S. Yamashita
Nagoya Math. J.
Vol. 34 (1969), 77–87

ON LEVEL CURVES OF HARMONIC AND ANALYTIC FUNCTIONS ON RIEMANN SURFACES

SHINJI YAMASHITA[1)]

1. In this note we shall denote by R a hyperbolic Riemann surface. Let $HP'(R)$ be the totality of harmonic functions u on R such that every subharmonic function $|u|$ has a harmonic majorant on R. The class $HP'(R)$ forms a vector lattice under the lattice operations:

$$u \vee v = (\text{the least harmonic majorant of } \max(u, v));$$

$$u \wedge v = -(-u) \vee (-v)$$

for u and v in $HP'(R)$. Following Parreau [4] we shall call an element u in $HP'(R)$ quasi-bounded on R if

$$\lim_{\alpha \to +\infty} (Mu) \wedge \alpha = Mu,$$

where α's are positive numbers and

$$Mu = u \vee 0 - u \wedge 0.$$

A subharmonic function v on R is said to be quasi-bounded on R if v is of the form:

$$v = v^{\wedge} - p,$$

where $v^{\wedge}$ is a quasi-bounded harmonic function on R and $p \geq 0$ is a Green's potential on R ([8]).

For any finite real-valued function f on R and for any finite real number α, we denote by $L(f; \alpha)$ the set of points z in R such that $f(z) = \alpha$ holds. We shall call $L(f; \alpha)$ the α-level set or the α-level curve of f on R. Especially, if $f = |g|$, where g is an analytic function (i.e., pole-free) on R, then we shall call $L(|g|; \alpha)$ the α-level curve of an analytic function

Received November 1, 1967

1) The present author wishes to express his warmest thanks to Prof. Fumi-Yuki Maeda for kind suggestions and valuable conversations.

g on R. For $\alpha > 0$, the α-level curve of an analytic function g on R is the counter image of the circle of radius α by g.

For any closed subset F of R and for any fixed point t in R, we denote

$$1_F(t) = \inf_s s(t),$$

where s runs over all non-negative superharmonic functions on R such that $s \geq 1$ quasi-everywhere (quasi überall) on F ([1]).

A function $\Phi(r)$ defined for $r \geq 0$ is said to be strongly convex if $\Phi(r)$ is a non-negative monotone non-decreasing convex function defined for $r \geq 0$ satisfying the condition:

$$\lim_{r \to +\infty} \Phi(r)/r = +\infty.$$

First we shall prove the following

THEOREM. *Let v be a non-negative continuous subharmonic function on a hyperbolic Riemann surface R and assume that v has a harmonic majorant on R. Then the following three conditions are mutually equivalent.*

(1) *v is quasi-bounded on R.*

(2) *There exists a strongly convex function Φ depending on v such that*

$$\lim_{\alpha \to +\infty} \Phi(\alpha)\, 1_{L(v;\,\alpha)}(t) = 0$$

for some (and hence for any) point t in R.

(3) $\liminf_{\alpha \to +\infty} \alpha\, 1_{L(v;\,\alpha)}(t) = 0$

for some (and hence for any) point t in R.

In section 3 we shall prove the following extension of Nakai's theorem ([3])[2] as an application of Theorem.

COROLLARY 1. *Let R be a hyperbolic Riemann surface. For an element u in $HP'(R)$, the following three conditions are mutually equivalent.*

(4) *u is quasi-bounded on R.*

(5) *There exist two strongly convex functions Φ and Ψ depending on u such that*

[2] Cf. Lemma 1 in this note.

(5. 1) $\lim_{\alpha\to+\infty} \Phi(\alpha)\, 1_{L(u;\alpha)}(t) = 0$

and

(5. 2) $\lim_{\beta\to-\infty} \Psi(-\beta)\, 1_{L(u;\beta)}(t) = 0$

for some (and hence for any) point t in R.

(6) *The following*

(6. 1) $\liminf_{\alpha\to+\infty} \alpha\, 1_{L(u;\alpha)}(t) = 0$

and

(6. 2) $\liminf_{\beta\to-\infty} (-\beta)\, 1_{L(u;\beta)}(t) = 0$

are valid for some (and hence for any) point t in R.

