqq (1+\delta_1)H_\varepsilon(\tilde{\xi})$.

Proof. (i), (ii) and (iii) are obvious from the formula (4). For the proof of (iv), we define the number $\theta = \theta(\varepsilon)$ by the equation $\sum_{n=1}^{\infty} \min(\tilde{\lambda}_n, \theta^2) = \varepsilon^2$, and define the integer $p = p(\varepsilon)$ by $p = \max\{n;\ \tilde{\lambda}_n \geqq \theta^2\}$. Then from (4)

$$H_\varepsilon(\tilde{\xi}) = \frac{1}{2}\sum_{n=1}^{p} \log \frac{\tilde{\lambda}_n}{\theta^2}.$$

We define $\delta = \delta(\varepsilon)$ by the following equation,

$$(1+\delta)^2\varepsilon^2 = (N\theta^2 + \varepsilon^2),$$

i.e. $$(1+\delta)^2\varepsilon^2 = (p+N)\theta^2 + \sum_{n=p+N+1}^{\infty}\lambda_n.$$

Then $\delta = \delta(\varepsilon)\downarrow 0$ as $\varepsilon\downarrow 0$ and by (4) we have

(15) $$H_{(1+\delta)\varepsilon}(\xi) = \frac{1}{2}\sum_{n=1}^{p+N}\log\frac{\lambda_n}{\theta^2} = H_\varepsilon(\tilde{\xi}) + \frac{1}{2}\sum_{n=1}^{N}\log\frac{\lambda_n}{\theta^2}.$$

Therefore in order to prove the lemma it is enough to show that

(16) $$\sum_{n=1}^{N}\log\frac{\lambda_n}{\theta^2} = o(H_\varepsilon(\tilde{\xi})) \quad \text{as} \quad \varepsilon\to 0.$$

Noting that $p(\varepsilon)\uparrow\infty$ as $\varepsilon\downarrow 0$, for any integer $k>0$, we can take $\varepsilon>0$ such as $p = p(\varepsilon) > kN$. It follows that

$$H_\varepsilon(\tilde{\xi}) = \frac{1}{2}\sum_{n=1}^{p}\log\frac{\tilde{\lambda}_n}{\theta^2} \geqq \frac{1}{2}\sum_{n=1}^{kN}\log\tilde{\lambda}_n - \frac{kN}{2}\log\theta^2.$$

Therefore,

$$\frac{\frac{1}{2}\sum_{n=1}^{N}\log\frac{\lambda_n}{\theta^2}}{H_\varepsilon(\tilde{\xi})} \leqq \frac{\sum_{n=1}^{N}\log\lambda_n - N\log\theta^2}{\sum_{n=1}^{kN}\log\tilde{\lambda}_n - kN\log\theta^2} \to \frac{1}{k} \quad \text{as } \varepsilon\to 0.$$

So we have the relation (16), since k is arbitrary. Thus the lemma is proved.

Proof of Theorem 3. Using Theorem 2, for any $0<\delta<1$ there is an integer n_0 such that

$$\frac{1}{(1+\delta)^2}\lambda_{n+n_0} \leqq \tilde{\lambda}_n \leqq \frac{1}{(1-\delta)^2}\lambda_{n-n_0}, \qquad n > n_0.$$

Define the sequences $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ as follows:

$$\lambda_{1,n} = \begin{cases} (1-\delta)^2\tilde{\lambda}_n, & n \leqq n_0, \\ \lambda_{n-n_0}, & n > n_0, \end{cases}$$

$$\lambda_{2,n} = \begin{cases} (1+\delta)^2\bar{\lambda}_n, & n \leqq n_0, \\ \lambda_{n+n_0}, & n > n_0. \end{cases}$$

We shall construct Gaussian processes ξ_1 and ξ_2, whose eigenvalues are $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$. For this we set

$$\varphi_n(t) = \begin{cases} \dfrac{1}{\sqrt{|T|}}, & n = 1, \\ \sqrt{\dfrac{2}{|T|}} \cos \dfrac{2\pi m}{|T|} t, & n = 2m+1, \ m = 1, 2, \cdots, \\ \sqrt{\dfrac{2}{|T|}} \sin \dfrac{2\pi m}{|T|} t, & n = 2m, \ m = 1, 2, \cdots, \end{cases}$$

($|T|$ denotes the length of the interval T) and

$$r_i(s, t) = \sum_{n=1}^{\infty} \lambda_{i,n} \varphi_n(s) \varphi_n(t), \qquad i = 1, 2.$$

