H. Röhrl
Nagoya Math. J.
Vol. 34 (1969), 35–76

A CATEGORICAL SETTING FOR DETERMINANTS AND TRACES

Helmut Röhrl[0)]

The aim of this paper is to investigate some applications of a certain universal problem. The universal problem deals with categories $\boldsymbol{C}$ which for every object A have some "structure" on the set $\boldsymbol{C}(A, A)$ of endomorphisms of A and calls for a universal solution, relative to this structure, that is associated with $\boldsymbol{C}(A, A)$ and centralizes the set of automorphisms of A. The commutative version of this universal problem asks for a universal solution, relative to the said structure, that abelianizes the canonical monoid structure of $\boldsymbol{C}(A, A)$.

In §1 the general case is discussed. A number of existence theorems, all versions of the Special Adjoint Functor Theorem (see [11]) are stated and various structure theorems concerning the universal solution are proved. §2 deals with presheaves and the corresponding universal problem. It is shown that the universal problem for presheaves may be solved pointwise and that, under fairly weak assumptions, the stalk functor commutes with the universal solution. It is also asserted that under appropriate hypotheses any recollatement of a sheaf leaves the universal solution unchanged. In §3 the trace for endomorphisms in an R-additive category is defined as a special instance of the universal problem of §1. Here, the previously mentioned structure is that of a left R-module. The existence of the trace (for the endomorphisms) of any object A is easily obtained. It turns out to be the canonical morphism from End A to H_0 (Aut A, End A) with Aut A operating on End A by conjugation. Moreover it is shown that the trace of an endomorphism of a finite direct sum "is the sum of the diagonal entries" in the matrix description of that endomorphism. In §4 we restrict ourselves to the study of the trace for endomorphisms of unital R-modules, R being

Received April 9, 1968

0) Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant AF-AFOSR-920-65.

an associative, commutative ring, though many of the results of this section remain valid for R-additive categories with multiplication. The topics discussed are the behavior of the trace under forming the transpose, under tensoring, under change of rings, and under restriction of scalars. However, the most important aspect of this section is the following localization principle: If the R-module M is finitely presentable then for every prime ideal $\boldsymbol{p}$ of R the commutative trace of $M_{\boldsymbol{p}}$ equals the localization at $\boldsymbol{p}$ of the commutative trace of M. Hence the notion of trace as defined here performs well on the category of finitely presentable module, a category which dominates the theory of coherent sheaves. The localization principle is used in order to prove that for finitely generated projective R-modules our notion of commutative trace coincides with the classical one (see [5]). §5 is concerned with another special instance of the universal problem of §1, namely the notion of predeterminant for endomorphisms in an R-monoidal category. Here, the previously mentioned structure is that of a left R-monoid. The structure of an R-monoid being more cumbersome and intricate than the structure of an R-module indicates that fewer results will be available on predeterminants than on traces. Again the existence of the predeterminant is easily established. It is shown that elementary automorphisms of a finite direct sum $\overset{k}{\oplus} A, k>2$, are mapped into the unit element by the predeterminant map; if End A contains "good" units then the restriction $k>2$ becomes unnecessary. One of the consequences of these properties is that for a possibly non-commutative field R the predeterminant for endomorphisms of $\overset{k}{\oplus} R$, $k>2$, is the classical (Dieudonné-) determinant; if R is different from $\boldsymbol{Z}_2$ then the restriction $k>2$ becomes unnecessary. A similar result is valid for R a commutative principal ideal domain having 1 as a stable range. Hence we obtain in these cases a characterization of the determinant by general (universality) properties rather than by properties explicitly referring to the particular nature, that is square-matrix-shape, of the endomorphisms involved (see e.g. [14] and [19]). For euclidean domains this characterization was obtained in [18]. §6 continues the discussion of §5 for the R-monoidal category associated with the category of unital R-modules, being an associative, commutative ring. It is shown that an endomorphism of a finitely generated projective R-module M is an isomorphism if and only if its image under the predeterminant map is a unit; this implies that for such a module the group of units of the predeterminant monoid is isomorphic to the factor commutator group of $\mathrm{Aut}_R M$. As in §4 the localization principle is established for finitely presentable R-modules. In the final section we outline additional applications. We also indicate a characterization of the determinant over a large class of commutative unital rings as the solution of a universal problem closely related to the one dealt with in §1.

§1. The Universal Problem.

Let $\boldsymbol{C}$ be a category. For a given object A of $\boldsymbol{C}$ we shall denote the set $\boldsymbol{C}(A, A)$ by $\boldsymbol{C}(A)$. The invertible elements of $\boldsymbol{C}(A)$ form a group $\boldsymbol{C}^*(A)$, the composition in $\boldsymbol{C}^*(A)$ being the obvious one. $\boldsymbol{C}^*(A)$ operates on $\boldsymbol{C}(A)$ by conjugation, that is by the assignment

$$\boldsymbol{C}^*(A) \times \boldsymbol{C}(A) \ni (\alpha, \mu) \longmapsto \alpha\mu\alpha^{-1} \in \boldsymbol{C}(A).$$

This operation and its orbit set $\boldsymbol{C}(A)/\boldsymbol{C}^*(A)$ shall be used in the sequel without any further reference.

Consider the following universal problem whose data are a category $\boldsymbol{C}$, a category $\boldsymbol{L}$, a map $l\colon \mathrm{Ob}\, \boldsymbol{C} \longrightarrow \mathrm{Ob}\, \boldsymbol{L}$, and a faithful, set-valued covariant functor $|\quad|\colon \boldsymbol{L} \longrightarrow \boldsymbol{S}$ subject to the condition

$$\boldsymbol{C}(A) = |l(A)| \qquad \text{for all objects } A \text{ of } \boldsymbol{C},$$

the universal problem being to find for a given object A of $\boldsymbol{C}$ an object U_A of $\boldsymbol{L}$ and a morphism u_A from $l(A)$ to U_A such that

(i) $|u_A|$ factors through $\boldsymbol{C}(A) \longrightarrow \boldsymbol{C}(A)/\boldsymbol{C}^*(A)$

(ii) for any morphism v with domain $l(A)$ such that $|v|$ factors through $\boldsymbol{C}(A) \longrightarrow \boldsymbol{C}(A)/\boldsymbol{C}^*(A)$ there exists uniquely a morphism $\hat{v}$ from U_A to the codomain of v satisfying $v = \hat{v} \cdot u_A$.

In case of existence the universal problem furnishes a unique factorization

$$\begin{array}{ccc} \boldsymbol{C}(A) & \xrightarrow{\;|u_A|\;} & |U_A| \\ {\scriptstyle q}\downarrow & \nearrow{\scriptstyle \lambda_A} & \\ \boldsymbol{C}(A)/\boldsymbol{C}^*(A) & & \end{array}$$

and we obtain trivially

(1. 1) Corollary. *In case of existence the pair* (u_A, U_A) *is unique up to isomorphism.*

(1. 2) Corollary. *In case of existence* $\varphi = |\hat{v}| \circ \lambda_A$ *where* $|v| = \varphi \circ q$.

In order to facilitate the statement of the following existence theorem we denote for any object A of $\boldsymbol{C}$ the class

$$\{v\colon \mathrm{dom}(v) = l(A) \ \&\ |v| \text{ factors through } \boldsymbol{C}(A) \longrightarrow \boldsymbol{C}(A)/\boldsymbol{C}^*(A)\}$$

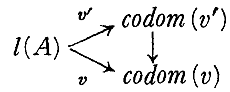
by Δ_A. Then we have (see [11], p. 84)

(1. 3) THEOREM. *Suppose that*

(i) $|\ \ |$ *preserves products and monomorphisms*

(ii) L *is well-powered and complete*

(iii) *for the object A of $\boldsymbol{C}$ there is a set S_A of objects of $\boldsymbol{L}$ such that for any $v \in \Delta_A$ there exists a $v' \in \Delta_A$ and a morphism codom$(v') \longrightarrow$ codom(v) such that*

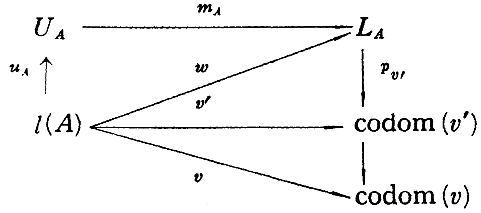
codom$(v') \in S_A$ and  *commutes.*

Then the universal problem for A possesses a solution.

Proof. For the sake of completeness we repeat the proof given in [11], p. 84–85. Form the product

$$L_A = \Pi\{\mathrm{codom}(v')\colon\ v' \in \Delta_A \ \&\ \mathrm{codom}(v') \in S_A\}.$$

Its existence is guaranteed by the assumption that $\boldsymbol{L}$ is complete. Denoting by $p_{v'}$ the v'. projection $L_A \longrightarrow \mathrm{codom}(v')$ there exists uniquely a $w\colon l(A) \longrightarrow L_A$ such that $p_{v'} \cdot w = v'$ for all $v' \in \Delta_A$ satisfying codom$(v') \in S_A$. Since $|\ \ |$ preserves products an easy argument shows that $w \in \Delta_A$ holds. L being well-powered and complete implies further that there is a minimal subobject $U_A \xrightarrow{m_A} L_A$ of L_A through which w factors. This factorization shall be written as $w = m_A \cdot u_A$. We claim that the pair (u_A, U_A) is a solution of the universal problem for A. First we observe that $|u_A|$ factors through $\boldsymbol{C}(A) \longrightarrow \boldsymbol{C}(A)/\boldsymbol{C}^*(A)$ since w does and since $|m_A|$ is a monomorphism in $\boldsymbol{S}$ (i.e. an injection). Next, given $v \in \Delta_A$ the condition (iii) of (1. 3) furnishes a commutative diagram as indicated there whence



commutes, establishing the existence of the desired factorization. Uniqueness of this factorization is then an immediate consequence of the minimality of U_A.

Browsing through [11], p. 87–89, and adopting the statements and reasonings to our situation we find the following results:

(1. 4) PROPOSITION. *Suppose that*

(i) $|\ \ |$ *preserves products and monomorphisms*

(ii) $\boldsymbol{C}$ *is complete, cocomplete, well-powered, and cowell-powered*

(iii) $\boldsymbol{L}$ *is a full, replete subcategory of* $\boldsymbol{C}$ *that is closed under the formation of products and subobjects.*

Then the universal problem possesses a solution for every object of $\boldsymbol{C}$.

(1. 5) PROPOSITION. *Suppose that*

(i) $|\ \ |$ *preserves products and monomorphisms*

(ii) $\boldsymbol{L}$ *is well-powered and complete*

(iii) *every object of* $\boldsymbol{L}$ *generates through the identity functor of* $\boldsymbol{L}$ *at most a set of non-isomorphic objects of* $\boldsymbol{L}$.

Then the universal problem possesses a solution for every object of $\boldsymbol{C}$.

(1. 6) PROPOSITION. *Suppose that*

(i) $|\ \ |$ *preserves products and monomorphisms*

(ii) $\boldsymbol{L}$ *is well-powered and complete and possesses a cogenerator.*

Then the universal problem possesses a solution for every object of $\boldsymbol{C}$.

(1. 7) COROLLARY. *Under the assumptions of either* (1. 3), (1. 5), *or* (1. 6) *the morphism* u_A *is an epimorphism.*

That much for general existence theorems.

For the sake of convenience we shall denote by Γ the quadruple $(C, L, l, |\ \ |)$ of the data described previously and subject to the above requirements. Let Γ and Γ' be two such quadruples and let $\Psi = (\mathscr{F}, \mathscr{G}, \mathscr{H})$ be a triple of covariant functors (with appropriate domain and codomain) such that

$$\begin{array}{ccccc} \mathrm{Ob}\,\boldsymbol{C} & \xrightarrow{\iota} & \boldsymbol{L} & \xrightarrow{|\ |} & \boldsymbol{S} \\ \mathscr{F}\big\downarrow & & \mathscr{G}\big\downarrow & & \mathscr{H}\big\downarrow \\ \mathrm{Ob}\,\boldsymbol{C}' & \xrightarrow{\iota'} & \boldsymbol{L}' & \xrightarrow{|\ |'} & \boldsymbol{S} \end{array}$$

commutes and that for every object A of $\boldsymbol{C}$—notice that $\mathscr{H}\boldsymbol{C}(A) = \boldsymbol{C}'(\mathscr{F}A)$ holds—there is a (necessarily unique) factorization

$$\boldsymbol{C}'(\mathscr{F}A) \longrightarrow \boldsymbol{C}'(\mathscr{F}A)/\boldsymbol{C}'^{*}(\mathscr{F}A) \xrightarrow{\Psi_A} \mathscr{H}(\boldsymbol{C}(A)/\boldsymbol{C}^{*}(A)) \ .$$

One checks immediately that these triples Ψ form a category, the composition of such triples being the obvious one. It should also be remarked that the last factorization implies that for every object A of $\boldsymbol{C}$, $\mathscr{G}$ maps Δ_A into $\Delta_{\mathscr{F}A}$.

(1. 8) THEOREM. *Let $\Psi : \Gamma \to \Gamma'$ be a morphism and suppose that for the object A of $\boldsymbol{C}$ both (u_A, U_A) and $(u_{\mathscr{F}A}, U_{\mathscr{F}A})$ exist. Then there exists a unique factorization $\mathscr{G}u_A = g_A \cdot u_{\mathscr{F}A}$, and for the resulting morphism g_A the relation $(\mathscr{H}\lambda_A) \circ \Psi_A = |g_A| \circ \lambda_{\mathscr{F}A}$ is valid.*

Proof. Straight forward.

(1. 9) PROPOSITION. *Let $\Psi : \Gamma \to \Gamma'$ be a morphism and suppose that for the object A of $\boldsymbol{C}$*

(i) *the universal problem for A possesses a solution (u_A, U_A)*

(ii) *u_A is an epimorphism and $\mathscr{G}$ preserves epimorphisms*

(iii) *$\mathscr{G}$ is a surjection from Δ_A onto $\Delta_{\mathscr{F}A}$.*

Then $(\mathscr{G}u_A, \mathscr{G}U_A)$ is a solution of the universal problem for $\mathscr{F}A$.

Proof. The existence of the required factorization follows easily from (iii) by pulling back. Uniqueness of the required factorization is an immediate consequence of (ii).