In section 4 we shall be concerned mainly with α-level curves of analytic functions on R. The following corollary will play a fundamental role.

COROLLARY 2. *Let $\psi(r)$ be a non-negative finite real-valued continuous function defined for $a < r < b$ (where $a = -\infty$ and $b = +\infty$ are admissible) and $\psi(r) \to +\infty$ strictly increasingly as $r \searrow a$* (resp. $r \nearrow b$). *Let $v(z)$ be a continuous function defined on a hyperbolic Riemann surface R such that $a < v(z) < b$ and the function $\psi(v)$ is a quasi-bounded subharmonic function on R. Then there exists a strongly convex function Φ depending on $\psi(v)$ such that*

(7) $\lim_{\beta\to a} \Phi(\psi(\beta))\, 1_{L(v;\beta)}(t) = 0$

(resp. $\lim_{\beta\to b} \Phi(\psi(\beta))\, 1_{L(v;\beta)}(t) = 0$)

for some (and hence for any) point t in R.

2. To prove Theorem we shall need the following two lemmas.

LEMMA 1. (*Nakai's theorem* ([3])) *Let u be a non-negative harmonic function on a hyperbolic Riemann surface R. Then the following three conditions are mutually equivalent.*

(8) *u is quasi-bounded on R.*

(9) $\lim_{\alpha\to+\infty} \alpha\, 1_{L(u;\alpha)}(t) = 0$

for some (and hence for any) point t in R.

(10) $\liminf_{\alpha\to+\infty} \alpha 1_{L(u;\alpha)}(t) = 0$

for some (and hence for any) point t in R.

LEMMA 2. *Let v be a non-negative quasi-bounded subharmonic function on a hyperbolic Riemann surface R. Then there exists a strongly convex function Φ depending on v such that the subharmonic function Φ(v) is quasi-bounded on R.*

Proof. First, by Lemma 2 in [8], there exists a strongly convex function Φ depending on v such that the subharmonic function $\Phi(v)$ has a harmonic majorant on R. Next, we define a function $\varphi(r)$ for $-\infty < r < +\infty$ by the following:

$$\varphi(r) = \begin{cases} \Phi(r) & \text{for } 0 \leq r, \\ \Phi(0) & \text{for } r < 0. \end{cases}$$

Then the subharmonic function v and the convex function $\varphi(r)$ satisfy the conditions in Lemma 3 in [8]. Therefore by (E) of Lemma 3 in [8], we can conclude that the least harmonic majorant of the subharmonic function $\varphi(v) = \Phi(v)$ is quasi-bounded on R, or equivalently, the subharmonic function $\Phi(v)$ is quasi-bounded on R.

Proof of Theorem.

Proof of (1) $\Longrightarrow$ (2). By Lemma 2 there exists a strongly convex function Φ depending on v such that the subharmonic function $w = \Phi(v)$ is quasi-bounded on R, that is, w is of the form:

$$w = w^{\wedge} - p,$$

where $w^{\wedge}$ is a non-negative quasi-bounded harmonic function on R and $p \geq 0$ is a Green's potential on R. Obviously, $w \leq w^{\wedge}$.

For a non-negative finite real-valued function g on R and for a positive finite constant α, we shall denote by $S(g; \alpha)$ the set of points z in R such that $g(z) \geq \alpha$ holds.

