ON OSCULATING SYSTEMS OF DIFFERENTIAL EQUATIONS OF TYPE (N, 1, 2)

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The main subject in the present article has the origin in the following quite primitive question: *Linear systems of ordinary differential equations form a nice family. Then, from the projective point of view, what does correspond to linear systems?*

An osculating system of ordinary differential equations of type $(N, 1, 2)$ means a system of differential equations

$$(*) \quad \det \begin{pmatrix} y_{\alpha_0}, & y_{\alpha_1} \\ \frac{dy_{\alpha_0}}{du}, & \frac{dy_{\alpha_1}}{du} \end{pmatrix} = F_{\alpha_0, \alpha_1}(u, y_0, \ldots, y_N) \qquad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$$

such that F_{α_0, α_1} $(0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$ are quadratic forms in $y_0, \ldots, y_N$. If a vector $(\varphi_0, \ldots, \varphi_N)$ is a solution of $(*)$, then for any holomorphic function ϕ the vector $(\phi\varphi_0, \ldots, \phi\varphi_N)$ is also a solution $(*)$. Hence the map: $u \to (\varphi_0(u), \ldots, \varphi_N(u))$ into the projective N-space $\boldsymbol{P}_N$ has a nice meaning. We shall call such a map a projective solution of $(*)$. From the projective point of view, roughly speaking, the system $(*)$ is equivalent to the following systems

$$(**) \qquad \frac{d\frac{y_\alpha}{y_\beta}}{du} = F_{\alpha, \beta,}(u, y_0, \ldots, y_N) \qquad (0 \leqslant \alpha, \beta \leqslant N)$$

where $F_{\alpha,\beta} + F_{\beta,\alpha} = 0$ $(0 \leqslant \alpha, \beta \leqslant N)$. The initial variety $\boldsymbol{W}_{u_0}^{(F)}$ at a regular point u_0 for $(*)$ means the set of all the point x in the projective N-space $\boldsymbol{P}_N$ such that there exists a holomorphic projective solution of $(*)$ with the initial point x at $u = u_0$.

Then the following comparative table shows that osculating systems of type $(N, 1, 2)$ together with their projective solutions give an answer to our primitive question.

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THE COMPARATIVE TABLE

Linear system	Osculating system of type $(N, 1, 2)$
$$\frac{dy_\alpha}{du} = \sum_{\lambda=1}^{N} a_{\alpha,\lambda}(u) y_\lambda \quad (1 \leqslant \alpha \leqslant N).$$	$$\det \begin{pmatrix} y_{\alpha_0}, & y_{\alpha_1} \\ \frac{dy_{\alpha_0}}{du}, & \frac{dy_{\alpha_1}}{du} \end{pmatrix} = F_{\alpha_0,\alpha_1}(u, y_0, \dots, y_N) \quad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N).$$
A linear transformation $$y_\alpha \to \sum_{\lambda=1}^{N} p_{\alpha,\lambda}(u) y_\lambda \quad (1 \leqslant \alpha \leqslant N)$$ maps a linear system to a linear system.	A projective automorphism $$y_\alpha \to \sum_{\lambda=0}^{N} p_{\alpha,\lambda}(u) y_\lambda \quad (0 \leqslant \alpha \leqslant N)$$ maps an osculating system of type $(N, 1, 2)$ to an osculating system of type $(N, 1, 2)$.
The singularities of a solution are the singularities of the coefficients $a_{\alpha,\beta}(u)$ $(1 \leqslant \alpha, \beta \leqslant N)$.	The singularities of a projective solution are the singularities of the coefficients in $F_{\alpha,\beta}$ $(0 \leqslant \alpha, \beta \leqslant N)$.
For each regular point u_0 there exists a holomorphic map ψ into $\boldsymbol{GL}(N)$[1] such that $\varphi(u) = \psi(u)x$ is the unique holomorphic solution with the initial point x at $u = u_0$.	For each regular point u_0 there exist a neighbourhood U of u_0 and a holomorphic map Φ: $U \times \boldsymbol{W}_{u_0}^{(F)} \to \boldsymbol{P}_N$ such that (i) for a fixed x_0 in $\boldsymbol{W}_{u_0}^{(F)}$ the map: $u \to \Phi(u, x_0)$ is a unique holomorphic projective solution with the initial point x_0 at $u = u_0$ and (ii) for a fixed regular point u_1 in U the map: $x \to \Phi(u_1, x)$ is a biregular birational map of the initial variety $\boldsymbol{W}_{u_0}^{(F)}$ onto the initial variety $\boldsymbol{W}_{u_0}^{(F)}$.
If the coefficient matrix $A = (a_{\alpha,\beta}(u))$ is a constant matrix, then the map ψ is the exponential homomorphism: $u \to e^{Au}$.	If the coefficients in $F_{\alpha,\beta}$ $(0 \leqslant \alpha, \beta \leqslant N)$ are constants, there exists an analytic homomorphism ρ of the additive group $\boldsymbol{C}$ into a commutative algebraic transformation group acting on $\boldsymbol{W}_{u_0}^{(F)}$[2] such that $$\Phi(u, x) = \rho(u)x.$$