(1. 10) PROPOSITION. *Let $\Psi : \Gamma \to \Gamma'$ be a morphism and suppose that for the object A of $\boldsymbol{C}$*

(i) *the universal problem for $\mathscr{F}A$ possesses a solution $(u_{\mathscr{F}A}, U_{\mathscr{F}A})$*

(ii) *there is a covariant functor* $\mathscr{G}'$: $\boldsymbol{L}' \to \boldsymbol{L}$ *such that* $\mathscr{G}' \circ \mathscr{G}$ *is the identity on* $\boldsymbol{L}$ *and that* $\mathscr{G} \circ \mathscr{G}'$ *is isomorphic to the identity on* $\boldsymbol{L}'$ *via an isomorphism* m *satisfying* $m_{l'(\mathscr{F}A)} = l'(\mathscr{F}A)$

(iii) $|u_{\mathscr{F}A}|'$ *factors through* $\boldsymbol{C}'(\mathscr{F}A) \to \mathscr{H}(\boldsymbol{C}(A)/\boldsymbol{C}^*(A))$

(iv) $\mathscr{H}$ *is full and faithful.*

Then the universal problem for A *has* $(\mathscr{G}'u_{\mathscr{F}A}, \mathscr{G}'U_{\mathscr{F}A})$ *as a solution.*

Proof. Let v be in Δ_A. Then $\mathscr{G}v$ is in $\Delta_{\mathscr{F}A}$ and therefore there exists a factorization $\mathscr{G}v = \hat{v}' \cdot u_{\mathscr{F}A}$. The first condition of (ii) implies then that $v = \mathscr{G}'\hat{v}' \cdot \mathscr{G}'u_{\mathscr{F}A}$. In case v belongs to Δ_A and $v = \hat{v} \cdot \mathscr{G}'u_{\mathscr{F}A}$ we obtain $\mathscr{G}v = \mathscr{G}\hat{v} \cdot \mathscr{G}\mathscr{G}'u_{\mathscr{F}A}$ whence the second part of (ii) establishes the uniqueness of $\mathscr{G}\hat{v}$. The first condition of (ii) then establishes the uniqueness of $\hat{v}$. In order to show that $\mathscr{G}'u_{\mathscr{F}A} \in \Delta_A$ holds we observe that there is a factorization $\mathscr{G}\mathscr{G}'u_{\mathscr{F}A} = w \cdot u_{\mathscr{F}A}$ whence we have a commutative diagram

$$\begin{array}{ccc}
\mathscr{H}(\boldsymbol{C}(A)) & \xrightarrow{\mathscr{H}(|\mathscr{G}'u_{\mathscr{F}A}|)} & \mathscr{H}(|\mathscr{G}'U_{\mathscr{F}A}|) \\
\Vert & & \uparrow |w|' \\
\mathscr{H}(\boldsymbol{C}(A)) & \xrightarrow{|u_{\mathscr{F}A}|'} & |U_{\mathscr{F}A}|' \\
\downarrow & \nearrow & \\
\mathscr{H}(\boldsymbol{C}(A)/\boldsymbol{C}^*(A)) & &
\end{array}$$

Therefore the fullness of $\mathscr{H}$ asserts the existence of a diagram

$$\begin{array}{ccc}
\boldsymbol{C}(A) & \xrightarrow{|\mathscr{G}'u_{\mathscr{F}A}|} & |\mathscr{G}'U_{\mathscr{F}A}| \\
\downarrow & \nearrow & \\
\boldsymbol{C}(A)/\boldsymbol{C}^*(A) & &
\end{array}$$

Its commutativity follows from the commutativity of the previous diagram and the faithfulness of $\mathscr{H}$.

It is clear from the proof of (1. 10) that, simultaneously, the second condition of (ii) can be weakened to:

there is an isomorphism w such that $\mathscr{G}\mathscr{G}'u_{\mathscr{F}A} = w \cdot u_{\mathscr{F}A}$

and condition (iv) can be weakened to:

$\mathscr{H}$ is a bijection from $\boldsymbol{S}(\boldsymbol{C}(A)/\boldsymbol{C}^*(A), |\mathscr{G}'u_{\mathscr{F}A}|)$ to $\boldsymbol{S}(\mathscr{H}(\boldsymbol{C}(A)/\boldsymbol{C}^*(A)), \mathscr{H}(|\mathscr{G}'u_{\mathscr{F}A}|))$.

Next we shall give several statements relating the solutions of the universal problem for various objects.

(1. 11) **Proposition.** *Let A and A' be objects of $\boldsymbol{C}$ for both of which the universal problem has a solution. Suppose that there is a morphism t from $l(A)$ to $l(A')$ such that*

$$\begin{array}{ccc} \boldsymbol{C}(A) & \xrightarrow{|t|} & \boldsymbol{C}(A') \\ \downarrow & & \downarrow \\ \boldsymbol{C}(A)/\boldsymbol{C}^*(A) & \xrightarrow{|\tau|} & \boldsymbol{C}(A')/\boldsymbol{C}^*(A') \end{array}$$

commutes. Then there exists uniquely a morphism u_t from U_A to $U_{A'}$ rendering

$$\begin{array}{ccc} l(A) & \xrightarrow{t} & l(A') \\ {\scriptstyle u_A}\downarrow & & \downarrow{\scriptstyle u_{A'}} \\ U_A & \xrightarrow{u_t} & U_{A'} \end{array}$$

commutative. If, in addition, t is an isomorphism and τ is a bijection, then u_t is an isomorphism.

Proof. Since $|u_{A'}|$ factors through $\boldsymbol{C}(A') \longrightarrow \boldsymbol{C}(A')/\boldsymbol{C}^*(A')$ our hypothesis shows that $|u_{A'} \cdot t|$ factors through $\boldsymbol{C}(A) \longrightarrow \boldsymbol{C}(A)/\boldsymbol{C}^*(A)$. Hence the universality property of u_A furnishes the desired morphism u_t. The remainder follows quickly from the uniqueness part of the universality property.

Suppose that for the objects A_1 and A_2 of $\boldsymbol{C}$ the product $A_1 \amalg A_2$ exists. Then we have canonical maps

$$\boldsymbol{C}(A_1) \times \boldsymbol{C}(A_2) \longrightarrow \boldsymbol{C}(A_1 \amalg A_2, A_1) \times \boldsymbol{C}(A_1 \amalg A_2, A_2) \longrightarrow \boldsymbol{C}(A_1 \amalg A_2)$$

the composition of which sends (u_1, u_2) into the unique morphism w satisfying

$$p_{A_1} w = u_1 p_{A_1} \qquad \text{and} \qquad p_{A_2} w = u_2 p_{A_2},$$

p_{A_i} being the projection onto A_i. An easy computation shows that this map is a homomorphism of monoids. Since the images of $\boldsymbol{C}(A_1) \times \{A_2\}$ and $\{A_1\} \times \boldsymbol{C}(A_2)$ in $\boldsymbol{C}(A_1 \amalg A_2)$ commute we obtain a commutative diagram

(1. 12)
$$\begin{array}{ccc} \boldsymbol{C}(A_1) \times \boldsymbol{C}(A_2) & \longrightarrow & \boldsymbol{C}(A_1 \amalg A_2) \\ \downarrow & & \searrow \\ \boldsymbol{C}(A_1)/\boldsymbol{C}^*(A_1) \times \boldsymbol{C}(A_2)/\boldsymbol{C}^*(A_2) & \longrightarrow & \boldsymbol{C}(A_1 \amalg A_2)/\boldsymbol{C}^*(A_1 \amalg A_2). \end{array}$$

(1. 13) PROPOSITION. *Let A_1, A_2 be objects of $\boldsymbol{C}$ for both of which one universal problem admits a solution. Assume that*

(i) *the products $A_1 \amalg A_2$ and $l(A_1) \amalg l(A_2)$, and the coproducts $l(A_1) \perp\!\!\!\perp l(A_2)$ and $U_{A_1} \perp\!\!\!\perp U_{A_2}$ exist*

(ii) *there is a morphism $m: l(A_1) \amalg l(A_2) \longrightarrow l(A_1 \amalg A_2)$ that, via $|\ \ |$, lies above the canonical map $\boldsymbol{C}(A_1) \times \boldsymbol{C}(A_2) \longrightarrow \boldsymbol{C}(A_1 \amalg A_2)$*

(iii) *there is a morphism $k: l(A_1) \perp\!\!\!\perp l(A_2) \longrightarrow l(A_1) \amalg l(A_2)$ which renders the diagrams*

$$\begin{array}{ccc} \boldsymbol{C}(A_i) \xrightarrow{|j_i|} |l(A_1) \perp\!\!\!\perp l(A_2)| & \xrightarrow{|k|} & \boldsymbol{C}(A_1) \times \boldsymbol{C}(A_2) \\ \downarrow & & \downarrow \\ \boldsymbol{C}(A_i)/\boldsymbol{C}^*(A_i) & \longrightarrow & \boldsymbol{C}(A_1)/\boldsymbol{C}^*(A_1) \times \boldsymbol{C}(A_2)/\boldsymbol{C}^*(A_2) \end{array}$$

commutative, where j_i: $l(A_i) \longrightarrow l(A_1) \perp\!\!\!\perp l(A_2)$ are the canonical injections

(iv) *the universal problem for $A_1 \amalg A_2$ possesses a solution.*

Then there exists uniquely a morphism $g: U_{A_1} \perp\!\!\!\perp U_{A_2} \longrightarrow U_{A_1 \amalg A_2}$ such that

$$\begin{array}{ccc} l(A_1) \perp\!\!\!\perp l(A_2) \xrightarrow{k} l(A_1) \amalg l(A_2) & \xrightarrow{m} & l(A_1 \amalg A_2) \\ u_{A_1 \perp\!\!\!\perp A_2} \downarrow & g & \downarrow u_{A_1 \amalg A_2} \\ U_{A_1} \perp\!\!\!\perp U_{A_2} & \longrightarrow & U_{A_1 \amalg A_2} \end{array}$$

commutes.

Proof. (1. 12) together with the hypotheses implies that $u_{A_1 \amalg A_2} mkj_i$ belongs to Δ_{A_i}. Hence there exists a factorization

$$u_{A_1 \amalg A_2} mkj_i = g_i u_{A_i} \qquad i = 1, 2,$$

the morphisms g_i having $U_{A_1 \amalg A_2}$ as common codomain. g_1 and g_2 therefore determine canonically a morphism g from $U_{A_1} \perp\!\!\!\perp U_{A_2}$ to $U_{A_1 \amalg A_2}$ which has the desired properties, as is checked easily.

(1. 14) PROPOSITION. *Let A' and A be objects of $\boldsymbol{C}$ for both of which the universal problem possesses a solution. Let furthermore k: $l(A') \longrightarrow l(A)$ be a morphism such that for some map κ the diagram*

$$\begin{array}{ccc} \boldsymbol{C}(A') & \xrightarrow{|k|} & \boldsymbol{C}(A) \\ \downarrow & & \downarrow \\ \boldsymbol{C}(A')/\boldsymbol{C}^*(A') & \xrightarrow{\kappa} & \boldsymbol{C}(A)/\boldsymbol{C}^*(A) \end{array}$$

commutes. Assume that

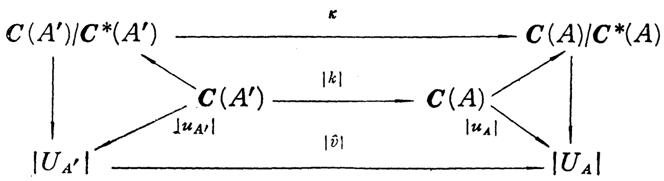
(i) $|\ \ |$ *reflects epimorphisms (resp. monomorphisms resp. isomorphisms)*

(ii) κ *is a surjection (resp. injection resp. bijection)*

(iii) $\boldsymbol{C}(A)/\boldsymbol{C}^*(A) \longrightarrow |U_A|$ *is a surjection (resp.* $\boldsymbol{C}(A')/\boldsymbol{C}^*(A') \longrightarrow |U_{A'}|$ *is a surjection and* $\boldsymbol{C}(A)/\boldsymbol{C}^*(A) \longrightarrow |U_A|$ *is an injection resp.* $\boldsymbol{C}(A')/\boldsymbol{C}^*(A') \longrightarrow |U_{A'}|$ *is a surjection and* $\boldsymbol{C}(A)/\boldsymbol{C}^*(A) \longrightarrow |U_A|$ *is a bijection).*

Then there is a natural epimorphism (resp. monomorphism resp. isomorphism) $U_{A'} \longrightarrow U_A$.

Proof. From the commutative diagram in (1. 14) we take that $u_A k$ belongs to $\Delta_{A'}$. Hence there is a unique factorization $u_A k = \hat{v} u_{A'}$. Consider the commutative diagram



From it we conclude that under the various hypotheses (ii) and (iii), $|\hat{v}|$ is a surjection (resp. injection resp. bijection) whence (i) finishes the proof.

There is a "commutative companion" to the universal problem posed at the beginning of this section. It is gotten by sharpening the condition (i), namely that $|u_A|(\alpha\mu) = |u_A|(\mu\alpha)$ for all automorphisms α of A and all endomorphisms μ of A, to

(i^c) $|u_A^c|(\mu_1\mu_2) = |u_A^c|(\mu_2\mu_1)$ for all endomorphisms μ_1, μ_2 of A

and by replacing (ii) by the correspondingly altered condition (ii^c). Clearly we have

(1. 15) PROPOSITION. *If for some object A of $\boldsymbol{C}$ both, u_A and u_A^c, exist then there is a unique factorization $u_A^c = w_A \cdot u_A$.*

Existence of a solution of this altered universal problem can be established under the conditions of either (1. 3)—with the solution set condition (iii) appropriately modified—or (1. 4) resp. (1. 5) resp. (1. 6). And, as before we obtain

(1. 16) COROLLARY. *Under the assumptions of either* (1. 3) *with the modified solution set condition, or of* (1. 5) *resp.* (1. 6) *the morphism w_A is an epimorphism.*

It should be observed that all results of this section that were stated for the "non-commutative" universal problem hold, mutatis mutandis, also for the "commutative" universal problem.

§2. The Universal Problem for Presheaves.

Let $\boldsymbol{P}(X, \boldsymbol{C})$ be the category of presheaves over X with values in the category $\boldsymbol{C}$. For each inclusion $W \subset V$ of open subsets of X we obtain a canonical covariant functor from $\boldsymbol{P}(V, \boldsymbol{C})$ to $\boldsymbol{P}(W, \boldsymbol{C})$ which sends every presheaf $\mathscr{P}$ over V into the restriction $\mathscr{P}|W$. It gives canonically rise to natural maps

$$[\mathscr{P}|V] \longrightarrow [\mathscr{P}|W], \tag{2. 1}$$

$[\mathscr{P}|V]$ being the set of endomorphisms of $\mathscr{P}|V$ in $\boldsymbol{P}(V, \boldsymbol{C})$. Clearly these maps constitute a set-valued presheaf $[\mathscr{P}|\ \ ]$. Since (2. 1) is a homomorphism with respect to the canonical monoid structures there are induced natural maps between orbit sets

$$[\mathscr{P}|V]/[\mathscr{P}|V]^* \longrightarrow [\mathscr{P}|W]/[\mathscr{P}|W]^*,$$

the operations being the ones discussed in §1. Again we obtain a set-valued presheaf $[\mathscr{P}|\ \ ]/[\mathscr{P}|\ \ ]^*$, and the maps which assign to each point its orbit constitute a natural transformation from $[\mathscr{P}|\ \ ]$ to $[\mathscr{P}|\ \ ]/[\mathscr{P}|]^*$. One checks easily that both $[\mathscr{P}|\ \ ]$ and $[\mathscr{P}|\ \ ]/[\mathscr{P}|\ \ ]^*$ are sheaves whenever $\mathscr{P}$ itself is a sheaf.

Assume that the following data are given: a category $\boldsymbol{C}$, a category $\boldsymbol{L}$, a map l: Ob $\boldsymbol{P}(X, \boldsymbol{C}) \longrightarrow$ Ob $\boldsymbol{P}(X, \boldsymbol{L})$, and a faithful, set-valued covariant functor $|\ \ |: \boldsymbol{L} \longrightarrow \boldsymbol{S}$ that reflects identities. Suppose that

$$\boldsymbol{P}(X, |\ \ |)l(\mathscr{P}) = [\mathscr{P}|\ \ ] \qquad \text{for all presheaves } \mathscr{P}.$$

Again we can pose the universal problem asking for the existence of a presheaf $\mathscr{U}_{\mathscr{P}}$ over X with values in $\boldsymbol{L}$ and a morphism of presheaves $u_{\mathscr{P}}$ from $l(\mathscr{P})$ to $\mathscr{U}_{\mathscr{P}}$ such that

(i_p) $\boldsymbol{P}(X, |\ \ |)u_{\mathscr{P}}$ factors through $[\mathscr{P}|\ \ ] \longrightarrow [\mathscr{P}|\ \ ]/[\mathscr{P}|\ \ ]^*$

(ii_p) for any morphism v of presheaves with domain $l(\mathscr{P})$ such that $\boldsymbol{P}(X, |\ \ |)v$ factors through $[\mathscr{P}|\ \ ] \longrightarrow [\mathscr{P}|\ \ ]/[\mathscr{P}|\ \ ]^*$ there exists uniquely a morphism $\hat{v}$ from $\mathscr{U}_{\mathscr{P}}$ to the codomain of v satisfying $v = \hat{v} \cdot u_{\mathscr{P}}$.

