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ON MULTIVARIATE WIDE-SENSE MARKOV PROCESSES*

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1. Introduction: The idea of multivariate wide-sense Markov processes has been recently used by F.J. Beutler [1]. In his paper, he shows that the solution of a linear vector stochastic differential equation in a wide-sense Markov process. We obtain here a characterization of such processes and as its consequence obtain the conditions under which it satisfies Beutler's equation. Furthermore, in stationary Gaussian case we show that these are precisely stationary Gaussian Markov processes studied by J. Doob [5].

In their remarkable papers, T. Hida [6] and H. Cramér [2], [3] have studied the representation of a purely non-deterministic (not necessarily stationary) second order processes. We obtain such a representation for wide-sense Markov processes directly, by using their theory. The interesting part of our representation is that we are able to show that the multiplicity of q-dimensional wide-sense Markov processes does not exceed q, as, in general, even one-dimensional (not necessarily stationary) processes could have infinite multiplicity (see H. Cramér [2] and T. Hida [6]). We also show that the kernel splits (see Theorem 6. 1). As a consequence of this, we obtain the classical representation of Doob [5].

The paper is divided into 7 sections. The next section is devoted to the introduction of terminology and notation used in the rest of the paper.

2. Direct-product Hilbert-spaces: In this section we want to introduce the idea of direct-product Hilbert-spaces as in [10]. If H is a Hilbert-space we shall mean by $H^{(q)}$ the space of all vectors $\underline{h} = (h_1, h_2, \cdots, h_q)$ where for each i, $h_i \in H$. In $H^{(q)}$ is introduced a norm $|||\underline{h}||| = \sqrt{\sum_1^q \| h_i \|_H^2}$ and an inner product given by the Gramian matrix $[h, h^*] = \{< h_i, h_j >_H\}$.

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A linear manifold in $H^{(q)}$ is a non-void subset $\mathscr{M}$ of $H^{(q)}$ such that if $\underline{h}$, $\underline{h}' \in \mathscr{M}$ then $A\underline{h} + B\underline{h}' \in \mathscr{M}$ for all $q \times q$ matrices A, B. A subspace of $H^{(q)}$ is a linear manifold closed under the topology $|||\ \ |||$. We recall here a lemma due to N. Wiener and P. Masani [10] which proves the existence of the projection of an element $\underline{h}$ and gives its structure.

LEMMA WM (*Lemma* 5. 8 [10]). (a) *If $\mathscr{M}$ is a subspace of $H^{(q)}$ then there exists a subspace M of H such that $\mathscr{M} = M^{(q)}$, where $M^{(q)}$ denotes the Cartisian product $M \times \cdots \times M$ with q-factors. M is a set of all components of all elements in $\mathscr{M}$.*

(b) *If $\mathscr{M}$ is a subspace of $H^{(q)}$ and $\underline{h} \in H^{(q)}$, then there is a unique $\underline{h}' \in \mathscr{M}$ such that $||| \underline{h} - \underline{h}' ||| \leq ||| \underline{h} - \underline{g} |||$ for all $\underline{g} \in \mathscr{M}$. For this $\underline{h}'$, $h_i' = P_M h_i$, M being as in* (a). *An element $\underline{h}'$ satisfies preceding condition iff $[\underline{h} - \underline{h}', \underline{g}] = 0$ for all $\underline{g} \in \mathscr{M}$.*

The part (c), (d), and (e) of the original lemma are omitted since they won't be referred to here.

DEFINITION 2. 1. The unique element $\underline{h}'$ of Lemma WM (b) is called the orthogonal projection of $\underline{h}$ onto $\mathscr{M}$ and is denoted by $(\underline{h} | \mathscr{M})$.

Let (Ω, F, P) be a probability space. By a q-variate second-order stochastic process on (Ω, F, P), we mean a family of random vectors $\{\underline{x}(t), -\infty < t < +\infty\}$ where for each t, $\underline{x}(t) \in L_2^{(q)}(\Omega)$, $L_2(\Omega)$ denoting the Hilbert-space of complex-valued square-integrable random variables $L_2(\Omega)$. The past of the process up to s, $L_2(\underline{x}; s)$ is defined to be the subspace of $L_2(\Omega)$ generated by $\{\underline{x}^{(i)}(\tau)\ \tau \leq s\ i = 1, z, \cdots, q\}$ with $\underline{x}(t) = \{x^{(1)}(t), \cdots, x^{(q)}(t)\}^*$. The following definition extends to q-variate case, the idea of wide-sense Markov process and that of wide-sense martingale [see Doob [4] pp. 90, 164].

DEFINITION 2. 2. (a) A q-variate process $\{\underline{x}(t)\}$ $(-\infty < t < +\infty)$ is a wide-sense martingale if for each t, $(\underline{x}(t) | L_2^{(q)}(\underline{x}; s)) = \underline{x}_s$ for $s < t$.