1) $\boldsymbol{GL}(N)$ means the general linear group of degree N.

§1. Osculating systems of differential equations

1.1 An osculating systems of type (N, r, s) means a system of ordinary differential equations

$$(1)\quad \det\begin{pmatrix} y_{\alpha_0}, & \ldots, & y_{\alpha_r} \\ \dfrac{dy_{\alpha_0}}{du}, & \ldots, & \dfrac{dy_{\alpha_r}}{du} \\ \vdots & & \vdots \\ \dfrac{d^r y_{\alpha_0}}{du^r}, & \ldots, & \dfrac{d^r y_{\alpha_r}}{du^r} \end{pmatrix} = F_{\alpha_0,\ldots,\alpha_r}(u, y_0, \ldots, y_N)$$

such that $F_{\alpha_0,\ldots,\alpha_r}$ $(0 \leqslant \alpha_0 < \ldots < \alpha_r \leqslant N)$ are homogeneous forms of degree s in $y_0, \ldots, y_N$. For each permutation π on $\{0, 1, \ldots, r\}$ we put

$$F_{\alpha_{\pi(0)},\ldots,\alpha_{\pi(r)}} = \operatorname{sign}(\pi) F_{\alpha_0,\ldots,\alpha_r} \qquad (0 \leqslant \alpha_0 < \ldots < \alpha_r \leqslant N)$$

and, if $\beta_0, \ldots, \beta_r$ are not all different, then $F_{\beta_0,\ldots,\beta_r} = 0$.

PROPOSITION 1. *Solutions of an osculating system* (1) *satisfy the following system of algebraic and differential equations*

$$(2)_0\quad \sum_{\lambda=0}^{r+1} (-1)^\lambda y_{\alpha_\lambda} F_{\alpha_0,\ldots,\alpha_{\lambda-1},\alpha_{\lambda+1},\ldots,\alpha_{r+1}} = 0,$$

$$(2)_1\quad \sum_{\lambda=0}^{r+1} (-1)^\lambda \frac{dy_{\alpha_\lambda}}{du} F_{\alpha_0,\ldots,\alpha_{\lambda-1},\alpha_{\lambda+1},\ldots,\alpha_{r+1}} = 0,$$

$$\vdots$$

$$(2)_r\quad \sum_{\lambda=0}^{r+1} (-1)^\lambda \frac{d^r y_{\alpha_\lambda}}{du^r} F_{\alpha_0,\ldots,\alpha_{\lambda-1},\alpha_{\lambda+1},\ldots,\alpha_{r+1}} = 0,$$

$$(0 \leqslant \alpha_0 < \ldots < \alpha_{r+1} \leqslant N)$$

$$(3)\quad \sum_{\lambda=0}^{r+1} (-1)^\lambda F_{\alpha_1,\ldots,\alpha_r,\beta_\lambda} F_{\beta_0,\ldots,\beta_{\lambda-1},\beta_{\lambda+1},\ldots,\beta_{r+1}} = 0,$$

$$\begin{pmatrix} 0 \leqslant \alpha_1 < \ldots < \alpha_r \leqslant N \\ 0 \leqslant \beta_0 < \ldots < \beta_{r+1} \leqslant N \end{pmatrix}.$$

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[2] $W_0^{(F)}$ means the initial variety at $u=0$. The initial varieties for the system with constant coefficients are coincide with $W_0^{(F)}$.