(2. 2) THEOREM. *Let* $\mathscr{P}$ *be a presheaf over* X *with values in* $\boldsymbol{C}$ *and assume that for every open subset* V *of* X *the pair* (u_V, U_V) *solves the universal problem for all morphisms* v *in* $\boldsymbol{L}$ *with domain* $l(\mathscr{P})(V)$ *such that* $|v|$ *factors through* $[\mathscr{P}|V]/[\mathscr{P}|V]^*$. *Then the previous universal problem for* $\mathscr{P}$ *admits a solution* $(u_{\mathscr{P}}, \mathscr{U}_{\mathscr{P}})$ *satisfying* $u_{\mathscr{P}V} = u_V$ *and* $\mathscr{U}_{\mathscr{P}}(V) = U_V$ *for all open sets* V.

Proof. Straight forward (see e.g. [16], p. 64).

Clearly we obtain

(2. 3) COROLLARY. *The hypothesis of* (2. 2) *is satisfied for every presheaf* $\mathscr{P}$ *provided that the assumption of either* (1. 3)—*with* (iii) *replaced by an appropriate solution set condition—or* (1. 4) *resp.* (1. 5) *resp.* (1. 6) *are valid. Under the assumptions of either* (1. 3)—*with* (iii) *modified as stated—or* (1. 5) *resp.* (1. 6) $u_{\mathscr{P}}$ *is pointwise an epimorphism.*

Before we go on a remark is in order. Via the canonical imbedding $\boldsymbol{C} \subset \boldsymbol{P}(X, \boldsymbol{C})$ the map l assigns to each object A of $\boldsymbol{C}$ a presheaf $l(A)$ with values in $\boldsymbol{L}$. By assumption we have $\boldsymbol{P}(X, |\ \ |)l(A) = [A|\ \ ]$. Since $[A|\ \ ]$ is a constant functor the requirement that $|\ \ |$ reflects identites implies that $l(A)$ is a constant presheaf. Hence l gives rise to a canonical map from Ob $\boldsymbol{C}$ to Ob $\boldsymbol{L}$ which shall again be denoted by l. This map evidently satisfies $|l(A)| = \boldsymbol{C}(A)$ which is just the relation imposed in §1 on the data of the universal problem.

Next we shall discuss the connection between the universal problem for presheaves and the universal problem of §1 when stated for individual stalks of those presheaves. This, of course, will only be possible under the assumption that both $\boldsymbol{C}$ and $\boldsymbol{L}$ are cocomplete.

We require now of all presheaves $\mathscr{P}$ that for every $x \in X$ and all open sets V containing x there is a morphism $\lambda(\mathscr{P})_x^V\colon l(\mathscr{P})(V) \longrightarrow l(\mathscr{P}_x)$ that is natural in V. Hitting this morphism with $|\ \ |$ we obtain a map $|\lambda(\mathscr{P})_x^V|\colon [\mathscr{P}|V] \longrightarrow \boldsymbol{C}(\mathscr{P}_x)$. We require, in addition, that the maps $\lambda(\mathscr{P})_x^V$ are homomorphisms of the canonical monoid structures. Then they give rise to commutative diagrams

$$\begin{array}{ccc} [\mathscr{P}|V] & \longrightarrow & \boldsymbol{C}(\mathscr{P}_x) \\ \downarrow & & \downarrow \\ [\mathscr{P}|V]/[\mathscr{P}|V]^* & \longrightarrow & \boldsymbol{C}(\mathscr{P}_x)/\boldsymbol{C}^*(\mathscr{P}_x)\,. \end{array} \tag{2. 4}$$

Suppose that one of the alternate conditions of (2. 3) is satisfied. Then the universal problem for $\mathscr{P}_x$ formulated in §1 has a solution $(u_{\mathscr{P}_x}, U_{\mathscr{P}_x})$ and the above commutative diagram shows that $|u_{\mathscr{P}_x} \cdot \lambda(\mathscr{P})_x^V| = |u_{\mathscr{P}_x}| \cdot |\lambda(\mathscr{P})_x^V|$ factors through $[\mathscr{P}|V]/[\mathscr{P}|V]^*$. Hence (2. 2) furnishes a unique factorization

$$u_{\mathscr{P}_x} \cdot \lambda(\mathscr{P})_x^V = \hat{v}_{\mathscr{P}V} \cdot u_{\mathscr{P}V}.$$

Passage to the direct limit furnishes then a commutative diagram

$$\begin{array}{ccc} l(\mathscr{P})_x & \xrightarrow{\;u_{\mathscr{P}x}\;} & \mathscr{U}_{\mathscr{P}x} \\ {\scriptstyle \lambda(P)_x}\downarrow & & \downarrow{\scriptstyle \hat{v}_{\mathscr{P}x}} \\ l(\mathscr{P}_x) & \xrightarrow{\;u_{\mathscr{P}_x}\;} & U_{\mathscr{P}x} \end{array}$$

(2. 5) PROPOSITION. *Assume that both* $\boldsymbol{C}$ *and* $\boldsymbol{L}$ *are cocomplete, that one of the alternate conditions of* (2. 3) *is satisfied, and that the natural morphisms* $\lambda(\mathscr{P})_x^V$ *render homomorphisms* $|\lambda(\mathscr{P})_x^V|$ *of the canonical monoid structures. Assume furthermore that* $|\quad|$ *preserves direct limits. Then* $\lambda(\mathscr{P})_x$ *being an isomorphism implies that* $\hat{v}_{\mathscr{P}x}$ *is an isomorphism.*

Proof. First we observe that the direct limit $u_{\mathscr{P}x}: l(\mathscr{P})_x \longrightarrow \mathscr{U}_{\mathscr{P}x}$ is characterized by the following universality property. Given any family of commutative diagrams

$$\text{(2. 6)}\qquad \begin{array}{ccc} l(\mathscr{P})(V) & \xrightarrow{\;u_{\mathscr{P}V}\;} & \mathscr{U}_{\mathscr{P}}(V) \\ \downarrow & & \downarrow \\ L & \longrightarrow & M \end{array}$$

that is natural in V there exists uniquely a pair of morphisms $l(\mathscr{P})_x \longrightarrow L$ and $\mathscr{U}_{\mathscr{P}x} \longrightarrow M$ rendering all diagrams

$$\begin{array}{ccccc} & l(\mathscr{P})(V) & \longrightarrow & \mathscr{U}_{\mathscr{P}}(V) & \\ \nearrow & \mid & & \nearrow & \mid \\ l(\mathscr{P})_x & \longrightarrow & \mathscr{U}_{\mathscr{P}x} & & \mid \\ \searrow & \mid & & \searrow & \mid \\ & L & \longrightarrow & M & \end{array}$$

commutative. Since $\lambda(\mathscr{P})_x: l(\mathscr{P})_x \longrightarrow l(\mathscr{P}_x)$ is an isomorphism by assumption the morphisms $l(\mathscr{P})(V) \longrightarrow l(\mathscr{P})_x \xrightarrow{\lambda(\mathscr{P})_x} l(\mathscr{P}_x)$ constitute a direct limit-diagram. The hypothesis on $|\quad|$ then implies that $[\mathscr{P}|V] \longrightarrow \boldsymbol{C}(\mathscr{P}_x)$ is also a direct limit-diagram. Thus (2. 4) shows that $[\mathscr{P}|V]/[\mathscr{P}|V]^* \longrightarrow$

$\boldsymbol{C}(\mathscr{P}_x)/\boldsymbol{C}^*(\mathscr{P}_x)$ is a direct limit-diagram too. Hence (2. 4) and (2. 6) give rise to the commutative diagram

$$\begin{array}{ccccc}
[\mathscr{P}|V] & \longrightarrow & [\mathscr{P}|V]/[\mathscr{P}|V]^* & \longrightarrow & |\mathscr{U}_{\mathscr{P}}(V)| \\
\big| & \searrow \quad \boldsymbol{C}(\mathscr{P}_x) & \longrightarrow & \boldsymbol{C}(\mathscr{P}_x)/\boldsymbol{C}^*(\mathscr{P}_x) & \big| \\
|L| & & \longrightarrow & & |M|
\end{array}$$

and our last remark shows the existence of a map $\boldsymbol{C}(\mathscr{P}_x)/\boldsymbol{C}^*(\mathscr{P}_x) \longrightarrow |M|$ rendering the right upper corner commutative. This means that the two possible compositions from $[\mathscr{P}|V]$ to $|M|$ "passing through $\boldsymbol{C}(\mathscr{P}_x)$" coincide. $\boldsymbol{C}(\mathscr{P}_x)$ being a direct limit implies therefore that the bottom diagram commutes, i.e. that $\boldsymbol{C}(\mathscr{P}_x) \longrightarrow |L| \longrightarrow |M|$ factors through $\boldsymbol{C}(\mathscr{P}_x)/\boldsymbol{C}^*(\mathscr{P}_x)$. Hence the universality property of $u_{\mathscr{P}_x}: l(\mathscr{P}_x) \longrightarrow \mathscr{U}_{\mathscr{P}_x}$ furnishes a unique morphism $\mathscr{U}_{\mathscr{P}_x} \longrightarrow M$ such that

$$\begin{array}{ccc}
l(\mathscr{P}_x) & \longrightarrow & \mathscr{U}\mathscr{P}_x \\
\downarrow & & \downarrow \\
L & \longrightarrow & M
\end{array}$$

commutes. The remaining required commutativity relations now follow trivially.

The assumptions of (2. 5) are tailored so as to fit the case of $\mathscr{P}$ being a coherent sheaf and l being the map that assigns to $\mathscr{P}$ the sheaf of endomorphisms of $\mathscr{P}$. The meaning of (2. 5) is, of course, that the stalks of the universal solution for the presheaf can be computed stalkwise.

There is another result that is of interest for sheaves. For the purpose of formulating it we recall that a presheaf $\mathscr{P}'$ over X is a recollatement of the presheaf $\mathscr{P}$ if there exists an open covering $\{V_i\}_{i\in I}$ of X and isomorphisms $\alpha_i: \mathscr{P}'|V_i \longrightarrow \mathscr{P}|V_i$. These isomorphisms canonically induce natural bijections $\beta_{iV}: [\mathscr{P}'|V] \longrightarrow [\mathscr{P}|V]$ (for $V \subset V_i$) by the rule $\beta_{iV}(\mu') = (\alpha_i|V)\cdot\mu'\cdot(\alpha_i|V)^{-1}$, and these bijections are homomorphisms of the canonical monoid structures. Altogether we obtain the isomorphisms of presheaves β_i: $[(\mathscr{P}'|V_i)|\quad] \longrightarrow [(\mathscr{P}|V_i)|\quad]$.

(2. 7) PROPOSITION. *Basic assumptions as in* (2. 2). *Suppose, in addition, that* $\mathscr{P}'$ *is a recollatement of* $\mathscr{P}$ *and that* $l(\mathscr{P}')$ *is a recollatement of* $l(\mathscr{P})$ *with local isomorphisms* λ_i: $l(\mathscr{P}')|V_i \longrightarrow l(\mathscr{P})|V_i$ *whose images under* $\boldsymbol{P}(V_i, |\quad|)$ *are the induced isomorphisms* β_i: $[(\mathscr{P}'|V_i)|\quad] \longrightarrow [(\mathscr{P}|V_i)|\quad]$. *Suppose furthermore that*

both $l(\mathscr{P}')$ *and* $\mathscr{U}_{\mathscr{P}}$ *are sheaves. Then the universal problem for the presheaf* $\mathscr{P}'$ *admits a solution* $(u_{\mathscr{P}'}, \mathscr{U}_{\mathscr{P}'})$ *and there is an isomorphism* $\mathscr{U}_{\mathscr{P}'} \xrightarrow{\simeq} \mathscr{U}_{\mathscr{P}}$ *such that for every* $i \in I$ *the diagram*

$$\begin{array}{ccc} l(\mathscr{P}')|V_i & \xrightarrow{\ u_{\mathscr{P}'}|V_i\ } & \mathscr{U}_{\mathscr{P}'}|V_i \\ \lambda_i \downarrow & & \downarrow \simeq \\ l(\mathscr{P})|V_i & \xrightarrow{\ u_{\mathscr{P}}|V_i\ } & \mathscr{U}_{\mathscr{P}}|V_i \end{array}$$

commutes.

Proof. Due to our assumptions $u: l(\mathscr{P}')|V_i \longrightarrow \mathscr{L}$ factors through $[(\mathscr{P}'|V_i)|\ \ ] \longrightarrow [(\mathscr{P}'|V_i)|\ \ ]/[(\mathscr{P}'|V_i)|\ \ ]^*$ if and only if $u \cdot \lambda_i^{-1}$ factors through $[(\mathscr{P}|V_i)|\ \ ] \longrightarrow [(\mathscr{P}|V_i)|\ \ ]/[(\mathscr{P}|V_i)|\ \ ]^*$. Since (2. 2) implies that $(u_{\mathscr{P}|V_i}, \mathscr{U}_{\mathscr{P}|V_i}) = (u_{\mathscr{P}}|V_i, \mathscr{U}_{\mathscr{P}}|V_i)$ we conclude that $((u_{\mathscr{P}}|V_i) \cdot \lambda_i, \mathscr{U}_{\mathscr{P}}|V_i)$ solves the universal problem for all morphisms u in $\boldsymbol{P}(V_i, \boldsymbol{L})$ with domain $l(\mathscr{P}')|V_i$ such that $\boldsymbol{P}(V_i, |\ \ |)u$ factors through $[(\mathscr{P}'|V_i)|\ \ ] \longrightarrow [(\mathscr{P}'|V_i)|\ \ ]/[\mathscr{P}'|V_i)|\ \ ]^*$. For $V \subset V_i \cap V_j$ we have the commutative diagram

$$\begin{array}{ccccc} [\mathscr{P}'|V] & \xrightarrow{\ \beta_{jV}\ } & [\mathscr{P}|V] & \longrightarrow & [\mathscr{P}|V]/[\mathscr{P}|V]^* \\ \| & & \downarrow \beta_{ijV} & & \| \\ [\mathscr{P}'|V] & \xrightarrow{\ \beta_{iV}\ } & [\mathscr{P}|V] & \longrightarrow & [\mathscr{P}|V]/[\mathscr{P}|V]^* \end{array} \tag{2. 8}$$

where

$$\beta_{ijV}(\mu) = ((\alpha_i|V) \cdot (\alpha_j|V)^{-1}) \cdot \mu \cdot ((\alpha_i|V) \cdot (\alpha_j|V)^{-1})^{-1} \qquad \text{for all } \mu \in [\mathscr{P}|V].$$

Clearly the β_{ijV} are functorial in V. Hence the commutativity of (2. 8) implies the commutativity of

$$\begin{array}{ccc} l(\mathscr{P}')|V_i \cap V_j & \xrightarrow{\ (u_{\mathscr{P}}|V_i) \cdot \lambda_i|V_i \cap V_j\ } & \mathscr{U}_{\mathscr{P}}|V_i \cap V_j \\ \| & & \| \\ l(\mathscr{P}')|V_i \cap V_j & \xrightarrow{\ (u_{\mathscr{P}}|V_j) \cdot \lambda_j|V_i \cap V_j\ } & \mathscr{U}_{\mathscr{P}}|V_i \cap V_j\ . \end{array} \tag{2. 9}$$

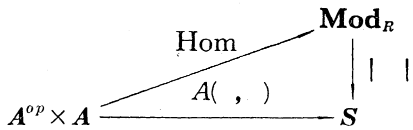
Since both $l(\mathscr{P}')$ and $\mathscr{U}_{\mathscr{P}}$ were assumed to be sheaves (2. 9) shows that the local morphisms $(u_{\mathscr{P}}|V_i) \cdot \lambda_i: l(\mathscr{P}')|V_i \longrightarrow \mathscr{U}_{\mathscr{P}}|V_i$ match up to a morphism $u_{\mathscr{P}'}: l(\mathscr{P}') \longrightarrow \mathscr{U}_{\mathscr{P}}$ which evidently solves the universal problem for $\mathscr{P}'$ and renders the required diagrams commutative.