Obviously the sets $S(w; \alpha)$ and $S(w^{\wedge}; \alpha)$ are closed subsets of R. On the other hand, the level set $L(w; \alpha)$ (resp. $L(w^{\wedge}; \alpha)$) is closed and hence by Satz 4. 8 in [1] we have

$$1_{L(w;\alpha)}(t) = 1_{S(w;\alpha)}(t) \text{ (resp. } 1_{L(w^{\wedge};\alpha)}(t) = 1_{S(w^{\wedge};\alpha)}(t))$$

for any point t in $R-S(w^{\wedge};\alpha)$. This means that

$$\lim_{\alpha\to+\infty}\alpha\, 1_{L(w;\alpha)}(t)=\lim_{\alpha\to+\infty}\alpha\, 1_{S(w;\alpha)}(t) \tag{11}$$

$$\left(\text{resp. } \lim_{\alpha\to+\infty}\alpha\, 1_{L(w^{\wedge};\alpha)}(t)=\lim_{\alpha\to+\infty}\alpha\, 1_{S(w^{\wedge};\alpha)}(t)\right)$$

for an arbitrary fixed point t in R, if either the right hand side or the left hand side of (11) has the meaning, since $R=\bigcup_{\alpha>0}(R-S(w^{\wedge};\alpha))$.

By $w\leq w^{\wedge}$, we have $S(w;\alpha)\subset S(w^{\wedge};\alpha)$ and from this it follows that

$$0\leq 1_{S(w;\alpha)}(t)\leq 1_{S(w^{\wedge};\alpha)}(t)$$

or

$$0\leq\alpha\, 1_{S(w;\alpha)}(t)\leq\alpha\, 1_{S(w^{\wedge};\alpha)}(t) \tag{12}$$

for any point t in R.

Now we apply Lemma 1 to the non-negative quasi-bounded harmonic function $w^{\wedge}$. Then by (9) in Lemma 1, we have

$$\lim_{\alpha\to+\infty}\alpha\, 1_{L(w^{\wedge};\alpha)}(t)=0 \tag{13}$$

for some (and hence for any) point t in R. By (11), (12) and (13) we have

$$\lim_{\alpha\to+\infty}\alpha\, 1_{L(w;\alpha)}(t)=0$$

or

$$\lim_{\alpha\to+\infty}\alpha\, 1_{L(\Phi(v);\alpha)}(t)=0 \tag{14}$$

for some (and hence for any) point t in R.

Since Φ is strictly increasing from sufficiently large r on, we have $L(\Phi(v);\alpha)=L(v;\Phi^{-1}(\alpha))$ for sufficiently large α. Therefore, by exchanging α in (14) for $\Phi(\alpha)$, we have

$$\lim_{\alpha\to+\infty}\Phi(\alpha)\, 1_{L(v;\alpha)}(t)=0$$

for some (and hence for any) point t in R.

Proof of (2) $\Longrightarrow$ (3) is obvious since $\Phi(\alpha)>\alpha$ for sufficiently large $\alpha>0$.

Proof of (3) $\Longrightarrow$ (1). Let $v=v^{\wedge}-q$ be the F. Riesz decomposition of v on R, where $v^\wedge$ is the least harmonic majorant of v on R and $q \geq 0$ is a Green's potential on R. Obviously q is continuous. By the same reason as in the proof of (1) $\Longrightarrow$ (2), we have

$$\text{(15)} \qquad \liminf_{\alpha\to+\infty} (\alpha/2)\, 1_{L(v;\alpha/2)}(t) = \liminf_{\alpha\to+\infty} (\alpha/2)\, 1_{S(v;\alpha/2)}(t)$$

$$\text{(resp. } \liminf_{\alpha\to+\infty} \alpha 1_{L(v^\wedge;\alpha)}(t) = \liminf_{\alpha\to+\infty} \alpha\, 1_{S(v^\wedge;\alpha)}(t))$$

for an arbitrary fixed point t in R, if either the right hand side or the left hand side of (15) has the meaning.

