John A. Beekman
Nagoya Math. J.
Vol. 35 (1969), 133–150

GREEN'S FUNCTIONS FOR GENERALIZED SCHROEDINGER EQUATIONS*

JOHN A. BEEKMAN

I. Introduction. The purpose of this paper is to discuss functions defined on the continuous sample paths of Gaussian Markov processes which serve as Green's functions for pairs of generalized Schroedinger equations. The results extend the author's earlier paper [2] to a forward time version, and consider different boundary conditions.

The approach is similar to another paper, "Feynman-Cameron Integrals," [4]. The restrictions on the potential $V[y,t]$ needed to prove existence theorems are so strict as to rule out most practical potentials. Hence, the first few theorems are designed to lead the reader to an expression which in many cases can be shown to satisfy the equations and boundary conditions.

A sequential approach is used for theoretical reasons, and because it would lead naturally into finite-dimensional approximations similar to those of [4]. It also may serve as background for numerical analysis work similar to that done for the conditional Wiener integral by L.D. Fosdick, [9]. It also reveals interesting facts about the conditional Wiener and Gaussian Markov processes. In particular, we will calculate the mean function for the conditioned Gaussian Markov process, and the covariance function for the conditioned Wiener process.

The wave function for the forced harmonic oscillator is calculated. This is one of the few time-dependent potentials considered in the numerous function space integral papers on this subject. It should be mentioned that Feynman and Hibbs [8] consider this example in a different manner. Itô [10] considered the regular harmonic oscillator. Donsker and Lions considered time-dependent potentials in [7].

Received July 15, 1968
* This research was partially supported by the National Science Foundation through grant NSF GP-7639.

Let $\{X(\tau),\ s\leq\tau\leq t\}$ be a Gaussian Markov process with transition density function

$$(1.1)\qquad p(x,s;\ y,t)=\frac{\partial}{\partial y}P[X(t)\leq y\,|\,X(s)=x]$$
$$=\{2\pi A(s,t)\}^{-1/2}\exp\left\{-\frac{[y-(v(t)/v(s))x]^2}{2A(s,t)}\right\}$$

where

$$(1.2)\qquad A(s,t)=\left[u(t)v(t)-\frac{u(s)}{v(s)}v^2(t)\right],\quad s\leq t$$

$$(1.3)\qquad u(\tau)\geq 0,\quad v(\tau)>0,\quad s\leq\tau\leq t$$

$$(1.4)\qquad u''(\tau),\ v''(\tau)\ \text{are continuous},\quad s\leq\tau\leq t$$

$$(1.5)\qquad [v(\tau)u'(\tau)-u(\tau)v'(\tau)]>0,\quad s\leq\tau\leq t.$$

In [5], a complex variance parameter λ was used to consider an analytic Feynman integral. We will now introduce a parameter λ which serves essentially the same purpose. For the moment, $\lambda>0$.

Let ${}^*u(\tau)=u(\tau)/\sqrt{\lambda}$, ${}^*v(\tau)=v(\tau)/\sqrt{\lambda}$, $s\leq\tau\leq t$. These functions satisfy (1.3), (1.4), and (1.5). Let $A^*(s,t)=A(s,t)/\lambda$.

So the ${}^*u$ and ${}^*v$ functions serve for another Gaussian-Markov process. Let its transition density function be

$$p^*(x,s;\ y,t)=\left(\frac{\lambda}{2\pi A(s,t)}\right)^{1/2}\exp\left\{-\frac{\lambda\left[y-\frac{v(t)}{v(s)}x\right]^2}{2A(s,t)}\right\}$$
$$=(2\pi A^*(s,t))^{-1/2}\exp\left\{-\frac{\left[y-\frac{v(t)}{v(s)}x\right]^2}{2A^*(s,t)}\right\}.$$

It too will have $X(s)=x$, $X(t)=y$ with probability one since $\lim_{t\to s+}P\Big[[X(t)\leq y\,|\,X(s)=x\Big]=\begin{matrix}1, & y>x\\ 0, & y<x.\end{matrix}$

Condition (1.4) insures the continuity of almost all sample functions. Denote the expected value of a functional $F[X]$ for this process by $E^*\Big\{F[X]\,|\,X(s)=x,\ X(t)=y\Big\}$.

Assume that $0<g<v(\tau)\leq G$, $s\leq\tau\leq t$.

II. Sequential integrals

To allow λ to be complex, we now wish to relate $E^*\Big\{F[X]\,|\,X(s)=x,\ X(t)=y\Big\}$ and $E\Big\{F[X]\,|\,X(s)=x, X(t)=y\Big\}$ by using sequential Gaussian Markov integrals. The sequential concept will be useful for other reasons also.

Let $\tau \equiv [\tau_1, \cdots, \tau_n]$ be a variable vector of a variable number of dimensions whose components form a subdivision of $[s, t]$ so that

$$\tau_0 \equiv s < \tau_1 < \tau_2 < \cdots < \tau_n = t.$$

Let $\|\tau\| = \max_{j=1,\ldots,n}(\tau_j - \tau_{j-1})$. Let $\xi \equiv [\xi_1, \cdots, \xi_{n-1}]$ denote an unrestricted real vector, where n is determined by τ, let $\xi_0 \equiv x$, and $\xi_n \equiv y$.

Let $\Psi_{\tau,\xi}(\tau_i) = \xi_i$, $i = 0, 1, \cdots, n$ and $\Psi_{\tau,\xi}$ be linear on $[\tau_{i-1}, \tau_i]$.

Then we define the sequential Gaussian Markov integral

$$E_\lambda^s\left\{F[X] | X(s) = x,\ X(t) = y\right\} = \lim_{\|\tau\| \to 0} \int_{R_{n-1}} G_\lambda(\tau, \xi) F(\Psi_{\tau,\xi}) d\xi \tag{2.1}$$

where

$$G_\lambda(\tau, \xi) = \frac{1}{p^*(x, s;\ y, t)} \prod_{i=1}^{n} p^*(\xi_{i-1}, \tau_{i-1};\ \xi_i, \tau_i). \tag{2.2}$$

Let $C[x, s;\ y, t]$ denote the space of continuous functions with x and y endpoints. For $x \in C[x, s;\ y, t]$, let $\|X\| = \sup_{s \le \tau \le t} |X(\tau)|$. Let $C_p[x, s; y, t]$ denote the set of all polygonal functions in $C[x, s;\ y, t]$.

A subset S of $C[x, s;\ y, t]$ is a Borel set if it is a member of the smallest σ-ring containing the quasi-intervals

$$\left\{X \in C[x, s;\ y, t]\colon\ \alpha_i < X(\tau_i) < \beta_i,\ \ i = 1, 2, \cdots, n\right\}$$

where τ ranges over all subdivision vectors of $[s, t]$ and α_i, β_i range over the extended reals. $F[X]$ is a Borel functional if it is measurable with respect to the σ-ring of Borel measurable subsets of $C[x, s;\ y, t]$.

