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FORMAL FOUNDATION OF ANALYTICAL DYNAMICS BASED ON THE CONTACT STRUCTURE

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Dedicated to Prof. K. Ono in celebration of his 60th birthday

A systematic treatment of analytical dynamics was given by E. Cartan in [1], where the 1-form $\sum_i p_i dq_i - Hdt$ plays the fundamental role. We give here a further investigation. One of our main purposes is to clarify relations between dynamical systems and Finsler spaces and the other is to formulate an intrinsic bundle structure of the systems. This paper is closely related to my previous papers [4] [5].

1. A contact structure on the dynamical system

The phase space in analytical dynamics can be stated mathematically as follows. Let M be an n-dimensional differentiable manifold of class C^∞ and local coordinates of a point x in M be $x^1, \cdots\cdots, x^n$. Let p be a vector in the dual tangent space whose components with respect to the natural frame are $p_1, \cdots, p_n$. The dual tangent bundle ${}^cT(M)$ of M consists of points (x, p) and ${}^cT(M)$ is nothing but a phase space. By the coordinate transformation

$$\bar{x}^i = \varphi^i(x^1, \cdots\cdots, x^n) \qquad (i = 1, 2, \cdots, n) \tag{1. 1}$$

p_i's are transformed as

$$\bar{p}_i = p_j \frac{\partial x^j}{\partial \bar{x}^i}. \tag{1. 2}$$

We omit summation symbols in this paper, as is usual in the tensor calculus. We denote the time interval $-\infty < t < \infty$ by τ. A function $H = H(x, p, t)$ of class C^∞ on ${}^cT(M) \times \tau$ with the assumption

$$\text{the rank of the matrix } \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) = n \tag{1. 3}$$

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defines a dynamical system on M. Hereafter we use notations H_{x^i}, H_{p_i}, $H_{p_ip_j}$ for the derviatives $\partial H/\partial x^i$, $\partial H/\partial p_i$, $\partial^2 H/\partial p_i \partial p_j$.

In general a 1-form ω on a $2n+1$ dimensional manifold such that $\omega \wedge (d\omega)^n \neq 0$ is said to define a contact structure. Here we have

THEOREM 1. *A 1-form $\Omega = p_i dx^i - H dt$ defines a contact structure on ${}^cT(M) \times \tau$, where exceptional points form a set without inner points.*

This can be verified as follows. We get by calculation

$$\Omega \wedge (d\Omega)^n = (-1)^{\frac{(n+1)(n+2)}{2}} (p_i H_{p_i} - H) dp_1 \wedge \cdots \wedge dp_n \wedge dx^1 \wedge \cdots \wedge dx^n \wedge dt.$$

If $p_i H_{p_i} - H$ vanishes on a open set we get by differentiation $p_i H_{p_i p_j} = 0$ which contradicts to (1. 3).

Next we put

$$\theta^i = dx^i - H_{p_i} dt, \qquad \rho_i = dp_i + H_{x^i} dt \qquad (i = 1, \cdots, n). \tag{1. 4}$$

Then we get a fundamental relation

$$d\Omega = \rho_i \wedge \theta^i. \tag{1. 5}$$

A curve $x = x(t)$, $p = p(t)$ on ${}^cT(M)$ is called a *path* if it satisfies $\theta^i = 0$, $\rho_i = 0$ $(i = 1, \cdots, n)$. Then we get

THEOREM 2. *We take a family C of curves $x(t)$, $p(t)$ $(t_1 \leqq t \leqq t_2)$ with $x^{(1)} = x(t_1)$, $x^{(2)} = x(t_2)$, where $x^{(1)}$ and $x^{(2)}$ are fixed points in M.*
The integral $\int \Omega$ is stationary for the path in the above sense among the family C.

The proof rums as follows as is essentially given in [1]. We take a 1-parametric family $x(t, \varepsilon)$, $p(t, \varepsilon)$ from C and assume that $\varepsilon = 0$ for the path. Then we have $d\Omega(\partial/\partial t, \partial/\partial \varepsilon) = \rho_i(\partial/\partial t)\, \theta^i(\partial/\partial \varepsilon) - \rho_i(\partial/\partial \varepsilon)\, \theta^i(\partial/\partial t)$ and for $\varepsilon = 0$ $\theta^i(\partial/\partial t) = 0$, $\rho_i(\partial/\partial t) = 0$, and so

$$d\Omega\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \varepsilon}\right) = \frac{\partial}{\partial t}\Omega\left(\frac{\partial}{\partial \varepsilon}\right) - \frac{\partial}{\partial \varepsilon}\Omega\left(\frac{\partial}{\partial t}\right) = 0.$$

