THE CONTINUATION OF SECTIONS OF TORSION-FREE COHERENT ANALYTIC SHEAVES

HIROTAKA FUJIMOTO

§1. **Introduction.** The purpose of this paper is to extend the continuation theorem of holomorphic functions, especially the generalized Hartogs-Osgood's theorem given in the previous papers [6] and [8], to the case of sections of torsion-free coherent analytic sheaves over a reduced complex space. In the following, we restrict ourselves only to reduced complex spaces.

For the local continuation, we have

THEOREM. *Let D be an open subset of a complex space X with structure sheaf $\mathcal{O}$ which is $*$-strongly s-concave at x. If a coherent $\mathcal{O}$-module $\mathcal{F}$ is torsion-free at x and a universal denominator u for $\mathcal{F}_x$ satisfies* l. $\dim_x(\mathcal{F}/u\mathcal{F})\geqq s$, *every section of $\mathcal{F}$ over D is uniquely continuable to a neighborhood of x* (see Def. 2. 8 in [6], p. 56, Def. 3.3 and Def. 4.1).

The proof is essentially due to W. Thimm [12], though he studied only the continuation of holomorphic functions on an open subset of an analytic subset in C^n to an analytically thin set. We shall state here the outline of the proof. We can regard $\mathcal{F}$ as a sub-$\mathcal{O}$-Module of the tensor product $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{M}$, where $\mathcal{M}$ denotes the sheaf of germs of meromorphic functions. While, $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{M}$ is isomorphic to the direct sum of several subsheaves of $\mathcal{M}$ with suitable conditions (§2). Therefore, any section of $\mathcal{F}$ over D can be identified with a system of meromorphic functions (f_i), which may be assumed to have a common denominator u depending on $\mathcal{F}_x$ only (§3). Roughly speaking, we call such u a universal denominator for $\mathcal{F}_x$. We continue the numerator of each f_i to a neighborhood of x as weakly holomorphic functions and obtain a system of meromorphic functions $(\tilde{f}_i)$. We shall show that such $(\tilde{f}_i)$ determines uniquely a continuation of f to a neighborhood of x under the assumption in the above theorem (§4).

For the global continuation, we can prove

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THEOREM. *Let X be a Stein space and $\mathscr{F}$ be a torsion-free coherent $\mathcal{O}$-Module over X with the property that for any $x \in X$ there exists a universal denominator u for $\mathscr{F}_x$ with* $\mathrm{l.\,dim}_x (\mathscr{F} / u\mathscr{F}) \geqq 1$. *If an open subset D of X and a compact subset K of D satisfy the condition that each irreducible component of D is irreducible in $D - K$, then every section of $\mathscr{F}$ over $D - K$ is uniquely continuable to D.*

We have also the analogous continuation theorems for a considerably general, not necessarily Stein complex space X if we assume the local irreducibility of X and for more general X's if we assume the suitable convexity of the domain D(§5).

Our assumption on $\mathscr{F}_x$ in the above theorems are closely related to the notion of homological codimension and also "starke homologische Codimension" in the sense of Scheja [11], p. 89. As to the continuation of sections of torsion-free coherent $\mathcal{O}$-Modules, our results are generalizations of the results in [3], [11] and [12] etc.

§2. The tensor product $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{M}$. Let X be a complex space with structure sheaf $\mathcal{O}$ and $\mathscr{F}$ be a coherent $\mathcal{O}$-Module over X. We consider the tensor product $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{M}$, where $\mathscr{M}$ denotes the sheaf of all germs of meromorphic functions on X.

Tensorizing the inclusion map $\mathcal{O} \to \mathscr{M}$ by $\mathscr{F}$, we get the canonical map $\tau: \mathscr{F} \to \mathscr{F} \otimes_{\mathcal{O}} \mathscr{M}$.

(2. 1) *The kernel $T(\mathscr{F})$ of τ is a coherent sub-$\mathcal{O}$-Module of $\mathscr{F}$ and so the image $\tau(\mathscr{F})$ is a coherent sub-$\mathcal{O}$-Module of $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{M}$* (c.f. Andreotti [2], Proposition 6, p. 14 and Houzel [5], Exposé 20, p. 2).

In fact, $T(\mathscr{F})$ is the sub-$\mathcal{O}$-Module of $\mathscr{F}$ such that each stalk $T(\mathscr{F})_x$ ($x \in X$) is the intersection of primary sub-$\mathcal{O}_x$-modules of $\mathscr{F}_x$ associated with minimal prime ideals in $\mathcal{O}_x$. Putting $\Pi = \{$all minimal prime Ideals in $\mathcal{O}\}$, we see easily it is nothing but the Π-component of a subsheaf (0) of $\mathscr{F}$ in the sense of Fujimoto [7]. The coherency of $T(\mathscr{F})$ is an immediate consequence of Theorem 6.3 in [7].

We say $\mathscr{F}$ is torsion-free at $x \in X$ if an $\mathcal{O}_x$-module $\mathscr{F}_x$ is torsion-free, namely, for any non-zero divisor u in $\mathcal{O}_x$ $uf = 0$ ($f \in \mathscr{F}_x$) implies $f = 0$ and $\mathscr{F}$ is torsion-free on X if $\mathscr{F}$ is torsion-free everywhere on X. Obviously, $\mathscr{F}$ is torsion-free at x if and only if $T(\mathscr{F})_x = (0)$. By (2. 1), the set $\{x \in X; \mathscr{F}$ is torsion-free at $x\}$ is open.

Suppose that X is irreducible at $x \in X$. Then $(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M})_x = \mathscr{F}_x \otimes_{\mathscr{O}_x} \mathscr{M}_x$ is considered as a module over the field $\mathscr{M}_x$ and any maximal $\mathscr{O}_x$-linearly independent family in $\mathscr{F}_x$ defines a base of $(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M})_x$ over $\mathscr{M}_x$. Therefore $\text{rank}_x \mathscr{F} = \dim_{\mathscr{M}_x} (\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M})_x$ is the number of elements of a maximal $\mathscr{O}_x$-linearly independent family in $\mathscr{F}_x$. Moreover, if X is locally irreducible and connected, $\text{rank}_x \mathscr{F}$ is constant for any $x \in X$. In this case, the rank of $\mathscr{F}$ over X is defined as $\text{rank}_x \mathscr{F}$ with an arbitrarily fixed $x \in X$. For an irreducible but not necessarily locally irreducible complex space we define $\text{rank}_X \mathscr{F} = \text{rank}_x \mathscr{F}$ with an arbitrarily fixed $x \in X - S(X)$, where $S(X)$ denotes the totality of all singularities of X.