(2. 7) states that "compatible" recollatements on the level of $\mathscr{P}$ and $l(\mathscr{P})$ have the "same" universal solution. Applying this to the situation indicated at the end of the proof of (2. 5) we conclude that, for example, the universal solution for an arbitrary vector bundle coincides with the one for a trivial vector bundle.

(2. 10) *Remark.* The definitions and remarks preceding (2. 2) as well as (2. 2) and (2. 3) when appropriately modified carry over to arbitrary functor categories (replacing $\boldsymbol{P}(X, \boldsymbol{C})$ and $\boldsymbol{P}(X, \boldsymbol{L})$). The same holds true for (2. 5) when "passage to the stalk at x" is replaced by "passage to a certain colimit".

§3. Traces for Endomorphisms in $\boldsymbol{R}$-additive categories.

Let R be an associative ring. As usual we mean by an R-additive category $\boldsymbol{A}$ a category with finite direct sums together with a covariant "structural" Hom-functor Hom: $\boldsymbol{A}^{op} \times \boldsymbol{A} \longrightarrow \mathbf{Mod}_R$ rendering



commutative where $|\ \ |: \mathbf{Mod}_R \longrightarrow \boldsymbol{S}$ is the standard forgetful functor from the category $\mathbf{Mod}_R$ of left R-modules to the category $\boldsymbol{S}$ of sets.

In order to set up the universal problems dealt with in §1 we choose for $\boldsymbol{C}$ the R-additive category $\boldsymbol{A}$, for $\boldsymbol{L}$ the category $\mathbf{Mod}_R$, for l the map given by

$$l(A) = \operatorname{End} A (= \operatorname{Hom}(A, A)),$$

and for $|\ \ |$ the above forgetful functor $|\ \ |: \mathbf{Mod}_R \longrightarrow \boldsymbol{S}$. For notational purposes we denote by Aut A the group $\boldsymbol{A}^*(A)$ when viewed as a subset of End A.

(3. 1) PROPOSITION. *Let $\boldsymbol{A}$ be an R-additive category. Then both the universal problem of* §1 *and its commutative version admit a solution for every object of $\boldsymbol{A}$.*

Proof. (1. 5).

Given the object A of $\boldsymbol{A}$ we shall denote the solution (u_A, U_A) of the universal problem of §1 (resp. the solution (u_A^c, U_A^c) of its commutative version) by $(\boldsymbol{A}\text{-tr}_A, \boldsymbol{A}\text{-Tr}_A)$ (resp. $(\boldsymbol{A}\text{-tr}_A^c, \boldsymbol{A}\text{-Tr}_A^c)$) and call it the trace (resp. the commutative trace) on A. Whenever the reference to the R-additive category $\boldsymbol{A}$ is clear the prefix $\boldsymbol{A}$- shall be dropped.

(3. 2) COROLLARY. *For every object A of $\mathbf{A}$ the morphisms tr_A, tr_A^c, and w_A: $Tr_A \longrightarrow Tr_A^c$ are surjections.*

Proof. (1. 7).

(3. 3) PROPOSITION. *Suppose that μ is contained in the submodule of End A that is generated by Aut A. Then for every element ν of End A,*

$$tr_A(\mu\nu) = tr_A(\nu\mu).$$

Proof. If $\mu = \sum r_i\alpha_i$ where the α_i are suitable elements of Aut A then

$$\mathrm{tr}_A(\nu\mu) = \sum r_i \mathrm{tr}(\nu\alpha_i) = \sum r_i \mathrm{tr}_A(\alpha_i^{-1}\alpha_i\nu\alpha_i) = \sum r_i \mathrm{tr}_A(\alpha_i\nu) = \mathrm{tr}_A(\mu\nu).$$

(3. 4) COROLLARY. *Let A be a finite direct sum of at least two copies of some object A'. Then $w_A: Tr_A \longrightarrow Tr_A^c$ is an isomorphism (i.e. the trace on A equals the commutative trace on A).*

Proof. A simple matrix computation reveals that End A is generated by Aut A. Hence (3. 3) implies that the trace is commutative. An obvious universality argument then shows that w_A is an isomorphism.

(3. 5) PROPOSITION. *Let $A = A_1 \oplus \cdots \oplus A_k$ be a finite direct sum with i_κ: $A_\kappa \longrightarrow A$ the canonical injections and $p_\kappa: A \longrightarrow A_\kappa$ the canonical projections. Then for every element μ of End A,*

$$tr_A(\mu) = \sum\{tr_A(i_\kappa p_\kappa \mu i_\kappa p_\kappa): \kappa = 1, \cdots, k\}$$
$$tr_A^c(\mu) = \sum\{tr_A^c(i_\kappa p_\kappa \mu i_\kappa p_\kappa): \kappa = 1, \cdots, k\}.$$

Proof. Since tr_A is a homomorphism from End A to Tr_A we obtain

$$\mathrm{tr}_A(\mu) = \sum\{\mathrm{tr}_A(i_\lambda p_\lambda \mu i_\kappa p_\kappa): \kappa, \lambda = 1, \cdots, k\}.$$

Let $\nu_{\kappa\lambda} \in \mathrm{Hom}(A_\kappa, A_\lambda)$, $\kappa < \lambda$, and denote by ν the (unique) endomorphism of A satisfying

$$p_\kappa \nu i_\kappa = \mathrm{id}(A_\kappa), \quad p_\lambda \nu i_\kappa = \nu_{\kappa\lambda}, \quad \text{and} \quad p_{\lambda'} \nu i_{\kappa'} = 0 \quad \text{otherwise,}$$

and by α the (unique) automorphism of A satisfying

$$p_{\kappa'} \alpha i_{\kappa'} = \mathrm{id}(A_{\kappa'}) \text{ for all } \kappa', \; p_\lambda \alpha i_\kappa = \nu_{\kappa\lambda}, \text{ and } p_{\lambda'} \alpha i_{\kappa'} = 0 \text{ otherwise.}$$

One checks easily that

$$p_\kappa \alpha^{-1} \nu \alpha i_\kappa = \mathrm{id}(A_\kappa), \quad p_{\lambda'} \alpha^{-1} \nu \alpha i_{\kappa'} = 0 \text{ otherwise.}$$

Hence we have

$$\operatorname{tr}_A(i_\kappa p_\kappa) = \operatorname{tr}_A(\alpha^{-1}\nu\alpha) = \operatorname{tr}_A(\nu) = \operatorname{tr}_A(i_\kappa p_\kappa) + \operatorname{tr}_A(i_\lambda \nu_{\kappa\lambda} p_\kappa)$$

and therefore

$$\operatorname{tr}_A(i_\lambda \nu_{\kappa\lambda} p_\kappa) = 0 \qquad \text{for } \kappa < \lambda .$$

Since we could equally well assume $\kappa > \lambda$, the first formula of (3. 5) follows from the first formula appearing in this proof. A similar argument establishes the second formula.

(3. 5) generalizes the well-known fact (or definition if one so wishes) that the trace of a square matrix is the sum of the diagonal entries. More specifically we refer to (3. 8).

(3. 6) PROPOSITION. *For every object A we have*

$$Tr_A = End\ A/\{\mu - \alpha\mu\alpha^{-1}\colon \mu \in End\ A\ \&\ \alpha \in Aut\ A\}^{1)}$$

$$Tr_A^c = End\ A/R\{\mu_1\mu_2 - \mu_2\mu_1\colon \mu_1, \mu_2 \in End\ A\}$$

with tr_A *and* tr_A^c *being the obvious quotient homomorphisms. In particular,* $Tr_A = H_0(Aut\ A,\ End\ A\ )$.

Proof. Every R-homomorphism End $A \longrightarrow L$ that factors through $A(A)/A^*(A)$ has a kernel containing every element of the form $\mu - \alpha\mu\alpha^{-1}$, $\mu \in$ End A & $\alpha \in$ Aut A, and hence the submodule generated by these elements. Since tr_A: End $A \longrightarrow \operatorname{Tr}_A$ is an epimorphism due to (3. 2) our claim concerning Tr_A follows. A similar argument applies to Tr_A^c. The fact that Tr_A is the indicated homology group is trivial since H_0(Aut A, End A) = End $A/I\cdot$End A where I is the augmentation ideal of the group algebra R(Aut A) with respect to the unit augmentation (see [8], p. 183).

(3. 7) COROLLARY. *Suppose that the objects* $A_1, \cdots, A_k$ *satisfy* $Hom(A_\kappa, A_\lambda) = 0$ *for all* $\kappa \neq \lambda$. *Then there is a commutative diagram*

$$\begin{array}{ccc} \oplus\ \mathrm{End}\ A_\kappa & \xrightarrow{\oplus tr_{A_\kappa}} & \oplus\ Tr_{A_\kappa} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{End}\oplus A_\kappa & \xrightarrow{tr_{\oplus A_\kappa}} & Tr_{\oplus A_\kappa} \end{array}$$

the isomorphisms in it being canonical. Similarly for the commutative trace.

1) By $R\{\cdots\}$ is meant the R-submodule generated by $\cdots$

Proof. Straight forward using (3. 6).

(3. 8) THEOREM. *Suppose that R possesses a unit. Let A be an object for which End A is a monogenic R-module and thus isomorphic to $R/\boldsymbol{a}$ for some left ideal $\boldsymbol{a}$ of R. Then for every natural number k there exists a commutative diagram*

$$\begin{array}{ccc} End \overset{k}{\oplus} A & \xrightarrow{tr^c_{\overset{k}{\oplus}A}} & Tr^c_{\overset{k}{\oplus}A} \\ \simeq\downarrow & & \downarrow\simeq \\ (R/\boldsymbol{a})^{(k,k)} & \xrightarrow{sp} & R/\boldsymbol{a} + R\{r_1r_2 - r_2r_1 : r_1, r_2 \in R\} \end{array}$$

where $(R/\boldsymbol{a})^{(k,k)}$ stands for the algebra of k-by-k matrices over $R/\boldsymbol{a}$ and sp stands for the canonical image of the sum of the diagonal entries.

Proof. For $k=1$ this is an immediate consequence of (3. 6). For $k>1$ we obtain due to (3. 5), putting $B = \overset{k}{\oplus} A$,

$$\mathrm{tr}^c_B(\mu) = \sum\{\mathrm{tr}^c_B(\mu_\kappa) : \kappa = 1, \cdots, k\}$$

where μ_κ is the (unique) endomorphism of B given by the relations

$$p_\kappa \mu_\kappa i_\kappa = p_\kappa \mu i_\kappa, \quad p_{\kappa'} \mu_\kappa i_{\lambda'} = 0 \qquad \text{otherwise.}$$

Our assumption implies that id_A generates End A as an R-module. Hence an easy computation shows that for some $r_\kappa \in R$, $\mu_\kappa = r_\kappa \cdot i_\kappa p_\kappa$ holds. The equivalence class of r_κ in $R/\boldsymbol{a}$ is uniquely determined by the relation $\mathrm{cl}(r_\kappa) = p_\kappa \mu i_\kappa(\mathrm{cl}(1))$. Using permutation-of-summands automorphisms of B one checks easily that all $i_\kappa p_\kappa$ define the same equivalence class in $\boldsymbol{A}(A)/\boldsymbol{A}^*(A)$ whence we finally obtain

$$\mathrm{tr}^c_B(\mu) = \sum\{r_\kappa : \kappa = 1, \cdots, k\} \cdot \mathrm{tr}^c_B(i_1 p_1)\,.$$

This together with (3. 2) shows that Tr^c_B is a monogenic R-module and therefore isomorphic to $R/\boldsymbol{a}'$ where $\boldsymbol{a}'$ is some left ideal. Clearly we have the inclusion

$$\boldsymbol{a} + R\{r_1r_2 - r_2r_1 : r_1, r_2 \in R\} \subset \boldsymbol{a}'. \tag{3. 9}$$

Since the canonical bijection $\mathrm{End} \overset{k}{\oplus} A \longrightarrow (R/\boldsymbol{a})^{(k,k)}$ is an isomorphism of the canonical monoid structures and since the map sp is "commutative" the universality of tr^c_B furnishes a unique homomorphism rendering the diagram in (3. 8) commutative. It remains to be shown that this homomorphism is in fact an isomorphism. Since $i_1 p_1$ corresponds under the canonical bijection to the matrix with cl(1) in the left upper corner and all other entries zero we conclude that the image of $\mathrm{tr}_B^c(i_1 p_1)$ equals cl(1). This in conjunction with (3. 9) shows that we have indeed equality in (3. 9), that is that the previous homomorphism is an isomorphism.

The bottom arrow in (3. 8) is, of course, the usual definition of the trace for an endomorphism of a finite free module (see [13], p. 113 for the non-commutative case).

At this point we should like to offer some examples without giving proofs. Since the trace depends on End A and the properties of the category $\mathbf{Mod}_R$ rather than on the properties of the category $\boldsymbol{A}$ we shall choose for $\boldsymbol{A}$ the category $\mathbf{Mod}_R$ itself, R now being assumed a commutative, unital ring, and for Hom the functor Hom_R.

(3. 10) Examples. a) For any infinitely generated free R-module M, $\mathrm{Tr}_M=0$.

b) In case $R=\boldsymbol{Z}$ we have

$$\mathrm{Tr}_{\oplus\{\boldsymbol{Z}_p:\ p \text{ all primes}\}} = \oplus\{\boldsymbol{Z}_p:\ p \text{ all primes}\}.$$

c) Let M be a reduced p-primary ($p>2$) $\boldsymbol{Z}$-module that is a direct sum of countable $\boldsymbol{Z}$-modules. Then $w_M\colon \mathrm{Tr}_M \longrightarrow \mathrm{Tr}_M^c$ is an isomorphism due to (3. 3) as in this case every endomorphism of M is the sum of two automorphisms ([9]).

d) [2]Let M be a finitely generated $\boldsymbol{Z}$-module. Denote for every prime p the p-primary part of M by M_p and the free part of M by M_∞. Using a few elementary matrix operations and the universality of the trace we see that there is a communtative diagram

$$\begin{array}{ccc} \mathrm{End}_{\boldsymbol{Z}}(M_\infty \oplus \oplus M_p & \rightleftarrows & \mathrm{End}_{\boldsymbol{Z}} M_\infty \oplus \mathrm{End}_{\boldsymbol{Z}}(\oplus M_p) \\ \mathrm{tr}_M \big\downarrow & & \big\downarrow \mathrm{tr}_{M_\infty} \oplus \mathrm{tr}_{\oplus M_p} \\ \mathrm{Tr}_M & \underset{\eta}{\overset{\chi}{\rightleftarrows}} & \mathrm{Tr}_{M_\infty} \oplus \mathrm{Tr}_{\oplus M_p} \end{array}$$

with χ and η isomorphisms. By (3. 7) we have

$$\mathrm{Tr}_{\oplus M_p} = \oplus\, \mathrm{Tr}_{M_p}.$$

In order to determine Tr_{M_p}, let N be a finitely geerated p-primary $\boldsymbol{Z}$-module. The we have the a canonical decomposition

[2] This example is due to Mr. E. L. Lady of UCSD.

$$N = \oplus\{n_k \boldsymbol{Z}_{p^{q_k}}:\ k = 1, \cdots, n\}.$$

We may assume $q_1 < \cdots < q_n$. Denote $n_k\boldsymbol{Z}_{p^{q_k}}$ by N_k. For $\gamma \in \mathrm{End}_Z N$, let $c_k = \mathrm{tr}_{N_k}(p_k \cdot \gamma \cdot i_k) \in \boldsymbol{Z}_{p^{q_k}}$, computed in accordance with (3. 8). Then a matrix computation leads to

$$\mathrm{Tr}_N = \boldsymbol{Z}_{p^{q_1}} \oplus \boldsymbol{Z}_{p^{r_2}} \oplus \cdots \oplus \boldsymbol{Z}_{p^{r_n}},\quad r_k = q_k - q_{k-1}$$

and
$$\mathrm{tr}_N(\gamma) = (\sum^n c_k,\ \sum_2^n c_k,\ \cdots,\ c_n),$$

were, by abuse of notation, we have written c_k for the image of c_k under the canonical homorphism $\boldsymbol{Z}_{p^{q_k}} \to \boldsymbol{Z}_{p^{r_j}}$, $j \leq k$, as appropriate.