Hence we get

$$\left[\frac{\partial}{\partial \varepsilon}\int_{t_1}^{t_2} \Omega\left(\frac{\partial}{\partial t}\right) dt\right]_{\varepsilon=0} = \int_{t_1}^{t_2}\left[\frac{\partial}{\partial \varepsilon}\Omega\left(\frac{\partial}{\partial t}\right)\right]_{\varepsilon=0} dt$$

$$= \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left[\Omega\left(\frac{\partial}{\partial \varepsilon}\right)\right]_{\varepsilon=0} dt = \left[\Omega\left(\frac{\partial}{\partial \varepsilon}\right)_{\varepsilon=0}\right]_{t_1}^{t_2} = 0$$

Next we take a 1-form

$$\omega = p_i dx^i \tag{1.6}$$

on ${}^cT(M)$. We restrict this to a submanifold generated by a family of paths $x = x(t, \alpha^1, \cdots, \alpha^k)$, $p = p(t, \alpha^1, \cdots, \alpha^k)$. By virtue of the relation $\theta^i(\partial/\partial t) = 0$, $\rho_i(\partial/\partial t) = 0$ we get by the same process as in the proof of theorem 1 $d\Omega(\partial/\partial t, \partial/\partial \alpha^l) = 0$. Hence $d\Omega$, which is an exterior differential form of second order, does not contain terms $dt \wedge d\alpha^l$. We put $d\Omega = m_{hl} d\alpha^h \wedge d\alpha^l$ $(m_{hl} = -m_{lh})$. Then by $d(d\Omega) = 0$ we see that $d\Omega$ does not contain t. When we put $t = \text{const.}$, $d\Omega$ reduces to $dp_i \wedge dx^i$ and we get the well known theorem.

THEOREM 3. *The form $d\omega = dp_i \wedge dx^i$ on ${}^cT(M)$ is invariant for a shift of points (x, p) along the path through each point.*

This leads to an invariance of

$$(d\omega)^{n-1} = (-1)^{\frac{(n-1)(n-2)}{2}} (n-1)! \sum_i dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

where $\widehat{\ }$ means a lack of the terms dp_i and dx^i. This is a volume element for a set of paths in ${}^cT(M)$.

2. Finsler space

Let M be an n-dimensional differentiable manifold with a point x, whose local coordinates are $x^1, \cdots, x^n$. We denote the components of a vector y in $T(M)$ at x by $y^1, \cdots, y^n$ with respect to the natural frame. A Finsler structure F on M is defined by a function $F = F(x, y)$ on $T(M)$ $(y \neq 0)$ or its subspace, satisfying the following conditions.

(1) $F(x, y)$ is positively homogeneous of degree 1 in $y^1, \cdots, y^n$

(2) rank of the matrix $\left(\frac{\partial^2 F}{\partial y^i \partial y^j}\right)$ is $n-1$.

When we put

$$p_i = \frac{\partial F}{\partial y^i} \qquad (i = 1, \cdots, n), \tag{2.1}$$

we can define a fundamental mapping $(x, y) \to (x, p)$. The image of the mapping is a hypersurface N which we called *p-manifold* in [4] [5]. A 1-form $\omega = p_i dx^i$ on N gives a contact structure, and when we express N locally by

$$p_n = -H(x^1, \cdots, x^n, p_1, \cdots, p_{n-1}) \tag{2. 2}$$

and put $\theta^a = dx^a - H_{p_a}dx^n$, $\rho_a = dp_a + H_{x^a}dx^n$, we get $d\omega = \rho_a \wedge \theta^a$. Here the index a runs as $a = 1, 2, \cdots, n-1$. The solution curves of $\theta^a = 0$, $\rho_a = 0$ are lifts to ${}^cT(M)$ of extremals on M of our Finsler structure. (The sign of ρ_a is different form that in [5].) The form ω is invariant under a dilatation (or a geodesic flow c.f. [4] p. 93). Hence the invariance of

$$\omega \wedge (d\omega)^{n-1} = (-1)^{\frac{n(n-1)}{2}}(n-1)! \sum_i (-1)^{i-1} p_i dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n \wedge dx^1 \wedge \cdots \wedge dx^n \tag{2. 3}$$

follows.