LEMMA 2. 2. *If X is an irreducible Stein space and a coherent $\mathscr{O}$-Module $\mathscr{F}$ over X is of rank $k (\geqq 1)$, we can find k sections $f_1, \ldots, f_k \in \Gamma(X, \mathscr{F})$ such that for each $x \in X$*

1^0, *$\{f_1, \ldots, f_k\}$ defines an $\mathscr{O}_x$-linearly independent family in $\mathscr{F}_x$,*

2^0, *there exists a non-zero divisor a in $\mathscr{O}_x$ with $a \cdot \mathscr{F}_x \subseteqq \mathscr{O}_x(f_1, \ldots, f_k)$.*

Proof. Firstly, we choose a normal point x_0 of X arbitrarily. Since X is Stein, $\mathscr{F}_{x_0}$ is generated by several elements in $\Gamma(X, \mathscr{F})$ as an $\mathscr{O}_{x_0}$-module. By a suitable choice of elements of these global sections, we can find $f_1, \ldots, f_k$ in $\Gamma(X, \mathscr{F})$ which define a maximal $\mathscr{O}_{x_0}$-linearly independent family in $\mathscr{F}_{x_0}$. The family $\{f_1, \ldots, f_k\}$ gives a base of the $\mathscr{O}_{x_0}$-module $(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M})_{x_0}$ and therefore satisfies the conditions 1^0 and 2^0 in Lemma 2. 2 for a special point $x_0 \in X$. We want to prove this family $\{f_1, \ldots, f_k\}$ satisfies the same conditions for every $x \in X$.

To examine the condition 1^0, we define the $\mathscr{O}$-Homomorphism $\tilde{f} := (f_1, \ldots, f_k)$: $\mathscr{O}^k \to \mathscr{F}$ as usual and consider the sheaf $\mathscr{N} := \text{Ker}(\tilde{f})$. Since $\mathscr{N}$ is a sub-$\mathscr{O}$-module of $\mathscr{O}^k$, the analytic set $|\mathscr{N}| := \{x \in X; \mathscr{N}_x \neq (0)\}$ has the non-empty open kernel if $|\mathscr{N}| \neq \phi$. By the assumption of the irreducibility of X, $|\mathscr{N}|$ must be equal to X or empty. While, it cannot happen to be $|\mathscr{N}| = X$ because $x_0 \notin |\mathscr{N}|$. Thus, $|\mathscr{N}| = \phi$ or $\mathscr{N} = (0)$. This shows that for each $x \in X$ $\tilde{f}$ gives the injective $\mathscr{O}_x$-homomorphism $\tilde{f}_x$: $\mathscr{O}_x^k \to \mathscr{F}_x$ and so $\{f_1, \ldots, f_k\}$ defines an $\mathscr{O}_x$-linearly independent family in $\mathscr{F}_x$.

Next, we shall confirm the condition 2^0. Let X^* be the normalization of X with projection $\mu: X^* \to X$. We denote the sheaf of germs of holomorphic functions on X^* by $\mathcal{O}^*$ and of meromorphic functions on X^* by $\mathcal{M}^*$. The inverse image $\mathcal{F}^* := \mu^*(\mathcal{F})$ of $\mathcal{F}$ by μ is a coherent $\mathcal{O}^*$-Module over X^*. Since X is normal at x_0, the section $f_1^* := \mu^*(f_1), \dots, f_k^* := \mu^*(f_k)$ in $\Gamma(X^*, \mathcal{F}^*)$ give a maximal $\mathcal{O}_{y_0}^*$-linearly independent family in $\mathcal{F}_{y_0}^*$ for $y_0 = \mu^{-1}(x_0)$. Therefore, by the same argument as above, $\{f_1^*, \dots, f_k^*\}$ defines an $\mathcal{O}_y^*$-linearly independent family in $\mathcal{F}_y^*$ for each $y \in X^*$. Moreover, since $\mathrm{rank}_y \mathcal{F}^*$ is constantly equal to $\mathrm{rank}_{y_0} \mathcal{F}^* = \mathrm{rank}_{x_0} \mathcal{F} = k, \{f_1^*, \dots, f_k^*\}$ gives a base of $(\mathcal{F}^* \otimes_{\mathcal{O}^*} \mathcal{M}^*)_y$ over the field $\mathcal{M}_y^*$. Consequently, for each $y \in X^*$ there exists a non-zero divisor $a^{(y)} \in \mathcal{O}_y^*$ with $a^{(y)} \cdot \mathcal{F}_y^* \subseteqq \mathcal{O}_y^*(f_1^*, \dots, f_k^*)$.

Now, let x be a point in X. There exists an arbitrarily small neighborhood U of x with the irreducible decomposition

$$U = U_1 \cup \dots \cup U_t$$

where U_i is irreducible at x. By y_i denoting the point in X^* which corresponds to U_i, we may assume $\mu^{-1}(U)$ is the disjoint union of connected neighborhoods V_i of y_i with $a^{(y_i)} \in \Gamma(V_i, \mathcal{O}^*)$ $(1 \leqq i \leqq t)$ and moreover there exists a universal denominator on U, namely, a holomorphic function u on U not vanishing identically on each U_i such that $u \cdot \mu_*(\mathcal{O}^*) \subseteqq \mathcal{O}$ on U for the direct image $\mu_*(\mathcal{O}^*)$ of $\mathcal{O}^*$ by μ. Putting $\tilde{a} = a^{(y_i)}$ on V_i, we define a section $\tilde{a} \in \Gamma(\mu^{-1}(U), \mathcal{O}^*) = \Gamma(U, \mu_*(\mathcal{O}^*))$. Thus we get a section $a' := u\tilde{a} \in \Gamma(U, \mathcal{O})$. On the other hand, there exists a non-zero divisor v in $\mathcal{O}_x$ such that $v \cdot T(\mathcal{F})_x = 0$. To complete the proof, it suffices to take the non-zero divisor $a = va'$, which will be shown to satisfy the condition $a \cdot \mathcal{F}_x \subseteqq \mathcal{O}_x(f_1, \dots, f_k)$. Take an arbitrary $f \in \mathcal{F}_x$. Since $a^{(y_i)} \cdot \mathcal{F}_{y_i}^* \subseteqq \mathcal{O}_{y_i}^*(f_1^*, \dots, f_k^*)$, we can write

$$a^{(y_i)} \mu^*(f) = b_1^{(i)} f_1^* + \cdots + b_k^{(i)} f_k^*$$

in $\mathcal{F}_{y_i}^*$ for suitable $b_l^{(i)} \in \mathcal{O}_{y_i}^*$ $(1 \leqq i \leqq t,\ 1 \leqq l \leqq k)$. Again, taking a sufficiently small neighborhood U' of x with $\mu^{-1}(U') = V_1' \cup \dots \cup V_t'$ and $b_l^{(i)} \in \Gamma(V_i', \mathcal{O}^*)$, we define $\tilde{b}_l \in \Gamma(\mu^{-1}(U'), \mathcal{O}^*) = \Gamma(U', \mu_*(\mathcal{O}^*))$ so as to be $\tilde{b}_l = b_l^{(i)}$ on V_i. Then, for $b_l := u\tilde{b}_l \in \Gamma(U', \mathcal{O})$ we have

$$a' \mu^*(f) = b_1 f_1^* + \cdots + b_k f_k^*$$

in $\Gamma(\mu^{-1}(U'), \mathscr{F}^*)$. This shows the canonical image of $g := a'f - (b_1 f_1 + \cdots + b_k f_k) \in \Gamma(U', \mathscr{F})$ into $\Gamma(\mu^{-1}(U'), \mathscr{F}^*)$ is equal to zero. While, by the following lemma, $\Gamma(\mu^{-1}(U'), \mathscr{F}^*) = \Gamma(U', \mu_*(\mu^*(\mathscr{F})))$ is isomorphic to $\Gamma(U', \mathscr{F} \otimes_{\mathscr{O}} \mu_*(\mathscr{O}^*))$ and $\mu_*(\mathscr{O}^*)$ is considered as a subsheaf of $\mathscr{M}$ canonically. Thus $\tau(g)$ is also equal to 0 in $(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{M})_x$ and so $vg = 0$ because v is chosen as $v \cdot T(\mathscr{F})_x = (0)$. This asserts

$$af = (vb_1)f_1 + \cdots + (vb_k)f_k,$$

which is contained in $\mathscr{O}_x(f, \ldots, f_k)$. Consequently $a \cdot \mathscr{F}_x \subseteqq \mathscr{O}_x(f_1, \ldots, f_k)$.