An analogous discription can be found for finitely generated modules over a principal ideal domain or a Dedekind domain R.

§4. Traces for Endomorphisms of R-modules.

In this section we shall deal with a special case of §3: We assume that R is a commutative unital ring, $\boldsymbol{A}$ is the category $\mathbf{Mod}_R$ of unital R-modules, and Hom is the usual Hom-functor Hom_R. Some of the constructions and results of this section carry over to more general situations (e.g. R-additive categories with multiplication) as can be seen easily.

Let M be an R-module and denote by M^* its dual. Then there is a canonical homomorphism $\mathrm{End}_R M \longrightarrow \mathrm{End}_R M^*$ that sends each endomorphism μ into its transpose ${}^t\mu$. One checks quickly that the composition $\mathrm{End}_R M \longrightarrow \mathrm{End}_R M^* \longrightarrow \mathrm{Tr}_{M^*}$ factors through $\mathrm{End}_R M/\mathrm{Aut}_R M$. Hence there exists uniquely a homomorphism d_M: $\mathrm{Tr}_M \longrightarrow \mathrm{Tr}_{M^*}$ rendering

(4. 1)
$$\begin{array}{ccc} \mathrm{End}_R M & \xrightarrow{\ {}^t(\ )\ } & \mathrm{End}_R M^* \\ \mathrm{tr}_M \downarrow & & \downarrow \mathrm{tr}_{M^*} \\ \mathrm{Tr}_M & \xrightarrow{\ d_M\ } & \mathrm{Tr}_{M^*} \end{array}$$

commutative. Similarly for the commutative trace.

(4. 2) Proposition. *Let M be a reflexive R-module. Then the homomorphism d_M is an isomorphism.*

Proof. Stick (4. 1) together with the analogous diagram for M^* and M^{**} and identify M^{**} canonically with M. Then the universality property of the trace implies that $d_{M^*}d_M$ is the identity. Since M^* is reflexive $d_{M^{**}}d_{M^*}$ is also the identity. Hence d_{M^*}, and therefore d_M, is an isomorphism.

(4. 3) COROLLARY. *Let M be a finitely generated projective R-module. Then d_M is an isomorphism.*

Proof. Any finitely generated projective R-module is reflexive (see [5], p. 72).

Let M and N be R-modules. Clearly there is a commutative diagram of R-homomorphisms

$$\begin{array}{ccc} \mathrm{End}_R M \otimes_R \mathrm{End}_R N & \xrightarrow{\otimes_R} & \mathrm{End}_R(M \otimes_R N) \\ {\scriptstyle \mathrm{tr}_M \otimes_R \mathrm{tr}_N} \downarrow & & \downarrow {\scriptstyle \mathrm{tr}_{M \otimes_R N}} \\ \mathrm{Tr}_M \otimes_R \mathrm{Tr}_N & \xrightarrow{t_{M,N}} & \mathrm{Tr}_{M \otimes_R N} \end{array}$$

where $t_{M,N}$ has all the expected functorial properties.

(4. 4) PROPOSITION. *Let M be a finitely generated projective R-module. Then $t_{M,N}$ is an epimorphism.*

Proof. If M is a finitely generated projective then $\mathrm{End}_R M \otimes_R \mathrm{End}_R N \longrightarrow \mathrm{End}_R(M \otimes_R N)$ is an isomorphism (see [5], p. 113). Hence (3. 2) implies our claim.

Additional results concerning the trace on tensor products will come up in connection with localization. Before, however, we shall deal briefly with change of rings.

Let $\rho: R \longrightarrow S$ be a unital homomorphism of commutative rings. For every R-module M we can form the S-module $\rho^* M = S \otimes_R M$. One checks easily that there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{End}_R M & \xrightarrow{\eta} & \rho^* \mathrm{End}_R M & \xrightarrow{\omega} & \mathrm{End}_S \rho^* M \\ & & & & \downarrow {\scriptstyle \mathrm{tr}_{\rho^* M}} \\ {\scriptstyle \mathrm{tr}_M} \downarrow & & & & \mathrm{Tr}_{\rho^* M} \\ & & & & \downarrow {\scriptstyle \xi} \\ \mathrm{Tr}_M & & \longrightarrow & & \rho_* \mathrm{Tr}_{\rho^* M} \end{array}$$

where η is defined by $\eta(\mu) = 1 \otimes \mu$, ω is the canonical homomorphism, ξ is the canonical $\mathbf{Z}$-homomorphism, and the bottom homomorphism is defined by the universality of the trace. This bottom homomorphism gives rise to the composite S-homomorphism (see [5], p. 121)

$$e(\rho): \rho^* \mathrm{Tr}_M \xrightarrow{\rho^* e(\rho)} \rho^* \rho_* \mathrm{Tr}_{\rho^* M} \longrightarrow \mathrm{Tr}_{\rho^* M}$$

describing the influence of change of rings on the trace. Similar definitions hold for the commutative trace, in which case the corresponding composite S-homomorphism shall be denoted by $e^c(\rho)$.

(4. 5) PROPOSITION. *Suppose that either S is a finitely generated projective R-module or that M is a finitely generated projective R-module. Then $e(\rho)$ as well as $e^c(\rho)$ is an epimorphism.*

Proof. One checks easily that the following diagram commutes

$$\begin{array}{ccc} \rho^*\mathrm{End}_R M & \xrightarrow{\quad\omega\quad} & \mathrm{End}_S \rho^* M \\ {\scriptstyle \rho^*\mathrm{tr}_M}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_{\rho^*M}} \\ \rho^*\mathrm{Tr}_M & \xrightarrow{\quad e(\rho)\quad} & \mathrm{Tr}_{\rho^*M} \end{array}$$

Since ω is an isomorphism under the stated assumptions (see [5], p. 124) and since tr_{ρ^*M} is an epimorphism due to (3. 2) our claim concerning $e(\rho)$ follows. Similarly for $e^c(\rho)$.

It is of interest to know how to compute $e(\sigma\rho)$ where $\rho\colon R \longrightarrow S$ and $\sigma\colon S \longrightarrow T$ are unital homomorphisms. Denoting the canonical isomorphism $(\sigma\rho)^*N \longrightarrow \sigma^*\rho^*N$ by γ_N we obtain

(4. 6) PROPOSITION. *Let $\rho\colon R \longrightarrow S$ and $\sigma\colon S \longrightarrow T$ be unital homomorphisms of commutative rings. Then*

$$e(\sigma\rho) = \gamma'_M \cdot e(\sigma) \cdot \sigma^* e(\rho) \cdot \gamma_{Tr_M} \quad \textit{and}$$
$$e^c(\sigma\rho) = \gamma^{c\,\prime}_M \cdot e^c(\sigma) \cdot \sigma^* e^c(\rho) \cdot \gamma_{Tr^c_M},$$

γ'_M being the canonical isomorphism $Tr_{\sigma^*\rho^*M} \longrightarrow Tr_{(\sigma\rho)^*M}$ (and similarly $\gamma^{c\,\prime}_M$).

Proof. We have the following commutative diagram

$$\begin{array}{ccccccc} (\sigma\rho)^*\mathrm{End}_R M & \xrightarrow{\gamma_{\mathrm{End}_R M}} & \sigma^*\rho^*\mathrm{End}_R M & \xrightarrow{\sigma^*\omega} & \sigma^*\mathrm{End}_S\rho^*M & \longrightarrow \\ {\scriptstyle (\sigma\rho)^*\mathrm{tr}_M}\downarrow & & {\scriptstyle \sigma^*\rho^*\mathrm{tr}_M}\downarrow & & {\scriptstyle \sigma^*\mathrm{tr}_{\rho^*M}}\downarrow & \\ (\sigma\rho)^*\mathrm{Tr}_M & \xrightarrow{\gamma_{\mathrm{Tr}_M}} & \sigma^*\rho^*\mathrm{Tr}_M & \xrightarrow{\sigma^* e(\rho)} & \sigma^*\mathrm{Tr}_{\rho^*M} & \longrightarrow \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\omega} & \mathrm{End}_T\sigma^*\rho^*M & \xrightarrow{\mathrm{Hom}_T(\gamma_M, \gamma_M^{-1})} & \mathrm{End}_T(\sigma\rho)^*M \\ & {\scriptstyle \mathrm{tr}_{\sigma^*\rho^*M}}\downarrow & & {\scriptstyle \mathrm{tr}_{(\sigma\rho)^*M}}\downarrow \\ \xrightarrow{e(\sigma)} & \mathrm{Tr}_{\sigma^*\rho^*M} & \xrightarrow{\quad\gamma'_M\quad} & \mathrm{Tr}_{(\sigma\rho)^*M} \end{array}$$

in which, due to functoriality, the top row equals the canonical homomorphism $(\sigma\rho)^*\mathrm{End}_R M \longrightarrow \mathrm{End}_T(\sigma\rho)^* M$.

A few words concerning restriction of scalars. Let again $\rho: R \longrightarrow S$ be a unital homomorphism of rings. For every S-module N we can form the R-module $\rho_* N$ obtained from N by restriction of scalars to R. Again there is a commutative diagram

$$\begin{array}{ccc} \rho_*\mathrm{End}_S N & \xrightarrow{\ \xi\ } & \mathrm{End}_R \rho_* N \\ {\scriptstyle \rho_*\mathrm{tr}_N}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_{\rho^* N}} \\ \rho_*\mathrm{Tr}_N & \xrightarrow{\ r(\rho)\ } & \mathrm{Tr}_{\rho_* N} \end{array}$$

where ξ is the canonical $\boldsymbol{Z}$-homomorphism and the bottom homomorphism $r(\rho)$ is defined by the universality of the trace. Similarly for the com mutative trace, in which case the bottom homomorphism shall be denoted by $r^c(\rho)$. Here we have

(4. 7) PROPOSITION. *Suppose that ρ: $R \longrightarrow S$ is an epimorphism. Then both $r(\rho)$ and $r^c(\rho)$ are isomorphisms.*

(4. 8) PROPOSITION. *Let $\rho: R \longrightarrow S$ and $\sigma: S \longrightarrow T$ be unital homomorphisms of commutative rings. Then*

$$r(\sigma\rho) = \bar{\gamma}'_M \cdot r(\rho) \cdot \rho_* r(\sigma) \cdot \bar{\gamma}_{Tr_M} \quad \text{and}$$
$$r^c(\sigma\rho) = \bar{\gamma}^{c\,\prime}_M \cdot r^c(\rho) \cdot \rho_* r^c(\sigma) \cdot \bar{\gamma}_{Tr^c_M}$$

with the adorned $\bar{\gamma}$'s defined in a fashion analogous to (4. 6).

Next we shall deal with the behavior of trace under localization. Let S be a multiplicatively closed subset of R. Then the universality property of the trace as well as of the localization at S furnishes a unique $S^{-1}R$-homomorphism $l_{S,M}$: $S^{-1}\mathrm{Tr}_M \longrightarrow \mathrm{Tr}_{S^{-1}M}$ rendering

$$\begin{array}{ccccc} & \swarrow & \mathrm{End}_R M & \searrow & \\ & & \downarrow{\scriptstyle \mathrm{tr}_M} & & \\ S^{-1}\mathrm{End}_R M & & \longrightarrow & & \mathrm{End}_{S^{-1}R} S^{-1} M \\ {\scriptstyle S^{-1}\mathrm{tr}_M}\downarrow & \swarrow & \mathrm{Tr}_M & \searrow & \downarrow{\scriptstyle \mathrm{tr}_{S^{-1}M}} \\ S^{-1}\mathrm{Tr}_M & & \xrightarrow{\ l_{S,M}\ } & & \mathrm{Tr}_{S^{-1}M} \end{array}$$

commutative in which all undesignated homomorphisms are canonical. A similar definition holds for the commutative trace, in which case the bottom homomorphism shall be denoted by $l^c_{S,M}$.

(4. 9) THEOREM. *Suppose that the canonical homomorphism* φ: $S^{-1}\mathrm{End}_R M \longrightarrow \mathrm{End}_{S^{-1}R} S^{-1}M$ *is an isomorphism. Then* $l^c_{S,M}$ *is also an isomorphism. In particular this is true if* M *is finitely presentable.*

Proof. From (3. 2) and the above commutative diagram we take that $l^c_{S,M}$ is an epimorphism whenever φ is an epimorphism. Hence it remains to be shown that $l^c_{S,M}$ is a monomorphism. Let m be in the kernel of $l^c_{S,M}$ and choose $\frac{\mu}{s} \in S^{-1}\mathrm{End}_R M$ such that $m = (S^{-1}\mathrm{tr}^c_M)\left(\frac{\mu}{s}\right)$. Then we have $\mathrm{tr}^c_{S^{-1}M}\varphi\left(\frac{\mu}{s}\right) = 0$ whence there are endomorphisms $\bar{\mu}'_i, \bar{\mu}''_i$ of $S^{-1}M$ such that

$$\varphi\left(\frac{\mu}{s}\right) = \sum\{\bar{\mu}'_i\bar{\mu}''_i - \bar{\mu}''_i\bar{\mu}'_i\}.$$

By assumption these endomorphisms can be written in the form

$$\bar{\mu}'_i = \varphi\left(\frac{\mu'_i}{s'_i}\right) \qquad \text{and} \qquad \bar{\mu}''_i = \varphi\left(\frac{\mu''_i}{s''_i}\right)$$

with μ'_i, μ''_i suitable elements of $\mathrm{End}_R M$ and s'_i, s''_i suitable elements of S. Since φ is an isomorphism we obtain

$$\frac{\mu}{s} = \sum\left\{\frac{\mu'_i}{s'_i}\cdot\frac{\mu''_i}{s''_i} - \frac{\mu''_i}{s''_i}\cdot\frac{\mu'_i}{s'_i}\right\}$$

and thus

$$m = (S^{-1}\mathrm{tr}^c_M)\left(\frac{\mu}{s}\right) = \sum\frac{\mathrm{tr}^c_M(\mu'_i\mu''_i - \mu''_i\mu'_i)}{s'_i s''_i} = 0,$$

which proves the assertion in the general case. However, if M is finitely presentable then φ is known to be an isomorphism (see [6], p. 98).

(4. 10) COROLLARY. *Suppose that the* R*-module* M *is finitely presentable. If the prime ideal* $\mathfrak{p}$ *of* R *is not in the support of* M *then* $(Tr^c_M)_\mathfrak{p} = 0$.

Proof. Since $M_\mathfrak{p} = 0$ and since $\mathrm{tr}^c_{M_\mathfrak{p}}$: $\mathrm{End}_{R_\mathfrak{p}} M_\mathfrak{p} \longrightarrow \mathrm{Tr}^c_{M_\mathfrak{p}}$ is an epimorphism we conclude that $\mathrm{Tr}^c_{M_\mathfrak{p}}$ is the null module. Hence (4. 9) finishes the proof.

(4. 11) PROPOSITION. *Let* M *be a finitely generated projective* R*-module. Then* tr^c_M *equals the composition*

$$\mathit{End}_R M \xrightarrow{\psi_M^{-1}} M^* \otimes_R M \xrightarrow{\varepsilon_M} \mathit{im}\, \varepsilon_M$$

where ε_M *is the evaluation map and* ψ_M *is the canonical isomorphism. Moreover,* $(Tr_M^c)_p$ *is isomorphic to* R_p *for every prime ideal* p *contained in the support of* M.