We can construct conversely a Finsler structure from a hypersurface in ${}^cT(M)$ given by (2. 2), where the rank of the matrix $(H_{p_a p_b})$ is equal to $n-1$. $(a, b = 1, \cdots, n-1)$ This can be done as follows. We put $z^a = H_{p_a}$. By the above assumption p_a's are functions of $x^1, \cdots, x^n, z^1, \cdots, z^{n-1}$ locally. We put $z^a = y^a/y^n$ and define F by

$$F = p_a y^a - H(x^1, \cdots, x^n, p_1, \cdots, p_{n-1})y^n. \tag{2. 4}$$

When we consider F as a function of $x^1, \cdots, x^n, y^1, \cdots, y^n$ it is homogeneous of degree 1 in $y^1, \cdots, y^n$. Moreover we get

$$\frac{\partial F}{\partial y^a} = p_a + y^b \frac{\partial p_b}{\partial y^a} - y^n \frac{\partial H}{\partial p_b}\frac{\partial p_b}{\partial y^a} = p_a,$$

$$\frac{\partial F}{\partial y^n} = \frac{\partial p_a}{\partial y^n} y^a - \frac{\partial H}{\partial p_a}\frac{\partial p_a}{\partial y^n} y^n - H = -H = p_n.$$

Finally by the differentiation of $y^a = y^n H_{p_a}$ with respect to y^c we get $\delta_{ac} = y^n H_{p_a p_b} F_{y^b y^c}$. Hence the rank of the matrikx $(F_{y^b y^c})$ is the same with that of $(H_{p_a p_b})$, namely $n-1$. Thus we have proved

THEOREM 4. *For the hypersurface* (2. 2), *where the rank of the matrix* $(H_{p_a p_b})$ *is equal to* $n-1$, *we can define a Finsler structure by* (2. 4) *on* M, *whose p-manifold is* (2. 2).

3. Relations between dynamical systems and Finsler spaces

By theorem 4 we can construct a Finsler structure on the dynamical system. We only put $x^{n+1} = t$, $p_{n+1} = -H$. Then applying theorem 4 to this case we get

THEOREM 5. *For a given dynamical system defined in section* 1 *we can construct a Finsler structure* F *on* $M\times\tau$ *in such a way that the lifts of extremals of* F *to* ${}^cT(M\times\tau)$ *correspond to the paths of the dynamical systems.*

As an example we take up the fundamental case

$$H=\frac{1}{2}g^{ij}(x)(p_i-c_i(x))(p_j-c_j(x))+U(x),$$

where $x=(x^1,\cdots,x^n)$ and $\det(g^{ij})\neq 0$. We put $(g_{ij})=(g^{ij})^{-1}$. We have $z^i=H_{p_i}=g^{ij}(p_j-c_j)$, $p_i-c_i=g_{ij}z^j$. Putting $z^i=y^i/y^{n+1}$ we get $F=p_iy^i-Hy^{n+1}=(2y^{n+1})^{-1}g_{ij}y^iy^j+c_iy^i-Uy^{n+1}$. And so the paths of our dynamical system correspond to the extremals of the integral

$$\int F(x,\dot{x})dt=\int\Big(\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j+c_i\dot{x}-U(x)\Big)dt.$$

Next we take up a case of an autonomous system. This means that $H(x,p,t)$ does not contain t. In this case along each path, namely a solution curve of $\theta^i=0$, $\rho_i=0$, H is constant, as is well known. We take up a hypersurface

$$H(x,p)=E \text{ (const.)}. \tag{3. 1}$$

We assume $\mathrm{grad}_p H=(H_{p_1},\cdots,H_{p_n})\neq(0,\cdots,0)$. (The set of points such that $\mathrm{grad}_p H=0$ is a one without inner points.) Then we can assume $H_{p_n}\neq 0$ without loss of generality and we can put

$$p_n=-h(x^1,\cdots,x^n,\ p_1,\cdots,p_{n-1}) \tag{3. 2}$$

locally. If we know

$$\det\Big(\frac{\partial^2 h}{\partial p_a\partial p_b}\Big)\neq 0 \quad (a,b=1,\cdots,n-1.),$$

we can introduce a Finsler structure on M based on the equienergy surface (3. 1) by theorem 4. We have by (3. 1) and (3. 2)

$$H(x^1,\cdots,x^n,\ p_1,\cdots,p_{n-1},\ -h)=E. \tag{3. 3}$$

Now we take $x^1,\cdots,x^n$, $p_1,\cdots,p_{n-1}$, E as independent variables and differentiate (3. 3) with respect to p_a. Then we get $\partial h/\partial p_a=H_{p_a}/H_{p_n}$. Again we differentiate this with respect to p_b and we get

$$H_{p_ap_b}-H_{p_ap_n}\frac{\partial h}{\partial p_b}-H_{p_bp_n}\frac{\partial h}{\partial p_a}+H_{p_np_n}\frac{\partial h}{\partial p_a}\frac{\partial h}{\partial p_b}-H_{p_n}\frac{\partial^2 h}{\partial p_a\partial p_b}=0.$$

Putting $Y_a = H_{p_n} X_a$, $Y_n = -H_{p_a} X_a = -H_{p_a} Y_a / H_{p_n}$ we get

$$\frac{\partial^2 h}{\partial p_a \partial p_b} X_a X_b = (H_{p_n})^{-3} (H_{p_a p_b} Y_a Y_b + 2 H_{p_a p_n} Y_a Y_n + H_{p_n p_n} Y_n^2) = (H_{p_n})^{-3} H_{p_i p_j} Y_i Y_j.$$

Hence we get

THEOREM 6. *If the quadratic form $H_{p_i p_j} Y_i Y_j$ is of rank $n-1$ for Y_i satisfying $H_{p_i} Y_i = 0$, we can introduce a Finsler structure F based on the equienergy surface* (3. 1) *as in theorem* 4.