Proof. The relation (1) implies

$$0=\det\begin{pmatrix} y_{\alpha_0}, & \ldots, & y_{\alpha_{r+1}} \\ y_{\alpha_0}, & \ldots, & y_{\alpha_{r+1}} \\ \dfrac{dy_{\alpha_0}}{du}, & \ldots, & \dfrac{dy_{\alpha_{r+1}}}{du} \\ \vdots & & \vdots \\ \dfrac{d^r y_{\alpha_0}}{du^r}, & \ldots, & \dfrac{d^r y_{\alpha_{r+1}}}{du^r} \end{pmatrix} = \sum_{\lambda=0}^{r+1}(-1)^\lambda F_{\alpha_0,\ldots,\alpha_{\lambda-1},\alpha_{\lambda+1},\ldots,\alpha_{r+1}}$$

$$(0\leqslant \alpha_0<\ldots<\alpha_{r+1}\leqslant N).$$

Similarly we can prove $(2)_1, \ldots, (2)_r$. The relations (3) are the Grassmann relations between $(r+1)\times(r+1)$-minor determinants of the $(r+1)\times(N+1)$ matrix

$$\begin{pmatrix} y_0, & \ldots, & y_N \\ \dfrac{dy_0}{du}, & \ldots, & \dfrac{dy_N}{du} \\ \vdots & & \vdots \\ \dfrac{d^r y_0}{du^r}, & \ldots, & \dfrac{d^r y_N}{du^r} \end{pmatrix}.$$

LEMMA 1. *If the system* (1) *is type* $(N, r, r+1)$ *and* $(\varphi_0, \ldots, \varphi_N)$ *is a solution of* (1), *then for any holomorphic function* ψ *the vector* $(\psi\varphi_0, \ldots, \psi\varphi_N)$ *is also a solution of the system* (1).

Proof. From the definitions it follows

$$\det\begin{pmatrix} \psi\varphi_{\alpha_0}, & \ldots, & \psi\varphi_{\alpha_r} \\ \dfrac{d(\psi\varphi_{\alpha_0})}{du}, & \ldots, & \dfrac{d(\psi\varphi_{\alpha_r})}{du} \\ \vdots & & \vdots \\ \dfrac{d^r(\psi\varphi_{\alpha_0})}{du^r}, & \ldots, & \dfrac{d^r(\psi\varphi_{\alpha_r})}{du^r} \end{pmatrix} = \psi^{r+1}\det\begin{pmatrix} \varphi_{\alpha_0}, & \ldots, & \varphi_{\alpha_r} \\ \dfrac{d\varphi_{\alpha_0}}{du}, & \ldots, & \dfrac{d\varphi_{\alpha_r}}{du} \\ \vdots & & \vdots \\ \dfrac{d^r\varphi_{\alpha_0}}{du^r}, & \ldots, & \dfrac{d^r\psi_{\alpha_r}}{du^r} \end{pmatrix}$$

$$=\psi^{r+1}F_{\alpha_0,\ldots,\alpha_r}(u, \varphi_0, \ldots, \varphi_r) = F_{\alpha_0,\ldots,\alpha_r}(u, \psi\varphi_0, \ldots, \psi\varphi_N)$$

$$(0\leqslant \alpha_0<\ldots<\alpha_r\leqslant N).$$

DEFINITION 1. *A projective solution* of an osculating system (1) of type $(N, r, r+1)$ is a map φ: $u\to(\varphi_0(u), \ldots, \varphi_N(u))$ into the projective N-space $\boldsymbol{P}_N(\boldsymbol{C})$ such that $(\varphi_0(u), \ldots, \varphi_N(u))$ is a solution of the osculating system (1).