Next we prove

$$\text{(16)} \qquad \lim_{\alpha\to+\infty} (\alpha/2)\, 1_{S(q;\alpha/2)}(t) = 0$$

or

$$\text{(16)}' \qquad \lim_{\alpha\to+\infty} \alpha\, 1_{S(q;\alpha)}(t) = 0.$$

To prove (16)$'$ we take $\alpha_0 > 0$ so large that a fixed point x is in $R - S(q;\alpha)$ for any $\alpha > \alpha_0$. Let $\alpha > \alpha_0$ and $R_{x,\alpha}$ be the connected component of the open set $R - S(q;\alpha)$ containing the point x. Then we have $\bigcup_{\alpha>\alpha_0} R_{x,\alpha} = R$. For any point t in $R_{x,\alpha}$ we have

$$q(t) \geq q_{x,\alpha}(t) \geq \alpha\, 1_{S(q;\alpha)}(t) \geq 0,$$

where $q_{x,\alpha}$ is the greatest harmonic minorant of q in the domain $R_{x,\alpha}$, since by the definition of $1_{S(q;\alpha)}$,

$$q(t) \geq \alpha\, 1_{S(q;\alpha)}(t) \geq 0$$

for any point t in $R_{x,\alpha}$. On the other hand,

$$q_{x,\alpha}(t) \searrow 0 \quad \text{as} \quad \alpha \to +\infty,$$

for any point t in R since q is a Green's potential on R and $\{R_{x,\alpha}\}_{\alpha>\alpha_0}$ exhausts R. Therefore we have

$$\limsup_{\alpha\to+\infty} \alpha\, 1_{S(q;\alpha)}(t) = 0$$

for any point t in R, or we have (16)$'$.

Now by $v^\wedge = v + q$ we obtain

$$S(v^\wedge;\alpha) \subset S(v;\alpha/2) \cup S(q;\alpha/2).$$

From this it follows that

$$0 \leq 1_{S(v^{\wedge};\alpha)}(t) \leq 1_{S(v;\alpha/2)}(t) + 1_{S(q;\alpha/2)}(t)$$

or

$$(17) \qquad 0 \leq \alpha\, 1_{S(v^{\wedge};\alpha)}(t) \leq 2[(\alpha/2)\, 1_{S(v;\alpha/2)}(t) + (\alpha/2)\, 1_{S(q;\alpha/2)}(t)]$$

for any point t in R. Assume (3) in the theorem. Then

$$(18) \qquad \liminf_{\alpha\to+\infty} (\alpha/2)\, 1_{L(v;\alpha/2)}(t) = 0$$

for some (and hence for any) point t in R. Therefore by (15), (16), (17) and (18), we have

$$\liminf_{\alpha\to+\infty} \alpha\, 1_{L(v^{\wedge};\alpha)}(t) = 0$$

for some (and hence for any) point t in R. We apply Lemma 1 to the non-negative harmonic function $v^{\wedge}$. Then $v^{\wedge}$ is quasi-bounded on R and therefore v is a quasi-bounded subharmonic function. We have completely proved the theorem.

Remark. By applying Lemma 2 to a non-negative continuous quasi-bounded subharmonic function v repeatedly and using (1) $\Longrightarrow$ (2) of Theorem, we have the following: There exists a sequence $\{\Phi_m\}_{m=1}^{\infty}$ of strongly convex functions depending on v such that for any fixed number m, we have

$$\lim_{\alpha\to+\infty} [\Phi_m(\Phi_{m-1}(\cdots(\Phi_1(\alpha))\cdots))]\, 1_{L(v;\alpha)}(t) = 0$$

for some (and hence for any) point t in R.

3. In this section we give

Proof of Corollary 1.

Proof of (4) $\Longrightarrow$ (5). Since u is quasi-bounded on R, $u \vee 0$ as well as $-u \wedge 0$ is quasi-bounded on R. By inequalities

$$\max(u, 0) \leq u \vee 0$$

and

$$\max(-u, 0) \leq (-u) \vee 0 = -u \wedge 0,$$

the subharmonic functions $\max(u, 0)$ and $\max(-u, 0)$ are quasi-bounded on

R. We apply $(1) \Longrightarrow (2)$ of Theorem to $\max(u, 0)$ and $\max(-u, 0)$. Then there exist two strongly convex functions Φ and Ψ depending on $\max(u, 0)$ and $\max(-u, 0)$ respectively (and hence depending on u) such that

$$\lim_{\alpha \to +\infty} \Phi(\alpha)\, 1_{L(\max(u,0);\, \alpha)}(t) = 0 \tag{19}$$