Lemma 2. 3. *Let* X, Y *be complex spaces with structure sheaves* $\mathscr{O}_X, \mathscr{O}_Y$ *respectively and* $\nu: Y \to X$ *be a proper nowhere degenerate holomorphic mapping. For an* $\mathscr{O}_X$*-Module* $\mathscr{F}$ *and an* $\mathscr{O}_Y$*-Module* $\mathscr{G}$ *the canonical* $\mathscr{O}_X$*-Homomorphism* $\mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}) \to \nu_*(\nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G})$ *is an isomorphism.*

Proof. The canonical $\mathscr{O}$-Homomorphism is given as the composition of canonical $\mathscr{O}$-Homomorphisms $\rho: \mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}) \to \nu_*(\nu^*(\mathscr{F})) \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G})$ and $\theta: \nu_*(\nu^*(\mathscr{F})) \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}) \to \nu_*(\nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G})$ (c.f. [9], Chap. 0, §4). It suffices to prove that for each $x \in X$ the stalk $(\mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}))_x$ is isomorphic to $(\nu_*(\nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}))_x$ canonically. We put $\nu^{-1}(x) = \{y_1, \ldots, y_t\}$. There exists an arbitrarily small neighborhood U of x such that $\nu^{-1}(U)$ is the disjoint union of neighborhoods V_i of y_i. For such a neighborhood U of x we have isomorphisms

$$\begin{aligned}\Gamma(U, \mathscr{F}) &\otimes_{\Gamma(U, \mathscr{O}_X)} \Gamma(U, \nu_*(\mathscr{G})) \\ &\cong \Gamma(U, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} \Gamma(\nu^{-1}(U), \mathscr{G}) \\ &\cong \Gamma(U, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} (\oplus_{i=1}^t \Gamma(V_i, \mathscr{G})) \\ &\cong \oplus_{i=1}^t \Gamma(U, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} \Gamma(V_i, \mathscr{G}).\end{aligned}$$

As the inductive limit of these isomorphisms, we see $(\mathscr{F} \otimes_{\mathscr{O}_X} \nu_*(\mathscr{G}))_x$ is isomorphic to $\oplus_{i=1}^t (\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_{y_i})$. On the other hand,

$$\begin{aligned}\Gamma(U, \nu_*(\nu^*(\mathscr{F})) \otimes_{\mathscr{O}_Y} \mathscr{G})) &= \Gamma(\nu^{-1}(U), \nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}) \\ &\cong \oplus_{i=1}^t \Gamma(V_i, \nu^*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}).\end{aligned}$$

Therefore we obtain

$$(\nu_*(\nu^*(\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{G}))_x\cong\oplus_{i=1}^t(\nu^*(\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{G})_{y_i},$$

which is also isomorphic to $\oplus_{i=1}^t(\mathscr{F}_x\otimes_{\mathscr{O}_{X,x}}\mathscr{O}_{Y,y_i})\otimes_{\mathscr{O}_{Y,y_i}}\mathscr{G}_{y_i}$ by definition and so to $\oplus_{i=1}^t(\mathscr{F}_x\otimes_{\mathscr{O}_{X,x}}\mathscr{G}_{y_i})$. It is easy to see these isomorphisms give the above canonical mapping. This completes the proof.

REMARK 1. In Lemma 2. 2, we can find a not identically vanishing $a\in\Gamma(X,\mathscr{O})$ with $a\cdot\mathscr{F}\subseteqq\mathscr{O}(f_1,\dots,f_k)$. In fact, the condition 2^0 shows the coherent Ideal $\mathscr{O}(f_1,\dots,f_k):\mathscr{F}$ with the stalks $(\mathscr{O}(f_1,\dots,f_k):\mathscr{F})_x=\{a\in\mathscr{O}_x;\ a\cdot\mathscr{F}_x\subseteqq\mathscr{O}_x(f_1,\dots,f_k)\}\ (x\in X)$ is not equal to zero. By the assumption that X is Stein, there exists a not identically vanishing $a\in\Gamma(X,\mathscr{O}(f_1,\dots,f_k):\mathscr{F})$, which is a desired section.

REMARK 2. In case of $\mathrm{rank}_X\mathscr{F}=0$, Lemma 2. 2 is also valid in the sense that there exists a not identically vanishing $a\in\Gamma(X,\mathscr{O})$ with $a\cdot\mathscr{F}=(0)$.

Now, we study $\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}$ for a coherent $\mathscr{O}$-Module $\mathscr{F}$ over a not necessarily irreducible complex space. Let $X=\cup_\nu X_\nu$ be the irreducible decomposition and $i_\nu: X_\nu\to X$ be the canonical inclusion maps. We denote the sheaf of germs of holomorphic functions over X_ν by $\mathscr{O}_\nu$ and of meromorphic functions over X_ν by $\mathscr{M}_\nu$. The direct image $\tilde{\mathscr{M}}_\nu:=(i_\nu)_*(\mathscr{M}_\nu)$ is considered as an $\mathscr{M}$-Module over X. The inclusion maps $\tilde{\mathscr{M}}_\nu\to\mathscr{M}$ induce the canonical map $\iota:\oplus_\nu\tilde{\mathscr{M}}_\nu\to\mathscr{M}$.

(2. 4) *The map ι is an $\mathscr{M}$-Isomorphism.*

The problem is of local nature. For the proof, see Abhyankar [1], p. 157.

For each X_ν we can consider the rank of $(i_\nu)^*(\mathscr{F})$ over X_ν. We call it the rank of $\mathscr{F}$ over X_ν and denote it by $\mathrm{rank}_{X_\nu}\mathscr{F}$.

