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UNITS AND CLASS NUMBERS OF REAL QUADRATIC FIELDS

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Dedicated to Professor Katuzi Ono on his 60-th birthday

§1

The aim of this paper is to prove the following main theorem:

THEOREM. *For the discriminant $d>0$ of a real quadratic field $\boldsymbol{Q}(\sqrt{d})$, let $(x, y)=(t, u)$ be the least positive integral solution of Pell's equation $x^2-dy^2=4$ and put $\varepsilon_d=\frac{1}{2}(t+u\sqrt{d})$, and denote by h_d the ideal class number. Then, at least one of the following two assertions is true:*

(i) *For an arbitrarily given positive number $\delta>0$ there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ of which discriminant d satisfies the inequality $h_d<\delta\sqrt{d}$.*

(ii) *There exists a positive constant $\kappa>0$ such that the inequality $\varepsilon_d\leqq(d-4)^{\kappa}$ holds for the discriminant d of any real quadratic field $\boldsymbol{Q}(\sqrt{d})$ except $\boldsymbol{Q}(\sqrt{5})$.*

Recently, A. Baker [1] and H. M. Stark [5] proved independently that there exist exactly only nine imaginary quadratic fields of class number one. On the other hand, C.F. Gauss conjectured that there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ of class number one, moreover that there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{p})$ of class number one, of which discriminant p is prime and congruent to 1 mod 4.

If we assume that this Gauss' conjecture concerning the class number of real quadratic fields is true, then it is proved that the assertion (i) of the main theorem is true (Proposition 1). From this fact and an already known general estimation[1])

$$\frac{1}{2}\log d<\log\varepsilon_d<\sqrt{d}\left(\frac{1}{2}\log d+1\right)$$

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1) Cf. L.K. Hua [2], M. Newman [4].

concerning the fundamental unit of the real quadratic fields $Q(\sqrt{d})$, it is probable that the assertion (i) is true and the assertion (ii) is not true.

§2

In order to prove the main theorem we must prepare the following two lemmas:

LEMMA 1. *Let* $\{d_n\}$ *be a sequence of discriminants* $d_n > 0$ *of real quadratic fields satisfying* $\lim_{n\to\infty} d_n = \infty$, *and let* $r > 0$, a, b *three real constants such that the inequality* $\varepsilon_{d_n} > a(d_n - b)^r$ *holds for this sequence. Then, for an arbitrarily given positive number* $\varepsilon > 0$ *there exists a natural number* $n_0 = n_0(\varepsilon)$ *such that the inequality* $h_{d_n} < \left(\frac{1}{2r} + \varepsilon\right)\sqrt{d_n}$ *holds for any natural number* n *bigger than* n_0.

Proof. From the class number formula[2)]

$$h_d = \frac{\sqrt{d}}{\log \varepsilon_d} L(d);\ L(d) = \sum_{k=1}^{\infty}\left(\frac{d}{k}\right)\frac{1}{k}$$

concerning real quadratic field $Q(\sqrt{d})$ and L.K. Hua's estimation[3)]

$$L(d) < \frac{1}{2}\log d + 1$$

concerning L-function $L(d)$ of $Q(\sqrt{d})$, we have

$$h_{d_n} < \frac{\frac{1}{2}\log d_n + 1}{\log a(d_n - b)^r}\sqrt{d_n} \cdots\cdots\cdots\cdots (1).$$

On the other hand, the assumption $\lim_{n\to\infty} d_n = \infty$ implies $\lim_{n\to\infty}\frac{\log d_n + 2}{\log a(d_n - b)^r} = \frac{1}{r}$. Therefore, for an arbitrarily given positive number $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon)$ such that the inequality

$$\frac{\log d_n + 2}{\log a(d_n - b)^r} < \frac{1}{r} + 2\varepsilon \cdots\cdots\cdots\cdots (2)$$

holds for any natural number n bigger than n_0. Hence, it follows from (1) and (2) that for any natural number n bigger than n_0 the relation

$$h_{d_n} < \frac{\frac{1}{2}\log d_n + 1}{\log a(d_n - b)^r}\sqrt{d_n} < \left(\frac{1}{2r} + \varepsilon\right)\sqrt{d_n}$$

2) Cf. e.g. E. Landau [3], p. 152.

3) Cf. L.K. Hua [2].

holds, which is our assertion.

LEMMA 2. *Let N be the set of the discriminants $d>0$ of all real quadratic fields $Q(\sqrt{d})$, and define the subset N_r of N for any real number $r\geqq 0$ by $N_r = \{d\in N;\ \varepsilon_d > (d-4)^r\}$. Then the set function N_r defined in $[0,\infty)$ has the following properties:*

(1) $0\leqq r_1 < r_2$ *implies* $N_{r_1} \supseteqq N_{r_2}$.

(2) $0\leqq r \leqq \frac{1}{2}$ *implies* $N_r = N$.

(3) N_1 *contains all prime numbers p congruent to* 1 *mod* 4.

