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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

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Dedicated to Professor Katuzi Ono on his 60th birthday

1. Let R^n be the n-dimensional Euclidean space, each point of which is denoted by its coordinate $x = (x_1, \cdots, x_n)$. The variable t is in the real half line $[0, \infty)$. We consider a differential operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial t} \tag{1}$$

in the $(n+1)$-dimensional Euclidean half space $R^n \times (0, \infty)$ and assume that the matrix (a_{ij}) is positive definite in $R^n \times (0, \infty)$. Suppose that for coefficients of L there exist constants $K_1 (> 0)$, $K_2 (\geqq 0)$, $K_3 (> 0)$ and $\lambda \in [0, 1]$ such that

$$|a_{ij}| \leqq K_1(|x|^2 + 1)^{1-\lambda}, \qquad 1 \leqq i, \ j \leqq n,$$
$$|b_i| \leqq K_2(|x|^2 + 1)^{1/2}, \qquad 1 \leqq i \leqq n,$$
$$|c| \leqq K_3(|x|^2 + 1)^{\lambda}.$$

Besala and Fife [1] investigated the asymptotic behavior of solutions of the Cauchy problem for such a parabolic differential operator L under a non-negative Cauchy data not identically equal to zero.

One of their result is as follows:

Let a continuous function $u(x, t)$ in $R^n \times [0, \infty)$ have the following properties;

i) *$Lu \leqq 0$ in $R^n \times (0, \infty)$ in the usual sense,*

ii) *$u(x, 0)$ is non-negative and not identically equal to zero*

and

iii) *there exist positive constants μ and ν such that*

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$$u(x,t) \geqq \begin{cases} -\mu e^{\nu(|x|^2+1)^\lambda}, & \lambda \in (0,1], \\ -\mu(|x|^2+1)^\nu, & \lambda = 0 \end{cases}$$

in $R^n \times (0,\infty)$.

If there exist a sufficiently large constant α *and a positive* β *satisfying*

$$4\alpha^2\lambda^2(|x|^2+1)^{2\lambda-2}\sum_{i,j=1}^n a_{ij}x_ix_j - 4\alpha\lambda(\lambda-1)(|x|^2+1)^{\lambda-2}\sum_{i,j=1}^n a_{ij}x_ix_j$$
$$-2\alpha\lambda(|x|^2+1)^{\lambda-1}\sum_{i=1}^n(a_{ii}+b_ix_i)+c \geqq \beta$$

in $R^n \times (0,\infty)$, *then* $u(x,t)$ *grows exponentially as* t *tends to* ∞ *and this exponential growth of* $u(x,t)$ *is uniform with respect to* $x \in R^n$.

In their proof of this result, the condition that α is sufficiently large, is essential. In this note we shall give a rather simple condition than that of Besala-Fife under a somewhat different condition for coefficients of the operator L.

2. In the following we assume that coefficients of the operator L in (1) satisfy the following condition in $R^n \times (0,\infty)$ for some $\lambda \in (0,1]$:

$$(2)\qquad \begin{cases} k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \leqq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \\ \qquad\qquad\qquad\text{for any real vector } \xi=(\xi_1,\cdots,\xi_n), \\ |b_i| \leqq K_2(|x|^2+1)^{1/2}, \qquad 1\leqq i\leqq n, \\ -k_3(|x|^2+1)^\lambda + k_3' \leqq c \leqq K_3(|x|^2+1)^\lambda, \end{cases}$$

where $k_1(>0)$, K_1, $K_2(\geqq 0)$, $k_3(>0)$, $k_3'(\geqq 0)$ and $K_3(>0)$ are constants.

First we construct a function of the form $H(x,t)=\exp\{-\alpha(t)(|x|^2+1)^\lambda + \beta(t)\}$ satisfying $LH \geqq 0$ in $R^n \times (0,\infty)$, where $\alpha(t)$ and $\beta(t)$ are positive and differentiable once in $(0,\infty)$.