PROPOSITION 2. 5. *Let X be a Stein space and $\mathscr{F}$ be a coherent $\mathscr{O}$-Module over X. Then there exists an $\mathscr{M}$-Isomorphism $\varphi:\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}\to\oplus_\nu\tilde{\mathscr{M}}_\nu^{k_\nu}$, where $k_\nu=\mathrm{rank}_{X_\nu}\mathscr{F}$ and $\tilde{\mathscr{M}}_\nu^{k_\nu}$ denotes the k_ν-fold direct sum of $\tilde{\mathscr{M}}_\nu$.*

Proof. Firstly, we prove Proposition 2. 5 in case that X is irreducible. To this end, we take sections $f_1,\dots,f_k\in\Gamma(X,\mathscr{F})$ with the conditions in

Lemma 2. 2 and define an $\mathscr{O}$-Homomorphism $f=(f_1,\ldots,f_k)\colon \mathscr{O}^k\to\mathscr{F}$ as usual. It induces an $\mathscr{M}$-Homomorphism $\tilde{f}\colon \mathscr{M}^k\to\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}$, which is injective by the condition 1^0 and surjective by the condition 2^0. The $\mathscr{M}$-Isomorphism $\varphi:=\tilde{f}^{-1}\colon \mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}\to\mathscr{M}^k$ is a desired one.

Now, take a not necessarily irreducible space X with the irreducible decomposition $X=\cup_\nu X_\nu$. In the above notations $\mathscr{F}_\nu:=(i_\nu)^*(\mathscr{F})$ is a coherent $\mathscr{O}_\nu$-Module of rank k_ν over X_ν for each ν. By the above argument, there exists an $\mathscr{M}_\nu$-Isomorphism $\varphi_\nu\colon \mathscr{F}_\nu\otimes_{\mathscr{O}_\nu}\mathscr{M}_\nu \xrightarrow{\sim} \mathscr{M}_\nu{}^{k_\nu}$ and so $\tilde{\varphi}_\nu\colon (i_\nu)_*(\mathscr{F}_\nu\otimes_{\mathscr{O}_\nu}\mathscr{M}_\nu)\xrightarrow{\sim}\tilde{\mathscr{M}}_\nu{}^{k_\nu}$. On the other hand, by Lemma 2. 3, $(i_\nu)_*(\mathscr{F}_\nu\otimes_{\mathscr{O}_\nu}\mathscr{M}_\nu)$ is canonically isomorphic to $\mathscr{F}\otimes_{\mathscr{O}}\tilde{\mathscr{M}}_\nu$ and by (2. 4) $\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}$ is isomorphic to $\mathscr{F}\otimes_{\mathscr{O}}(\oplus_\nu\tilde{\mathscr{M}}_\nu)=\oplus_\nu\mathscr{F}\otimes_{\mathscr{O}}\tilde{\mathscr{M}}_\nu$. Consequently we obtain the $\mathscr{M}$-Isomorphism $\varphi\colon \mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}\xrightarrow{\sim}\oplus_\nu\tilde{\mathscr{M}}_\nu{}^{k_\nu}$ by the composition of these isomorphisms.

§3. A property of a torsion-free coherent $\mathscr{O}$-Module.

For a given complex space X, let $X=\cup_\nu X_\nu$ be the irreducible decomposition of X. As in the previous section, we put $\tilde{\mathscr{M}}_\nu:=(i_\nu)_*(\mathscr{M}_\nu)$ for the sheaf $\mathscr{M}_\nu$ of all germs of meromorphic functions of X_ν and the inclusion map $i_\nu\colon X_\nu\to X$.

LEMMA 3. 1. *Let $\mathscr{G}$ and $\mathscr{G}'$ be coherent sub-$\mathscr{O}$-Modules of $\oplus_\nu\tilde{\mathscr{M}}_\nu{}^{l_\nu}$ over a Stein space X, where l_ν are non-negative integers. If for each $x\in X$ there exists a non-zero divisor w in $\mathscr{O}_x$ with $w\mathscr{G}_x\subseteqq\mathscr{G}_x'$, then there exists a holomorphic function u with $u\cdot\mathscr{G}\subseteqq\mathscr{G}'$ on X which does not vanish identically on any X_ν.*

Proof. Firstly, we shall prove the sheaf $\mathscr{G}'\colon\mathscr{G}$ with the stalks $(\mathscr{G}'\colon\mathscr{G})_x:=\{u\in\mathscr{O}_x\,;\,u\cdot\mathscr{G}_x\subseteqq\mathscr{G}_x'\}$ $(x\in X)$ is a coherent Ideal over X. The problem is of local nature. We take an arbitrary point $x\in X$. By the assumption, we can find a non-zero divisor w in $\mathscr{O}_x$ with $w\cdot\mathscr{G}_x\subseteqq\mathscr{G}_x'$, which satisfies $w\cdot\mathscr{G}\subseteqq\mathscr{G}'$ on U and does not vanish identically on any irreducible component of U for a sufficiently small neighborhood U of x. For each $y\in U$, an element $a\in\mathscr{O}_x$ satisfies $a\cdot\mathscr{G}_y\subseteqq\mathscr{G}_y'$ if and only if $(aw)\mathscr{G}_y\subseteqq w\cdot\mathscr{G}_y'$ because w gives a non-zero in $\oplus_\nu(\mathscr{M}_\nu)_y^{l_\nu}$. This shows $\mathscr{G}'\colon\mathscr{G}=w\cdot\mathscr{G}'\colon w\cdot\mathscr{G}$, which is coherent over X since both $w\cdot\mathscr{G}$ and $w\cdot\mathscr{G}'$ are coherent sub-$\mathscr{O}$-Modules of $\mathscr{G}'$.

By the assumption, $(\mathscr{G}':\mathscr{G})_x$ is not equal to (0) for any $x\in X$. While, since X is Stein, $\mathscr{G}':\mathscr{G}$ is generated by global sections of $\mathscr{G}':\mathscr{G}$ over X as an $\mathscr{O}$-Module. For each X_ν we choose a point $x_\nu\in X_\nu$ arbitrarily. We can find a suitable $u_\nu\in\Gamma(X,\mathscr{G}':\mathscr{G})$ which does not vanish identically on a neighborhood of x_ν. Then u_ν does not vanish identically on any open subset of X_ν by the theorem of identity. Again, using the assumption that X is Stein, we can take a holomorphic function v_ν over X which vanishes identically on each $X_\mu(\mu\neq\nu)$ but not on X_ν. Since $\{X_\nu\}$ is a locally finite family, we can define a holomorphic function $u=\sum_\nu v_\nu u_\nu$, which belongs to $\Gamma(X,\mathscr{G}':\mathscr{G})$. Ovbiously, u satisfies the desired conditions.

Now, let X_ν^* be the normalization of each irreducible component X_ν of X with projection $\mu_\nu: X_\nu^*\to X_\nu$ and structure sheaf $\mathscr{O}_\nu^*$. The direct image $(\mu_\nu)_*(\mathscr{O}_\nu^*)$ is nothing but the sheaf of all germs of weakly holomorphic functions on X_ν. We put $\tilde{\mathscr{O}}_\nu := (i_\nu)_*(\mu_\nu)_*(\mathscr{O}_\nu^*)$.