THEOREM 1. *Let $F[X]$ be Borel measurable over $C[x, s;\ y, t]$ and continuous in the uniform topology almost everywhere (in the Gaussian sense) on the space $C[x, s;\ y, t]$.*

Let $\phi(w)$ be a positive monotone increasing function such that $\phi\left(G\sqrt{\frac{u(t)}{v(t)} - \frac{u(s)}{v(s)}}\, w + |x| \frac{G}{g}\right) \exp[-w^2/2]$ is integrable on $[0, \infty)$.

Then if $|F(X)| \le \phi(\|X\|)$ on $C_p[x, s;\ y, t]$, the sequential Gaussian Markov integral of F exists for $\lambda = 1$ and equals the Gaussian Markov integral

$$E_1^s\left\{F[X] | X(s) = x, X(t) = y\right\} = E\left\{F[X] | X(s) = x, X(t) = y\right\}. \tag{2.3}$$

Proof. Let τ be a subdivision vector of $[s, t]$. Let X_τ denote the polygonal function which has the same values as X for $\tau = \tau_0, \tau_1, \cdots, \tau_n$ and is linear in between. Then $F(X_\tau)$ depends on $X(\tau_1), \cdots, X(\tau_n)$ only and hence

$$E\left\{F(X_\tau) \mid X(s) = x, X(t) = y\right\} = \int_{R_{n-1}} G(\tau, z) F(\Psi_{\tau, z}) dz. \tag{2. 4}$$

By the continuity of F and of X, we have for almost all $X \in C[x, s; y, t]$,

$$\lim_{\text{norm } \tau \to 0} F(X_\tau) = F(X). \tag{2. 5}$$

Moreover $|F(X_\tau)| \leq \phi(\|X_\tau\|) \leq \phi(\|X\|)$.
Now by Lemma 2 of [2], and the triangle inequality,

$$E\Big[\phi(\|X\|) \mid X(s) = x\Big] \leq E\Big[\phi\left(\|X\| + |x| \frac{G}{g}\right) \mid X(s) = 0\Big]$$

$$= \int_{c[0,1]}^{w} \phi\left\{\left\| v(\cdot) \sqrt{\frac{u(t)}{v(t)} - \frac{u(s)}{v(s)}}\, X\left(\frac{\frac{u(\cdot)}{v(\cdot)} - \frac{u(s)}{v(s)}}{\frac{u(t)}{v(t)} - \frac{u(s)}{v(s)}}\right)\right\| + |x| \frac{G}{g}\right\} d_w X$$

where the latter is a Wiener integral. Denote it by I.

In general, we have by well-known results for the Wiener process, ($g(x) \geq 0$ for $0 \leq x < \infty$),

$$\int_{C[0,1]}^{w} g[\|X\|] d_w X \leq \int_{C[0,1]}^{w} g\Big[\max_{0 \leq t \leq 1} X(t)\Big] d_w X + \int_{C[0,1]}^{w} g\Big[\max_{0 \leq t \leq 1} (-X(t))\Big] d_w X$$

$$= 2 \int_{C[0,1]}^{w} g\Big[\max_{0 \leq t \leq 1} X(t)\Big] d_w X$$

$$= 4 \int_{C[0,1]}^{w} g(X(1)) d_w X.$$

Then

$$I \leq \frac{4}{\sqrt{2\pi}} \int_0^\infty \phi\left(G \sqrt{\frac{u(t)}{v(t)} - \frac{u(s)}{v(s)}}\, w + |x| \frac{G}{g}\right) \exp\{-w^2/2\} dw < \infty.$$

But

$$E\Big[\phi(\|X\|) \mid X(s) = x\Big] = \int_{-\infty}^{\infty} E\Big[\phi(\|X\|) \mid X(s) = x, X(t) = y\Big] p(x, s; y, t) dy.$$

Hence, $E\Big[\phi(\|X\|) \mid X(s) = x, X(t) = y\Big] < \infty$ except possibly for a y set of measure zero. But this finiteness holds everywhere in y by continuity in y.

Hence (2. 3) follows from (2. 1), (2. 4), (2. 5), and the above finiteness by dominated convergence.

EXAMPLE. For $s \leq \theta \leq t$,

$$E\left\{X(\theta) | X(s) = x,\ X(t) = y\right\} = x \frac{v(\theta)}{v(s)} \frac{A(\theta, t)}{A(s, t)} + y \frac{v(t)}{v(\theta)} \frac{A(s, \theta)}{A(s, t)}. \tag{2. 6}$$

Proof. By Theorem 1

$$\begin{aligned} &E\left\{X(\theta) | X(s) = x,\ X(t) = y\right\} \\ &= E_1^s \left\{X(\theta) | X(s) = x,\ X(t) = y\right\} \\ &= \lim_{||\tau|| \to 0} \int_{R_{n-1}} G(\tau, \xi) \Psi_{\tau, \xi} d\xi \ \text{ where} \end{aligned}$$

$$\Psi_{\tau, \xi}(\theta) = \xi_{i-1} + (\theta - \tau_{i-1}) \left(\frac{\xi_i - \xi_{i-1}}{\tau_i - \tau_{i-1}} \right), \quad \tau_{i-1} \leq \theta \leq \tau_i.$$

By repeated application of the Chapman-Kolmogorov equation, both before and after i,

$$\begin{aligned} &\int_{R_{n-1}} G(\tau, \xi) \Psi_{\tau, \xi} d\xi \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p(x, s;\ \xi_{i-1}, \tau_{i-1})\, p(\xi_{i-1}, \tau_{i-1};\ \xi_i, \tau_i)\, p(\xi_i, \tau_i;\ y, t)}{p(x, s;\ y, t)} \\ &\qquad \left[\xi_{i-1} \left(\frac{\tau_i - \theta}{\tau_i - \tau_{i-1}} \right) + \xi_i \left(\frac{\theta - \tau_{i-1}}{\tau_i - \tau_{i-1}} \right) \right] d\xi_{i-1} d\xi_i. \end{aligned}$$

Now consider

$$I = \int_{-\infty}^{\infty} \xi_k p(\xi_{k-1}, \tau_{k-1};\ \xi_k, \tau_k)\, p(\xi_k, \tau_k;\ \xi_{k+1}, \tau_{k+1}) d\xi_k.$$

By completing the square, one can show

$$I = \frac{\beta}{\alpha} p(\xi_{k-1}, \tau_{k-1};\ \xi_{k+1}, \tau_{k+1})$$

where

$$\beta = \frac{v(\tau_k)}{v(\tau_{k-1})} \xi_{k-1} A(\tau_k, \tau_{k+1}) + \frac{v(\tau_{k+1})}{v(\tau_k)} \xi_{k+1} A(\tau_{k-1}, \tau_k) \quad \text{and} \quad \alpha = A(\tau_{k-1}, \tau_{k+1}).$$

Using this fact, we get

$$\int_{R_{n-1}} G(\tau, \xi) \Psi_{\tau, \xi} d\xi = \frac{\tau_i - \theta}{\tau_i - \tau_{i-1}} h(\tau_{i-1}) + \frac{\theta - \tau_{i-1}}{\tau_i - \tau_{i-1}} h(\tau_i)$$

where

$$h(\tau_k) = x\frac{v(\tau_k)}{v(s)}\frac{A(\tau_k,\tau_{k+1})}{A(s,\tau_{k+1})} + x\frac{v(\tau_{k+1})}{v(\tau_k)}\frac{A(s,\tau_k)}{A(s,\tau_{k+1})}\frac{v(\tau_{k+1})}{v(s)}\frac{A(\tau_{k+1},t)}{A(s,t)}$$

$$+ y\frac{v(t)}{v(\tau_k)}\frac{A(s,\tau_k)}{A(s,t)}.$$

Since the u and v functions are continuous and v is bounded away from 0, for $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau\| < \delta$ implies that $i(\theta) - \varepsilon < h(\tau_k) < i(\theta) + \varepsilon$, $k = i-1, i$, where $i(\theta) = x\frac{v(\theta)}{v(s)}\frac{A(\theta,t)}{A(s,t)} + y\frac{v(t)}{v(\theta)}\frac{A(s,\theta)}{A(s,t)}$.