(b) A process $\{\underline{x}(t)\}$ is called wide-sense Markov if for each $s < t$, $(\underline{x}(t) | L_2^{(q)}(\underline{x}; s)) = A(t, s)\underline{x}(s)$.

3. Characterization of a wide-sense Markov process: The assumption (D) given below will be made through this paper.

(D. 1) $\underline{x}(t)$-process is continuous in q.m., i.e.,

$$\lim_{s \to t} ||| \underline{x}(t) - \underline{x}(s) ||| = 0.$$

(D. 2) For all t, s real the covariance matrix $\Gamma(t, s)$ is non-singular.

The assumption (D. 2) and the definition of wide-sense Markov process imply $(\underline{x}(t)|L_2^{(q)}(\underline{x}; s)) = A(t,s)\underline{x}_s$ where the matrix $A(t,s)$ is given by $A(t,s) = \Gamma(t,s)\Gamma^{-1}(s,s)$ for $s \leq t$. The function $A(t,s)$ is called a transition matrix function and is defined only for $s \leq t$. Observe that if $\underline{x}(t)$ is wide-sense Markov, then for $s \leq t \leq u$ $A(u,s) = A(u,t)A(t,s)$.

The following is the main theorem of this section.

THEOREM 3. 1. *A q-variate second order continuous parameter process satisfying* (D) *is wide-sense Markov if and only if* $\underline{x}(t) = \underline{\Phi}(t)\underline{u}_t$ *with probability one, where for each* t, $\underline{\Phi}(t)$ *is a non-singular* $q \times q$ *matrix and* $\underline{u}_t$ *is a q-dimensional wide-sense martingale such that* $L_2(\underline{u}; t) = L_2(\underline{x}; t)$.

Proof. *Sufficiency.* Let $\underline{x}(t) = \underline{\Phi}(t)\underline{u}_t$ where $\underline{\Phi}(t)$ and $\underline{u}_t$ are as described above. Then for $s \leq t$ $(\underline{x}(t)|L_2^{(q)}(\underline{x}: s)) = (\underline{\Phi}(t)\underline{u}_t|L_2^{(q)}(xs)) = (\underline{\Phi}(t)\underline{u}_t|L_2^{(q)}(\underline{u}; s)) = \underline{\Phi}(t)\underline{u}_s$. Since $\underline{u}_s = \underline{\Phi}^{-1}(s)\underline{x}_s$ with probability one, we obtain that the transition matrix function $A(t,s) = \underline{\Phi}(t)\underline{\Phi}^{-1}(s)$.

Necessity. Let $\underline{x}(t)$-process be wide-sense Markov. Then denoting by $A(t,s)$ the transition matrix function we have for $s \leq t$,

(3. 1) $$(\underline{x}(t)|L_2^{(q)}(\underline{x}; s)) = A(t,s)\underline{x}_s \qquad \text{with probability one.}$$

(3. 2) $$A(u,s) = A(u,t)A(t,s) \qquad \text{for } s \leq t \leq u.$$

Let us now define, following Hida [6], for every real t, the function

$$\begin{aligned} \underline{\Phi}(t) &= A(t, s_0) \quad \text{if } s_0 \leq t \\ &= A^{-1}(s_0, t) \quad \text{if } t < s_0 \end{aligned}$$

where s_0 is a fixed real number. We now show that for all s, $t (s \leq t)$ real

(3. 3) $$A(t,s) = \underline{\Phi}(t)\underline{\Phi}^{-1}(s).$$

First of all, if $s < s_0 \leq t$, (3. 3) is a restatement of (3. 2): i.e., $A(t,s) = A(t,s_0)A(s_0,s)$. If $s_0 \leq s < t$ from (3. 2) we have $A(t,s)A(s,s_0) = A(t,s_0)$, i.e., $A(t,s) = A(t,s_0)A^{-1}(s,s_0)$ giving (3. 3) again. Finally if $s < t \leq s_0$ we again get from (3. 3), $A(s_0,s) = A(s_0,t)A(t,s)$ and hence $A(t,s) = \underline{\Phi}(t)\underline{\Phi}^{-1}(s)$. $\underline{\Phi}(t)$ is non-singular since $A(t,s)$ is. Therefore from (3. 1) and (3. 3) we have for $s < t$,

(3. 4) $$(\underline{x}(t)|L_2^{(q)}(\underline{x}; s)) = \underline{\Phi}(t)\underline{\Phi}^{-1}(s)\underline{x}_s \text{ with probability one.}$$

Hence $\underline{u}_t = \underline{\Phi}^{-1}(t)\underline{x}(t)$ is a martingale and $L_2(u; t) = L_2(\underline{x}; t)$. The proof of the Theorem is now complete.

The characterization of Theorem 3. 1 will be used later to study purely non-deterministic wide-sense Markov processes and their multiplicity.