Proposition 3. 2. *If X is Stein and a coherent $\mathscr{O}$-Module $\mathscr{F}$ over X is torsion-free, then there exists an injective $\mathscr{O}$-Homomorphism $\chi:\mathscr{F}\to\oplus_\nu\tilde{\mathscr{O}}_\nu^{k_\nu}$ $(k_\nu:=$* $\mathrm{rank}_{X_\nu}\mathscr{F})$ *such that $u\cdot(\oplus_\nu\tilde{\mathscr{O}}_\nu^{k_\nu})\subseteqq\chi(\mathscr{F})$ on X for a suitable $u\in\Gamma(X,\mathscr{O})$ not identically vanishing on any X_ν.*

Proof. Since $\mathscr{F}$ is torsion-free on X, the canonical map $\tau:\mathscr{F}\to\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M}$ is injective and so $\psi:=\varphi\cdot\tau$ is an injective $\mathscr{O}$-Homomorphism $\mathscr{F}\to\oplus_\nu\tilde{\mathscr{M}}_\nu^{k_\nu}$, where φ is an $\mathscr{M}$-Isomorphism given in Proposition 2. 5. For each ν, we take sections $g_1^{(\nu)},\dots,g_{k_\nu}^{(\nu)}$ in $\Gamma(X,\mathscr{F}\otimes_{\mathscr{O}}\mathscr{M})$ whose φ-images give a base of an $\mathscr{M}_y$-module $(\tilde{\mathscr{M}}_\nu)_y^{k_\nu}$ $(y\in X)$. We consider the coherent $\mathscr{O}$-Modules $\mathscr{G}_i^{(\nu)}:=\mathscr{O}(g_i^{(\nu)})$ $(1\leqq i\leqq k_\nu)$. Obviously, for each $x\in X$ there exists a non-zero divisor w in $\mathscr{O}_x$ with $w\cdot(\mathscr{G}_i^{(\nu)})_x\subseteqq\tau(\mathscr{F})_x$. According to Lemma 3. 1, there exists a holomorphic function u on X with $u\cdot\mathscr{G}_i^{(\nu)}\subseteqq\tau(\mathscr{F})$ not identically vanishing on any X_ν. This shows that $ug_i^{(\nu)}\subseteqq\Gamma(X,\tau(\mathscr{F}))$ and so there exists $f_i^{(\nu)}\in\Gamma(X,\mathscr{F})$ with $\tau(f_i^{(\nu)})=u\,g_i^{(\nu)}$ $(1\leqq i\leqq k_\nu)$. Thus we get a system of sections $\{f_1^{(\nu)},\dots,f_{k_\nu}^{(\nu)}\}$ in $\Gamma(X,\mathscr{F})$ whose ψ-image gives a base of an $\mathscr{M}_y$-module $(\tilde{\mathscr{M}}_\nu)_y^{k_\nu}$ for each $y\in X$. Renewing φ suitably if necessary, we may assume $\psi(f_i^{(\nu)})$ is the identity element in each $\tilde{\mathscr{M}}_\nu$.

Now, we consider two coherent sub-$\mathscr{O}$-Modules $\mathscr{G} := \psi(\mathscr{F})$ and $\mathscr{G}' := \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ of $\oplus_\nu(\tilde{\mathscr{M}}_\nu)^{k_\nu}$. We shall prove $\mathscr{G}$ and $\mathscr{G}'$ satisfy the assumption in Lemma 3. 1. Take an arbitrary point $x \in X$. There exists a system of generators $\{g_1, \dots, g_s\}$ of an $\mathscr{O}_x$-module $\mathscr{F}_x$. Since there exists only finitely many ν's with $x \in X_\nu$, say $X_1, \dots, X_t$, we can write

$$\psi(g_j) = \sum_{1 \leqq \nu \leqq t} \sum_{1 \leqq i \leqq k_\nu} c_{i,j}^{(\nu)} \psi(f_i^{(\nu)})$$

where each $c_{i,j}^{(\nu)}$ is a section of $\tilde{\mathscr{M}}_\nu$ over a neighborhood U of x and considered as a meromorphic function on U vanishing identically on each $X_\mu \cap U (\mu \neq \nu)$. Moreover, taking sufficiently small U, we may assume that all $c_{i,j}^{(\nu)}$ have the common denominator w in $\mathscr{O}_x$ which does not vanish identically on any irreducible component of U. Obviously, $w \cdot \psi(g_j) \in \Gamma(U, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})$ and hence for each $f \in \mathscr{F}_x$, if we write $f = a_1 g_1 + \cdots + a_s g_s$ $(a_j \in \mathscr{O}_x)$,

$$w\psi(f) = a_1 w\psi(g_1) + \cdots + a_s w\psi(g_s) \in (\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})_x .$$

This shows $w\psi(\mathscr{F}) \subseteqq \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$. Therefore we can apply Lemma 3. 1 to the coherent sheaves $\mathscr{G}$ and $\mathscr{G}'$. We obtain a holomorphic function u' with $u' \cdot \psi(\mathscr{F}) \subseteqq \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ on X which does not vanish identically on any X_ν.

Multiplying each component of elements in $(\oplus_\nu \tilde{\mathscr{M}}_\nu)^{k_\nu}$ by the above u', we define an $\mathscr{M}$-Isomorphism $\tilde{u}' : \oplus_\nu (\tilde{\mathscr{M}}_\nu)^{k_\nu} \to \oplus (\tilde{\mathscr{M}}_\nu)^{k_\nu}$ on X. Then, the composite $\chi = \tilde{u}'\psi$ is an injective $\mathscr{O}$-Homomorphism of $\mathscr{F}$ into $\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$, because $\chi(\mathscr{F}) = u'\psi(\mathscr{F}) \subseteqq \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$. We shall prove such a map χ satisfies the condition in Proposition 3. 2. For our purpose, it suffices to show that the coherent sub-$\mathscr{O}$-Modules $\chi(\mathscr{F})$ and $\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ of $\oplus_\nu \tilde{\mathscr{M}}_\nu^{k_\nu}$ satisfy the assumption in Lemma 3. 1, namely, for each $x \in X$ there exists a non-zero divisor w in $\mathscr{O}_x$ with $w \cdot (\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})_x \subseteqq \chi(\mathscr{F})_x$. To see this, we take a universal denominator w' in $\mathscr{O}_x$ and define a non-zero divisor $w = w'u'$. Since $\psi(f_i^{(\nu)})$ $(1 \leqq i \leqq k_\nu)$ give the canonical base of the $\mathscr{M}$-module $\oplus_\nu \tilde{\mathscr{M}}_\nu^{k_\nu}$, each $h \in (\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})_x \subseteqq \oplus_\nu \tilde{\mathscr{M}}_\nu^{k_\nu}$ can be written as

$$h = \sum_\nu \sum_{1 \leqq i \leqq k_\nu} h_i^{(\nu)} \psi(f_i^{(\nu)})$$

where $h_i^{(\nu)} \in (\tilde{\mathcal{O}}_\nu^{k_\nu})_x$. By the definition of universal denominator, each $\tilde{h}_i^{(\nu)} := w' h_i^{(\nu)}$ belongs to $\mathcal{O}_x$. Thus we have

$$w'u'h = \sum_\nu \sum_{1 \leqq i \leqq k} w' h_i^{(\nu)} u' \psi(f_i^{(\nu)})$$
$$= \sum_\nu \sum_i \tilde{h}_i^{(\nu)} \chi(f_i^{(\nu)})$$
$$\in \mathcal{O}(\ldots, \chi(f_1^{(\nu)}), \ldots, \chi(f_{k_\nu}^{(\nu)}), \ldots) \subseteqq \chi(\mathscr{F}).$$

This shows $w \cdot (\oplus_\nu \mathcal{O}_\nu^{k_\nu})_x \subseteqq \chi(\mathscr{F})_x$. The proof is completely accomplished.