Explicit calculation of F runs as follows. We put $y^i = \lambda H_{p_i}$, from which we get $p_i = K_i(x, z)$ where $z^i = y^i/\lambda$. We put these into $H(x, p) - E = 0$ and we get λ locally, which is possible when

$$\frac{\partial}{\partial \lambda} H\Big(x, K\Big(x, \frac{y}{\lambda}\Big)\Big) = H_{p_i} \frac{\partial K_i}{\partial z^j} \Big(-\frac{y^j}{\lambda^2}\Big) = -\frac{1}{\lambda} \frac{\partial K_i}{\partial z^j} H_{p_i} H_{p_j} \neq 0. \tag{3. 4}$$

This can be verified as follows. We have $z^i = H_{p_i}(x, K(x, z))$ and by differentiation with respect to z^j we get $\delta_{ij} = H_{p_i p_k} \partial K_k / \partial z^j$. Hence $(\partial K_i / \partial z^j)$ is an inverse to $(H_{p_i p_j})$. Now the assumption in theorem 6 reduces to

$$\begin{vmatrix} H_{p_1 p_1} \cdots\cdots\cdots H_{p_1 p_n} & H_{p_1} \\ \cdots\cdots\cdots & \\ H_{p_n p_1} \cdots\cdots\cdots H_{p_n p_n} & H_{p_n} \\ H_{p_1} \quad \cdots\cdots\cdots H_{p_n} & 0 \end{vmatrix} \neq 0,$$

which proves (3. 4), where the exceptional points form a set without inner points.

The application of theorem 6 to the case

$$H = \frac{1}{2} g^{ij}(x)(p_i - c_i(x))(p_j - c_j(x)) + U(x)$$

gives $$F = p_i y^i = \lambda^{-1} g_{ij} y^i y^j + c_i y^i = \pm \sqrt{2(E-U) g_{ij} y^i y^j} + c_i y^i, \tag{3. 5}$$

which is known as Maupertuis's principle. (c.f. [3] p. 225)

Next we consider a relation between the invariant volume element

$$dV = (-1)^{\frac{n(n-1)}{2}} (d\omega)^n / n! = dp_1 \wedge \cdots \wedge dp_n \wedge dx^1 \wedge \cdots \wedge dx^n \tag{3. 6}$$

on ${}^cT(M)$ and that on the equienergy surface (3. 1). By theorem 6 a Finsler structure is introduced on M corresponding to (3. 1). The contact structure associated with it can be given by $\omega = p_i dx^i$ (E = const.) and the invariant volume element on it is given by a constant multiple of (2. 3), namely

$$dV_E = \sum (-1)^{i-1} p_i dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n \wedge dx^1 \wedge \cdots \wedge dx^n. \tag{3. 7}$$

We have by (3. 3)

$$dp_n = -(H_{p_n})^{-1}(H_{x^i} dx^i + H_{p_a} dp_a + dE) \quad (i=1, \cdots, n;\ a=1, \cdots, n-1).$$

Putting this into (3. 6) (3,7) we get the following relation.

THEOREM 6. $$dV = (p_i H_{p_i})^{-1} dE \wedge dV_E.$$

4. Structure equations of dynamical systems

We consider a dynamical system with a function $H = H(x, p, t)$. Putting

$$\theta^j = dx^i - H_{p_i} dt, \quad \rho_i = dp_i + H_{x^i} dt \tag{4. 1}$$

we have got for $\Omega = p_i dx^i - Hdt$

$$d\Omega = \rho_i \wedge \theta^i.$$

Now we take up a coordinate transformation

$$\bar{x}^i = \varphi^i(x^1, \cdots\cdots, x^n) \qquad (i = 1, \cdots, n). \tag{4. 2}$$

Then we get by virtue of the relation $\bar{p}_i = p_j \partial x^j / \partial \bar{x}^i$

$$H_{x^i} = H_{\bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^i} + H_{\bar{p}_j} p_k \frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial \bar{x}^j}\right), \quad H_{p_i} = H_{\bar{p}_j} \frac{\partial x^i}{\partial \bar{x}^j}.$$

When we put $\bar{\theta}^i = d\bar{x}^i - H_{\bar{p}_i} dt$, $\bar{\rho}_i = d\bar{p}_i + H_{\bar{x}^i} dt$, we get

$$\bar{\theta}^i = \frac{\partial \bar{x}^i}{\partial x^j} \theta^j, \quad \bar{\rho}_i = \frac{\partial x^j}{\partial \bar{x}^i} \rho_j + p_k \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^h} \theta^h. \tag{4. 3}$$