However, as a first application we show that if $\underline{x}_0 = 0$ and $\underline{\phi}(t)$ is differentiable, then it satisfies the following differential equation with probability one.

$$\overset{\circ}{\underline{x}}(t) = A(t)\underline{x}(t) + M(t)\underline{\eta}(t) \quad t \geq 0 \tag{3. 5}$$

where $\eta(\cdot)$ is a multivariate "white noise" random process and

$A(t) = \overset{\circ}{\underline{\phi}}(t)\underline{\phi}^{-1}(t)$, $M(t) = \underline{\phi}(t)$. The equation (3. 5) is to be interpreted as $\underline{x}(t) = \int A(t)\,\underline{x}(t)\,dt + \int M(t)d\underline{u}(t)$, $\underline{\eta}_t$ being the "fictitious derivative" of $\underline{u}_t$.

THEOREM 3. 2. *Let $\{\underline{x}(t),\ 0 \leq t < \infty\}$ be a wide-sense Markov process satisfying (D). If further $x_0 = 0$ and $\underline{\phi}(t)$ of Theorem 3. 1 is continuously differentiable then $\underline{x}(t)$ satisfies equation (3. 5) for $t \geq 0$ where $\underline{\eta}_t$ is a q-variate white noise process and the matrix function $A(t) = \overset{\circ}{\underline{\phi}}(t)\underline{\phi}^{-1}(t)$, $M(t) = \underline{\phi}(t)$.*

The proof of the Theorem follows by substituting in (3. 5) $\underline{x}_t = \underline{\phi}(t)\underline{u}_t$.

We now take up the study of covariance function of a stationary wide-sense Markov process.

DEFINITION 3. 2. We say that a q-variate second order process $\{\underline{x}(t),\ -\infty < t < +\infty\}$ is stationary if $\Gamma(t,s) = [\underline{x}(t),\ \underline{x}(s)] = R(t-s)$ for $s < t$.

By Theorem 3. 1 and the definition of wide-sense martingale we get for $h \geq 0$,

$$R(h) = [\underline{x}(t+h),\ \underline{x}(t)] = \underline{\phi}(t+h)J(t,t)\underline{\phi}^*(t), \quad \text{where } J(t,s) = [\underline{u}(t),\ \underline{u}(s)]_{L_2^{(q)}(\Omega)}. \tag{3. 6}$$

Let $h = 0$, we get

$$R(0) = \underline{\phi}(t)J(t,t)\underline{\phi}^*(t). \tag{3. 7}$$

With $t = 0$ in (3. 6), one has

$$R(h) = \underline{\phi}(h)J(0,0)\underline{\phi}^*(0). \tag{3. 8}$$

Relations (3. 6) and (3. 8) imply $h \geq 0$, $t \geq 0$

$$R(h) = R(t+h)R^{-1}(t)R(0). \tag{3. 9}$$

With $R_1(t) = R(t)R^{-1}(0)$, (3. 9) reduces to

$$R_1(t+h) = R_1(t)R_1(h). \tag{3. 10}$$

We prove now the following theorem.

THEOREM 3. 3. *Let* $\{\underline{x}(t)\}$ $(-\infty < t < \infty)$ *be a q-dimensional stationary process satisfying assumption (D). Then it is wide-sense Markov if and only if the transition matrix function* $B(t) = e^{tQ}$ *for* $t \geq 0$ *where* $B(t) = A(t, 0)$ *and* Q *is uniquely determined constant* $q \times q$ *matrix none of whose eigenvalues has positive real part.*

Proof. Necessity. We have already shown that for $R_1(t) = R(t)R^{-1}(0)$ the equation (3. 10) holds. Further, from (D. 1) it follows that $R_1(t)$ is a continuous function and therefore $R_1(t) = e^{tQ}(t \geq 0)$ is the solution of (3. 10) where Q is a $q \times q$ constant matrix (see E. Hille and R.S. Phillips [11]). The assumption (D. 2) implies that $R_1(t)$ is non-singular and hence Q is uniquely determined by $R_1(t)$. Since $B(t) = R(t)R^{-1}(0)$ for $t \geq 0$ we have $B(t) = e^{tQ}$. Due to the fact that $\lambda(t) = \max_{j \leq q} \lambda_j(t)$ (where $\lambda_j(t)$ is j^{th} eigenvalue of $B(t)$) satisfies for all t, $|\lambda(t)| \leq \mathrm{tr}[R^{-1}(0)(R^{-1}(0))^*](\sum_{i=1}^{q} E|x_i(0)|^2)^2$ it follows that the eigenvalues of $Q = \lim_{t \to 0} \dfrac{B(t) - I}{t}$ has non-negative real parts.

The above result was first proved by J.L. Doob [5] for Stationary Gaussian Markov processes. It was reproved by Beutler [1] for wide-sense Markov processes. We have proved it because our proof is based directly on the characterization of Theorem 3. 1. Furthermore it brings out the form of $\underline{\phi}(t)$ in stationary case which will be utilized in Theorem 5. 1. It is also interesting to note that the fact that $R(t-s) = \phi(t)J(s, s)\phi^*(s)$ could enable one to obtain a general form for the covariance function of stationary wide-sense Markov processes (see Kalmykov [8]).

