T. Tugué and S. Nagata
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# A REMARK ON PEIRCE'S LAW

TOSIYUKI TUGUÉ and SHÛRÔ NAGATA

Before stating the purpose, we explain some propositional logics treated in this paper. The logical symbols we use are: implication $\rightarrow$, conjunction $\wedge$, disjunction $\vee$, and the propositional constant $\curlywedge$ denoting contradiction. The axioms for the intuitionistic propositional logic (denoted by ***LJS***) are:

(I) $p \rightarrow (q \rightarrow p)$, $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,

(C) $(p \wedge q) \rightarrow p$, $(p \wedge q) \rightarrow q$, $(r \rightarrow p) \rightarrow ((r \rightarrow q) \rightarrow (r \rightarrow (p \wedge q)))$,

(D) $p \rightarrow (p \vee q)$, $q \rightarrow (p \vee q)$, $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r))$,

(F) $\curlywedge \rightarrow p$.

The rules of inference are *modus ponens* and *substitution*. The system characterized by the axioms (I) we call the primitive propositional logic (denoted by ***LOS***), which is the propositional part of the primitive logic ***LO*** introduced in Ono [3]. ***LOS*** is also known as the positive implicational logic. The axioms (I), (C), (D) characterize the (full) positive propositional logic (denoted by ***LPS***). Not all classically true formulas expressible in ***LOS*** are derivable from (I); they are derivable from (I) together with the axiom known as *Peirce's law*:

(P) $((p \rightarrow q) \rightarrow p) \rightarrow p$.

The axioms (I), (C), (D), (P) are sufficient for the derivation of all classically true formulas expressible in ***LPS***. Moreover, the axioms (I), (C), (D), (F), (P) characterize the classical propositional logic (denoted by ***LKS***). Indeed, all classically true propositional formulas are provable in ***LKS***. Finally, by deleting the axiom (F) from ***LJS***, we obtain Johansson's minimal propositional logic (denoted by ***LMS***). It is easy to see that the following formula

(M) $((p \rightarrow \curlywedge) \rightarrow p) \rightarrow p$

is equivalent to *the law of the excluded middle* in ***LMS***. Furthermore, it

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should be remarked that (M) is not equivalent to (P) in ***LMS***, as is well-known; in ***LJS***, however, (M) is equivalent to (P).

Now, let us consider the following formula:

(P*) $((p \to (((q \to r) \to q) \to q)) \to p) \to p$.

It is shown in Troelstra [4] as well as in Nagata [2] that (P*) is strictly weaker than (P) in ***LOS***. This fact suggests us a method for weakening Peirce's law, and this is really carried out in Miura and Nagata [1]. Indeed, in [1] (and also in [2] and [4]), various formulas of the type:

(*) $((p \to A) \to p) \to p$

are given and shown that these are strictly weaker than (P) (in ***LOS***, for example). Now, we remark that the formulas $A$ in (*) appearing in [1], [2], and [4] are all classically true. Thus, it would be a natural course of matter to raise the following question: Is the formula (*) equivalent to Peirce's law (P) in some propositional logic if a formula $A$ is not classically true? The answer is obviously "no." In fact, take $q \to p$ as $A$, then (*) is evidently provable in ***LOS***; hence, though $q \to p$ is not classically true, the formula (*) is not equivalent to (P) in any propositional logic. If we assume, however, that $p \to A$ is not classically true, then we can assert that the formula (*) is equivalent to (P) in some propositional logic, and vice versa. The purpose of this paper is to prove the following theorem.

THEOREM. *For any propositional variable $p$ and for any formula $A$ expressible in **LOS** (in **LPS**, or in **LJS**), the formula* (*) *is equivalent to Peirce's law* (P) *in **LOS** (in **LPS**, or in **LJS**) if and only if $p \to A$ is not classically true.*

*Remark.* The part "if" of Theorem does not hold for ***LMS***. In fact, (M) is not equivalent to (P) in ***LMS***, but $p \to \curlywedge$ is not classically true.

At first, we state two lemmas which are helpful to prove our theorem. One of them is a lemma proved in Tugué [5] (p. 304). Following [5], we denote by $v$ an evaluating function of the ordinary two-valued truth table whose values are 0 (truth) and 1 (falsity). Let $A$ be a formula and $p_1, \cdots, p_n$ be all propositional variables occurring in $A$. Given $n$-tuple $v(p_1), \cdots, v(p_n)$ of values of $p_1, \cdots, p_n$, we shall denote, as convention, the variables assigned the value 0 by $r_1, \cdots, r_u$, the rest by $s_1, \cdots, s_v$. Then, the following lemma holds for ***LOS*** (or ***LPS***).

LEMMA 1. *Let $A$ be a formula expressible in* **LOS** *(or in* **LPS***). For the given $n$-tuple $v(p_1), \cdots, v(p_n)$ of values of variables occurring in $A$,*

$$r_1, \cdots, r_u, \ s_1 \to s_2, \ s_2 \to s_3, \ \cdots, \ s_v \to s_1 \underset{\substack{\boldsymbol{LOS} \\ (\boldsymbol{LPS})}}{\vdash} A,$$

*or*

$$r_1, \cdots, r_u, \ s_1 \to s_2, \ s_2 \to s_3, \ \cdots, \ s_v \to s_1 \underset{\substack{\boldsymbol{LOS} \\ (\boldsymbol{LPS})}}{\vdash} A \to s_1,$$

*according as $v(A) = 0$ or 1.*

Lemma 1 is restricted to the negationless (or positive) propositional logics. We can extend this to the propositional logic with negation concept. That is, the following lemma holds for **LJS**, where $v(\curlywedge) = 1$.