Proof. Since M is a finitely generated projective module there exists an R-module M' satisfying $M \oplus M' = \overset{q}{\oplus} R$. There is a canonical homomorphism $\mathrm{End}_R M \longrightarrow \mathrm{End}_R \overset{q}{\oplus} R$ which sends every endomorphism μ of M into $\mu \oplus O_{M'}$. Using (3. 8) we then obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_R M & \xrightarrow{\oplus O_{M'}} & \mathrm{End}_R \overset{q}{\oplus} R \\ {\scriptstyle \mathrm{tr}_M^c} \downarrow & & \downarrow {\scriptstyle \mathrm{tr}^c_{\overset{q}{\oplus} R}} \\ \mathrm{Tr}_M^c & \longrightarrow & R \end{array} .$$

If p is in the support of M then M_p is a free (see [6], p. 143) R_p-module of positive rank. Since a finitely generated projective module is finitely presentable (see [6], p. 36) we obtain, using (4. 9), the commutativity of the diagram

$$\begin{array}{ccccccc} \mathrm{End}_{R_p} M_p & \xrightarrow{\simeq} & (\mathrm{End}_R M)_p & \xrightarrow{\oplus O} & (\mathrm{End}_R \overset{q}{\oplus} R)_p & \xrightarrow{\simeq} & \mathrm{End}_{R_p} \overset{q}{\oplus} R_p \\ {\scriptstyle \mathrm{tr}^c_{M_p}} \downarrow & & {\scriptstyle (\mathrm{tr}_M^c)_p} \downarrow & & {\scriptstyle (\mathrm{tr}^c_{\overset{q}{\oplus} R})_p} \downarrow & & {\scriptstyle \mathrm{tr}^c_{\overset{q}{\oplus} R_p}} \downarrow \\ R_p & \xrightarrow{\simeq} & (\mathrm{Tr}_M^c)_p & \longrightarrow & R_p & \xrightarrow{\simeq} & R_p \end{array}$$

One checks easily that the composition of the homomorphisms on top equals $\oplus O_{M'_p}$. Since M_p is free and a non-trivial direct summand of $\overset{q}{\oplus} R_p$ we conclude from (3. 8) that there is an element in $\mathrm{End}_{R_p} M_p$ that via the "upper dogleg" of the last diagram hits the unit element of R_p. Hence the upper dogleg is an epimorphism, and so is $(\mathrm{Tr}_M^c)_p \longrightarrow R_p$. However, $(\mathrm{Tr}_M^c)_p$ is isomorphic to the canonical R_p-module R_p, a claim that is incorporated in the last diagram. Hence a straight forward argument shows that $(\mathrm{Tr}_M^c)_p \longrightarrow R_p$ is indeed an isomorphism, which proves one of our claims. In order to complete our proof consider the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_R M & \xrightarrow{\psi_M^{-1}} & M^* \otimes_R M \\ {\scriptstyle \mathrm{tr}_M^c} \downarrow & & \downarrow {\scriptstyle \varepsilon_M} \\ \mathrm{Tr}_M^c & \xrightarrow{\lambda} & \mathrm{im}\, \varepsilon_M \end{array}$$

By localizing it at the prime ideal $\boldsymbol{p}$ we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_{R_p}M_p & \xrightarrow{\psi_{M_p}^{-1}} & M_p^*\otimes_{R_p}M_p \\ (\mathrm{tr}_M^c)_p \downarrow & & \downarrow \varepsilon_{M_p} \\ (\mathrm{Tr}_M^c)_p & \xrightarrow{\lambda_p} & \mathrm{im}\,\varepsilon_{M_p} \end{array}$$

If $\boldsymbol{p}$ is not in the support of M then (4. 10) implies that λ_p is an isomorphism. If $\boldsymbol{p}$ is in the support of M then our previous argument asserts that $(\mathrm{Tr}_M^c)_p$ is isomorphic to R_p, as is $\mathrm{im}\,\varepsilon_{M_p}$. Since the upper dogleg is again an epimorphism it follows that λ_p is an epimorphism. Hence, as before, we conclude that λ_p is in fact an isomorphism. Therefore a well known globalization theorem (see [6], p. 114) implies that λ itself is an isomorphism.

(4. 11) states that our definition of trace coincides with the customary one for finitely generated projective modules (see [5]). In this connection it should be recalled that a finitely generated module M is faithful if and only if ε_M is an epimorphism (cf. [2], p. 133).

Note that the upper dogleg in the first diagram of the proof of (4. 11) coincides with the definition of trace used in [17]. Since the proof of (4. 11) implies that the bottom homomorphism in this diagram is an isomorphism we conclude that our notion of commutative trace is equivalent to the one used in [17].

(4. 12) PROPOSITION. *Suppose that M is a finitely presentable R-module and that N is a finitely generated projective R-module. Then*

$$t_{M,N}^c: Tr_M^c\otimes_R Tr_N^c \longrightarrow Tr_{M\otimes_R N}^c$$

is an isomorphism.

Proof. Obviously $t_{M,N}$ is an isomorphism whenever N is a free R-module of rank $\leqq 1$. For N a free R-module of rank >1 our claim is implicit in (3. 5) and (3. 8). In the general case consider the pertinent commutative diagram

$$\begin{array}{ccc} \mathrm{End}_R M\otimes_R \mathrm{End}_R N & \xrightarrow{\otimes_R} & \mathrm{End}_R(M\otimes_R N) \\ \mathrm{tr}_M^c\otimes_R \mathrm{tr}_N^c \downarrow & & \downarrow \mathrm{tr}_{M\otimes_R N}^c \\ \mathrm{Tr}_M^c\otimes_R \mathrm{Tr}_N^c & \xrightarrow{t_{M,N}^c} & \mathrm{Tr}_{M\otimes_R N}^c \end{array}$$

Since the tensor product of finitely presentable modules is finitely presentable (4. 9) implies for every prime ideal $\boldsymbol{p}$ of R the commutativity of the following diagram

$$\begin{array}{ccccc}
\mathrm{End}_{R_p}M_p\otimes_{R_p}\mathrm{End}_{R_p}N_p & \xrightarrow{\cong} & (\mathrm{End}_R M\otimes_R \mathrm{End}_R N)_p & \xrightarrow{(\otimes_R)_p} & (\mathrm{End}_R M\otimes_R N)_p \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Tr}^c_{M_p}\otimes_{R_p}\mathrm{Tr}^c_{N_p} & \xrightarrow{\cong} & (\mathrm{Tr}^c_M\otimes_R\mathrm{Tr}^c_N)_p & \xrightarrow{(t_{M,N})_p} & (\mathrm{Tr}^c_{M\otimes_R N})_p
\end{array}$$

$$\begin{array}{c}
\xrightarrow{\cong} \mathrm{End}_{R_p}M_p\otimes_{R_p}N_p \\
\downarrow \\
\xrightarrow{\cong} \mathrm{Tr}^c_{M_p\otimes_{R_p}N_p}
\end{array}$$

The composition of the top homomorphisms equals $\otimes_{R_p}$ as is seen easily. Hence the composition of the bottom homomorphisms equals t_{M_p,N_p}. Now, if $\boldsymbol{p}$ is not in the support of N then clearly $(t_{M,N})_p$ has domain and codomain the null module and is thus an isomorphism. If $\boldsymbol{p}$ is in the support of M then N_p is a finitely generated free R_p-module of positive rank and our previous remarks concerning the free case imply that $(t_{M,N})_p$ is again an isomorphism. Hence a well known globalization theorem shows that $t_{M,N}$ is indeed an isomorphism.

(4. 13) COROLLARY. *Suppose that M is a finitely presentable R-module and that N is a finitely generated, faithful projective R-module. Then there exists an isomorphism $t: Tr^c_M \longrightarrow Tr^c_{M\otimes_R N}$ such that the diagram*

$$\begin{array}{ccc}
End_R M & \xrightarrow{\otimes id_N} & End_R(M\otimes_R N) \\
{\scriptstyle tr^c_M}\downarrow & & \downarrow{\scriptstyle tr^c_{M\otimes_R N}} \\
Tr^c_M & \xrightarrow{tr^c_N(id_N)\cdot t} & Tr^c_{M\otimes_R N}
\end{array}$$

commutes.

Proof. (4. 11) and the remark following the proof of (4. 11) imply that for N a finitely generated, faithful projective R-module tr^c_N equals the composition

$$\mathrm{End}_R N \xrightarrow{\psi_N^{-1}} N^*\otimes_R N \xrightarrow{\varepsilon_N} R\ .$$

It is then clear that $t = t_{M,N}$ makes the above diagram commutative. Hence (4. 12) proves our claim.

(4. 14) PROPOSITION. *Let M be a finitely generated projective R-module. Then for any homomorphism $\rho: R \longrightarrow S$ of unital rings $e^c(\rho): \rho^* Tr_M^c \longrightarrow Tr_{\rho^* M}^c$ is an isomorphism.*

The proof of this last statement follows the pattern of previous proofs using (4. 9) and shall therefore be omitted.

In view of (4. 9) it would be quite interesting to determine the trace modules Tr_M^c for finitely presentable modules M over local rings. However, only partial results are available here.

(4. 15) THEOREM. *Let R be a (not necessarily noetherian) local ring with maximal ideal $\boldsymbol{m}$. Let $M \neq 0$ be a finitely generated R-module such that char$(R/\boldsymbol{m})$ does not divide the minimal number of generators of M. Then there is a canonical epimorphism $Tr_M^c \longrightarrow R/\boldsymbol{m}$.*

Proof. The canonical homomorphism $\mathrm{End}_R M \longrightarrow \mathrm{End}_{R/\boldsymbol{m}} M/\boldsymbol{m}M$ gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_R M & \longrightarrow & \mathrm{Tr}_M^c \\ \downarrow & & \downarrow \\ \mathrm{End}_{R/\boldsymbol{m}} M/\boldsymbol{m}M & \longrightarrow & \mathrm{Tr}_{M/\boldsymbol{m}M}^c \end{array}$$

where the map on the right hand side is an R-homomorphism with respect to the obvious structures. $M/\boldsymbol{m}M$ is a free $R/\boldsymbol{m}$-module of rank equal the minimal number n of generators of M. Since $\mathrm{Tr}_{M/\boldsymbol{m}M}^c$ is isomorphic to $R/\boldsymbol{m}$ due to (3. 8) it remains to be shown that the lower dogleg in the above diagram is an epimorphism. Let $s \in R/\boldsymbol{m} \cong \mathrm{Tr}_{M/\boldsymbol{m}M}^c$. By assumption we can form $n^{-1}s$. Evidently the trace of the homothety of $M/\boldsymbol{m}M$ by $n^{-1}s$ equals s. This homothety, however, is the image of the homothety of M by any element r of R satisfying $cl(r) = n^{-1}s$. Hence our claim.

(4. 16) COROLLARY. *Let M be a finitely representable R-module such that for some prime ideal $\boldsymbol{p}$ of R in the support of M, char$(R_{\boldsymbol{p}}/\boldsymbol{p}R_{\boldsymbol{p}})$ does not divide the minimal number of generators of $M_{\boldsymbol{p}}$. Then Tr_M^c is not the null module.*

Proof. (4. 9) and (4. 15).

§5. Predeterminants for Endomorphisms in R-monoidal Categories.

By a monoid with zero is meant an associative, unital multiplicative system M in which there exists a (then unique) element 0 such that $0 \cdot m$

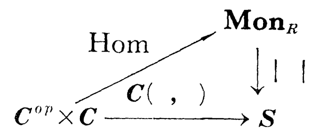
$= m \cdot 0 = 0$ holds for all elements m of M. If R is an associative unital ring then the multiplicative structure of R furnishes an example of a monoid $R^{\times}$ with zero. By a homomorphism between monoids with zero is meant a map that preserves all products as well as the zero and the unit element.

Let R be an associative unital ring. Then we shall speak of a left R-monoid M if M is endowed with the structure of a monoid with zero and with a left operation $\omega: R \times M \longrightarrow M$ satisfying—with rm standing for $\omega(r, m)$—

$$(r_1 r_2)m = r_1(r_2 m) \ , \quad r(m_1 m_2) = (rm_1)m_2 = m_1(rm_2)$$
$$1m = m \quad , \quad 0m = r0 = 0 \quad .$$

If A is an associative left R-algebra with unit then the multiplicative structure of A furnishes an example of a left R-monoid $A^{\times}$. A left monoid with zero is in the obvious manner a $\boldsymbol{Z}_2$-monoid.

By an R-monoidal category is meant a category $\boldsymbol{C}$ together with a covariant "structural" Hom-functor Hom: $\boldsymbol{C}^{op} \times \boldsymbol{C} \longrightarrow \mathbf{Mon}_R$, $\mathbf{Mon}_R$ being the category of left R-monoids and their obvious homomorphisms, rendering



commutative where $| \ |: \mathbf{Mon}_R \longrightarrow \boldsymbol{S}$ is the standard forgetful functor. The prime example for such categories are furnishes by partially forgetting the structure of R-additive categories $\boldsymbol{A}$: in this case End A carries the structure of an associative unital left R-algebra whence $(\text{End } A)^{\times}$ is a left R-monoid.

We can now formulate the universal problem, and its commutative companion, by substituting $\mathbf{Mon}_R$ for $\mathbf{Mod}_R$ in the definition given in §3.

(5. 1) PROPOSITION. *Let $\boldsymbol{C}$ be an R-monoidal category. Then both the universal problem of* §1 *and its commutative version admit a solution for every object of $\boldsymbol{C}$.*

Proof. An easy argument involving $R^{\times}$ shows that every monomorphism in $\mathbf{Mon}_R$ is an injection. Since $\mathbf{Mon}_R$ obviously satisfies the additional assumptions of (1. 5) our claim is verified.

Given the object A of $\boldsymbol{C}$ we shall denote the solution (u_A, U_A) of this universal problem (resp. the solution (u_A^c, U_A^c) of its commutative version) by $(\boldsymbol{C}\text{-}pdt_A,\ \boldsymbol{C}\text{-}pDt_A)$ (resp. $(\boldsymbol{C}\text{-}pdt_A^c,\ \boldsymbol{C}\text{-}pDt_A^c)$) and call it the predeterminant (resp.

the commutative predeterminant) on A. Whenever the reference to the R-monoidal category $\boldsymbol{C}$ is clear the prefix $\boldsymbol{C}$- shall be dropped.

(5. 2) COROLLARY. *For every object A the morphisms pdt_A, pdt_A^c, and w_A: $pDt_A \longrightarrow pDt_A^c$ are surjections.*

Proof. (1. 7) together with the fact that epimorphisms in $\mathbf{Mon}_R$ are surjections (an assertion that can be obtained in the same manner as in the category of groups).

At this point a word of justification for the term "predeterminant". For R a commutative unital ring, M a finitely generated, projective, unital R-module, and μ an endomorphism of M the determinant $\det \mu \in R$ has been defined in [12]. The resulting map det: $(\mathrm{End}_R M)^\times \longrightarrow R$ has the properties set forth in condition (ii^c) for the commutative version of the universal problem in §1. Hence, if Det_M stands for the R-monoid $\det(\mathrm{End}_R M)^\times$ there exists a unique epimorphism (and thus surjection) of R-monoids p_M^c: $pDt_M^c \longrightarrow \mathrm{Det}_M$ rendering

$$\begin{array}{ccc} (\mathrm{End}_R M)^\times & \xrightarrow{\ pdt_M^c\ } & pDt_M^c \\ & \searrow^{\det} & \downarrow p_M^c \\ & & \mathrm{Det}_M \end{array}$$

commutative. Hence pdt_M as well as pdt_M^c furnishes a description of the multiplicative structure of $\mathrm{End}_R M$ that is at least as discerning as the one furnished by det. In general p_M^c cannot be expected to be an isomorphism: If $R = \boldsymbol{Z}_2$ and $M = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ then $\mathrm{Det}_M = \boldsymbol{Z}_2$ while pDt_M^c consists of three elements, say, 0, 1, and 2 with 0 being the zero, 1 being the unit, and 2 satisfying $2 \cdot 2 = 2$ (note that precisely the elementary automorphisms are mapped to 2 by pdt_M^c).