4. Multiplicity of purely non-deterministic wide-sense Markov processes: A second order q-variate process is called purely non-deterministic if $\bigcap_t L_2(\underline{x}; t) = \{0\}$ where $L_2(\underline{x}; t)$ is as defined in Section 2. Let us denote by $E_x(t)$ the projection operators from $L_2(\underline{x})$ (the subspace generated by $\bigcup_t L_2(\underline{x}; t)$) onto $L_2(\underline{x}; t)$. Then under assumption (D. 1) of Section 3 and pure non-determinism, we obtain (see H. Cramér [3])

(4. 1)

(i) $L_2(\underline{x})$ is separable

(ii) $E_x(+\infty) = I \;\; E_x(-\infty) = 0$

(iii) $E_x(t)E_x(s) = E_x(s)E_x(t) = E_x(\min(s, t))$

(iv) $E_x(t+0) = \lim_{n\to\infty} E_x\left(t + \dfrac{1}{n}\right) = E_x(t) = E_x(t-0) = \lim_{n\to\infty} E_x\left(t - \dfrac{1}{n}\right).$

In other words $\{E_x(t), -\infty < t < +\infty\}$ is a resolution of the identity in $L_2(\underline{x})$. A subset $A \subset L_2(\underline{x})$ is called a generating subset of $L_2(\underline{x})$ with respect to E if $L_2(\underline{x})$ is generated by $\{E(\Delta)f, f \in A$ and Δ a Borel subset of the real line$\}$. The idea of generating set is certainly not unique. However, it is known (see Yosida [12] p. 321), that such sets A can be ordered through their cardinality and there exists one with minimal cardinality. This minimal cardinal number, which because of 4. 1 (i) is at most countable, is called the multiplicity of E. Following H. Cramér [3] and T. Hida [6] we call this multiplicity the multiplicity of $\underline{x}(t)$. Our first result here is to show that under assumption (D) every q-variate wide-sense Markov process has multiplicity not exceeding q. For this purpose we need the following Lemma.

LEMMA 4. 1. *Let H be a separable Hilbert-space with H_1, H_2 be two subspaces of H such that $H_1 \perp H_2$ and $H = H_1 \oplus H_2$. Suppose $\{E_1(t)\}$ is a resolution of the identity in H_1 and $\{E_2(t)\}$ be a resolution of the identity in H_2 such that $E(t) = E_1(t) + E_2(t)$ is a resolution of the identity in H. If N_i is the multiplicity of $E_i (i = 1, 2)$ then multiplicity of E does not exceed $N_1 + N_2$.*

Proof. We are given $H_i = \mathfrak{S}\{E_i(\Delta)f, f \in A, \Delta$ a Borel subset of the real line$\}$, where $\mathfrak{S}$ denotes the "subspace generated by." Since card $A_1 = N_1$, card $A_2 = N_2$, and $E(\Delta)f = E_i(\Delta)f$ for $f \in A_i$ we get that $H = \mathfrak{S}\{E(\Delta)f, f \in A_1 \cup A_2, \Delta$ a Borel subset of real line$\}$. Thus multiplicity of $E \leq \text{card}\,(A_1 \cup A_2) \leq N_1 + N_2$ completing the proof.

LEMMA 4. 2. *Every purely non-deterministic univariate process $\{v(t); -\infty < t < +\infty\}$ with orthogonal increments has unit multiplicity.*

Proof. Let $\rho(\Delta) = \mathfrak{S}|v(t) - v(s)|^2$, $\Delta = [s, t]$. It is well-known (see Doob [4] Ch. IX) that $L_2(v) = \left\{\int_{-\infty}^{+\infty} f(t)v(dt), f \in L_2(\rho)\right\}$ where $\int_{-\infty}^{+\infty} f(t)v(dt)$ is a stochastic integral in the sense of Doob ([4] Ch. IX). Let $f \in L_2(\rho)$, where f is positive almost everywhere with respect to measure ρ. Then $\int_{-\infty}^{+\infty} f(t)v(dt) = f_0$ generates $L_2(v)$ completing the proof.

THEOREM 4. 1. *The multiplicity of a q-variate wide-sense Markov process satisfying assumption (D) does not exceed q.*