Obviously the condition (2) implies

$$\begin{aligned} \frac{LH}{H} &\geqq 4\alpha^2(t)\lambda^2k_1(|x|^2+1)^{\lambda-1}|x|^2 - 2\alpha(t)\lambda nK_1 \\ &\quad - 2\alpha(t)\lambda nK_2(|x|^2+1)^\lambda - k_3(|x|^2+1)^\lambda + k_3' \\ &\quad + \alpha'(t)(|x|^2+1)^\lambda - \beta'(t) \\ &\geqq (|x|^2+1)^\lambda[4\alpha^2(t)\lambda^2k_1 - 2\alpha(t)\lambda nK_2 - k_3 + \alpha'(t)] \\ &\quad - 2\alpha(t)\lambda nK_1 + k_3' - 4\alpha^2(t)\lambda^2k_1 - \beta'(t). \end{aligned}$$

We can easily verify that the function

$$\alpha(t)=\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1},\quad \gamma_0=\left(k_3+\frac{n^2K_2^2}{4k_1}\right)^{1/2}$$

is a solution of the differential equation

$$4\alpha^2(t)\lambda^2k_1-2\alpha(t)\lambda nK_2-k_3+\alpha'(t)=0$$

of the Riccati type in $(0,\infty)$ and that for this $\alpha(t)$ the function

$$\beta(t)=\left\{\frac{n\gamma_0}{\sqrt{k_1}}(K_1+K_2)-\frac{n^2K_2}{2k_1}(K_1+K_2)-k_3+k_3'\right\}t$$
$$-\frac{n(K_1+K_2)}{2\lambda k_1}\log(e^{4\gamma_0\lambda\sqrt{k_1}t}-1)+\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}$$

satisfies

$$-2\alpha(t)\lambda nK_1+k_3'-4\alpha^2(t)\lambda^2k_1-\beta'(t)=0$$

in $(0,\infty)$. Hence we see $LH\geqq 0$ in $R^n\times(0,\infty)$ for the function

$$H(x,t)=(e^{4\gamma_0\lambda\sqrt{k_1}t}-1)^{-\frac{n(K_1+K_2)}{2\lambda k_1}}\exp\left\{\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}\right\}\times$$
$$\times\exp\left\{-\left(\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\right)(|x|^2+1)^\lambda\right. \tag{3}$$
$$\left.+\left[\frac{n\gamma_0}{\sqrt{k_1}}(K_1+K_2)-\frac{n^2K_2}{2k_1}(K_1+K_2)-k_3+k_3'\right]t\right\},$$

where $\gamma_0=\left(k_3+\frac{n^2K_2^2}{4k_1}\right)^{1/2}$. Since

$$\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}>0\quad\text{and}\quad\frac{\gamma_0}{\lambda\sqrt{k_1}}\{1-(|x|^2+1)^\lambda\}<0,\quad x\neq 0,$$

it holds that

$$\lim_{t\downarrow 0}H(x,t)=0\quad\text{for } x\neq 0. \tag{4}$$

3. Suppose that the function $u(x,t)$ non-negative and continuous in $R^n\times[0,\infty)$ has the following property:

$$\begin{cases}\text{i)}\ \ Lu\leqq 0 \text{ in } R^n\times(0,\infty) \text{ in the usual sense,}\\ \text{ii)}\ \ u(x,0)\,(\geqq 0) \text{ is not identically equal to zero.}\end{cases} \tag{5}$$

Here L is a differential operator of the form (1) with coefficients satisfying (2) and

$$(6)\qquad -2\Big(\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)\lambda K_1 n-4\Big(\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)^2\lambda^2 k_1+k_3'>0,$$

$$\gamma_0=\Big(k_3+\frac{n^2K_2^2}{4k_1}\Big)^{1/2}.$$

We can find a positive number ε so small that

$$(7)\qquad -2\Big(\varepsilon+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)\lambda K_1 n-4\Big(\varepsilon+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)^2\lambda^2 k_1+k_3'>0.$$

Let T be a positive number such that

$$0<\frac{\gamma_0}{\lambda\sqrt{k_1}}\,\frac{1}{e^{2\gamma_0\lambda\sqrt{k_1}T}-1}<\varepsilon.$$