and

$$\lim_{\alpha \to +\infty} \Psi(\alpha)\, 1_{L(\max(-u,0);\, \alpha)}(t) = 0 \tag{20}$$

for some (and hence for any) point t in R. On the other hand,

$$L(\max(u, 0);\, \alpha) = L(u;\, \alpha) \tag{21}$$

and

$$L(\max(-u, 0);\, \alpha) = L(-u;\, \alpha) = L(u;\, \beta) \tag{22}$$

for $\alpha > 0$, where we put $\beta = -\alpha$. By (19) and (21) (resp. (20) and (22)) we have (5. 1) (resp. (5. 2)).

Proof of $(5) \Longrightarrow (6)$ is obvious.

Proof of $(6) \Longrightarrow (4)$. Combining (21) and (6. 1) (resp. (22) and (6. 2)) and using Theorem, $(3) \Longrightarrow (1)$, we can easily show that the subharmonic function $\max(u, 0)$ (resp. $\max(-u, 0)$) is quasi-bounded on R. Hence $u \vee 0$ as well as $(-u) \vee 0$ is a quasi-bounded harmonic function on R. Therefore $u = u \vee 0 + u \wedge 0 = u \vee 0 - (-u) \vee 0$ is quasi-bounded on R. This completes the proof of Corollary 1.

4. Before proving Corollary 2, we shall give some examples of functions v and ψ stated in Corollary 2.

Example 1. Let $H_p(R)$ (for $p > 0$) be the Hardy class on R, that is, the totality of analytic functions f on R such that every subharmonic function $|f|^p$ has a harmonic majorant on R. Then, by Theorem 2 in [8], an analytic function f on R belongs to $H_p(R)$ if and only if the subharmonic function $|f|^p$ has a quasi-bounded harmonic majorant on R, or equivalently, $|f|^p$ is a quasi-bounded subharmonic function on R. In this case,

$$v = |f|$$

and

$$\psi(r) = \begin{cases} 0 & \text{for} \quad a < r < 0, \\ r^p & \text{for} \quad 0 \leq r < +\infty, \end{cases}$$

where a is an arbitrary negative number. Obviously $\psi(r) \nearrow +\infty$ as $r \nearrow +\infty$.

We have: There exists a strongly convex function Φ such that

$$\lim_{\beta \to +\infty} \Phi(\beta^p) 1_{L(|f|;\beta)}(t) = 0$$

for some (and hence for any) point t in R.

EXAMPLE 2. By Theorem 1 in [8], an analytic function f on R is in the Smirnov class $S(R)$ (cf., e.g., [8]) if and only if the subharmonic function $\log^+|f|$ has a quasi-bounded harmonic majorant on R, or equivalently, $\log^+|f|$ is a quasi-bounded subharmonic function on R. In this case,

$$v = |f|$$

and

$$\psi(r) = \begin{cases} 0 & \text{for} \quad a < r < 1, \\ \log r & \text{for} \quad 1 \leq r < +\infty, \end{cases}$$

where a is an arbitrary negative number. We have $\psi(r) \nearrow +\infty$ as $r \nearrow +\infty$.

EXAMPLE 3. Let f be an analytic function on R such that $w = f(z)$ takes only the values in the angular domain: $|\arg w| < \delta$ $(0 < \delta < \pi)$. Then, for any constant p, where $0 < p < \pi/2\delta$, the function f is in the Hardy class $H_p(R)$. This can be proved as follows.[3] By

$$f(z) = |f(z)| e^{i \arg f(z)}$$

we have

$$|f(z)|^p = \frac{\Re[(f(z))^p]}{\cos(p \arg f(z))} < \frac{\Re[(f(z))^p]}{\cos p\delta},$$

if $0 < p < \pi/2\delta$. Hence f is in $H_p(R)$ so that the subharmonic function $|f|^p$ is quasi-bounded on R for any p, $0 < p < \pi/2\delta$. Therefore this is a special case of Example 1.