Proof. By Theorem 3. 1, $L_2(\underline{u}; t) = L_2(\underline{x}; t)$ for all t and hence in particular $L_2(\underline{x}) = L_2(\underline{u})$. Therefore $E_x(t)$, the projection from $L_2(\underline{x})$ onto $L_2(\underline{x}; t)$ is the same operator as $E_u(t)$ from $L_2(\underline{u})$ onto $L_2(\underline{u}; t)$. Therefore by definition of multiplicity, multiplicity of the process $\underset{\sim}{x}(t)$ is the same as that of $\underset{\sim}{u}(t)$. For the sake of simplicity, we shall establish that the multiplicity of a 2-variate wide-sense martingale does not exceed two. The general case being similar, this will conclude the proof. Define $v_1(t)=u_1(t)$, $v_2(t) = (I - P_{L_2(u_1)})u_2(t)$. Since $L_2(u_1) = L_2(u_1, t) \oplus \{u_1(\tau) - u_1(t) \ \tau \geq t\}$ by martingale property. But since $u_2(t) \perp \{u_1(\tau) - u_1(t), \ \tau \geq t\}$ we obtain that $v_2(t) = (I - P_{L_2(u_1\,;\,t)})u_2(t)$. Hence $L_2(\underset{\sim}{u}\,;\,t) = L_2(v_1\,;\,t) \oplus L_2(v_2\,;\,t)$. It can be easily seen that $v_2(t) = P_{L_2(v_2\,;\,t)}u_2(t)$. This implies that both $\{v_1(t) \ \ -\infty < t < +\infty\}$ and $\{v_2(t) \ \ -\infty < t < +\infty\}$ are mutually orthogonal processes with orthogonal increments. Hence each has multiplicity one by Lemma 4. 2. But $E_u(t) = E_{v_1}(t) + E_{v_2}(t)$ and $L_2(\underset{\sim}{u}) = L_2(v_1) \oplus L_2(v_2)$ and hence by Lemma 4. 1 we get multiplicity of $E_u \leq 2$. Q.E.D.

Before we conclude this section we want to recall here some ideas of Hida-Cramér theory. They are directly taken from G. Kallianpur and V. Mandrekar [7]. The following theorem of Hellinger-Hahn is well-known (see T. Hida [6]).

THEOREM H-H. *Let $L_2(\underset{\sim}{x})$ be the separable Hilbert-space and $E(t)$ be any resolution of the identity in $L_2(\underset{\sim}{x})$ (i.e., satisfies* 4. 1 (ii), (iii), (iv)) *then*

(i) $L_2(\underset{\sim}{x}) = \sum_{1}^{M} \oplus \mathscr{M}_f(i)$ *where* $\mathscr{M}_f(i) = \mathfrak{S}\{E(\Delta)f^{(i)}$, *$\Delta$ a Borel subset of the real line*$\}$.

(ii) *If $\rho_f(i)$ is the measure denoted by $\rho_f(i)(\Delta) = \|E(\Delta)f^{(i)}\|^2$ for each Borel set Δ, then $\rho_f(1) \gg \rho_f(2) \gg \cdots$.*

(iii) $\mathscr{M}_f(i) = \left\{\int_{-\infty}^{+\infty} f(u)Z_i(du)\,;\, f \in L_2(\rho_f(i))\right\}$ *where $Z_i(t) = E(t)f^{(i)}$ $(-\infty < t < +\infty,\ i = 1, 2, \cdots, M)$ are mutually orthogonal processes with orthogonal increments.*

(iv) $\{f^{(1)}, \cdots, f^{(M)}\}$ *is the minimal generating sequence.*

The processes with orthogonal increments are defined in Doob [4] Chapter IX.

The above theorem is essentially the main theorem of Hida [6] and Cramér [3]. It is quoted here in the form as to bring out the connection of multiplicity as defined by us and the multiplicity of a representation as defined by Hida and Cramér.

Applying the above theorem we get

(4. 2) $$x^{(i)}(t)=\sum_{j=1}^{M}\int_{-\infty}^{+\infty}F_{ij}(t,u)Z_j(du) \text{ where } \sum_{1}^{M}\int_{-\infty}^{+\infty}|F_{ij}(t,u)|^2\rho_f(j)(du)<\infty.$$

Equation (4. 2) gives the Hida-Cramér representation of a stochastic process where M is its multiplicity. It has the property $(s<t)$

(4. 3) $$E_x(s)x_i^{(i)}(t)=\sum_{1}^{M}\int_{-\infty}^{s}F_{ij}(t,u)Z_j(du).$$

A representation satisfying (4. 3) is called a canonical representation. A canonical representation is called proper canonical if

(4. 4) $$L_2(\underset{\sim}{x};t)=\sum_{j=1}^{M}\oplus L_2(Z_j;t).$$

Note that $L_2(Z_j;t)=\mathscr{M}_f(i)(t)=\mathfrak{S}\{E(\varDelta)f^{(i)},\ \varDelta$ a Borel subset of $(-\infty,t)\}$. It is proved by Kallianpur and Mandrekar ([7] Theorem 3. 1) that every canonical representation can be assumed to be proper canonical.

Now by Theorem 4. 1 we get that for wide-sense Markov process $M\leq q$. Hence one can write representation (4. 3) in the form of a vector stochastic integral. In the next section we define this concept following M. Rosenberg [9] and obtain an analytic characterization so that a canonical representation be proper canonical.