From the assumption for $u(x,t)$ we see by the strong maximum principle due to Nirenberg [5] that $u(x,t)>0$ in $R^n\times(0,\infty)$. So the value $m=\min\limits_{\substack{|x|=r\\ t\in[\delta,T]}}u(x,t)$ is positive for an arbitrary $r(>0)$ and for any $\delta(>0)$ fixed sufficiently small. We may assume that $\frac{T}{2}<T-\delta$. For these r and δ, clearly $0<M_1=\max\limits_{\substack{|x|=r\\ t\in[\delta,T]}}H(x,t-\delta)<\infty$, where H is the function given by (3). Put

$$w(x,t)=\frac{m}{M_1}H(x,t-\delta)-u(x,t).$$

Evidently we have $Lw\geqq 0$ in $\Omega\times(\delta,T)$, where Ω is the set of all points $x\in R^n$ such that $r<|x|$. Moreover, $w(x,t)$ is continuous on $\bar\Omega\times[\delta,T]$, $w(x,\delta)\leqq 0$ for $|x|\geqq r$ and $w(x,t)\leqq 0$ for $|x|=r$ and $t\in[\delta,T]$. Bodanko's maximum principle [2] implies that $w(x,t)\leqq 0$ in $\bar\Omega\times[\delta,T]$. Therefore we get

$$\frac{m}{M_1}H(x,T-\delta)\leqq u(x,T)$$

for $|x|\geqq r(>0)$. As is seen easily, there is a positive constant M_2 such that $M_2H(x,T-\delta)\leqq u(x,T)$ in $|x|\leqq r$. Hence by putting $M_3=\min\Big(\frac{m}{M_1},\ M_2\Big)$ we have $M_3H(x,T-\delta)\leqq u(x,T)$ at every point $x\in R^n$. Since $\frac{T}{2}<T-\delta$, we obtain

$$u(x,T) \geqq M_3 H(x, T-\delta)$$
$$\geqq M_4 \exp\left\{-\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)(|x|^2+1)^\lambda\right\}$$

in R^n for some positive constant M_4.

4. Now we can prove the following theorem.

THEOREM 1. *Let L be a parabolic differential operator of the form* (1) *with coefficients satisfying* (2) *and* (6). *Assume that the function $u(x,t)$ continuous in $R^n \times [0,\infty)$ satisfies* (5) *and $u(x,t) \geqq -\mu e^{\nu(|x|^2+1)^\lambda}$ for some positive constants μ and ν. Then $u(x,t)$ grows to infinity exponentially as t tends to ∞ and this exponential growth of $u(x,t)$ is uniform in any compact subset of R^n.*

Proof. Bodanko's maximum principle shows that $u(x,t) \geqq 0$ in $R^n \times [0,\infty)$. As was shown in §3, for a positive number ε satisfying (7) there exist a positive number T and a positive constant M_4 such that

$$u(x,T) \geqq M_4 \exp\left\{-\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)(|x|^2+1)^\lambda\right\}$$
$$\equiv M_4 H_0(x), \qquad \text{say.}$$

From (7) we can take a positive number β_0 which satisfies

$$-2\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)\lambda K_1 n - 4\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)^2 \lambda^2 k_1 + k_3' - \beta_0 > 0.$$

Putting

$$h(x,t) = M_4 H_0(x) e^{\beta_0(t-T)}$$

and $v(x,t) = u(x,t) - h(x,t)$ in $R^n \times (T,\infty)$, we see

$$Lv \leqq -Lh$$
$$= -h[4\alpha_0^2\lambda^2(|x|^2+1)^{2\lambda-2}\sum_{i,j=1}^n a_{ij}x_i x_j$$
$$-4\alpha_0\lambda(\lambda-1)(|x|^2+1)^{\lambda-2}\sum_{i,j=1}^n a_{ij}x_i x_j$$
$$-2\alpha_0\lambda(|x|^2+1)^{\lambda-1}\sum_{i=1}^n (a_{ii}+b_i x_i) + c - \beta_0]$$

in $R^n \times (T,\infty)$, where $\alpha_0 = \varepsilon + \dfrac{\gamma_0}{2\lambda\sqrt{k_1}} + \dfrac{nK_2}{4\lambda k_1}$. Hence it follows from (2) that

$$Lv \leqq -h[(|x|^2+1)^\lambda\{4\alpha_0^2\lambda^2k_1-2\alpha_0\lambda K_2n-k_3\}$$
$$-2\alpha_0\lambda K_1n-4\alpha_0^2\lambda^2k_1+k_3'-\beta_0].$$