Hence, for that δ,

$$\frac{\tau_i - \theta}{\tau_i - \tau_{i-1}}[i(\theta) - \varepsilon] + \frac{\theta - \tau_{i-1}}{\tau_i - \tau_{i-1}}[i(\theta) - \varepsilon]$$

$$< \int_{R_{n-1}} G(\tau,\xi)\Psi_{\tau,\xi}d\xi$$

$$< \frac{\tau_i - \theta}{\tau_i - \tau_{i-1}}[i(\theta) + \varepsilon] + \frac{\theta - \tau_{i-1}}{\tau_i - \tau_{i-1}}[i(\theta) + \varepsilon], \text{ or}$$

$$i(\theta) - \varepsilon < \int_{R_{n-1}} G(\tau,\xi)\Psi_{\tau,\xi}d\xi < i(\theta) + \xi.$$

Letting $\varepsilon \to 0$ gives the desired result.

Remark. In order that the reader won't be delayed from the body of the paper, the second example of this theorem will be presented as an Appendix.

III. Conditioned Integrals

THEOREM 2. Subject to the hypotheses of Theorem 1,

$$E\left\{F[X] \mid X(s) = x, X(t) = y\right\} \tag{3.1}$$

$$= E\left\{F\left[X(\cdot) + x\frac{v(\cdot)}{v(s)}\right] \mid X(s) = 0,\ X(t) = y - x\frac{v(t)}{v(s)}\right\}.$$

Proof. Express the left hand side as a sequential integral by Theorem 1.

$$E\left\{F[X] \mid X(s) = x, X(t) = y\right\}$$

$$= \lim_{\|\tau\|\to 0}\int_{R_{n-1}} G(\tau,\xi)F(\Psi_{\tau,\xi})d\xi_1 \cdots d\xi_{n-1}$$

$$= \lim_{\|\tau\|\to 0}\int_{R_{n-1}} F(\Psi_{\tau,\xi})\frac{p(x,s;\xi_1,\tau_1)\prod_{i=2}^{n-1} p(\xi_{i-1},\tau_{i-1};\xi_i,\tau_i)p(\xi_{n-1},\tau_{n-1};y,t)}{p(x,s;y,t)}$$

$$d\xi_1 \cdots d\xi_{n-1}.$$

Let $w_i = \xi_i - x \dfrac{v(\tau_i)}{v(s)}$, $i = 1, 2, \cdots, n-1$. Then

$$\xi_i - \frac{v(\tau_i)}{v(\tau_{i-1})}\xi_{i-1} = w_i - \frac{v(\tau_i)}{v(\tau_{i-1})} w_{i-1}, \quad i = 2, \cdots, n-1, \text{ and}$$

$$y - \frac{v(t)}{v(\tau_{n-1})}\xi_{n-1} = y - \frac{v(t)}{v(\tau_{n-1})} w_{n-1} - x\frac{v(t)}{v(s)}.$$

$$\Psi_{\tau, w}(\tau_i) = \begin{cases} w_i + x\dfrac{v(\tau_i)}{v(s)}, & i = 1, 2, \cdots, n-1 \\ 0 & , i = 0 \\ y - x\dfrac{v(t)}{v(s)} & , i = n. \end{cases}$$

The Jacobian is one. Hence, the preceding equals

$$\lim_{\|\tau\| \to 0} \int_{R_{n-1}} F(\Psi_{\tau, w}) \frac{p(0, s; w_1, \tau_1) \prod_{i=2}^{n-1} p(w_{i-1}, \tau_{i-1}; w_i, \tau_i) p\left(w_{n-1}, \tau_{n-1}; y - x\dfrac{v(t)}{v(s)}, t\right)}{p(x, s; y, t)} dw_1 \cdots dw_{n-1}.$$

Since $p(x, s; y, t) = p\left(0, s; y - x\dfrac{v(t)}{v(s)}, t\right)$, the above equals the right hand side of (3. 1).

Theorem 3. *For $\lambda > 0$,*

$$E^{r*}\left\{F[X] \mid X(s) = x, X(t) = y\right\} p^*(x, s; y, t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu\left[y - x\frac{v(t)}{v(s)}\right]} E^r\left[F\left[\frac{1}{\sqrt{\lambda}} X(\cdot) + x\frac{v(\cdot)}{v(s)}\right] e^{i\frac{\mu}{\sqrt{\lambda}}X(t)} \mid X(s) = 0\right] d\mu. \tag{3. 2}$$

Proof. By Theorem 2,

$$E^{r*}\left\{F[X] \mid X(s) = x, X(t) = y\right\} p^*(x, s; y, t)$$

$$= E^{r*}\left\{F\left[X(\cdot) + x\frac{v(\cdot)}{v(s)}\right] \mid X(s) = 0,\ X(t) = y - x\frac{v(t)}{v(s)}\right\} p^*\left(0, s; y - x\frac{v(t)}{v(s)}, t\right)$$

since $p^*(x, s; y, t) = p^*\left(0, s; y - x\dfrac{v(t)}{v(s)}, t\right)$.

Since the resulting process not subject to the condition on $X(t)$ has a zero mean function, we can apply Donsker and Lions, [7], pages 150 and 154, and M. Kac [11], p. 172, to obtain

$$E^{r*}\Big\{F[X] \mid X(s) = x, X(t) = y\Big\}\, p^*(x, s;\ y, t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu\left[y - x\frac{v(t)}{v(s)}\right]} E^{r*}\Big\{F\Big[X(\cdot) + x\frac{v(\cdot)}{v(s)}\Big] e^{i\mu X(t)} \mid X(s) = 0\Big\} d\mu.$$

Now apply the Cameron Donsker reference, [6], p. 23.

Remark. This method of calculating function space integrals conditioned at both time extremities has the advantage of using all known results for integrals conditioned at the first time only.