LEMMA 2. *For the given $n$-tuple $v(p_1), \cdots, v(p_n)$ of values of variables occurring in $A$,*

$$r_1, \cdots, r_u, \ s_1 \to \curlywedge, \ \cdots, \ s_v \to \curlywedge \underset{\boldsymbol{LJS}}{\vdash} A,$$

*or*

$$r_1, \cdots, r_u, \ s_1 \to \curlywedge, \ \cdots, \ s_v \to \curlywedge \underset{\boldsymbol{LJS}}{\vdash} A \to \curlywedge,$$

*according as $v(A) = 0$ or 1.*

Now, by making use of Lemma 1, we can prove the following:

LEMMA 3. *For any propositional variable $p$ and for any formula $A$ expressible in* **LOS** *(or in* **LPS***), if $p \to A$ is not classically true, then the formula* (*) *is equivalent to* (P) *in* **LOS** *(or in* **LPS***).*

*Proof.* It is obvious that (*) is derivable from (P) in **LOS** (**LPS**). So, we have only to show that (P) is derivable from (*) in **LOS** (**LPS**) under the assumption that $p \to A$ is not classically true. If $p \to A$ is not classically true, $v(p \to A)$ is not identically equal to 0. Let $p_1, \cdots, p_n$ be all variables occurring in $A$. Then, for some $n$-tuple $v(p_1), \cdots, v(p_n)$, $v(A) = 1$. Let us fix an $n$-tuple $v(p_1), \cdots, v(p_n)$ such that $v(A) = 1$. Now, consider a formula $A^*$ obtained from $A$ by substituting $p$ for all variables $p_i$ such that $v(p_i) = 0$, $q$ for all variables $p_j$ such that $v(p_j) = 1$. Then, $A^*$ is a formula expressible in **LOS** (**LPS**) in which no variables other than $p$ and $q$ occur. Moreover, $v(A^*) = 1$ when $v(p) = 0$ and $v(q) = 1$. Hence, by virtue of Lemma 1, we have $p, q \to q \underset{\substack{\boldsymbol{LOS}\\(\boldsymbol{LPS})}}{\vdash} A^* \to q$. From this, $((p \to A^*) \to p) \to p \underset{\substack{\boldsymbol{LOS}\\(\boldsymbol{LPS})}}{\vdash} ((p \to q) \to p) \to p$. Since $((p \to A^*) \to p) \to p$ is derivable from (*) in ***LOS*** (***LPS***), we can conclude that (P) is derivable from (*) in ***LOS*** (***LPS***).

Similarly, the following lemma is proved by making use of Lemma 2.

LEMMA 4. *For any propositional variable $p$ and for any formula $A$, if $p \to A$ is not classically true, then the formula* (*) *is equivalent to* (P) *in* ***LJS***.

*Proof.* Assume that $p \to A$ is not classically true. Since (*) is derivable from (P) in ***LJS*** and (P) is equivalent to (M) in ***LJS***, we have only to show that (M) is derivable from (*) in ***LJS***. We define $A^*$ as in the proof of Lemma 3; i.e., $A^*$ is obtained from $A$ by substituting $p$ for all variables $p_i$ such that $v(p_i) = 0$, $\curlywedge$ for all variables $p_j$ such that $v(p_j) = 1$. Then, $A^*$ contains only one variable $p$, and $v(A^*) = 1$ when $v(p) = 0$. By virtue of Lemma 2, we have $p \underset{\boldsymbol{LJS}}{\vdash} A^* \to \curlywedge$. From this, $((p \to A^*) \to p) \to p \underset{\boldsymbol{LJS}}{\vdash} ((p \to \curlywedge) \to p) \to p$. Hence, (M) is derivable from (*) in ***LJS***. So, (P) is derivable from (*) in ***LJS***.

Finally, we show the converse of Lemmas 3 and 4. This is stated as follows.

LEMMA 5. *For any propositional variable $p$ and for any formula $A$ expressible in* ***LOS*** *(in* ***LPS***, *or in* ***LJS***), *if the formula* (*) *is equivalent to* (P) *in* ***LOS*** *(in* ***LPS***, *or in* ***LJS***), *then $p \to A$ is not classically true.*

*Remark.* Lemma 5 holds also for ***LMS***.

In order to prove this, we use another evaluating function $v^*$ of the three-valued truth table whose values are 0, 1, 2. This table is defined as follows:

$$v^*(p \to q) = \begin{cases} v^*(q) & \text{if } v^*(p) < v^*(q), \\ 0 & \text{otherwise,} \end{cases}$$

$$v^*(p \wedge q) = max\,(v^*(p),\ v^*(q)),$$

$$v^*(p \vee q) = min\,(v^*(p),\ v^*(q)),$$

$$v^*(\curlywedge) = 2.$$

For this function $v^*$, we can easily see that

$$v^*(((p \to q) \to p) \to p) = \begin{cases} 1 & \text{if } v^*(p) = 1 \text{ and } v^*(q) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for a formula $A$, if $v^*(p \to A)$ is never equal to 2, then $v^*(((p \to A) \to p) \to p)$ is identically equal to 0. Using these facts, we can prove Lemma 5.

*Proof of Lemma* 5. Assume that the formula (*) (*i.e.* $((p \to A) \to p) \to p)$ is equivalent to (P) in ***LOS*** (***LPS***, ***LJS***). We wish to show that $p \to A$ is not classically true. Suppose that $p \to A$ is classically true. Then, $v^*(p \to A)$ is never equal to 2. Hence, $v^*(((p \to A) \to p) \to p)$ is identically equal to 0. Therefore, (P) is not derivable from (*) in ***LOS*** (***LPS***, ***LJS***). This contradicts to our assumption. Accordingly, we can conclude that $p \to A$ is not classically true.

Our theorem is immediate from Lemmas 3, 4, and 5.

## References

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*Nagoya University.*