Evidently α_0 and β_0 satisfy

$$4\alpha_0^2\lambda^2k_1-2\alpha_0\lambda K_2n-k_3>0 \text{ and } -2\alpha_0\lambda K_1n-4\alpha_0^2\lambda^2k_1+k_3'-\beta_0>0.$$

Therefore we have $Lv\leqq 0$ in $R^n\times(T,\infty)$. Further, we see $v(x,T)=u(x,T)-M_4H_0(x)\geqq 0$. Applying Bodanko's maximum principle again, we can see $v(x,t)\geqq 0$ in $R^n\times[T,\infty)$, so

$$M_4H_0(x)e^{\beta_0(t-T)}\leqq u(x,t) \text{ in } R^n\times[T,\infty).$$

From this we get the assertion of Theorem 1.

Example. Consider an operator

$$L_0=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}+(-k^2|x|^2+l)-\frac{\partial}{\partial t} \tag{8}$$

in $R^n\times(0,\infty)$. Let $u(x,t)$ continuous in $R^n\times[0,\infty)$ satisfy $L_0u\leqq 0$ and $u(x,t)\geqq -\mu e^{\nu|x|^2}$ in $R^n\times(0,\infty)$ for some positive μ,ν and let $u(x,0)$ be non-negative and not identically equal to zero. The condition (2) is satisfied for $\lambda=1$, $k_1=K_1=1$, $K_2=0$, $k_3=k^2$ and $k_3'=k^2+l$. Theorem 1 implies that, if the condition $l>kn$ corresponding to (6) is fulfilled, then $u(x,t)$ grows exponentially to infinity as t tends to infinity. This fact was essentially proved by Szybiak [6] although his theorem is false as Besala and Fife pointed out. Szybiak missed the condition $l>kn$ out of the statement of his theorem.

5. Assume $l<kn$ in (8). In this case, Krzyżański [4] proved the following by constructing the fundamental solution of the Cauchy problem for the equation $L_0u=0$: Let u be the solution of the Cauchy problem

$$L_0u=0 \quad\text{in } R^n\times(0,\infty),$$
$$u(x,0)=f(x)$$

for the Cauchy data $f(x)$ bounded in R^n. Then $u(x,t)$ tends to zero uniformly in $x\in R^n$ as t tends to infinity.

Recently Chen [3] treated an analogous problem for an operator of a general form and proved the following fact.

Let the differential operator L in (1) *satisfy the condition*

$$(9)\quad \begin{cases} k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \text{ for real vector } \xi, \\ |b_i| \leqq K_2(|x|^2+1)^{1/2}, \quad 1 \leqq i \leqq n, \\ c \leqq -k_3(|x|^2+1)^{\lambda} + k_3' \end{cases}$$

for some $\lambda \in (0,1]$, $k_1(>0)$, K_1, $K_2(\geqq 0)$, $k_3(>0)$ *and* k_3'. *Further, let* $u(x,t)$ *be a solution of the Cauchy problem* $Lu=0$ *in* $R^n \times (0,\infty)$, $u(x,0)=f(x)$ *in* R^n *and satisfy* $|u(x,t)| \leqq \mu e^{\nu(|x|^2+1)^{\lambda}}$ *for some constants* μ *and* ν *positive. If* $f(x)$ *is bounded in* R^n *and if*

$$\frac{1}{2K_1}[2K_1(1-\lambda)-k_1 n]\,(\sqrt{n^2K_2^2+4K_1k_3}-nK_2)+k_3'<0,$$

then $u(x,t)$ *tends to zero uniformly in* $x \in R^n$ *as* t *tends to infinity.*