THEOREM 4. *Assume* $V(y, \tau)$ *and* $V_y(y, \tau)$ *are bounded and continuous complex valued functions for* $0 \leq \tau \leq T < \infty$, $y \in \{(-\infty, \infty) - E\}$ *where* E *is a finite set of points. Assume* $\lambda > 0$. *Then* $r^*(x, s;\ y,\ t) = E^*\Big\{\exp\Big[-i\int_s^t V[X(\tau), \tau]\, d\tau\Big] \mid X(s) = x,\ X(t) = y\Big\}\, p^*(x, s;\ y, t)$ *uniquely satisfies the pair of equations*

$$\frac{A(t)}{\lambda}\frac{\partial^2 r^*}{\partial y^2} - B(t)\frac{\partial}{\partial y}[yr^*] - iV(y, t)r^* = \frac{\partial r^*}{\partial t} \tag{3. 3}$$

and

$$\frac{A(s)}{\lambda}\frac{\partial^2 r^*}{\partial x^2} + xB(s)\frac{\partial r^*}{\partial x} - iV(x,\ s)r^* = -\frac{\partial r^*}{\partial s}, \tag{3. 4}$$

where

$$A(t) = [v(t)u'(t) - u(t)v'(t)]/2,\ 0 \leq t \leq T, \tag{3. 5}$$

$$B(t) = v'(t)/v(t),\ \ 0 \leq t \leq T. \tag{3. 6}$$

It also satisfies the boundary conditions

(3. 7) $\lim_{t \to s+}\int_{-\infty}^{\infty} g(x)\, r^*(x, s;\ y, t) dx = g(y)$ *for every bounded continuous* g,

and

(3. 8) $\lim_{s \to t-}\int_{-\infty}^{\infty} g(y) r^*(x, s;\ y, t) dy = g(x)$ *for every bounded continuous* g, $\lim_{|y| \to \infty} r^* = 0$, $\lim_{|x| \to \infty} r^* = 0$ *and the continuity properties* $\partial r^*/\partial y$, $\partial^2 r^*/\partial y^2$, $\partial r^*/\partial t$ *continuous for* $0 \leq s < t < T$, $y \in (-\infty,\ \infty) - E$ *and* $\partial r^*/\partial x$, $\partial^2 r^*/\partial x^2$, $\partial r^*/\partial s$ *continuous for* $0 < s < t \leq T$, $x \in (-\infty, \infty) - E$ *and* $\partial r^*/\partial y$ *continuous for* $y \in E$, $\partial r^*/\partial x$ *continuous for* $x \in E$.

Proof. Use Theorem 1 of [3] with coefficients $A^*(t)$, $B^*(t)$. Then $A^*(t) = \frac{A(t)}{\lambda}$, $B^*(t) = B(t)$.

Remark. We now wish to let $\lambda = -i$. Before verifying the partial differential equations and conditions, one first of all has to decide on the meaning of r^* with a complex variance. As shown by Cameron, [5], the Wiener kernel with nonreal variance cannot be used to generate a countably additive measure. The same applies to the Gaussian Markov kernel. But as in [5], one can use it to define a sequential integral. For the unconditioned process, this was done in [1]. Looking ahead to (4. 1), one can show that the parameter λ can be "transferred" to the functional, and that for appropriate functionals, (4. 1) holds for complex λ. See [1] for an analogous theorem. For a restricted class of functionals, [5] used a formula similar to (4. 1) to verify the Schroedinger equation.

IV. Approximation of Conditioned Integrals

The following theorem was motivated by a paper of L.D. Fosdick [9]. It derives an approximation formula for the conditioned Wiener integral which could be used in electronic computer work. Assuming equality for complex λ, one could split the right hand side functional into real and imaginary parts and approximate the resulting integrals.

THEOREM 5. *For* $\lambda > 0$,

$$E^{r*}\left\{F[X] \mid X(s) = x, X(t) = y\right\}$$

$$\stackrel{e}{=} E^{w}\left\{F\left[\frac{v(\cdot)}{v(t)}\frac{\sqrt{A(s,t)}}{\sqrt{\lambda}}X\left(\frac{v^2(t)}{v^2(\cdot)}\frac{A(s,\cdot)}{A(s,t)}\right) + x\frac{v(\cdot)}{v(s)}\frac{A(\cdot,t)}{A(s,t)} + y\frac{v(t)}{v(\cdot)}\frac{A(s,\cdot)}{A(s,t)}\right] \mid X(0)=0, \quad X(1)=0\right\} \tag{4. 1}$$

where $\stackrel{e}{=}$ *means if one side exists so does the other and they are equal.*

Proof. Assume that the left hand side exists.
By (2. 6) and a calculation such as in Example 2, [3], page 33,

$$I \equiv E^{r*}\left\{F[X] \mid X(s) = x, X(t) = y\right\} \frac{\sqrt{\lambda}}{\sqrt{2\pi A(s,t)}}$$

$$= E^{r*}\left\{F\left[X(\cdot) + x\frac{v(\cdot)}{v(s)}\frac{A(\cdot,t)}{A(s,t)} + y\frac{v(t)}{v(\cdot)}\frac{A(s,\cdot)}{A(s,t)}\right] \mid X(s) = 0, X(t) = 0\right\} \frac{\sqrt{\lambda}}{\sqrt{2\pi A(s,t)}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} E^{r}\Big\{F\Big[\frac{1}{\sqrt{\lambda}}X(\cdot)+x\frac{v(\cdot)}{v(s)}\frac{A(\cdot,t)}{A(s,t)}+y\frac{v(t)}{v(\cdot)}\frac{A(s,\cdot)}{A(s,t)}\Big]e^{i\mu\frac{X(t)}{\sqrt{\lambda}}}\mid X(s)=0\Big\}\,d\mu$$

by Theorem 3. Then

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} E^{w}\Big\{F\Big[\frac{1}{\sqrt{\lambda}}\frac{v(\cdot)}{v(t)}\sqrt{A(s,t)}\;X\Big(\frac{v^2(t)A(s,\cdot)}{v^2(\cdot)A(s,t)}\Big)+x\frac{v(\cdot)}{v(s)}\frac{A(\cdot,t)}{A(s,t)}$$

$$+\,y\frac{v(t)}{v(\cdot)}\frac{A(s,\cdot)}{A(s,t)}\Big]e^{i\mu\frac{\sqrt{A(s,t)}}{\sqrt{\lambda}}X(1)}\mid X(0)=0\Big\}\,d\mu$$

by Doob and Cuthill (See Lemma 2 of [2]).
Next

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} E^{\tilde{w}}\Big\{F\Big[\frac{v(\cdot)}{v(t)}X\Big(\frac{v^2(t)A(s,\cdot)}{v^2(\cdot)A(s,t)}\Big)+x\frac{v(\cdot)}{v(s)}\frac{A(\cdot,t)}{A(s,t)}$$

$$+\,y\frac{v(t)}{v(\cdot)}\frac{A(s,\cdot)}{A(s,t)}\Big]e^{i\mu X(1)}\mid X(0)=0\Big\}d\mu$$

by Cameron-Donsker, [6], p. 23, where the covariance multiplier of $A(s,t)/\lambda$ is indicated by $E^{\tilde{w}}$. Then

$$I = E^{\tilde{w}}\Big\{F\Big[\frac{v(\cdot)}{v(t)}X\Big(\frac{v^2(t)A(s,\cdot)}{v^2(\cdot)A(s,t)}\Big)+x\frac{v(\cdot)}{v(s)}\frac{A(\cdot,t)}{A(s,t)}+y\frac{v(t)}{v(\cdot)}\frac{A(s,\cdot)}{A(s,t)}\Big]$$

$$\mid X(0)=0,\;\; X(1)=0\Big\}\frac{\sqrt{\lambda}}{\sqrt{2\pi A(s,t)}}$$

by Theorem 3 again. The result follows by using reference [6] again, and dividing by $\sqrt{\lambda}/\sqrt{2\pi A(s,t)}$.