Then we seek for

$$\mu_i = \rho_i + r_{ij} \theta^j \qquad (r_{ij} = r_{ji}) \tag{4. 4}$$

such that $$\bar{\mu}_i = \frac{\partial x^j}{\partial \bar{x}^i} \mu_j. \tag{4. 5}$$

(4. 4) assures the relation $d\Omega = \mu_i \wedge \theta^i$. We put

$$a_{ij} = H_{x^i x^j}, \quad b_i^j = H_{x^i p_j}, \quad h^{ij} = H_{p_i p_j}. \tag{4. 6}$$

(h^{ij}) is transformed as a tensor for the transformation (4. 2). We have

$$d\theta^i = -(b^i_j\theta^j + h^{ij}\rho_j)\wedge dt, \qquad d\rho_i = (a_{ij}\theta^j + b^j_i\rho_j)\wedge dt. \tag{4. 7}$$

By the frame transformation θ^i, ρ_i, $dt \to \bar{\theta}^i$, $\bar{\rho}_i$, dt in the dual tangent bundle of ${}^cT(M)\times\tau$, where μ_i's are given by (4. 4) we get

$$d\theta^i = -((b^i_j - h^{ik}r_{kj})\theta^j + h^{ij}\mu_j)\wedge dt. \tag{4. 8}$$

Putting
$$dh^{ij} = u^{ij}_k\theta^k + v^{ijk}\mu_k + w^{ij}dt \tag{4. 9}$$

we have
$$v^{ijk} = \frac{\partial h^{ij}}{\partial p_k}, \qquad w^{ij} = \frac{\partial h^{ij}}{\partial t} - \frac{\partial h^{ij}}{\partial p_k}H_{x^k} + \frac{\partial h^{ij}}{\partial x^k}H_{p_k}, \tag{4. 10}$$

$$u^{ij}_k = \frac{\partial h^{ij}}{\partial x^k} - r_{kh}v^{ijh}. \tag{4. 11}$$

With these preliminaries we prove

THEOREM 7. *For θ^i, ρ_i given by* (4. 1) *we can uniquely find*

$$\mu_i = \rho_i + r_{ij}\theta^j \qquad (r_{ij} = r_{ji}) \tag{M}$$

$$\lambda^j_i = P^j_{ik}\theta^k + Q^{jk}_i\mu_k + R^j_i dt, \tag{L}$$

which satisfy the relations

$$d\theta^i = \theta^j\wedge\lambda^i_j + m^{ik}_j\theta^j\wedge\mu_k - h^{ij}\mu_j\wedge dt, \tag{I}$$

$$d\mu_i = -\mu_j\wedge\lambda^j_i + n^k_{ij}\theta^j\wedge\mu_k + \frac{1}{2}z_{ijk}\theta^j\wedge\theta^k - K_{ij}\theta^j\wedge dt, \tag{II}$$

$$dh^{ij} + h^{ik}\lambda^j_k + h^{kj}\lambda^i_k = 0. \tag{III}$$

Remark. We assume $z_{ijk} = -z_{ikj}$. We have by (4. 4) $d\Omega = \mu_i\wedge\theta^i$. Hence $0 = d(d\Omega) = d\mu_i\wedge\theta^i - \mu_i\wedge d\theta^i$. Putting (I) (II) into this relation we get

$$m^{ik}_j = m^{ki}_j, \quad n^k_{ij} = n^k_{ji}, \quad K_{ij} = K_{ji},$$
$$z_{ijk} + z_{jki} + z_{kij} = 0.$$

proof of theorem 7. We will find such μ_i, λ^j_i. Considering the terms in (III) containing dt we get by (4. 9) (L)

$$w^{ij} + h^{ik}R^j_k + h^{kj}R^i_k = 0.$$

Putting (L) into (I) we get

$$d\theta^i = \theta^j\wedge(P^i_{jk}\theta^k + Q^{ik}_j\mu_k + R^i_j dt) + m^{ik}_j\theta^j\wedge\mu_k - h^{ij}\mu_j\wedge dt. \tag{4. 13}$$

We compare this with (4. 8). Firstly

$$R^i_j = -b^i_j + h^{ik} r_{kj} \tag{B}$$

and then we get

$$h^{ik} h^{jh} r_{kh} = \frac{1}{2} (h^{ik} b^j_k + h^{jk} b^i_k - w^{ij}). \tag{A}$$

Next we get by (4. 13) (4. 8)

$$P^i_{jk} = P^i_{kj}, \quad Q^{ik}_j = -m^{ik}_j. \tag{4. 14}$$

We get by (4. 9) (L)

$$u^{ij}_h + h^{ik} P^j_{kh} + h^{kj} P^i_{kh} = 0, \qquad v^{ijh} + h^{ik} Q^{jh}_k + h^{kj} Q^{ih}_k = 0. \tag{4. 15}$$

By (M) (4. 7) $d\mu_i$ does not contain terms $\mu_j \wedge \mu_k$ for the base θ^i, μ_i, dt.