In analogy to (3. 7) we have here

(5. 3) PROPOSITION. *Suppose that the objects $A_1, \cdots, A_k$ of $\boldsymbol{C}$ satisfy $Hom(A_\kappa, A_\lambda) = 0$ for all $\kappa \neq \lambda$. Then there is a commutative diagram*

$$\begin{array}{ccc} \Pi\, \mathrm{End}\, A_\kappa & \xrightarrow{\ \Pi\, pdt_{A_\kappa}\ } & \Pi\, pDt_{A_\kappa} \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{End} \oplus A_\kappa & \xrightarrow{\ pdt_{\oplus A_\kappa}\ } & pDt_{\oplus A_\kappa} \end{array}$$

the isomorphisms in it being canonical. Similarly for the commutative predeterminant.

Proof. Straight forward.

(5. 4) PROPOSITION. *Suppose that α is an automorphism of A. Then $pdt_A\alpha$ as well as $pdt_A^c\alpha$ is a unit (note that in a monoid with zero a unit is different from* 0 *if and only if* $1 \neq 0$ *holds).*

Proof. Since pdt_A is a surjection by (5. 2), $pdt_A(\mathrm{id}_A)$ is the unit element in pDt_A. Hence our claim follows routinely.

(5. 5) COROLLARY. *Suppose that the automorphism α of A is in the commutator subgroup of Aut A. Then $pdt_A\alpha$ as well as $pdt_A^c\alpha$ is the unit element in pDt_A resp. pDt_A^c.*

Proof. Immediate from (5. 4).

(5. 6) COROLLARY. *Suppose that every endomorphism of A that is different from* 0 *is in fact an automorphism. Then w_A: $pDt_A \longrightarrow pDt_A^c$ is an isomorphism and $pDt_A - \{0\}$ is canonically isomorphic to the factor commutator group of* End $A-\{0\}$.

Proof. Immediate from (5. 4).

We shall now deal with finite direct sums of the type $\overset{k}{\oplus} A$. Let us denote the canonical injection (resp. canonical projection) of the κ. summand into $\overset{k}{\oplus} A$ (resp. of $\overset{k}{\oplus} A$ onto the κ. summand) by i_κ (resp. p_κ). Then an element μ End $\overset{k}{\oplus} A$ is called elementary if there exist indices κ_0, λ_0 with $\kappa_0 \neq \lambda_0$ such that

$$p_\kappa \mu i_\kappa = \mathrm{id}_A \qquad \text{for all} \quad \kappa = 1, \cdots, k$$

$$p_\lambda \mu i_\kappa = 0 \qquad \text{for all} \quad \kappa \neq \kappa_0, \ \lambda \neq \lambda_0, \ \kappa \neq \lambda \ .$$

Clearly an elementary endomorphism of $\overset{k}{\oplus} A$ is indeed an automorphism. We shall now list a few results concerning elementary automorphisms.

(5. 7) PROPOSITION. *Let A be an object of the R-additive category* **A**. *Then for every elementary automorphism α of $\overset{k}{\oplus} A$ with $k > 2$, $pdt_{\overset{k}{\oplus} A}\alpha$ is the unit element.*

Proof. [3], Corollary (1. 5), (i), together with (5. 5).

The restriction $k > 2$ in (5. 7) is necessary as can be seen from the example following (5. 2). Yet in special cases the assertion of (5. 7) remains valid for $k = 2$:

(5. 8) PROPOSITION. *Let A be an object in the R-additive category $\mathbf{A}$. Assume that the R-algebra End A contains units u_1, u_2, v_1, v_2 such that the ideal generated by u_1+u_2 and v_1+v_2 equals End A. Then for every elementary automorphism ε of $A+A$, $pdt_{A\oplus A}\varepsilon$ is the unit element.*

Proof. If $1=a(u_1+u_2)+b(v_1+v_2)$ in End A then

$$\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}=\left[\begin{pmatrix}u_1 & rbu_2^{-1}v_2^{-1}\\ 0 & u_2\end{pmatrix},\begin{pmatrix}v_1 & -rau_2^{-1}v_2^{-1}\\ 0 & v_2\end{pmatrix}\right].$$

For this proof see also [3], Lemma (1. 6).

We shall now slightly modify a definition given in [3]. Let R be a (not necessarily commutative) unital ring. Then the positive integer n_0 is said to define a right-stable range for R (in [3]: for $GL(R)$) if for every $n>n_0$ and all $r_1,\cdots,r_n\in R$ satisfying $Rr_1+\cdots+Rr_n=R$ there exist $s_1,\cdots,s_{n-1}\in R$ such that $R(r_1+s_1r_n)+\cdots+R(r_{n-1}+s_{n-1}r_n)=R$ holds. Similarly one defines a left-stable range for R. Finally, n_0 is said to be a stable range for R if it is both a left-stable as well as a right-stable range for R. It is pointed out in [3] that for a semi-local (not necessarily noetherian) ring $n_0=1$ is a stable range; that for a Dedekind ring $n_0=2$ is a stable range; that for a coordinate ring of a d-dimensional affine algebraic variety $n_0=d+1$ is a stable range (see also [3], Theorem (11. 1)).

Let A be an object in the R-additive category $\mathbf{A}$. Just as in the case of the trace there is for every $n>m$ a canonical morphism of R-monoids $pDt^m_{\oplus A}\longrightarrow pDt^n_{\oplus A}$ rendering the diagram

$$\begin{array}{ccc}(\mathrm{End}\overset{m}{\bigoplus}A)^\times & \xrightarrow{\;pdt^m_{\oplus A}\;} & pDt^m_{\oplus A}\\ {\scriptstyle \oplus\overset{n-m}{\bigoplus}\mathrm{id}_A}\downarrow & & \downarrow\\ (\mathrm{End}\overset{n}{\bigoplus}A)^\times & \xrightarrow{\;pdt^n_{\oplus A}\;} & pDt^n_{\oplus A}\end{array}$$

commutative. Similarly for the commutative predeterminant.

(5. 9) PROPOSITION. *Let A be an object in the R-additive category $\mathbf{A}$. Assume that n_0 is a stable range for the ring End A and that End A is a principal ideal domain. Then for every $n>max(2,n_0)$ the canonical morphism $pDt^{n_0}_{\oplus A}\longrightarrow pDt^{n_0}_{\oplus A}$ is an epimorphism (and thus a surjection). Similarly for the commutative predeterminant.*

Proof. Denote End A by B. Then an endomorphism of $\overset{n}{\oplus} A$ is a $n \times n$ matrix μ with entries in B. Since, by assumption, B is a principal ideal domain $Bb_1 + \cdots + Bb_n = Bb$ together with $b_\nu = b'_\nu b$ implies $Bb'_1 + \cdots + Bb'_n = B$. Hence the procedure in [3], proof of Theorem (4. 2), a), shows the existence of a product ε of elementary automorphisms such that

$$\varepsilon\mu = \left(\begin{array}{c|c} X & O \\ \hline Y & \begin{matrix} c_{n_0+1} & & O \\ Z & \ddots & \\ & & c_n \end{matrix} \end{array}\right)$$

However,

$$\left(\begin{array}{c|c} X & O \\ \hline Y & \begin{matrix} c_{n_0+1} & & O \\ Z & \ddots & \\ & & c_n \end{matrix} \end{array}\right) = \left(\begin{array}{c|c} X & O \\ \hline Y & \begin{matrix} c_{n_0+1} & & O \\ Z & \ddots & \\ & & c_{n-1} & \\ & & & 1 \end{matrix} \end{array}\right) \cdot \begin{pmatrix} 1 & & & O \\ & \ddots & & \\ & & 1 & \\ O & & & c_n \end{pmatrix}.$$

Again by the same technique there exists an elementary automorphism ε' such that

$$\left(\begin{array}{c|c} X & O \\ \hline Y & \begin{matrix} c_{n_0+1} & & O \\ Z & \ddots & \\ & & c_{n-1} & \\ & & & 1 \end{matrix} \end{array}\right) \varepsilon' = \left(\begin{array}{c|c|c} X & O & O \\ \hline Y' & \begin{matrix} c_{n_0+1} & & O \\ Z' & \ddots & \\ & & c_{n-1} \end{matrix} & O \\ \hline O & O & 1 \end{array}\right).$$

Hence (5. 7) together with an obvious induction argument implies that $pdt^n_{\oplus A}\mu$ equals

$$\text{(5. 10)} \qquad pdt\left(\begin{array}{c|c} X & O \\ \hline O & 1 \end{array}\right) \cdot pdt\begin{pmatrix} 1 & & & & O \\ & \ddots & & & \\ & & c_{n_0+1} & & \\ & & & \ddots & \\ O & & & & 1 \end{pmatrix} \cdots\cdots pdt\begin{pmatrix} 1 & & & O \\ & \ddots & & \\ & & 1 & \\ O & & & c_n \end{pmatrix}.$$

Since every permutation matrix is a product of elementary matrices another application of (5. 7) leads to

$$pdt\begin{pmatrix} 1 & & & & O \\ & \ddots & & & \\ & & c_\nu & & \\ & & & \ddots & \\ O & & & & 1 \end{pmatrix} = pdt\begin{pmatrix} c_\nu & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{pmatrix}$$

which together with (5. 10) proves our assertion.

(5. 11) Corollary. *Let either R be a (possibly skew) field and $n \neq 2$ or R (possibly a skew) field other than $\boldsymbol{Z}_2$ and n arbitrary. Then, with $A = \boldsymbol{Mod}_R$ and Hom the obvious functor from $\boldsymbol{A}^{op} \times \boldsymbol{A}$ to $\boldsymbol{Ab}$, $pdt^n_{\oplus A}$ is the usual determinant (see* [1], [10]) *and $w^n_{\oplus R}: pDt^n_{\oplus A} \longrightarrow pDt^c_{\underset{\oplus R}{n}}$ is an isomorphism.*

Proof. First we observe that under the present assumptions the requirement $n > \max(2, n_0)$ in (5. 9) can be replaced by $n > 1$ as $n > 2$ was only needed for invoking (5. 7) which under our alternate assumptions may be replaced by (5. 8). Hence (5. 9) implies that every non-zero element of $pDt^n_{\oplus R}$ is the image of some matrix of the form

$$\begin{pmatrix} r & & O \\ & 1 & \\ O & & 1 \end{pmatrix} \tag{5. 12}$$

with $r \neq 0$. Since every non-zero element of R is a unit it is clear that $pdt^n_{\oplus R}$ is in fact commutative and hence $w^n_{\oplus R}$ is an isomorphism. Moreover the classical determinant assumes the same value on two matrices of the form (5. 12) if and only if the entries in the upper left corner are conjugate to each other. Thus two such matrices are mapped by $pdt^n_{\oplus R}$ into the same element if and only if the entries in the upper left corner are conjugate to each other. This implies that the canonical morphism $p^n_{\oplus R}$: $pDt^n_{\oplus R} \longrightarrow \mathrm{Det}^n_{\oplus R}$ is an injection. Since $\det^n_{\oplus R}$: $(\mathrm{End} \overset{n}{\oplus} R)^\times \longrightarrow \mathrm{Det}^n_{\oplus R}$ is also a surjection our assertion is proved.

(5. 13) COROLLARY. *Let R be a commutative principal ideal domain having* 1 *as a stable range. If $n \neq 2$ or if n is arbitrary while R satisfies the assumptions of* (5. 8) (*imposed there on End A*) *then the conclusions of* (5. 11) *are valid.*

Proof. Same as the proof of (5. 11).

In connection with (5. 13) see also [3], §5, Remark 3; and [7]; and also [15]. For R a euclidean domain (5. 13) has been proved in [18].

(5. 14) PROPOSITION. *Let R be a commutative unital ring. If $n \neq 2$ or if n is arbitrary while R satisfies the assumptions of* (5. 8) (*imposed there on End A*) *then the canonical morphism $p^n_{\oplus R}$: $pDt^n_{\oplus R} \longrightarrow \mathrm{Det}^n_{\oplus R} = R$ is an injection on the set of those elements which are images of triangular matrices.*

Proof. Using Lemma 1 of [18] the reasoning of the proof of (5. 10) shows that with τ a triangular matrix

$$pdt^n_{\oplus R}\tau = pdt^n_{\oplus R}\begin{pmatrix} \det\tau & & O \\ & 1 \cdot \, & \\ O & & \ddots \; 1 \end{pmatrix}$$

holds. Hence our claim follows immediately.

§6. Predeterminants for endomorphisms of *R*-modules.

As in §4 we assume that R is a commutative unital ring. We then deal with the special case of §5 in which $\boldsymbol{C}$ is the category $\mathbf{Mod}_R$ and Hom is the usual Hom-functor Hom_R.

(6. 1) PROPOSITION. *Suppose that M is a finitely generated projective R-module. Then the endomorphism μ of M is an automorphism if and only if $pdt_M(\mu)$ (resp. $pdt_M^c(\mu)$) is a unit.*

Proof. It is an easy consequence of [12], Proposition 1. 3, that both the set of automorphisms of M and the set of non-automorphisms of M are submonoids of $\mathrm{End}_R M$. Hence the map from $\mathrm{End}_R M$ to the augmented group[3] of the factor commutator group of $\mathrm{Aut}_R M$ that assigns to each automorphism of M its canonical image in the factor commutator group of $\mathrm{Aut}_R M$ and to each non-automorphism the zero element is in fact a homomorphism δ of monoids with zero. The ring R operates on this augmented group through the canonical homomorphism $R \longrightarrow \mathrm{End}_R M$, and with respect to this structure δ is a homomorphism of R-monoids. Therefore we have a canonical homomorphism of R-monoids from pDt_M (resp. pDt_M^c) to this augmented group. Now, if $pdt_M(\mu)$ (resp. $pdt_M^c(\mu)$) is a unit the canonical homomorphism will send it into the factor commutator group proper. Thus μ is an automorphism. The converse was already stated in (5. 4).

(6. 2) COROLLARY. *Suppose that M is a finitely generated projective R-module. Then the group of units of both pDt_M and pDt_M^c is canonically isomorphic to the factor commutator group of $Aut_R M$.*

Proof. Since every unit in both pDt_M and pDt_M^c is the image of automorphisms only the group of units in either monoid must be abelian by the universal property (i) resp. (i^c) of §1. Therefore the homomorphism δ constructed in the proof of (6. 1) must be an isomorphism on the group of units.

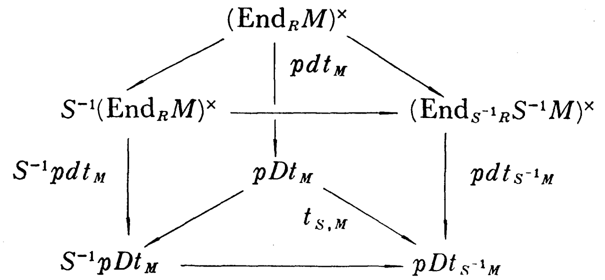
Let S be a multiplicatively closed subset of R and let M be an R-monoid. In the set $M \times S$ we have the well-known equivalence relation

"$(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists a $t \in S$ with $ts_2m_1 = ts_1m_2$".

[3] By the augmented group of the group G is meant the monoid with zero that is obtained from G by adjoining a zero element.

The set of equivalence classes is denoted by $S^{-1}M$. $S^{-1}M$ carries a canonical left $S^{-1}R$-monoid structure, and the assignment of $S^{-1}M$ to M extends in the obvious fashion to a covariant functor from $\mathbf{Mon}_R$ to $\mathbf{Mon}_{S^{-1}R}$. Moreover it is clear that the canonical map $M \longrightarrow S^{-1}M$ is a homomorphism of R-monoids.

As in §4 we have for every R-module M a commutative diagram



in which all the undesignated homomorphisms are canonical. A similar definition holds for the commutative predeterminant, in which case the bottom homomorphism shall be denoted by $t^c_{S,M}$.

(6. 3) THEOREM. *Suppose that the canonical homomorphism* φ: $S^{-1}(End_R M)^\times \longrightarrow (End_{S^{-1}R} S^{-1}M)^\times$ *is an isomorphism. Then* $t^c_{S,M}$ *is also an isomorphism (and thus a bijection). In particular this is true if* M *is finitely presentable.*

Proof. Clearly the proof for (4. 9) does not carry over to the present situation. We shall now give a proof which mutatis mutandis also works in the case discussed in (4. 9).