The reverse implication follows easily.

Remark. The parameter indicated by $(\cdot)$ will vary from s to t. Since $A(s,s)=0$, the sample function will proceed from $X(0)$ to $X(1)$.

V. Example. Forced Harmonic Oscillator

Consider $V(x,t)=x^2-f(t)x$ as per Feynman and Hibbs, [8], p. 64, 70. From p. 233, [8], we assume $f(0)=0$ and $f(T)=0$, $0\leq s<t\leq T$. From [6], Lemma 2, we ask $f(\tau)\in L_2[0,T]$. We assume, for simplicity, that $\hbar=1$ and mass is equal to 1. The following is motivated by Lemma 2, [6].

Let $x(\tau)=\sum_{k=1}^{\infty}\frac{\alpha_k u_k(\tau)}{\sqrt{\rho_k}}$, $s\leq\tau\leq t$ where the $\{\alpha_i\}$ are independent normal variates with zero means and unit variances, $\{\rho_k\}$ and $\{u_k(s)\}$ are eigenvalues and normalized eigenfunctions associated with the integral equation $\rho \int_s^t r(\tau, x)\, u(x)\, dx = u(\tau)$. See Kac and Siegert [13]. From Mercer's theorem and the definition of the reciprocal kernel, $R_r(s, t;\, \mu)$ and the Fredholm determinant,

$$D_r(\mu), r(a, b) = \sum_{k=1}^{\infty} \frac{u_k(a) u_k(b)}{\rho_k},$$

$$R_r(a, b;\, \mu) = \sum_{k=1}^{\infty} \frac{u_k(a) u_k(b)}{\mu - \rho_k}, \text{ and}$$

$$D_r(\mu) = \prod_{k=1}^{\infty} \left(1 - \frac{\mu}{\rho_k}\right).$$

Following [4], assume $(t - s) < \pi/\sqrt{8}$. Use Theorem 3, with $\lambda > 0$.

$$E^r \left\{ \exp\left[-i \int_s^t \left(\left[\frac{X(\tau)}{\sqrt{\lambda}} + x \frac{v(\tau)}{v(s)} \right]^2 - f(\tau) \left[\frac{X(\tau)}{\sqrt{\lambda}} + x \frac{v(\tau)}{v(s)} \right] \right) d\tau \right] e^{i\mu X(t)/\sqrt{\lambda}} \,|\, X(s) = 0 \right\}$$

$$= g(t)\, E\left\{ \exp\left[-\frac{i}{\lambda} \sum_{k=1}^{\infty} \frac{\alpha_k^2}{\rho_k} - i \sum_{k=1}^{\infty} \frac{\alpha_k}{\sqrt{\rho_k}} \int_s^t \left(\frac{2xv(\tau)}{\sqrt{\lambda} v(s)} - \frac{f(\tau)}{\sqrt{\lambda}} \right) u_k(\tau) d\tau + i \frac{\mu}{\sqrt{\lambda}} \sum_{k=1}^{\infty} \frac{\alpha_k u_k(t)}{\sqrt{\rho_k}} \right] \right\}$$

by the representation and Parseval's Theorem, and where

$$g(t) = \exp\left[-i \int_s^t \left\{ \left(x \frac{v(\tau)}{v(s)} \right)^2 - f(\tau)\, x \frac{v(\tau)}{v(s)} \right\} d\tau \right].$$

Then the preceding equals

$$g(t) \prod_{k=1}^{\infty} \left[\int_{-\infty}^{\infty} \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}} \exp\left\{ -\frac{i}{\lambda \rho_k} \left[\alpha^2 + \alpha\sqrt{\lambda}\sqrt{\rho_k} \int_s^t \left(\frac{2xv(\tau)}{v(s)} - f(\tau) \right) u_k(\tau) d\tau - \alpha\sqrt{\lambda}\, \mu u_k(t) \sqrt{\rho_k} \right] \right\} d\alpha \right]$$

by the independence of $\{\alpha_i\}$.

Let $C_k = \dfrac{-\int_s^t \left(\frac{2xv(\tau)}{v(s)} - f(\tau) \right) u_k(\tau) d\tau + \mu u_k(t)}{\sqrt{\rho_k}}$. Then the preceding equals

$$g(t) \prod_{k=1}^{\infty} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2(1+(2i/\lambda\rho_k)) + i\alpha/\sqrt{\lambda}\, C_k} d\alpha \right]$$

$$= g(t) \prod_{k=1}^{\infty}\Big[\Big(1+\frac{2i}{\lambda\rho_k}\Big)^{-1/2}\exp\Big\{-\frac{1}{2}\Big(1+\frac{2i}{\lambda\rho_k}\Big)^{-1}\frac{1}{\lambda}C_k^2\Big\}\Big]$$

by the following

LEMMA. *For* $Re\ \sigma>0$,

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}\exp\Big\{i\,t\,x-\frac{1}{2}\,\frac{(x-\mu)^2}{\sigma^2}\Big\}dx=\exp\Big\{i\,t\,\mu-\frac{1}{2}\sigma^2t^2\Big\}.$$

Proof. From Wilks, [14], p. 157, we know it is true for $\sigma>0$. Both sides are analytic for $Re\ \sigma>0$. Hence it is true for $Re\ \sigma>0$ by analytic continuation. It is also true for complex μ and complex t.

So the preceding quantity

$$= g(t)\Big(D_r\Big(-\frac{2i}{\lambda}\Big)\Big)^{-1/2}\exp\Big\{-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\rho_k}{\lambda\rho_k+2i}C_k^2\Big\}$$

$$= g(t)\Big(D_r\Big(-\frac{2i}{\lambda}\Big)\Big)^{-1/2}$$

$$\exp\Big\{-\sum_{k=1}^{\infty}\frac{\int_s^t h(\tau)u_k(\tau)\,d\tau\int_s^t h(x)u_k(x)dx-2\mu u_k(t)\int_s^t h(\tau)u_k(\tau)d\tau+\mu^2u_k^2(t)}{2(\lambda\rho_k+2i)}\Big\}$$

$$\Big(\text{where } h(\tau)=\frac{2xv(\tau)}{v(s)}-f(\tau)\Big)$$

$$= g(t)\Big(D_r\Big(-\frac{2i}{\lambda}\Big)\Big)^{-1/2}\exp\Big\{-\frac{1}{2}\int_s^t\int_s^t\sum_{k=1}^{\infty}\frac{u_k(\tau)u_k(x)h(\tau)h(x)}{2i+\lambda\rho_k}\,d\tau dx$$

$$+\mu\int_s^t\sum_{k=1}^{\infty}\frac{u_k(t)u_k(\tau)}{2i+\lambda\rho_k}\,h(\tau)d\tau-\frac{\mu^2}{2}\sum_{k=1}^{\infty}\frac{u_k{}^2(t)}{2i+\lambda\rho_k}\Big\}$$

$$= g(t)\Big(D_r\Big(-\frac{2i}{\lambda}\Big)\Big)^{-1/2}\exp\Big\{\frac{1}{2\lambda}\int_s^t\int_s^t R_r\Big(\tau,x;-\frac{2i}{\lambda}\Big)h(\tau)h(x)d\tau dx$$

$$-\frac{\mu}{\lambda}\int_s^t R_r\Big(\tau,t;-\frac{2i}{\lambda}\Big)h(\tau)d\tau+\frac{\mu^2}{2\lambda}R_r\Big(t,t;-\frac{2i}{\lambda}\Big)\Big\}.$$