Hence

$$Q^{jh}_i = Q^{hj}_i. \tag{4. 16}$$

We put $A^{ilj} = h^{ik} h^{lh} P^j_{kh}$ and $B^{ijh} = h^{ik} Q^{jh}_k$. Then we have by (4. 14) (4. 15) (4. 16)

$$A^{ilj} + A^{jli} = -h^{lk} u^{ij}_k, \quad A^{ilj} = A^{lij},$$
$$B^{ijh} + B^{jih} = -v^{ijh}, \qquad B^{ijh} = B^{ihj}.$$

From these relations we get A^{ijl} and B^{ijh}. Hence

$$h^{ik} h^{jh} P^l_{kh} = -\frac{1}{2} (h^{ik} u^{lj}_k + h^{jk} u^{li}_k - h^{lk} u^{ij}_k) \tag{C}$$

$$h^{ik} Q^{jh}_k = -\frac{1}{2} (v^{ihj} + v^{ijh} - v^{jhi}). \tag{D}$$

Putting $(h_{ij}) = (h^{ij})^{-1}$ we can resume (A) (B) (C) (D) as follows, where v^{ijk}, w^{ij} are given by (4. 10) and u^{ij}_k by (4. 11).

$$r_{ij} = \frac{1}{2} (h_{kj} b^k_i + h_{ik} b^k_j - h_{ik} h_{jh} w^{kh}), \tag{A$'$}$$

$$R^i_j = h^{ik} r_{kj} - b^i_j, \tag{B$'$}$$

$$P^k_{ij} = -\frac{1}{2} (h_{lj} u^{kl}_i + h_{il} u^{kl}_j - h^{kn} h_{il} h_{jm} u^{lm}_n) \tag{C$'$}$$

$$Q^{ij}_k = -m^{ij}_k = -\frac{1}{2} h_{kl} (v^{lij} + v^{lji} - v^{ijl}). \tag{D$'$}$$

Now we will verify that (M) (L) with (A′) (B′) (C′) (D′) satisfy (I) (II) (III). By (4. 9) (4. 15) we get (III). By (4. 13) (4. 14) (B) (4. 8) we get (I). As to (II) we proceed as follows. We put

$$dr_{ij} = s_{ijk}\theta^k + s^k_{ij}\mu_k + s_{ij}dt.$$

Then we have by (4. 1) (4. 4)

$$\left.\begin{aligned} s^k_{ij} &= \frac{\partial r_{ij}}{\partial p_k}, \quad s_{ijk} = \frac{\partial r_{ij}}{\partial x^k} - r_{kh}s^h_{ij}, \\ s_{ij} &= \frac{\partial r_{ij}}{\partial t} + \frac{\partial r_{ij}}{\partial x^k} H_{p_k} - \frac{\partial r_{ij}}{\partial p_k} H_{x^k}. \end{aligned}\right\} \qquad (4.\ 17)$$

We express $d\mu_i + \mu_j \wedge \lambda^j_i$ in θ^i, μ_i, dt by (4. 4) (4. 7) (L). We have

$$\begin{aligned} d\mu_i + \mu_j\wedge\lambda^j_i &= d(\rho_i + r_{ij}\theta^i) + \mu_j\wedge(P^j_{ik}\theta^k + Q^{jk}_i\mu_k + R^j_i dt) \\ &= (a_{ij}\theta^j + b^j_i(\mu_j - r_{jk}\theta^k))\wedge dt + (s_{ijk}\theta^k + s^k_{ij}\mu_k + s_{ij}dt)\wedge\theta^j \\ &\quad - r_{ij}(b^j_k\theta^k + h^{jk}(\mu_k - r_{kh}\theta^h))\wedge dt + \mu_j\wedge(P^j_{ik}\theta^k + R^j_i dt). \end{aligned}$$

These reduce to (II) when we put

$$\left.\begin{aligned} n^k_{ij} &= -P^k_{ij} - s^k_{ij}, \quad z_{ijk} = s_{ijk} - s_{ikj}, \\ K_{ij} &= -a_{ij} + r_{jk}b^k_i + r_{ik}b^k_j - h^{kh}r_{ki}r_{hj} + s_{ij}. \end{aligned}\right\} \qquad (\mathrm{E})$$

Thus we have proved theorem 7.