5. Vector stochastic integrals and analytic characterization of proper canonical representations: Let P, Q be $q\times M$ $(M\leq q)$ matrix-valued functions of real numbers. We say that (P,Q) is integrable with respect to an $M\times M$ hermitian-matrix-valued function ρ if the matrix $P\rho'Q^*$ is integrable elementwise with respect to $\mathrm{tr}\rho$ where ρ' denotes the matrix of densities of elements of ρ, with respect to $\mathrm{tr}\rho$. We then define $\int Pd\rho Q^*=\int P\rho'Q^*\mathrm{tr}\rho(du)$. P is said to be square-integrable $[\rho]$ if $\mathrm{tr}\left(\int Pd\rho P^*\right)$ is finite. If we denote by $\mathscr{L}_2(\rho)$ the class of all measurable P square integrable with respect to $[\rho]$ where functions P,Q such that $\{P(u)-Q(u)\}\rho'(u)=0$ a.e. $[\mathrm{tr}\rho]$ are identified. Then $\mathscr{L}_2(\rho)$ is a complete Hilbert space with gramian $[[P,Q]]=\mathrm{tr}\int Pd\rho Q^*$ and $\mathrm{tr}[[P,P]]=\text{norm } P$. We shall call $\underset{\sim}{\xi}$ an orthogonally scattered random vector-valued measure on the real line of dimension M if for each Borel set B, $\underset{\sim}{\xi}(B)\in L^{(M)}(\Omega)$ and for Borel sets A and B $[\underset{\sim}{\xi}(B),\ \underset{\sim}{\xi}(A)]=\rho(A\cap B)$ where ρ is a Hermitian-matrix-valued measure on the real line. With this setup, Rosenberg [9] defined the vector stochastic integral $\int_{-\infty}^{+\infty} P(u)\underset{\sim}{\xi}(du)$ for $P \in \mathscr{L}_2(\rho)$ in the same way as Doob does for the case $M = q$ ([4], p. 596). Further, if we denote by $\mathscr{L}_2(\underset{\sim}{\xi})$ the subspace of $L_2^{(M)}(\Omega)$ generated by $\{\xi(B),\ B \in \mathscr{B}\}$ with $q \times M$ matrices as coefficients, then he has the following theorem, with $\mathscr{B}$ denoting the Borel subsets on the real line.

THEOREM R. *The correspondence* $P \to \int_{-\infty}^{+\infty} P(u)\underset{\sim}{\xi}(du)$ *is an isomorphism from* $\mathscr{L}_2(\rho)$ *to* $\mathscr{L}_2(\underset{\sim}{\xi})$.

In our context $\underset{\sim}{Z}(B) = (Z_1(B), \cdots, Z_N(B))^*$ and $F(t,u)$ will be denoted by the matrix $\{F_{ij}(t,u)\}$. We then have from (4. 2) and Theorem 4. 1 that

$$\underset{\sim}{x}(t) = \int_{-\infty}^{t} F(t,u)\underset{\sim}{Z}(du);\ F \in \alpha_2(\rho) \text{ where } \rho(u) = \begin{bmatrix} \rho_f(1) & & 0 \\ & \ddots & \\ 0 & & \rho_f(M) \end{bmatrix}. \tag{5. 1}$$

If we denote by $\mathscr{L}_2(\underset{\sim}{Z}; t)$, the subspace of $\mathscr{L}_2(\underset{\sim}{Z})$ generated by $\{Z(B),\ B$ a Borel subset of $(-\infty,\ t)\}$, then we trivially have

$$L_2^{(q)}(\underset{\sim}{Z};\ t) = \mathscr{L}_2(\underset{\sim}{Z};\ t). \tag{5. 2}$$

We now give an analytical characterization of a proper canonical representation. This is a direct generalization of Theorem 1. 7 of [6].

THEOREM 5. 1. *A canonical representation* (5. 1) *is proper canonical if and only if*

$$\int_{-\infty}^{t} P(u)d\rho(u)F^*(t,u) = 0 \ \textit{for}\ t \leq t_0 \ \textit{implies}\ P(u) = 0 \ \textit{a.e.}\ [\rho] \ \textit{on}\ (-\infty, t) \tag{5. 3}$$

where $P \in \mathscr{L}_2(\rho)$.

Proof. Sufficiency. Let (5. 3) hold and suppose that there is a t_0 with $L_2(\underset{\sim}{Z};\ t_0) \neq L_2(\underset{\sim}{x};\ t_0)$. Since by canonical property $L_2^{(q)}(\underset{\sim}{x};\ t_0) \subseteq L_2(\underset{\sim}{Z};\ t_0)$ we have a $\underset{\sim}{V} \in \mathscr{L}_2(\underset{\sim}{Z};\ t)$ (see 5. 2) such that $[\underset{\sim}{V}, \underset{\sim}{x}(t)] = 0$ for $t \leq t_0$. By Theorem R we have $\underset{\sim}{V} = \int_{-\infty}^{t_0} P(u)Z(du)$ and $\neq 0$ and $\int_{-\infty}^{t} P(u)\,\underset{\sim}{Z}\,(du)\,F^*(t,\ u) = 0$ for $t \leq t_0$. But (5. 3) this implies $P(u) = 0$ a.e. $[\rho]$ contradicting $\underset{\sim}{V} \neq 0$.