For later uses, we give

DEFINITION 3. 3. For a coherent $\mathcal{O}$-Module $\mathscr{F}$ we shall say a non-zero divisor u in $\mathcal{O}_x$ to be a *universal denominator* for $\mathscr{F}_x$ $(x \in X)$ if for a suitable neighborhood U of x it satisfies the conditions as in Proposition 3. 2, that is, there exists an injective $\mathcal{O}$-Homomorphism $\chi: \mathscr{F}|U \to \oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}$ such that $u \cdot (\oplus_\nu \tilde{\mathcal{O}}_\nu^{k_\nu}) \subseteqq \chi(\mathscr{F})$ on U, where $\tilde{\mathcal{O}}_\nu$ is the sheaf defined as above for each irreducible component U_ν of U and k_ν is some non-negative integer.

As is easily proved, any k_ν satisfying the above conditions is necessarily equal to $\mathrm{rank}_{U_\nu} \mathscr{F}$ and we can discuss the existence of universal denominators for $\mathscr{F}_x$ only if $\mathscr{F}$ is torsion-free at x. While, if $\mathscr{F}$ is torsion-free at $x \in X$, there exists at least one universal denominator for $\mathscr{F}_x$. For, in this case, we can take a Stein neighborhood U of x such that $\mathscr{F}$ is torsion-free on U and apply Proposition 3. 2 to the space U.

§4. **Local Continuation Theorem.** Now, we can generalize the theorems on the continuation of holomorphic functions given in [6], [8] and [10] etc. to the case of sections of torsion-free $\mathcal{O}$-Modules.

To see this, we need

DEFINITION 4. 1. By $\mathrm{Ass}(\mathscr{F}_x)$, we denote the totality of all prime ideals associated with a reduced primary decomposition of sub-$\mathcal{O}_x$-module (0) of $\mathscr{F}_x$. If $k = \min\{\dim \mathcal{O}_x/\mathfrak{p}; \mathfrak{p} \in \mathrm{Ass}(\mathscr{F}_x)\}$, we shall say that $\mathscr{F}$ is *of lower dimension* k at x and indicate this by $\mathrm{l.\,dim}_x \mathscr{F} = k$.

The first main theorem is given as follows.

THEOREM 4. 2 *Let D be an open subset of X which is $*$-strongly s-concave at $x \in X$ (see Definition 2. 5 in [6]), and $\mathscr{F}$ be a coherent $\mathcal{O}$-Module over X. Suppose that $\mathscr{F}$ is torsion-free at x and a universal denominator u for $\mathscr{F}_x$ satisfies*

l. $\dim_x(\mathscr{F}/u\mathscr{F}) \geqq s$. *Then, there exists a neighborhood U of x such that every section of $\mathscr{F}$ over D is uniquely continuable to a section of $\mathscr{F}$ over $D \cup U$.*

Proof. In case that $\mathscr{F}_x = (0)$, Theorem 4. 2 is trivial. We assume $\mathscr{F}_x \neq (0)$. By Definition 3. 3, we can find a neighborhood V_1 of x such that $\mathscr{F}$ is torsion-free on V_1 and there exists an injective $\mathscr{O}$-Homomorphism $\chi: \mathscr{F}|V_1 \rightarrow \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ with $u \cdot (\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}) \subseteqq \chi(\mathscr{F})$ on V_1. Moreover, we may assume l. $\dim_y(\mathscr{F}/u\mathscr{F}) \geqq s$ for any $y \in V_1$. On the other hand, l. $\dim_x \mathscr{O} \geqq s+1$, or any irreducible component of X at x is of dimension $\geqq s+1$. Indeed, since u is a non-zero divisor in $\mathscr{O}_x$, for each $\mathfrak{p} \in \mathrm{Ass}(\mathscr{O}_x)$ $u \notin \mathfrak{p}$ and there exists some $\mathfrak{p}' \in \mathrm{Ass}(\mathscr{F}_x / u\mathscr{F}_x)$ with $\mathfrak{p} \cup (u) \subseteqq \mathfrak{p}'$. By definition, $\mathrm{depth}\, \mathfrak{p} \geqq s$ and hence $\mathrm{depth}\, \mathfrak{p}' \geqq s+1$ because $\mathfrak{p} \subsetneqq \mathfrak{p}'$. This shows l. $\dim_x \mathscr{O} \geqq s+1$. According to Proposition 6. 1 in [6], there exists a neighborhood V_2 of $x (V_2 \subseteqq V_1)$ such that every weakly holomorphic function on $D \cap V_2$ is uniquely continuable to V_2. Moreover, as in the proof of Proposition 6. 2 in [6], we can find another neighborhood V_3 of $x (V_3 \subseteqq V_2)$ with the property that any analytic set N of V_2 satisfies $N \cap D \neq \phi$ if $\dim N \cap V_3 \geqq s$. We shall prove that the neighborhood $U := V_3$ of x satisfies the desired condition.

Now, take an arbitrary $f \in \Gamma(D, \mathscr{F})$. The sheaf $\tilde{\mathscr{O}}_\nu$ can be regarded as the sheaf of all germs of weakly holomorphic functions vanishing identically on any irreducible component except the corresponding component X_ν. By the assumption of V_2, every section of $\tilde{\mathscr{O}}_\nu$ over $V_2 \cap D$ is uniquely continuable to V_2. Therefore, each component $g_i^{(\nu)}$ of $\chi(f) \in \Gamma(D \cap V_2, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})$ has the unique continuation $h_i^{(\nu)} \in \Gamma(V_2, \tilde{\mathscr{O}}_\nu)$. Thus we get a section $h = \sum_\nu \sum_i h_i^{(\nu)} \in \oplus_\nu \Gamma(V_2, \tilde{\mathscr{O}}_\nu)^{k_\nu} = \Gamma(V_2, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})$ with $h|V_2 \cap D = \chi(f)$. By definition, uh is contained in $\Gamma(V_2, \chi(\mathscr{F}))$. Since χ is injective, there exists a section $\tilde{h} \in \Gamma(V_2, \mathscr{F})$ with $\chi(\tilde{h}) = uh$ on V_2 and so $= u\chi(f) = \chi(u f)$ on $V_2 \cap D$. Again, using the injectivity of χ, we conclude $\tilde{h} = uf$ on $V_2 \cap D$. This shows the analytic set $N := |\mathscr{O} / u\mathscr{F} : (\tilde{h})|$ does not intersect D. While, in view of Lemma 2.4 in [7], N is of dimension $\geqq s$ everywhere on $V_2 \cap N$. By the property of U, $N \cap D = \phi$ implies $N \cap U = \phi$. Therefore, $u\mathscr{F} : (\tilde{h}) = \mathscr{O}$ on U or $\tilde{h} \in \Gamma(U, u\mathscr{F})$. Since u defines a non-zero divisor in $\mathscr{O}_y$ for any $y \in U$, there exists a section $\tilde{f} \in \Gamma(U, \mathscr{F})$ with $u\tilde{f} = \tilde{h} = uf$ on $U \cap D$, whence $\tilde{f} = f$ on $U \cap D$. This shows that $\tilde{f}$ is a continuation of f to U.