EXAMPLE 4. Let $f(z) = u(z) + iw(z)$ be an analytic function in the open unit disc U: $|z| < 1$ such that the real part $u(z)$ of $f(z)$ can be extended continuously to the closed disc $\bar{U}$: $|z| \leq 1$. Then, by Smirnov's theorem

[3] V.I. Smirnov [6] proved the case: $\delta = \pi/2$ (cf. [5]).

([6], cf., e.g., [2], p. 401, Theorem 7), the analytic function e^{if} is in the Hardy class $H_p(U)$ for any $p>0$, or $|e^{if}|^p = e^{-pw}$ is a quasi-bounded subharmonic function on U for any $p>0$. In this case,

$$v = w$$

and

$$\psi(r) = e^{-pr} \quad \text{for } -\infty < r < +\infty.$$

Obviously $\psi(r) \nearrow +\infty$ as $r \searrow -\infty$.

EXAMPLE 5.[4] A bounded Jordan domain G in the plane with rectifiable boundary is said to be a Smirnov domain if for some (and hence for any) one to one conformal mapping $\varphi(z)$ from the open unit disc U: $|z|<1$ onto G, the harmonic function $\log|\varphi'|$ is represented as the Poisson integral of its boundary values on the unit circle: $|z|=1$, or equivalently, it is a quasi-bounded harmonic function on U ([6], cf., e.g., [2] and [5]). We know that a bounded Jordan domain G in the plane with rectifiable boundary is a Smirnov domain if and only if for some (and hence for any) one to one conformal mapping φ from U onto G, the analytic function $1/\varphi'$ is in the class $S(U)$ (cf., e.g., [7]), or equivalently, the subharmonic function $\log^+|1/\varphi'|$ is quasi-bounded on U. In this case,

$$v = |\varphi'|$$

and

$$\psi(r) = \log^+(1/r) \quad \text{for } 0 < r < +\infty.$$

We have $\psi(r) \nearrow +\infty$ as $r \searrow 0$.

We give

Proof of Corollary 2. This is an immediate consequence of (1) $\Longrightarrow$ (2) of Theorem. In fact, by (2) in Theorem, we obtain a strongly convex function Φ depending on the quasi-bounded subharmonic function $\psi(v)$ such that

$$\lim_{\alpha\to+\infty} \Phi(\alpha)\, 1_{L(\psi(v);\,\alpha)}(t) = 0$$

---

[4] Tumarkin and Havinson [7] defined Smirnov domains of finite connectivity and obtained some analogous results as in the case of simply connected Smirnov domains.

for some (and hence for any) point t in R. Let β be near a (resp. b). Then by property of the function $\psi(r)$ we have the assertion.

REFERENCES

[1] C. Constantinescu and A. Cornea, Ideale Ränder Riemannsher Flächen; Springer, Berlin, (1963).

[2] G.M. Goluzin, Geometric theory of functions of a complex variable; 2nd ed., Izd. NAUKA, Moscow, (1966), (Russian), German transl. of the 1st ed., Deutscher Verlag der Wissenschaften, Berlin, (1957).

[3] M. Nakai, On a criterion of quasi-boundedness of positive harmonic functions; Proc. Japan Acad., **41** (1965), 215–217.

[4] M. Parreau, Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann; Ann. Inst. Fourier, **3** (1952), 103–197.

[5] I.I. Privalov, Boundary behaviours of analytic functions; GTTI, Moscow and Leningrad, (1950), (Russian), German transl., Deutscher Verlag der Wissenschaften, Berlin, (1956).

[6] V.I. Smirnov, Sur les formules de Cauchy et de Green et quelques problèmes qui s'y rattachent; Izv. AN SSSR, ser. mat., **3** (1932), 337–372.

[7] G.C. Tumarkin and S. Ya. Havinson, Analytic functions in multiply connected domains of V.I. Smirnov's class (of class S); Izv. AN SSSR, ser. mat., **22** (1958), 379–386, (Russian).

[8] S. Yamashita, On some families of analytic functions on Riemann surfaces; Nagoya Math. Journ., **31** (1968), 57–68.

Department of Mathematics
Faculty of Science
Hiroshima University