Necessity. Suppose that $L_2(\underset{\sim}{Z};\ t) = L_2(\underset{\sim}{x};\ t)$ for each t and let t_0 be a real number such that

$$\int_{-\infty}^{t} P(u)\underset{\sim}{Z}(du)F^*(t,u) = 0 \text{ for } t \leq t_0. \tag{5. 4}$$

Observe that from the proper canonical property $L_2^{(q)}(\underset{\sim}{x}; t_0) = H^{(q)}(\underset{\sim}{x}; t_0) = \mathscr{L}_2(\underline{Z}; t_0)$. Hence the vector $\underline{V} = \int_{-\infty}^{t} P(u)\underline{Z}(du)$ belongs to $L_2^{(q)}(\underset{\sim}{x}; t_0)$. But (5. 4) implies that $[\underline{V}, \underset{\sim}{x}(t)] = 0$ for all $t \leq t_0$. Hence $\underline{V} = 0$ giving $\underline{P}(u) = 0$ a.e. $[\rho]$. This proves the theorem.

In the next section we use this theorem to obtain the representation of purely non-deterministic processes.

6. Representation of a purely non-deterministic wide-sense Markov process and the result of Doob: In this section we obtain the representation of a purely-nondeterministic Markov process and as a consequence obtain the representation [(4. 3. 2 of [5]). The main theorem is as follows.

THEOREM 6. 1. *Let $\underset{\sim}{x}(t)$ be a continuous parameter purely non-deterministic process satisfying (D). Then it is wide sense Markov if and only if*

$$\underset{\sim}{x}(t) = \int_{-\infty}^{t} \underline{\bar{\phi}}(t)G(u)\underline{Z}(du) \tag{6. 1}$$

where

(i) $\underline{\bar{\phi}}(t)$ *is as in Theorem* 3. 1,

(ii) $\underline{Z}$ *is an orthogonally scattered vector random function with*

$$[\underline{Z}(B),\ \underline{Z}(A)] = \begin{pmatrix} \rho_1(A \cap B) & & 0 \\ & \ddots & \\ 0 & & \rho_M(A \cap B) \end{pmatrix} = \rho(A \cap B)$$

for A, B Borel subsets of the real line,

(iii) $G \in \mathscr{L}_2(\rho)$

(iv) $L_2(\underline{Z}; t) = L_2(x; t)$.

Proof. Sufficiency. Define $\underline{u}(t) = \int_{-\infty}^{t} G(u)\underline{Z}(du)$. Then it suffices to prove that $\underline{u}(t)$ is a wide-sense Martingale. As then by Theorem 3. 1 the result will follow. Consider $s < t$ and $L_2^{(q)}(\underline{u}; s)$. Then

$$(\underline{u}(t) \mid L_2^{(q)}(\underline{u}; s)) = (\underline{u}(t) \mid L_2^{(q)}(\underset{\sim}{x}; s)) = (\underline{u}(t) \mid L_2^{(q)}(\underline{Z}; s)), \tag{6. 2}$$

where the first equality follows from non-singularity of $\phi(t)$ and the second (iv) of the hypothesis. Hence

(6. 3) $$(\underline{u}(t)\,|\,L_2^{(q)}(\underline{u}\,;\,s)) = \left(\int_{-\infty}^{t} G(u)\underline{Z}(du)\,|\,L_2^{(q)}(\underline{Z}\,;\,s)\right) = \int_{-\infty}^{s} G(u)d\underline{Z}(u).$$

The last equality in (6. 3) is a consequence of the fact that $\int_s^t G(u)d\underline{Z}(u) \perp L_2^{(q)}(Z; s)$. Hence $\underline{u}(t)$ is a wide-sense Martingale completing the sufficiency part.

Necessity. By wide-sense Markov hypothesis we obtain that $s < t$

(6. 4) $$\underline{x}(t) - A(t,s)\underline{x}_s \perp L_2^{(q)}(\underline{x}\,;\,\sigma) \text{ for } \sigma \leq s.$$

Equivalently (6. 4) gives

(6. 5) $$\int_{-\infty}^{\sigma} [F(t,u) - A(t,s)F(s,u)]\rho(du)F^*(\sigma,u) = 0 \text{ for } \sigma \leq s.$$

By Theorem (5. 1) we have

(6. 6) $$F(t,u) = A(t,s)\,F(s,u) \text{ a.e. } [\rho] \text{ on } (-\infty, s).$$

However, as in Theorem 3. 1 $A(t,s) = \underline{\phi}(t)\underline{\phi}^{-1}(s)$ and hence $F(t,u) = \underline{\phi}(t)\underline{\phi}^{-1}(s)F(s,u)$ a.e. $[\rho]$ on $(-\infty, s)$ i.e.,