The uniqueness of the continuation is owing to the assumption that $\mathscr{F}$ is torsion-free on U. In fact, for any $x \in U$ every element in $\mathrm{Ass}(\mathscr{F}_x)$ is a minimal prime ideal of $\mathcal{O}_x$. Therefore, any section of $\mathscr{F}$ over U satisfies the theorem of identity on each irreducible component of U(c.f. [7], Theorem 5.1). The proof is carried out similarly to the case of holomorphic functions.

REMARK. If a torsion-free coherent $\mathcal{O}$-Module $\mathscr{F}$ satisfies $\mathrm{dih}_x \mathscr{F} \geqq s+1$ for a point $x \in X$ (see [3], p. 197), any non-zero divisor in $\mathcal{O}_x$, particularly any universal denominator for $\mathscr{F}_x$, satisfies $\mathrm{l.\,dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$. In fact, if $\mathrm{dih}_x \mathscr{F} \geqq s+1$, there exists an $\mathscr{F}_x$-sequence $u_1 := u, u_2, \dots, u_{s+1}$ in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_x$. For each $\mathfrak{p} \in \mathrm{Ass}(\mathscr{F}_x/u\mathscr{F}_x)$ we can find easily prime ideals $\mathfrak{p}_i \in \mathrm{Ass}(\mathscr{F}_x/(u_1, \dots, u_i)\mathscr{F}_x)$ with $\mathfrak{p}_1 = \mathfrak{p} \subsetneqq \mathfrak{p}_2 \subsetneqq \cdots\cdots \subsetneqq \mathfrak{p}_{s+1}$. Moreover, if the local ring $\mathcal{O}_x$ is normal and of dimension $\geqq s+1$, any non-zero divisor in $\mathcal{O}_x$ satisfies $\mathrm{l.\,dim}_x(\mathcal{O}/u\mathcal{O}) \geqq s$. Both of Proposition 6.1 and 6.2 in [6] are special cases of Theorem 4.2 in view of Proposition 3. 2.

We can prove also the following local continuation theorem.

THEOREM 4.3. *If $\mathscr{F}$ is torsion-free at x and there exists a universal denominator u for $\mathscr{F}_x$ such that* $\mathrm{l.\,dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$, *then there exists an arbitrarily small neighborhood U of x such that for any open subset V of U and any analytic set M in V of dimension $\leqq s-1$ every section of $\mathscr{F}$ over $V - M$ is uniquely continuable to V.*

Proof. We may assume $\mathscr{F}_x \neq (0)$. By the assumption any irreducible component of X at x is of dimension $\geqq s+1$ and so $\dim_y X \geqq s+1$ for any y in a neighborhood U of x. For any locally analytic set M in U of dimension $\leqq s-1$ M is of codimension $\geqq (s+1)-(s-1) = 2$. Therefore, every weakly holomorphic function on $V - M$ is uniquely continuable to V. While, any analytic set N with $\dim_x N \geqq s$ intersects $X - M$. From these facts, we can easily conclude Theorem 4. 3 by the same arguments as in the proof of Theorem 4. 2. We omit the details.

REMARK. As to the continuation of sections of torsion-free $\mathcal{O}$-Modules over a purely dimensional reduced complex space, Theorem 4. 3 is an improvement of the implication (1) $\rightarrow$ (3) of Satz 10 in [11], p. 90. In fact, in this case, if we put $k=1$ in [11], Satz 10, the assertion (1) means that $\mathscr{G}_x=(0)$, or $\mathscr{G}_x$ is torsion-free and "any" non-zero divisor u in $\mathscr{O}_x$ satisfies $\text{l. dim}_x(\mathscr{G}/u\mathscr{G}) \geqq \dim_x X-1$. Then, there exists a neighborhood U of x such that for any open set $V \subseteqq U$ and analytic set M in V of codimension $\geqq 2$ the restriction mapping $\Gamma(V, \mathscr{G}) \to \Gamma(V-M, \mathscr{G})$ is bijective by Theorem 4.3. This is the first half of the assertion (3) in [11] Satz 10.

§5. **Global Continuation Theorems.** As in the previous paper [6], using the local continuation theorems, we can show global continuation theorems of sections of a torsion-free coherent $\mathscr{O}$-module to sets with the non-empty open kernel.

We first generalize Theorem 7.6 in [6].

THEOREM 5.1. *Let X be a complex space where there exists a $*$-strongly s-convex function v and $\mathscr{F}$ be a torsion-free coherent $\mathscr{O}$-Module over X. Suppose that for any $x \in X$ there exists a universal denominator u for $\mathscr{F}_x$ with* $\text{l. dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$ *and an open subset B of X satisfies the conditions;*

1°. *$\bar{B} \cap \{v>\lambda\} \Subset X$ for any λ,*

2°. *for any $x \in \partial B$ $\{v>v(x)\}-B$ intersects any irreducible component of X at x.*

Then, every section of $\mathscr{F}$ over a connected neighborhood U of ∂B is uniquely continuable to $U \cup B$.

Proof. Using Theorem 4.2 and the theorem of identity for sections of $\mathscr{F}$, we can prove Theorem 5.1 by the analogous arguments as in the proof of Theorem 7.6 in [6]. The details are left to the reader.

Easily, we have

COROLLARY 5.2. *Suppose that there exists a $*$-strongly s-convex function v on X with $\{\lambda<v<\mu\} \Subset X$ for any $\lambda, \mu (\lambda<\mu)$. If a torsion-free coherent $\mathscr{O}$-Module $\mathscr{F}$ over X has a universal denominator u in $\mathscr{O}_x$ with* $\text{l. dim}_x(\mathscr{F}/u\mathscr{F}) \geqq s$ *for each $x \in X$, every section of $\mathscr{F}$ over $X_\lambda := X \cap \{v>\lambda\}$ is uniquely continuable to the total X.*

REMARK. A strongly s-convex function is obviously $*$-strongly s-convex. As to the continuation of sections of torsion-free coherent $\mathscr{O}$-Modules, Corollary 5.2 is a generalization of Theorem 15 in [3], p. 254.

Let X be a complex space where there exists a $*$-strongly s-convex function v and $\mathscr{F}$ be a torsion-free coherent $\mathcal{O}$-Module over X. If X is locally irreducible, $\mathscr{F}$ is a hard sheaf over X (see Definition 8.1 in [6]), because any section of $\mathscr{F}$ satisfies the theorem of identity. Moreover, if for any $x \in X$ $\mathscr{F}_x$ has a universal denominator $u \in \mathcal{O}_x^\circ$ with $\text{l.}\dim_x(\mathscr{F}/u\mathscr{F}) \geqq s$, then each $x \in X$ has a fundamental system of neighborhoods $\mathfrak{U} = \{U\}$ such that the restriction map $\Gamma(U, \mathscr{F}) \to \Gamma(U \cap \{v > v(x)\}, \mathscr{F})$ is surjective for each U. This shows that v is admissible for $\mathscr{F}$ if $\{\lambda \leqq v \leqq \mu\}$ is compact for any $\lambda, \mu (\lambda < \mu)$ (see [6], p. 81). Now, we can apply the method in [6], §8 to give global continuation theorems of sections of $\mathscr{F}$.