Let $$B=-\frac{1}{\lambda}\int_s^t R_r\Big(\tau,t;-\frac{2i}{\lambda}\Big)h(\tau)d\tau$$

and $$\theta=-\frac{1}{\lambda}R_r\Big(t,t;-\frac{2i}{\lambda}\Big).$$ Then

$$E^r\Big\{\exp\Big[-i\int_s^t\Big(\Big[\frac{X(\tau)}{\sqrt{\lambda}}+x\,\frac{v(\tau)}{v(s)}\Big]^2-f(\tau)\Big[\frac{X(\tau)}{\sqrt{\lambda}}+x\,\frac{v(\tau)}{v(s)}\Big]\Big)d\tau\, e^{i\mu\frac{X(t)}{\sqrt{\lambda}}}\,\Big|\,X(s)=0\Big\}$$

$$= g(t)\Big(D_r\Big(-\frac{2i}{\lambda}\Big)\Big)^{-1/2}\exp\Big\{\frac{1}{2\lambda}\int_s^t\int_s^t R_r\Big(\tau,x;-\frac{2i}{\lambda}\Big)h(\tau)h(x)\,d\tau\,dx\Big\}\cdot e^{\mu B-\frac{\mu^2}{2}\theta}.$$

The first part is independent of μ.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu\left[y-x\frac{v(t)}{v(s)}\right]} e^{-\frac{\mu^2}{2}\theta+\mu B} d\mu$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{-\frac{\theta}{2}\left[\mu^2-2\mu\frac{B}{\theta}+\frac{B^2}{\theta^2}\right]+i\mu\left[x\frac{v(t)}{v(s)}-y\right]\right\}d\mu\cdot e^{\frac{B^2}{2\theta}}.$$

This used the fact that $Re\ \theta>0$. The preceding equals

$$\exp\left\{i\left[x\frac{v(t)}{v(s)}-y\right]\frac{B}{\theta}-\frac{1}{2\theta}\left[x\frac{v(t)}{v(s)}-y\right]^2\right\}e^{\frac{B^2}{2\theta}}\sqrt{\frac{1}{\theta 2\pi}}.$$

So from Theorem 3, $r^*(x,s;\ y,t)$

$$=\frac{1}{\sqrt{2\pi\theta}}e^{\frac{-1}{2\theta}\left[Bi-\left(x\frac{v(t)}{v(s)}-y\right)\right]^2}$$

$$\cdot g(t)\left(D_r\left(-\frac{2i}{\lambda}\right)\right)^{-1/2}\exp\left\{\frac{1}{2\lambda}\int_s^t\int_s^t R_r\left(a,\ b;\ -\frac{2i}{\lambda}\right)h(a)h(b)da\,db\right\}.$$

Now to verify the partial differential equations, one must pick a specific process, let $\lambda=-i$, and try it as a solution.

Subcase 1. Wiener process.

$$D_w(2)=\cos\sqrt{2}\,(t-s).$$

$$R_w(a,\ b;\ 2)=\begin{cases}\dfrac{-\cos\sqrt{2}\,(t-b)\,\sin\sqrt{2}\,(a-s)}{\sqrt{2}\,\cos\sqrt{2}\,(t-s)}, & a\leq b\\[2ex] \dfrac{-\cos\sqrt{2}\,(t-a)\,\sin\sqrt{2}\,(b-s)}{\sqrt{2}\,\cos\sqrt{2}\,(t-s)}, & b\leq a.\end{cases}$$

$$\theta=-iR_w(t,t;\ 2)=\frac{i}{\sqrt{2}}\tan\sqrt{2}\,(t-s).$$

$$g(t)=\exp\left\{-ix^2(t-s)+ix\int_s^t f(\tau)d\tau\right\}.$$

$$h(\tau)=2x-f(\tau),\quad s\leq\tau\leq t.$$

$$B=-i\int_s^t R_w(\tau,t;\ 2)\,[2x-f(\tau)]d\tau$$

$$=-xi+\frac{xi}{\cos\sqrt{2}\,(t-s)}-\frac{i\int_s^t\sin\sqrt{2}\,(\tau-s)f(\tau)d\tau}{\sqrt{2}\,\cos\sqrt{2}\,(t-s)}.$$

$$r^*(x,s;\ y,t)=\frac{\sqrt[4]{2}}{\sqrt{2\pi i\sin\sqrt{2}\,(t-s)}}$$

$$\exp\left\{-\frac{1}{2\theta}[Bi-(x-y)]^2 - ix^2(t-s) + ix\int_s^t f(\tau)d\tau + \frac{i}{2}\int_s^t\int_s^t R_w(a,b;2)h(a)h(b)dadb\right\}.$$

One can show that

$$-\frac{1}{2\theta}[Bi-(x-y)]^2 = \frac{i}{\sqrt{2}\sin\sqrt{2}(t-s)}\left\{\left[x^2+\frac{1}{2}\left(\int_s^t \sin\sqrt{2}(\tau-s)f(\tau)d\tau\right)^2 - \sqrt{2}\,x\int_s^t \sin\sqrt{2}(\tau-s)f(\tau)d\tau\right]\Big/\cos\sqrt{2}(t-s) + y^2\cos\sqrt{2}(t-s)-2xy+\sqrt{2}\,y\int_s^t\sin\sqrt{2}(\tau-s)f(\tau)d\tau\right\}.$$

Also

$$\int_s^t\int_s^t R_w(a,b;2)dadb = \frac{\int_s^t\int_s^b \cos\sqrt{2}(t-b)\sin\sqrt{2}(a-s)dadb+\int_s^t\int_b^t\cos\sqrt{2}(t-a)\sin\sqrt{2}(b-s)dadb}{-\sqrt{2}\cos\sqrt{2}(t-s)}$$

$$=\frac{t-s}{2}-\frac{\tan\sqrt{2}(t-s)}{2\sqrt{2}};$$

$$\int_s^t\int_s^t R_w(a,b;2)f(a)dadb = \frac{1}{2}\int_s^t f(b)db - \frac{1}{2}\int_s^t \frac{f(b)\cos\sqrt{2}(t-b)}{\cos\sqrt{2}(t-s)}db.$$