Then each $r_i(s, t)$ is continuous in (s, t) and positive definite, therefore it determine a Gaussian process $\xi_i = \{\xi_i(t);\ t \in T\}$ whose covariance function is $r_i(s, t)$ having the eigenvalues $\{\lambda_{i,n}\}$. Here we apply the lemma to the processes ξ_1 and ξ_2. Using (i) and (ii) of the lemma, we get

$$H_{(1+\delta)\varepsilon}(\xi_2) \leqq H_\varepsilon(\xi) \leqq H_{(1-\delta)\varepsilon}(\xi_1),$$

and using (iii) and (iv), it follows that there exists $\delta_0 = \delta_0(\delta, \varepsilon)$ such that $\delta_0(\delta, \varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and that

$$H_\varepsilon(\xi_1) \leqq (1 + \delta_0) H_{(1-\delta_0)\varepsilon}(\xi),$$
$$H_\varepsilon(\xi_2) \geqq (1 - \delta_0) H_{(1+\delta_0)\varepsilon}(\xi).$$

These three inequalities give the relation (14). Thus we have proved Theorem 3.

From this theorem we can get the following result by simple calculation.

THEOREM 4. *In addition to the assumption of Theorem* 3, *suppose that* $H_\varepsilon(\xi)$ *satisfies*

$$\lim_{\varepsilon, \delta \to 0} \frac{H_{(1+\delta)\varepsilon}(\xi)}{H_\varepsilon(\xi)} = 1. \tag{17}$$

Then

(18) $$\lim_{\varepsilon \to 0} \frac{H_\varepsilon(\tilde{\xi})}{H_\varepsilon(\xi)} = 1.$$

Remark. There exist Gaussian processes ξ and $\tilde{\xi}$ which satisfy the relation (18) and are not equivalent (cf. Example 2).

EXAMPLE 1. Let $\xi = \{\xi(t);\ t \in T\}$ be the Brownian motion. Then $H_\varepsilon(\xi)$ is of the form

$$H_\varepsilon(\xi) = \frac{2|T|^2}{\pi^2}\frac{1}{\varepsilon^2} + o\left(\frac{1}{\varepsilon^2}\right), \quad \text{as } \varepsilon \to 0,$$

([2]) and it satisfies the condition (17). If a Gaussian process $\tilde{\xi} = \{\tilde{\xi}(t);\ t \in T\}$ is equivalent to ξ, then by Theorem 4 it holds that

$$H_\varepsilon(\tilde{\xi}) = \frac{2|T|^2}{\pi^2}\frac{1}{\varepsilon^2} + o\left(\frac{1}{\varepsilon^2}\right), \quad \text{as } \varepsilon \to 0.$$

For example, we present the Ornstein-Uhlenbeck Brownian motion as $\tilde{\xi}$.

EXAMPLE 2. Let $\xi = \{\xi(t);\ t \in T\}$ and $\tilde{\xi} = \{\tilde{\xi}(t);\ t \in T\}$ be stationary Gaussian processes with spectral density $f(\lambda)$ and $\tilde{f}(\lambda)$. Suppose that the following conditions (i) and (ii) are satisfied.

(i) $f(\lambda) = c\lambda^{-(1+\alpha)}(\log\lambda)^{-\beta}$, for sufficiently large λ,
where $0 < c$, $0 < \alpha$, and $-\infty < \beta < \infty$.

(ii) $$\lim_{\lambda \to \infty} \frac{\tilde{f}(\lambda)}{f(\lambda)} = 1.$$

From these assumptions we can obtain ([5]) that

$$H_\varepsilon(\tilde{\xi}) = c_0 \varepsilon^{-\frac{2}{\alpha}}\left(\log\frac{1}{\varepsilon}\right)^{-\frac{\beta}{\alpha}} + o\left(\varepsilon^{-\frac{2}{\alpha}}\left(\log\frac{1}{\varepsilon}\right)^{-\frac{\beta}{\alpha}}\right), \quad \text{as } \varepsilon \to 0,$$

where

$$c_0 = \frac{1+\alpha}{2\pi}\left(\frac{2^{1-\beta}(1+\alpha)|T|^{1+\alpha}}{\alpha^{1-\beta}}\right)^{\frac{1}{\alpha}}.$$

Therefore we have

$$H_\varepsilon(\tilde{\xi}) = H_\varepsilon(\xi) + o(H_\varepsilon(\xi)), \quad \text{as } \varepsilon \to 0.$$

This shows that ξ and $\tilde{\xi}$ satisfy the relation (18).

Consider an another condition:

(iii) $\tilde{f}(\lambda)=f(\lambda)+c'\lambda^{-(1+\alpha+\delta)}$, for sufficiently large λ,
where $0<c'$ and $0<\delta<\frac{1}{2}$. Then it is straightforward that (i) and (iii) imply (ii). Therefore under the assumptions (i) and (iii), the relation (18) is fulfilled, but by the theorem in [6] we see that $\tilde{\xi}$ is not equivalent to ξ.

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