(4) N_1 *contains all d in N such that Pell's equation $x^2-dy^2=-4$ is solvable.*

(5) *If N_r contains infinitely many elements for some positive number $r>0$,* i.e. $N_r = \{d_n \in N;\ d_1 < d_2 < \cdots < d_n < \cdots\}$, *then for an arbitrarily given positive number $\varepsilon>0$ there exists a natural number $n_0 = n_0(\varepsilon)$ such that the inequality $h_{d_n} < \left(\frac{1}{2r} + \varepsilon\right)\sqrt{d_n}$ holds for any natural number n bigger than n_0.*

(6) *If N_r contains only a finite number of elements for some positive number $r>0$, then there exists a real number r_0 bigger than r such that $N_{r'} = \{5\}$ for any real number r' bigger than r_0.*

Proof. (1) Since for any element d in N_{r_2} the inequality $\varepsilon_d>(d-4)^{r_2}$ holds and r_2 is bigger than r_1, the inequality $\varepsilon_d>(d-4)^{r_1}$ holds. Hence we get $d\in N_{r_1}$.

(2) If we notice that $u\geqq 1$ and $t^2=du^2+4>d$ hold in $\varepsilon_d=\frac{1}{2}(t+u\sqrt{d})$, then we get $\varepsilon_d>\sqrt{d}$. Hence the inequality $\varepsilon_d > (d-4)^{1/2}$ holds for any d in N.

(3) It is obtained by M. Newman [4] that the inequality $\varepsilon_p > p-3$ holds for any prime p congruent to 1 mod 4. From this fact our assertion follows immediately.

(4) If Pell's equation $x^2-dy^2=-4$ is solvable for d in N, then for the least positive integral solution $(x,y)=(t_0,u_0)$ we have $\varepsilon_d^- = \frac{1}{2}(t_0+u_0\sqrt{d}) > \sqrt{d-4}$, because $u_0\geqq 1$ and $t_0^2 = du_0^2 - 4 \geqq d-4$ hold. Hence we get $\varepsilon_d = (\varepsilon_d^-)^2 > d-4$, from which our assertion follows immediately.

(5) Since $N_r \ni d_n$ is equivalent to $\varepsilon_{d_n} > (d_n-4)^r$ for any natural number n, and $\lim_{n\to\infty} d_n = \infty$ holds, our assertion is clear by lemma 1.

(6) If we set $N_r=\{5<d_1<d_2<\cdots<d_s\}$, then the inequality $\varepsilon_{d_i}>(d_i-4)^r$ holds for $i=1, 2, \cdots, s$ and the inequality $\varepsilon_d \leqq (d-4)^r$ holds for any $d \neq 5$ in N different from d_i $(i=1, 2, \cdots, s)$. Hence, if we define the positive number r_i bigger than r by $\varepsilon_{d_i}=(d_i-4)^{r_i}$ and set $r_0=\underset{i=1,2,\cdots,s}{\mathrm{Max}}\ r_i$, then we have

$$\begin{cases} \varepsilon_{d_i}=(d_i-4)^{r_i} \leqq (d_i-4)^{r_0} \leqq (d_i-4)^{r'} \\ \varepsilon_d \leqq (d-4)^r < (d-4)^{r'} \end{cases}$$

for any r' bigger than r_0 and any d in N different from d_i $(i=1, 2, \cdots, s)$. Therefore, $N_{r'}=\{5\}$ for any r' bigger than r_0, which is our assertion.

Proof of the theorem. (i) If for any $r \geqq 0$ N_r contains infinitely many elements, we put $\varepsilon=\delta/2$ for an arbitrarily given positive number $\delta>0$. Then, since for any r satisfying $r \geqq 1/\delta$ N_r also contains infinitely many elements, it follows from (5) of lemma 2, that there exist infinitely many d in N_r such that the relation

$$h_d<\left(\frac{1}{2r}+\varepsilon\right)\sqrt{d} \leqq (\delta/2+\delta/2)\sqrt{d}=\delta\sqrt{d}$$

holds. Therefore, in this case the first assertion (i) of the main theorem is true.

(ii) If for some positive number $r>0$ N_r contains at most only a finite number of elements, then by (6) of lemma 2, there exists a real number r_0 such that $N_{r'}=\{5\}$ for any real number r' bigger than r_0. Therefore, in this case there exists a positive constant $\kappa>0$ such that the inequality $\varepsilon_d \leqq (d-4)^\kappa$ holds for any $d \neq 5$ in N, and the second assertion (ii) of the main theorem is true.

Thus, our main theorem is completely proved.

§3.

PROPOSITION. *If we assume that Gauss' conjecture concerning the class number of real quadratic fields is true, then for an arbitrarily given positive number* $\delta>0$ *there exist infinitely many real quadratic fields* $Q(\sqrt{d})$, *of which discriminant* d *satisfies the inequality* $h_d<\delta\sqrt{d}$.

Proof. If we assume that Gauss' conjecture concerning the class number of real quadratic fields is true, there exist infinitely many real quadratic fields $Q(\sqrt{d})$, of which class number is equal to one. Hence, even if for an arbitrarily given positive number $\delta > 0$ we may add a condition $d > 1/\delta^2$ moreover, there exist infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ which satisfy all these conditions. Therefore, for such infinitely many real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ the relation $h_d = 1 < \delta\sqrt{d}$ holds, which is the assertion in our proposition.

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