(6. 7) $$\underline{\phi}^{-1}(t)F(t,u) = \underline{\phi}^{-1}(s)F(s,u) \text{ a.e. } [\rho] \text{ on } (-\infty, s).$$

From equation (6. 7) and the fact that $\| F(t,u) - F(s,u) \|_{\mathscr{L}_2(\rho)} \to 0$ as $s \to t$. From (D. 1) we obtain that $G(u) = \underline{\phi}^{-1}(t)F(t,u)$ is independent of t. Hence $F(t,u) = \underline{\phi}(t)G(u)$ on $(-\infty, t)$ a.e. $[\rho]$. This completes the proof since (ii), (iii) and (iv) are consequences of the properties of proper canonical representation.

Now to obtain the result of Doob we appeal to the following theorem

THEOREM KM (*G. Kallianpur and V. Mandreker* [7] *Theorem* (5. 1)). *If $\underline{x}(t)$ is a q-variate purely non-deterministic stationary process satisfying* (D. 1), *then*

(6. 7) $$\underline{x}(t) = \int_{-\infty}^{t} K(t-u)\underline{\xi}(du) \quad \textit{where } L_2(\underline{x}\,;\,t) = L_2(\underline{\xi}\,;\,t);$$

(i) $\underline{\xi}(\Delta) = (\xi_1(\Delta), \xi_2(\Delta), \cdots, \xi_M(\Delta))$ *with*

$$\xi_i(\Delta) = \int_\Delta \left[\frac{d\rho_f(i)}{d\mu}\, u\right]^{-\frac{1}{2}} Z_i(du)$$

for each Borel set Δ on the real line,

(ii) $K(t - \cdot) \in \alpha_2(\rho)$,

(iii) *M is the multiplicity of* $\underset{\sim}{x}(t)$.

We would like to remark that as a consequence of (i) $[\underset{\sim}{\xi}(\Delta), \underset{\sim}{\xi}(\Delta')] = \mu(\Delta \cap \Delta')I$ where μ is the Lebesgue measure on the real line and Δ, Δ' are Borel subsets. I denotes $M \times M$ identity matrix. From (6. 1) and (6. 7) we obtain that

$$K(t-u) = \underset{-}{\bar{\phi}}(t)H(u) \qquad \text{a.e. } \mu \tag{6. 8}$$

where the equality is taken elementwise and $H(u) = G(u)\sum(u)$ where

$$\sum(u) = \begin{pmatrix} \left(\frac{d\rho_f(i)}{d\mu}\right)^{-\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 \quad {\scriptstyle M\times M} & & \left(\frac{d\rho_f(i)}{d\mu}\right)^{-\frac{1}{2}} \end{pmatrix}.$$

Without loss of generality we can assume that (6. 8) holds at $u = 0$, otherwise the change is multiplication by a constant non-singular matrix. Putting $u = 0$ in (6. 8) we get

$$F(t) = \underset{-}{\bar{\phi}}(t)H(0) \quad t \geq 0 \tag{6. 9}$$

i.e., $F(t) = e^{tQ}S$ where $S = \underset{-}{\bar{\phi}}(0)H(0)$ from Theorem 3. 3. We have thus,

THEOREM 6. 2. *Let* $\underset{\sim}{x}(t)$ *be a purely non-deterministic process satisfying* (D. 1). *It is widesense stationary Markov if and only if*

$$\underset{\sim}{x}(t) = \int_{-\infty}^{t} e^{(t-u)Q}S\underset{\sim}{\xi}(du) \tag{6. 10}$$

where Q *is as in Theorem* (3. 3), $[\underset{\sim}{\xi}(\Delta), \underset{\sim}{\xi}(\Delta')]_{L_2(q)(\underset{\sim}{x})} = \mu(\Delta \cap \Delta')I$. I *is an* $M \times M$ *identity matrix where* M *is the rank of the process.*

The fact that M is the rank of the process from the representation (6. 10).

In comparing (6. 10) to Doob ([5]) we observe that Doob does not use Gaussian hypothesis. If we denote by $\zeta(t) = \int_{-\infty}^{t} e^{uQ}S\underset{\sim}{\xi}(du)$ then $\underset{\sim}{\xi}(t)$ is the ζ-process of equation (4. 3. 14) of Doob. M will then correspond to the number of ones occuring in his diagonal matrix U (see (4. 3. 18) In [5]).

7. Concluding Remark and Acknowledgements:

Remark. Theorem 3. 1 opens up the question of what processes can be represented at $\sum_{1}^{N} \underset{-}{\bar{\phi}}_i(t)u_i(t)$ where $\underset{-}{\bar{\phi}}_i(t)$ are some matrix functions and

$\underline{u}_i(t)$ are widesense martingales. It has been established by the author [AMS (1965) Abstract] that these lead under suitable conditions to continuous parameter N-ple Markov processes. Extension to such processes of the analytic questions studied here are being investigated and will be published later.

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