6. Here we shall discuss the case when $\lambda \in [1,\infty)$ in Chen's theorem. Let L be an operator of the form (1) with coefficients satisfying (9) for $\lambda \in [1,\infty)$. For $H(x,t)=\exp\{-\alpha(t)(|x|^2+1)^{\lambda}+\beta(t)\}$ with $\alpha(t)(>0)$ and $\beta(t)$ differentiable once in $(0,\infty)$ we have

$$\begin{aligned}\frac{LH}{H} \leqq & (|x|^2+1)^{\lambda}[4\lambda^2K_1\alpha^2(t)+2\lambda nK_2\alpha(t)-k_3+\alpha'(t)] \\ & -2\lambda k_1 n\alpha(t)+k_3'-4\lambda^2K_1\alpha^2(t)-\beta'(t).\end{aligned}$$

Hence, if we take

$$(10)\qquad \alpha(t)=\gamma\ \tanh 4\lambda^2K_1\gamma t$$

and

$$\beta(t)=[-2\lambda k_1 n\gamma-4\lambda^2K_1\gamma^2+k_3']t+\frac{k_1 n}{2\lambda K_1}\log\frac{e^{8\lambda^2K_1\gamma t}}{e^{8\lambda^2K_1\gamma t}+1}-\frac{2\gamma}{e^{8\lambda^2K_1\gamma t}+1}$$

for the positive root γ of the quadratic equation $4\lambda^2K_1X^2+2\lambda nK_2X-k_3=0$, then we get $LH \leqq 0$ in $R^n \times (0,\infty)$. Clearly $H(x,0)=e^{\beta(0)}=e^{-\gamma-\frac{k_1 n}{2\lambda K_1}\log 2}$. Putting $w_{\pm}(x)=Me^{-\beta(0)}H(x,t)\pm u(x,t)$ for $M=\sup_{x\in R^n}|f(x)|$, where $u(x,t)$ is a solution of the Cauchy problem $Lu=0$ in $R^n\times(0,\infty)$, $u(x,0)=f(x)$ for the bounded Cauchy data $f(x)$ and satisfies $|u(x,t)| \leqq \mu e^{\nu(|x|^2+1)^{\lambda}}$ for some positive μ and ν, we have $Lw_{\pm} \leqq 0$ in $R^n\times(0,\infty)$ and $w_{\pm}(x,0)\geqq 0$. From Bodanko's maximum principle in the case of $\lambda\in[1,\infty)$ we get $w_{\pm}(x,t)\geqq 0$ in $R^n\times[0,\infty)$, so

$$
\begin{aligned}
|u(x,t)| &\leqq Me^{-\beta(0)}H(x,t) \\
&\leqq Me^{-\beta(0)}e^{\beta(t)} \leqq Me^{-\beta(0)}e^{(-2\lambda k_1 n\gamma - 4\lambda^2 K_1\gamma^2 + k_3')t}
\end{aligned}
$$

in $R^n \times [0,\infty)$. Therefore, if

$$
- 2\lambda k_1 n\gamma - 4\lambda^2 K_1 \gamma^2 + k_3' < 0, \tag{11}
$$

then $u(x,t)$ decays to zero exponentially as t tends to infinity. Thus we have the following

THEOREM 2. *Let L be a differential operator of the form* (1) *with coefficients satisfying* (9) *for some $\lambda \in [1,\infty)$ and let $u(x,t)$ be a solution of the Cauchy problem*

$$
\begin{gathered}
Lu = 0 \quad in\ R^n \times (0,\infty), \\
u(x,0) = f(x) \quad in\ R^n
\end{gathered}
$$

for a bounded continuous Cauchy data $f(x)$ in R^n. Assume that there exist positive constants μ and ν such that $|u(x,t)| \leqq \mu e^{\nu(|x|^2+1)^\lambda}$ in $R^n \times [0,\infty)$. If the condition (11) *is valid, then $u(x,t)$ decays to zero exponentially as t tends to infinity and this decay of $u(x,t)$ is uniform in R^n.*

7. We apply Theorem 2 to the operator (8). In this case we may take $\lambda = 1$, $k_1 = K_1 = 1$, $K_2 = 0$, $k_3 = k^2$, $k_3' = k^2 + l$ and γ in (10) equal to $\frac{k}{2}$. So (11) reduces to $kn > l$. This is nothing but the result of Krzyżański stated in §5.

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