So $\int_s^t\int_s^t R_w(a,b;2)h(a)h(b)dadb$

$$= 2x^2(t-s)-\sqrt{2}\,x^2\tan\sqrt{2}(t-s)-2x\int_s^t f(b)db + 2x\int_s^t\frac{f(b)\cos\sqrt{2}(t-b)}{\cos\sqrt{2}(t-s)}db + \int_s^t\int_s^t R_w(a,b;2)f(a)f(b)dadb.$$

Then $r^*(x,s,;\,y,t) = \dfrac{\sqrt[4]{2}}{\sqrt{2\pi i\sin\sqrt{2}(t-s)}}\cdot$

$$\exp\left\{\frac{i}{\sqrt{2}\sin\sqrt{2}(t-s)}\left\{\left[x^2\cos^2\sqrt{2}(t-s)+\frac{1}{2}\left(\int_s^t\sin\sqrt{2}(\tau-s)f(\tau)d\tau\right)^2 - \sqrt{2}\,x\int_s^t\sin\sqrt{2}(\tau-s)f(\tau)d\tau\right]\Big/\cos\sqrt{2}(t-s)+y^2\cos\sqrt{2}(t-s) - 2xy+\sqrt{2}\,y\int_s^t\sin\sqrt{2}(\tau-s)f(\tau)d\tau\right\} + ix\int_s^t\frac{f(b)\cos\sqrt{2}(t-b)}{\cos\sqrt{2}(t-s)}db+\frac{i}{2}\int_s^t\int_s^t R_w(a,b;2)f(a)f(b)dadb\right\}.$$

Using several trigonometric identities,

$$r^*(x,s;\,y,t) = \frac{\sqrt[4]{2}}{\sqrt{2\pi i\sin\sqrt{2}(t-s)}}\cdot$$

$$\exp\left\{\frac{i}{\sqrt{2}\ \sin\sqrt{2}\ (t-s)}\left\{(x^2+y^2)\cos\sqrt{2}\ (t-s)-2xy\right.\right.$$
$$+\sqrt{2}\ x\int_s^t f(b)\sin\sqrt{2}\ (t-b)\,db+\sqrt{2}\ y\int_s^t f(a)\sin\sqrt{2}\ (a-s)da$$
$$-\frac{1}{2}\int_s^t\int_s^b \sin\sqrt{2}\ (a-s)\sin\sqrt{2}\ (t-b)f(a)f(b)dadb$$
$$\left.\left.-\frac{1}{2}\int_s^t\int_b^t \sin\sqrt{2}\ (b-s)\sin\sqrt{2}\ (t-a)\,f(a)f(b)\,dadb\right\}\right\}.$$

Since the last term equals the next to last term, this agrees with Feynman and Hibbs, [8], p. 64, {with $w=\sqrt{2}$, $m=1$, $\hbar=1$, $t_b=t$, $t_a=s$, $x_b=y$, and $x_a=x$}. This also eliminates the need to verify the forward partial differential equation. It is easy to verify that the expression for $\frac{\partial^2 r}{\partial x^2}$ equals that for $\frac{\partial^2 r}{\partial y^2}$ with the roles of x and y, and s and t interchanged, and the expression for $\frac{\partial r^*}{\partial s}$ equals that for $-\frac{\partial r^*}{\partial t}$ with the roles of x and y, and s and t interchanged. Thus the backwards partial differential equation also holds.

Unfortunately, $\lim_{|y|\to\infty}|r^*|=\frac{\sqrt[4]{2}}{\sqrt{2\pi\sin\sqrt{2}\ (t-s)}}$, not 0. The same applies to the x limit. From Feynman and Hibbs, [8], pages 81, 34, 60, and 64, we know that (3. 7) and (3. 8) hold. Thus as [8] states on p. 82, $r^*(x,s;y,t)$ "is a kind of Green's function for the Schroedinger equation."

The special case $f(\tau)\equiv 0$, $0\leq\tau\leq T$, was considered through a function space integral approach by K. Itô, [10].

Subcase 2. Ornstein-Uhlenbeck process.

Since $r(a,b)=e^{-|b-a|}$, the eigenvalues and normalized eigenfunctions for this process for the interval $[0,T]$ are contained in the article [12], pages 66, 7 by M. Kac. Probably one could adjust these for the different time period, $[s,t]$. However, they are too complicated to use in $D_r(2)$, and $R_r(a,b;\ 2)$.

An alternate procedure is to use Theorem 5 and numerically approximate the right hand side of (4. 1) as in [9]. One could thus obtain a grid of values of the integral for various x, y, s and t values. It would be impossible to verify the partial differential equations with such a set of discrete values, but the author conjectures that these values multiplied by

$p^*(x, s; y, t)$ would satisfy the finite difference analogues of the Schroedinger equations.

For the Ornstein-Uhlenbeck process, $v(\tau) = e^{-\tau}$, $u(\tau) = e^{\tau}$, $A(a, b) = 1-e^{-2(b-a)}$. Hence the generalized Schroedinger equation would be (in the forward case) $i \frac{\partial^2 r^*}{\partial y^2} + \frac{\partial}{\partial y}[yr^*] - i[y^2 - f(t)y]r^* = \frac{\partial r^*}{\partial t}$. The right hand side of Theorem 5 becomes

$$E^w\Big\{F\Big[e^{-((\cdot)-t)}[1-e^{-2(t-s)}]^{1/2}\sqrt{i}\; X\Big(\frac{e^{2(\cdot)}-e^{2s}}{e^{2t}-e^{2s}}\Big)$$

$$+ x\frac{e^{-(\cdot)}-e^{-2t+(\cdot)}}{e^{-s}-e^{-2t+s}} + y\frac{e^{(\cdot)}-e^{-(\cdot)+2s}}{e^{t}-e^{-t+2s}}\Big] | X(0) = 0,\ X(1) = 0\Big\}$$

where

$$F[X] = \exp\Big[-i\int_s^t [X^2(\tau) - f(\tau)X(\tau)]d\tau\Big].$$

Repeating an earlier remark, the parameter indicated by $(\cdot)$ varies from s to t. Since $A(s, s) = 0$, the sample function will proceed from $X(0)$ to $X(1)$. This integral must be multiplied by $p^*(x, s; y, t)$

$$= \Big(\frac{i}{2\pi(1-e^{-2(t-s)})}\Big)^{1/2} \exp\Big\{\frac{i[y - e^{-(t-s)}x]^2}{2[1-e^{-2(t-s)}]}\Big\}$$

VI. Appendix. As an interesting by-product of Theorem 1, we will prove the following theorem.

THEOREM 6. *The Wiener process, conditioned by* $X(s) = x, X(t) = y$, *is Gaussian Markov with mean function.*

$$m(\theta) = E\Big\{X(\theta) | X(s) = x, X(t) = y\Big\} = x + \frac{\theta - s}{t-s}(y-x) \tag{6. 1}$$

and covariance function

$$E\Big\{[X(a) - m(a)][X(b) - m(b)] | X(s) = x, X(t) = y\Big\}$$

$$= \begin{cases} \frac{(a-s)(t-b)}{(t-s)}, & a < b \\ \frac{(b-s)(t-a)}{(t-s)}, & b < a. \end{cases} \tag{6. 2}$$

Proof. Since $u(\tau) = \tau$, $v(\tau) \equiv 1$ for the regular Wiener process, (6. 1) is obtained easily from (2. 6).