As a consequence we get the desired relation (4. 5), namely

THEOREM 8. *For the coordinate transformation* $\bar{x}^i = \varphi^i(x)$ $(i = 1, \cdots, n)$ θ^i *and* μ_i *are transformed as*

$$\bar{\theta}^i = \frac{\partial \bar{x}^i}{\partial x^j}\theta^j, \qquad \bar{\mu}_i = \frac{\partial x^j}{\partial \bar{x}^i}\mu_j.$$

Proof. For the coordinate transformation in question θ^i is transformed as in (4. 3). We take $\bar{\lambda}^j_i$ which is transformed as connection forms from λ^j_i. Then $d\theta^i - \theta^j\wedge\lambda^i_j$ is transformed into $d\bar{\theta}^i - \bar{\theta}^j\wedge\bar{\lambda}^i_j$ as a vector. Therefore, if we take $\bar{m}^{ik}_j$, $\bar{h}^{ij}$, $\bar{n}^k_{ij}$, $\bar{K}_{ij}$, $\bar{z}_{ijk}$ which are transformed from m^{ik}_j, h^{ij} etc as tensors and $\tilde{\mu}_i = \mu_j\partial x^j/\partial\bar{x}^i$, the relations (I) (II) (III) hold good for $\bar{\theta}^i$, $\tilde{\mu}_i$, $\bar{\lambda}^i_j$, $\bar{h}^{ij}$ etc.

Now by $\tilde{\mu}_i = (\rho_j + r_{jk}\theta^k)\partial x^j/\partial\bar{x}^i$ and (4. 3) we have $\tilde{\mu}_i = \bar{\rho}_i + t_{ij}\bar{\theta}^j$. On the other hand we have $d\Omega = \mu_i\wedge\theta^i = \tilde{\mu}_i\wedge\bar{\theta}^i$ and $d\Omega = \bar{\rho}_i\wedge\bar{\theta}^i$. Hence we get $t_{ij} = t_{ji}$. By theorem 7 $\bar{\lambda}^j_i$ and $\bar{\mu}_i = \bar{\rho}_i + \bar{r}_{ij}\bar{\theta}^j$ $(\bar{r}_{ij} = \bar{r}_{ji})$ are determined uniquely from $\bar{\theta}^i$, $\bar{\rho}_i$ so as to satisfy (I) (II) (III). Hence $\tilde{\mu}_i$ coincides with $\bar{\mu}_i$ determined from corresponding $\bar{\theta}^i$ and $\bar{\rho}_i$. Q.E.D.

Next we take up the fundamental case

$$H = \frac{1}{2} g^{ij}(x, t)(p_i - c_i(x, t))(p_j - c_j(x, t)) + U(x, t).$$

Then we have

$$h^{ij} = H_{p_i p_j} = g^{ij}, \qquad v^{ijk} = 0, \qquad u_k^{ij} = \frac{\partial g^{ij}}{\partial x^k}$$

$$b_j^i = H_{x^j p_i} = \frac{\partial}{\partial x^j}(g^{ik}(p_k - c_k)) = \frac{\partial g^{ik}}{\partial x^j}(p_k - c_k) - g^{ik}\frac{\partial c_k}{\partial x^j}$$

$$w^{ij} = \frac{\partial h^{ij}}{\partial t} + u_k^{ij} H_{p_k} = \frac{\partial g^{ij}}{\partial t} + \frac{\partial g^{ij}}{\partial x^k} g^{kl}(p_l - c_l).$$

We put $(g_{ij}) = (g^{ij})^{-1}$ and

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

We get from (A′) (B′) (C′)

$$P^k_{ij} = \Gamma^k_{ij}, \qquad Q^{ij}_k = 0, \qquad m^{ij}_k = 0,$$

$$r_{ij} = -\Gamma^k_{ij}(p_k - c_k) - \frac{1}{2}\left(\frac{\partial c_i}{\partial x^j} + \frac{\partial c_j}{\partial x^i}\right) + \frac{1}{2}$$

$$R^i_j = g^{hk}\Gamma^i_{jh}(p_k - c_k) + \frac{1}{2} g^{ih}\left(\frac{\partial c_h}{\partial x^j} - \frac{\partial c_j}{\partial x^h} + \frac{\partial g_{hj}}{\partial t}\right).$$

Hence by (E) and (4. 17) $\qquad n^k_{ij} = 0.$

Next we take up the simpler case

$$H = \frac{1}{2} g^{ij}(x) p_i p_j + U(x).$$

Then we have $P^k_{ij} = \Gamma^k_{ij}$, $r_{ij} = -\Gamma^k_{ij} p_k$, $R^i_j = g^{kh}\Gamma^i_{jh} p_k$ and

$$\theta^i = dx^i - g^{ij} p_j dt, \qquad \mu_i = dp_i - \Gamma^k_{ij} p_k dx^j + U_{x^i} dt.$$

We put $\quad U_{ij} = \dfrac{\partial^2 U}{\partial x^i \partial x^j} - \Gamma^k_{ij}\dfrac{\partial U}{\partial x^k}$, $R^i_{jkl} = \dfrac{\partial \Gamma^i_{jk}}{\partial x^l} - \dfrac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{hl}\Gamma^h_{jk} - \Gamma^i_{kh}\Gamma^k_{jl}$.