Let $\psi\colon S^{-1}(\mathrm{End}_R M)^\times \longrightarrow N$ be a $S^{-1}R$-homomorphism which satisfies $\psi(\mu_1\mu_2) = \psi(\mu_1\mu_2)$ for all μ_1, μ_2 in $S^{-1}(\mathrm{End}_R M)^\times$. Denoting the canonical R-homomorphism $(\mathrm{End}_R M)^\times \longrightarrow S^{-1}(\mathrm{End}_R M)^\times$ by ι we obtain a unique R-homomorphism χ': $pDt^c_M \longrightarrow N$ satisfying $\psi\iota = \chi' \cdot pdt^c_M$. Let m, m' be in pDt^c_M such that for some s, s' in S, $\frac{m}{s} = \frac{m'}{s'}$ holds in $S^{-1}pDt^c_M$. Then there exists an element t in S with $tsm' = ts'm$. Hence $ts\chi'(m') = ts'\chi'(m)$ and, as N is a $S^{-1}R$-monoid, $\frac{\chi'(m)}{s} = \frac{\chi'(m')}{s'}$. Therefore we can define a map χ: $S^{-1}pDt^c_M \longrightarrow N$ by putting $\chi\left(\frac{m}{s}\right) = \frac{\chi'(m)}{s}$. Evidently χ is a $S^{-1}R$-homomorphism satisfying $\chi' = \chi\iota'$ where ι': $pDt^c_M \longrightarrow S^{-1}pDt^c_M$ is the canonical R-homomorphism. Hence $\psi\iota = \chi' \cdot pdt^c_M = \chi\iota' \cdot pdt^c_M = \chi \cdot S^{-1}pdt^c_M \cdot \iota$. Since $(\mathrm{End}_R M)^\times$ generates $S^{-1}(\mathrm{End}_R M)^\times$ as a $S^{-1}R$-monoid we conclude that $\psi = \chi \cdot S^{-1}pdt^c_M$. Uniqueness of χ follows similarly. Since φ was assumed to be an isomorphism this means that $S^{-1}pdt^c_M \cdot \varphi^{-1}$ solves the commutative version of the universal problem for $S^{-1}M$. Hence our claim follows by standard argument.

(6. 4) COROLLARY. *Suppose that the R-module M is finitely presentable. If the prime ideal* $\mathfrak{p}$ *of R is not in the support of M then* $(pDt^c_M)_{\mathfrak{p}} = 0$.

Proof. Since $M_{\mathfrak{p}} = 0$ and since $pdt^c_{M_{\mathfrak{p}}}$: $(\mathrm{End}_{R_{\mathfrak{p}}} M_{\mathfrak{p}})^{\times} \longrightarrow pDt^c_{M_{\mathfrak{p}}}$ is an epimorphism we conclude that $pDt^c_{M_{\mathfrak{p}}}$ is the zero monoid. Hence (6. 3) finishes the proof.

(6. 5) COROLLARY. *Suppose that M is a finitely generated projective R-module. Suppose that the prime ideal* $\mathfrak{p}$ *of R is in the support of M and that either the* $\mathfrak{p}$*-rank of M is* $\neq 2$ *or that* $R_{\mathfrak{p}}$ *satisfies the condition of* (5. 8) (*stated there for End A*). *Then the group of units of* $(pDt^c_M)_{\mathfrak{p}}$ *is canonically isomorphic to the group of units of* $R_{\mathfrak{p}}$.

Proof. By (6. 3) the group of units of $(pDt^c_M)_{\mathfrak{p}}$ is isomorphic to the group of units of pDt^c_N where N is some finitely generated free $R_{\mathfrak{p}}$-module of positive rank. By (6. 2) the latter is isomorphic to the factor commutator group of $\mathrm{Aut}_R N$ which, due to (5. 7), (5. 8) and [3], Theorem (4. 2), b), is isomorphic to $R_{\mathfrak{p}}$.

(6. 6) COROLLARY. *Suppose that R is a principal ideal domain and that M is a finitely generated free R-module of positive rank. Then for every prime ideal* $\mathfrak{p}$ *of R in the support of M,* $(pDt^c_M)_{\mathfrak{p}} = R_{\mathfrak{p}}$.

Proof. Since every localization of a principal ideal domain is a principal ideal domain having 1 as a stable range, (5. 13) and (6. 3) imply our assertion.

It is clear that the aspects of §4 other than the localization principle (4. 9) carry over to pdt and pdt^c in various degrees. Details are left to the reader.

It should also be remarked that mutatis mutandis the considerations of §5 and §6 remain valid when the category $\mathbf{Mod}_R$ is replaced by the category $\mathbf{Mod}^*_R$ of null morphisms and isomorphisms of R-modules (Note that R is operating on $\mathbf{Mod}^*_R(M_1, M_2)$ as follows. If $r \in R$ is a unit then r operates on $\mathbf{Mod}^*_R(M_1, M_2)$ in the usual fashion; if $r \in R$ is a non-unit then $r \cdot \mu$ is always the null morphism).

(6. 7) THEOREM. *Let R be a (not necessarily noetherian) local ring with maximal ideal $\boldsymbol{m}$. Let $M \neq 0$ be a finitely generated R-module such that with n the minimal number of generators of M exponentiation by n maps $R/\boldsymbol{m}$ onto itself. Then there is a canonical epimorphism $pDt_M^c \longrightarrow (R/\boldsymbol{m})^\times$. In particular there is an epimorphism from the factor commutator group of $Aut_R M$ to $(R/\boldsymbol{m})^*$.*

Proof. The fact that the canonical morphism $pDt_M^c \longrightarrow (R/\boldsymbol{m})^\times$ is an epimorphism is proved along the lines of the proof of (4. 15). This argument then reveals that already $\mathrm{Aut}_R M$ maps onto $(R/\boldsymbol{m})^*$ via $(\mathrm{End}_R M)^\times \longrightarrow pDt_M^c \longrightarrow (R/\boldsymbol{m})^\times$. Hence our assertion follows.

(6. 8) COROLLARY. *Assumptions as in (6. 7). If $pdt_M^c(\mu)$ is a unit then μ is an epimorphism.*

Proof. Looking at the appropriate re-interpretation of the diagram in the proof of (4. 15) we conclude that the image of μ in $\mathrm{End}_{R/\boldsymbol{m}} M/\boldsymbol{m}M$ is an isomorphism. Hence an easy application of the Nakayama Lemma proves our claim.

(6. 9) COROLLARY. *Let M be a finitely presentable R-module such that for some prime ideal $\boldsymbol{p}$ of R in the support of M exponentiation by n maps $R_{\boldsymbol{p}}/\boldsymbol{p}R_{\boldsymbol{p}}$ onto itself, n being the minimal number of generators of $M_{\boldsymbol{p}}$. Then pDt_M^c is not the null monoid.*

Proof. (6. 3) and (6. 7).

§7. Concluding Remarks.

I. The universal problem expounded in §1 leads to another invariant for endomorphisms of modules. Let $\boldsymbol{C}$ be the category $\mathbf{Mod}_R$ with a commutative ring, $\boldsymbol{L}$ the category $\mathbf{Alg}_R$ of unital R-algebras, 1 the map that assigns to each R-module M the R-algebra $\mathrm{End}_R M$, and $|\ \ |$ the standard forgetful functor. Then the universal problem has a solution (u_M, U_M) for every module M. u_M clearly "combines" the properties of tr_M and pdt_M. One checks quickly that for every free R-module M of rank $\neq 1$ the universal solution U_M is the zero algebra. Since the localization principle (4. 9) carries over to this case one can prove that U_M is the zero algebra for every finitely generated projective R-module M for which every local rank is different from 1. Hence the universal solution U_M measures the deviation of the module M from being projective.

II. Let $\sum$ be a species of an algebraic structure (in the sense of [4], p. 42) in which for every principal term x_i one of the internal compositions, say $\top^i$, is singled out and in which the following is part of the axiom of $\sum$:

"for every i, $\top^i$ is defined throughout, is associative, and possesses a neutral element".

Given a structure of the species $\sum$ on the principal sets $S_1, \cdots, S_m$ and the auxiliary sets $T_1, \cdots, T_n$, one obtains a one-object category $\boldsymbol{C}_{\Sigma,S,T}$ whose set of morphisms equals $S_1 \times \cdots \times S_m$ and whose multiplication is determined, in each component, by the internal composition $\top^i$. If e_i is the neutral element for $\top^i$ then $E = (e_1, \cdots, e_m)$ is the sole object of this category. Evidently $\boldsymbol{C}_{\Sigma,S,T}(E) = S_1 \times \cdots \times S_m$ and $\boldsymbol{C}^{\times}_{\Sigma,S,T}(E)$ consists precisely of those m-tuples $(u_1, \cdots, u_m)$ for which every u_i is symmetric with respect to $\top^i$. $\boldsymbol{C}_{\Sigma,S,T}(E)$ carries also the original structure of species $\sum$.

Denote by $sc\sum$ the species that is obtained formally from $\sum$ by adding to the axiom of $\sum$

"for every i, each element that is symmetric with respect to $\top^i$ is central with respect to $\top^i$".

Let $\boldsymbol{C}$ be the category $\boldsymbol{C}_{\Sigma,S,T}$, $\boldsymbol{L}$ the category $\boldsymbol{C}_{sc\Sigma}$ of structures of species $sc\sum$ and their representations (cf. [4], p. 48), l the map that assigns to E the canonical structure of species $\sum$ on $\boldsymbol{C}_{\Sigma,S,T}(E)$ and $| \quad |$ the canonical forgetful functor. If $\boldsymbol{C}_{sc\Sigma}$ is complete and if $| \quad |$ preserves monomorphisms then we take from (1. 5) that the universal problem has a solution $(u_{\Sigma,S,T}, U_{\Sigma,S,T})$ for every structure of the species $\sum$ on the principal sets $S_1, \cdots, S_m$ and on the auxiliary sets $T_1, \cdots, T_n$. Representations between various such structures then give rise to representations between the associated universal solutions; this assignment obviously constitutes a covariant functor that is adjoint to the inclusion of $\boldsymbol{C}_{sc\Sigma}$ in the category $\boldsymbol{C}_{\Sigma}$ of structures of species $\sum$.

III. Let R be a commutative unital ring, let $\boldsymbol{C}$ be the R-monoidal category associated with $\mathbf{Mod}_R$, and let Hom be the usual Hom-functor Hom_R. For every R-module M we pose now the universal problem for all morphisms v in $\mathbf{Mon}_R$ with domain $(\mathrm{End}_R M)^{\times}$ satisfying:

$|v|(\mu_1) = |v|(\mu_2)$ whenever there exists a prime ideal $\boldsymbol{p}$ of R and an element $\alpha \in \mathrm{Aut}_{R_p} M_p$ such that $\frac{\mu_1}{1} = \alpha \frac{\mu_2}{1} \alpha^{-1}$ holds.

Again on has an obvious commutative companion to this universal problem.

As before one obtains in both cases the existence of a solution for every

R-module M. This solution shall be denoted by (ldt_M, lDt_M) resp. (ldt^c_M, lDt^c_M). A number of results concerning predeterminants carry over to the present situation. In particular the localization principle is valid for finitely presentable modules and the analogue to (5. 14) remains true. We claim that this suffices to prove for a commutative unital ring R whose nilradical is zero and which satisfies the assumptions of (5. 8) (imposed there on End A) that for any integer n the canonical morphism $l_{\overset{n}{\oplus} R}: lDt_{\overset{n}{\oplus} R} \longrightarrow \mathrm{Det}_{\overset{n}{\oplus} R} = R$ is an isomorphism. Obviously it suffices to show that the canonical morphism is an injection. Let $l \cdot ldt(\mu) = \det(\mu) = 0$. Then Lemma 3 of [18] implies the existence of a triangular matrix τ with $\det(\tau) \neq 0$ such that one column, say the first one, of vanishes. Hence $\mu\tau$ is the product of

$$\begin{pmatrix} 0_1 & \cdot & & O \\ & & \cdot & \\ O & & & \cdot\, 1 \end{pmatrix}$$

with a suitable matrix. Thus the analogue of (5. 14) shows that $ldt(\mu) \cdot ldt(\tau) = 0$. If p is a prime ideal in R not containing $\det(\tau)$ then $\frac{\tau}{1} \in \mathrm{End}_{R_p} \overset{n}{\oplus} R_p$ is invertible, which allows us to conclude $\frac{ldt(\mu)}{1} = 0$. By the localization principle we have then $ldt\left(\frac{\mu}{1}\right) = 0$ and hence the universal problem implies $ldt(\mu) = 0$. Next, let μ_1 and μ_2 be such that $l \cdot ldt(\mu_1) = l \cdot ldt(\mu_2) \neq 0$. If p is a prime ideal in R not containing $\det(\mu_1)$ then both $\frac{\mu_1}{1}$ and $\frac{\mu_2}{1} \in \mathrm{End}_{R_p} \overset{n}{\oplus} R_p$ are invertible and have the same determinant. Hence $\frac{\mu_1}{1} \cdot \left(\frac{\mu_2}{1}\right)^{-1}$ belongs to $SL(n, R_p)$ which equals $[GL(n, R_p), GL(n, R_p)]$ due to [3], Proposition (5. 1), a). Therefore $ldt\frac{\mu_1}{1} = ldt\frac{\mu_2}{1}$ and the universal problem implies $ldt(\mu_1) = ldt(\mu_2)$. Hence our claim. Clearly if we exclude $n = 2$, R need not to satisfy the assumptions of (5. 8) for our claim to remain valid.

Bibliography

[1] Artin, E., Geometric Algebra. Interscience Publ., New York 1957.
[2] Auslander, M. and O. Goldman, Maximal Orders. Trans. AMS **97** (1960), 1–24.
[3] Bass, H., K-Theory and Stable Algebra. Publ. math. IHES, No. **22** (1964), 489–544.
[4] Bourbaki, N., Eléments de Mathématique. Algèbre, Chap. 1, Hermann, Paris 1958.
[5] ——————, Eléments de Mathématique. Algèbre, Chap. 2, Hermann, Paris 1962.
[6] ——————, Eléments de Mathématique. Algèbre Commutative, Chap. 1 et 2, Hermann, Paris 1961.
[7] Brenner, J., The Linear Homogeneous Group, III. Ann. Math. **71** (1960), 210–223.

[8] Cartan, H. and S. Eilenberg, Homological Algebra. Princeton Univ. Press, Princeton 1956.
[9] Castagna, F., Sums of Automorphisms of a Primary Abelian Group. To appear.
[10] Dieudonné, J., Les Déterminants sur un Corps non Commutative. Bull. Soc. Math. France **71** (1943), 27–45.
[11] Freyd, P., Abelian Categories. Harper & Row, New York, Evanston, London 1964.
[12] Goldman, O., Determinants in Projective Modules. Nagoya Math. Journ. **18** (1961), 27–36.
[13] Hattori, A., Rank Element of a Projective Module. Nagoya Math. Journ. **25** (1965), 113–120.
[14] Hitotumatu, S., A Characterization of the Determinant. Comm. Math. Univ. Sancti Pauli **XIII** (1965), 45–50.
[15] Litoff, O.I., On the Commutator Subgroup of the General Linear Group. Proc. AMS **6** (1955), 465–470.
[16] Mitchell, B., Theory of Categories. Academic Press, New York and London, 1965.
[17] Ozeki, H., Chern Classes of Projective Modules. Nagoya Math. Journ. **23** (1963), 121–152.
[18] Taussky, O. and H. Wielandt, On the role of the determinant in semigroups of matrices. Quart. Journ. Math. **14** (1963), 123–130.
[19] Wolffhardt, K., Über eine Charakterisierung der Determinante. Math. Ztsch. **103** (1968) 259–267.

University of California, San Diego,
and Princeton University.