As a generalization of Theorem 8. 4 in [6], we have

THEOREM 5. 3. *Let X be a connected, locally irreducible complex space of dimension n and v be a $*$-strongly s-convex function on X with the property that $\{\lambda \leqq v \leqq \mu\}$ is compact for any $\lambda, \mu(\lambda < \mu)$ and v is represented as $v = v'\varphi$ by a suitable nowhere degenerate holomorphic mapping φ of X into a purely n-dimensional complex manifold Y and a real analytic function v' on Y. Suppose that a coherent $\mathcal{O}$-Module $\mathscr{F}$ over X is torsion-free on X and has a universal denominator $u \in \mathcal{O}_x^\circ$ with $\text{l.}\dim_x(\mathscr{F}/u\mathscr{F}) \geqq s$ for any $x \in X$. Then for any open set $B \Subset X$ with the connected boundary ∂B every section of $\mathscr{F}$ over a connected neighborhood U of ∂B is uniquely continuable to $B \cup U$.*

Proof. As is seen above, v is admissible for $\mathscr{F}$. Take an open set D and a compact set K with $K \subsetneqq B \subsetneqq D$ such that $U := D - K$ is connected. In view of Theorem 1 in [10], p. 299, it suffices to show that there exists an open set B' with $K \subsetneqq B' \subsetneqq D$ such that $\partial B'$ is good for v in the sense of Kasahara [10]. By the assumption of local irreducibility of X, the normalization X^* of X is homeomorphic to X. For the proof, see [8], §7, [10], §5, Lemma 4 and [6], §8, Lemma 8. 3.

For a Stein space X, we can prove the following continuation theorem without the assumption of the local irreducibility.

THEOREM 5. 4. *If X is Stein and a torsion-free coherent $\mathcal{O}$-Module $\mathscr{F}$ over X has a universal denominator u in $\mathcal{O}_x^\circ$ with $\text{l.}\dim_x(\mathscr{F}/u\mathscr{F}) \geqq 1$ for each $x \in X$, then the restriction mapping $\rho^D_{D-K} : \Gamma(D, \mathscr{F}) \to \Gamma(D-K, \mathscr{F})$ is bijective for an open set D and a compact subset K of D satisfying the condition that each irreducible component of D is irreducible in $D-K$.*

Proof. In the notations as in §3, there exists an injective $\mathscr{O}$-Homomorphism $\chi: \mathscr{F} \to \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}$ $(k_\nu = \mathrm{rank}_{X_\nu} \mathscr{F})$ and a holomorphic function u on X with $u\cdot(\oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}) \subseteqq \chi(\mathscr{F})$ which does not vanish identically on any irreducible component by virtue of Proposition 3.2. Take a section $f \in \Gamma(D-K, \mathscr{F})$. We put

$$\chi(f) = (\ldots, h_1^{(\nu)}, \ldots, h_{k_\nu}^{(\nu)}, \ldots) \in \Gamma(D-K, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu})$$

where $h_i^{(\nu)} \in \Gamma(D-K, \tilde{\mathscr{O}}_\nu)$. By the assumption, each irreducible component of X is of dimension $\geqq 2$. According to Kasahara [10], every weakly holomorphic function is uniquely continuable to D. Especially, each $h_i^{(\nu)}$ has a continuation $\tilde{h}_i^{(\nu)} \in \Gamma(D, \tilde{\mathscr{O}}_\nu)$. We put

$$\tilde{h} = (\ldots, \tilde{h}_1^{(\nu)}, \ldots, \tilde{h}_{k_\nu}^{(\nu)}, \ldots) \in \Gamma(D, \oplus_\nu \tilde{\mathscr{O}}_\nu^{k_\nu}).$$

By the property of u, $u\tilde{h} \in \Gamma(D, \chi(\mathscr{F}))$ and so there exists a section $g \in \Gamma(D, \mathscr{F})$ with $\chi(g) = u\tilde{h}$ on D and so $= u\chi(f) = \chi(uf)$ on $D-K$. Thus we have a section $g \in \Gamma(D, \mathscr{F})$ with $g = uf$ on $D-K$ by the injectivity of χ.

To complete the proof of the existence of the continuation, it suffices to show the existence of a section $\tilde{f} \in \Gamma(D, \mathscr{F})$ with $g = u\tilde{f}$ on D. For, such $\tilde{f}$ satisfies $\tilde{f} = f$ on $D-K$ because u defines a non-zero divisor in $\mathscr{O}_x$ for any $x \in X$. Now, since X is Stein, we know the existence of a strongly 1-convex function v on X. As usual, we consider the set $\Lambda = \{\lambda;$ there exists a section $\tilde{f}_\lambda$ of $\mathscr{F}$ over $D_\lambda := D \cap \{v > \lambda\}$ with $u\tilde{f}_\lambda = g$ on $D_\lambda\}$. Putting $\lambda_0 = \sup\{v(x); x \in K\}$, we see easily $\lambda_0 \in \Lambda$ and so $\Lambda \neq \phi$. Moreover, for $\lambda_1 = \inf \Lambda$ we have $\lambda_1 \in \Lambda$. We assume $\lambda_1 > -\infty$. In view of Theorem 4.2, each point $x \in \{v = \lambda_1\} \cap D$ has a neighborhood $U^{(x)}$ such that there exists a continuation $\tilde{f}^{(x)}$ of $\tilde{f}_{\lambda_1}$ to $U^{(x)}$. On $D-K$, $u\tilde{f}_{\lambda_1} = g = uf$ implies $\tilde{f}_{\lambda_1} = f$. For a point $x \in \{v = \lambda_1\} \cap (D-K)$ we can take $\tilde{f}^{(x)} = f$ on $U^{(x)} = D-K$. For a point $x \in \{v = \lambda_1\} \cap K$, taking $U^{(x)}$ satisfying that each connected component of $U^{(x)}$ intersects D_{λ_1}, we may assume $u\tilde{f}^{(x)} = g$ on $U^{(x)}$. Obviously, $\tilde{f}^{(x)} = \tilde{f}^{(x')}$ on $U^{(x)} \cap U^{(x')}$ if $U^{(x)} \cap U^{(x')} \neq \phi$. If we cover K by finitely many above $U^{(x)}$, it is easy to find $\lambda_2 \in \Lambda$ with $\lambda_2 < \lambda_1$. This is a contradiction. We conclude $\lambda_1 = -\infty$. This shows there exists a section $f \in \Gamma(D, \mathscr{F})$ with $g = u\tilde{f}$.

The uniqueness of the continuation is obvious by the theorem of identity for sections of torsion-free coherent $\mathscr{O}$-Modules. Theorem 5.4 is completely proved.

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Nagoya Institute of Technology