Then we get

$$z_{ijk} = \frac{1}{2} R^l_{ijk} p_l, \qquad K_{ij} = -(g^{km} R^l_{ijk} p_l p_m + U_{ij}).$$

5. Product structures and connections

By the preceding consideration we can naturally construct a product structure on ${}^cT(M)\times\tau$ when we decompose the dual tangent space of ${}^cT(M)$

$\times\tau$ into S_1 spanned by $\theta^1, \cdots, \theta^n$, S_2 spanned by $\mu_1, \cdots, \mu_n$ and T spanned by dt. If we put

$$P=\begin{pmatrix} \left(\dfrac{\partial \bar{x}^i}{\partial x^j}\right) & 0 & 0 \\ 0 & \left(\dfrac{\partial x^i}{\partial \bar{x}^j}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda=\begin{pmatrix} (\lambda_i^j) & 0 & 0 \\ 0 & -(\lambda_j^i) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

P constitutes the structure group and Λ gives an affine connection on ${}^cT(M)\times\tau$. Moreover we have

THEOREM 8. *The quadratic forms* $h_{ij}\theta^i\theta^j$, $h^{ij}\mu_i\mu_j$, $\theta^i\mu_i$ *are intrinsic forms on* ${}^cT(M)\times\tau$.

For an autonomous case $H=H(x,p)$ we have dx^i, $\sigma_i = dp_i + r_{ij}dx^j$ from θ^i, μ_i when we put $t=$ const. A product structure can be defined on ${}^cT(M)$ when we decompose the dual tangent spaces into S_1 spanned by $dx^1, \cdots, dx^n$ and S_2 spanned by $\sigma_1, \cdots, \sigma_n$. The fundamental quadradratic forms $h_{ij}dx^idx^j$, $h^{ij}\sigma_i\sigma_j$, $dx^i\sigma_i$ are given on ${}^cT(M)$, and an affine connection is also defined.

6. Mapping by the fiow

We shift each point (x,p) of ${}^cT(M)$ along the path, namely the solution curve of $\theta^i=0$ and $\rho_i=0$ (hence $\mu_i=0$). We fix a time interval t and we get a mapping of $(x,p)\to(x',p')$ of ${}^cT(M)$ into itself which we call a flow. We will show how the tangent spaces are mapped by the flow.

We take a vector field $T=\partial/\partial t$ along the flow. Then we get

$$\theta^i(T)=0, \quad \mu_i(T)=0.$$

Let X be a vector field which commutes with T. Then we get from (I) (II) in 4

$$\begin{aligned} T(\theta^i(X)) &= -\lambda_j^i(T)\theta^j(X) + h^{ij}\mu_j(X), \\ T(\mu_i(X)) &= \lambda_i^j(T)\mu_j(X) + K_{ij}\theta^j(X), \end{aligned}$$

and $\lambda_i^j(T)=R_i^j$. We put $X=u^i\dfrac{\partial}{\partial x^i}+v_i\dfrac{\partial}{\partial p_i}$ and we get

$$\theta^i(X)=u^i, \quad \mu_i(X)=(\rho_i+r_{ij}\theta^j)(X)=v_i+r_{ij}u^j.$$

Putting $z_i=\mu_i(X)$ we get

$$\frac{du^i}{dt} = -R^i_j u^j + h^{ij} z_j, \qquad \frac{dz_i}{dt} = K_{ij} u^j + R^j_i z_j. \tag{6. 1}$$

Here we have by (III)

$$\frac{dh^{ij}}{dt} = -h^{ik} R^j_k - h^{kj} R^i_k. \tag{6. 2}$$

The equations (6. 1) (6. 2) are fundamental and correspond to the Jacobi's equations in the calculus of variation. The invariance of the volume element (3. 6) can be seen from the vanishing of the trace of the matrix.

$$\begin{pmatrix} (-R^i_j) & (h^{ij}) \\ (K_{ij}) & (R^j_i) \end{pmatrix}.$$

We have from (6. 1) (6. 2)

$$\frac{d}{dt}(u^i z_i) = h^{ij} z_i z_j + K_{ij} u^i u^j, \quad \frac{d}{dt}(h^{ij} z_i z_j) = 2h^{il} K_{lj} u^j z_i,$$

$$\frac{d}{dt}(h_{ij} u^i u^j) = 2u^i z_i,$$

which wait for future applications.

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