To verify (6. 2) first assume that $s<a<b<t$. (The case $s<b<a<t$ is identical.) Then by Theorem 1,

$$E\left\{X(a)X(b)\,|\,X(s)=x,\ X(t)=y\right\}=E_1^s\left\{X(a)X(b)\,|\,X(s)=x,\ X(t)=y\right\}$$

$$\lim_{||\tau||\to 0}\int_{R_{n-1}}G(\tau,\xi)F(\Psi_{\tau,\xi})d\xi \text{ where}$$

$$\Psi_{\tau,\xi}(a)=\xi_{i-1}+(a-\tau_{i-1})\left(\frac{\xi_i-\xi_{i-1}}{\tau_i-\tau_{i-1}}\right),\quad \tau_{i-1}\leq a\leq \tau_i,$$

$$\Psi_{\tau,\xi}(b)=\xi_{k-1}+(b-\tau_{k-1})\left(\frac{\xi_k-\xi_{k-1}}{\tau_k-\tau_{k-1}}\right),\quad \tau_{k-1}\leq b\leq \tau_k,$$

and $F[x]=X(a)X(b)$.

With $a<b$, for sufficiently small $||\tau||$, $\tau_i<\tau_{k-1}$.

By repeated application of the Chapman-Kolmogorov equation, both before i, between i and $k-1$, and after k,

$$\int_{R_{n-1}}G(\tau,\xi)F(\Psi_{\tau,\xi})\,d\xi=\iiiint_{-\infty}^{\infty}\frac{p(x,s;\ \xi_{i-1},\tau_{i-1})}{p(x,s;\ y,t)}$$

$$p(\xi_{i-1},\tau_{i-1};\ \xi_i,\tau_i)p(\xi_i,\tau_i;\ \xi_{k-1},\tau_{k-1})p(\xi_{k-1},\tau_{k-1};\ \xi_k,\tau_k)p(\xi_k,\tau_k;\ y,t)$$

$$[AC\xi_{i-1}\xi_{k-1}+AD\xi_{i-1}\xi_k+BC\xi_i\xi_{k-1}+BD\xi_i\xi_k]d\xi_{i-1}d\xi_i d\xi_{k-1}d\xi_k$$

where $A=\dfrac{\tau_i-a}{\tau_i-\tau_{i-1}}$, $B=\dfrac{a-\tau_{i-1}}{\tau_i-\tau_{i-1}}$, $C=\dfrac{\tau_k-b}{\tau_k-\tau_{k-1}}$, $D=\dfrac{b-\tau_{k-1}}{\tau_k-\tau_{k-1}}$.

It is easy to verify that A, B, C, and D each approach 1/2 as $||\tau||\to 0$. One also makes the preliminary calculations

$$\int_{-\infty}^{\infty}bp(a,\tau_1;\ b,\tau_2)p(b,\tau_2;\ c,\tau_3)db=\frac{\beta}{\alpha}p(a,\tau_1;\ c,\tau_3)$$

and $\displaystyle\int_{-\infty}^{\infty}b^2p(a,\tau_1;\ b,\tau_2)p(b,\tau_2;\ c,\tau_3)db$

$$=\left\{\frac{(\tau_2-\tau_1)(\tau_3-\tau_2)}{\alpha}+\frac{\beta^2}{\alpha^2}\right\}p(a,\tau_1;\ c,\tau_3)$$

where $\alpha=\tau_3-\tau_1$ and $\beta=a(\tau_3-\tau_2)+c(\tau_2-\tau_1)$.

Using these results, the term with AC, as is true of the AD, BC, and BD terms, produces (in the limit, as $||\tau||\to 0$)

$$\frac{1}{4}x\frac{(b-a)}{(b-s)}\frac{[x(t-b)+y(b-s)]}{t-s}+\frac{1}{4}\frac{(a-s)}{(b-s)}\left\{\frac{(b-s)(t-b)}{t-s}+\frac{[y(b-s)+x(t-b)]^2}{(t-s)^2}\right\}.$$

The relation

$$E\left\{[X(a) - m(a)]\,[X(b) - m(b)]\,|\,X(s) = x,\, X(t) = y\right\}$$

$$E\left\{X(a)X(b)\,|\,X(s) = x,\, X(t) = y\right\} - m(a)m(b) \quad \text{gives} \quad (6.\ 2).$$

The fact that the finite dimensional distributions are Gaussian and that the covariance is factorable (see [2]) completes the proof.

Bibliography

[1] Beekman, John A., "Solutions to Generalized Schroedinger Equations via Feynman Integrals Connected with Gaussian Markov Stochastic Processes," Thesis, U. of Minnesota, 1963.

[2] Beekman, John A., "Gaussian Processes and Generalized Schroedinger Equations," *J. Math and Mech.* **14** (1965), 789–806.

[3] Beekman, John A., "Gaussian Markov Processes and a Boundary Value Problem," *Trans. Amer. Math. Soc.* **126** (1967), 29–42.

[4] Beekman, John A., "Feynman-Cameron Integrals," *J. Math. and Physics*, **46** (1967), 253–266.

[5] Cameron, R.H., "A Family of Integrals Serving to Connect the Wiener and Feynman Integrals," *J. of Math. and Physics*, **39** (1960), 126–140.

[6] Cameron, R.H. and Donsker, M.D., "Inversion Formulae for Characteristic Functionals of Stochastic Processes," *Ann. of Math.*, **69** (1959), 15–36.

[7] Donsker, M.D. and Lions, J.L., "Fréchet-Volterra Variational Equations, Boundary Value Problems, and Function Space Integrals," *Acta Mathematica*, **108** (1962), 142–228.

[8] Feynman, R.P. and Hibbs, A.R., *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

[9] Fosdick, Lloyd D., "Approximation of a Class of Wiener Integrals," *Math. of Computation*, **19** (1965), 225–233.

[10] Itô, Kiyosi, "Generalized Uniform Complex Measures in the Hilbertian Metric Space with their Application to the Feynman Integral," Proc. Fifth Berkeley Symposium on Math. Statistics and Probability, Vol. II, Part 1 (1967), Univ. of Calif. Press, 145–161.

[11] Kac, Mark, *Probability and Related Topics in Physical Sciences*, Interscience Publishers, New York, 1959.

[12] Kac, Mark, "Random Walk in the Presence of Absorbing Barriers," *Ann. Math. Statist.* **16**, 62–67.

[13] Kac, Mark and Siegert, A.J.F., "On the Theory of Noise in Radio Receivers with Square Law Detectors," *J. of Applied Physics*, **18** (1947), 383–397.

[14] Wilks, S.S., *Mathematical Statistics*, Wiley, New York, 1962.

Ball State University
Muncie, Indiana