Let σ be an element in $\boldsymbol{PGL}(N+1, \boldsymbol{C})$[3] and $(p_{\alpha,\beta}(\sigma))$ and $(p_{\alpha,\beta}(\sigma^{-1}))$ be the representatives of σ and σ^{-1} in $\boldsymbol{GL}(N+1, \boldsymbol{C})$ such that

$$(p_{\alpha,\beta}(\sigma^{-1})) = (p_{\alpha,\beta}(\sigma))^{-1}. \tag{4}$$

Putting

$$p_{\alpha_0,\dots,\alpha_r;\,\beta_0,\dots,\beta_r}(\sigma) = \det\begin{pmatrix} p_{\alpha_0,\beta_0}(\sigma), & \dots, & p_{\alpha_0,\beta_r}(\sigma) \\ \vdots & & \\ p_{\alpha_r,\beta_0}(\sigma), & \dots, & p_{\alpha_r,\beta_r}(\sigma) \end{pmatrix}$$

$$\begin{pmatrix} 0 \leqslant \alpha_0 < \dots < \alpha_r \leqslant N \\ 0 \leqslant \beta_0 < \dots < \beta_r \leqslant N \end{pmatrix}$$

and

$$\begin{aligned} &\sigma(F)_{\alpha_0,\dots,\alpha_r}(u, y_0, \dots, y_N) \\ &= \sum_{0 \leqslant \lambda_0 < \dots < \lambda_r \leqslant N} p_{\alpha_0,\dots,\alpha_r;\,\lambda_0,\dots,\lambda_r}(\sigma) \\ &F_{\lambda_0,\dots,\lambda_r}\Big(u, \sum_{\nu=0}^{N} p_{0,\nu}(\sigma^{-1})y_\nu, \dots, \sum_{\nu=0}^{N} p_{N,\nu}(\sigma^{-1})y_\nu\Big), \end{aligned} \tag{5}$$

we have an osculating system of the same type

$$\det\begin{pmatrix} y_{\alpha_0}, & \dots, & y_{\alpha_r} \\ \dfrac{dy_{\alpha_0}}{du}, & \dots, & \dfrac{dy_{\alpha_r}}{du} \\ \vdots & & \vdots \\ \dfrac{d^r y_{\alpha_0}}{du^r}, & \dots, & \dfrac{d^r y_{\alpha_r}}{du^r} \end{pmatrix} = \sigma(F)_{\alpha_0,\dots,\alpha_r}(u, y_0, \dots, y_N) \quad (0 \leqslant \alpha_0 < \dots < \alpha_r \leqslant N). \tag{6}$$

If the osculating system (1) is type $(N, r, r+1)$, the transformed osculating system (6) does not depend on the choice of the representative $(p_{\alpha,\beta}(\sigma))$ of σ in $\boldsymbol{GL}(N+1, C)$.

LEMMA 2. *Let* $(p_{\alpha,\beta}(\sigma))$ *be an element in* $\boldsymbol{GL}(N+1, C)$. *Then a vector* $(\varphi_0, \dots, \varphi_N)$ *is a solution of* (1) *if and only if* $\big(\sum_{\lambda=0}^{N} p_{0,\lambda}(\sigma)\varphi_\lambda, \dots, \sum_{\lambda=0}^{N} p_{N,\lambda}(\sigma)\varphi_\lambda\big)$ *is a solution of the transformed osculating system* (6).

Proof. From the definitions it follows

[3] $\boldsymbol{PGL}(N+1)$ means the projective transformation group acting on the projective N-space $\boldsymbol{P}_N$.

$$\det\begin{pmatrix} \sum_{\lambda=0}^{N} p_{\alpha_0,\lambda}(\sigma)y_\lambda, & \dots, & \sum_{\lambda=0}^{N} p_{\alpha_r,\lambda}(\sigma)y_\lambda \\ \frac{d}{du}(\sum_{\lambda=0}^{N} p_{\alpha_0,\lambda}(\sigma)y_\lambda), & \dots, & \frac{d}{du}(\sum_{\lambda=0}^{N} p_{\alpha_r,\lambda}(\sigma)y_\lambda) \\ \vdots & & \vdots \\ \frac{d^r}{du^r}(\sum_{\lambda=0}^{N} p_{\alpha_0,\lambda}(\sigma)y_\lambda), & \dots, & \frac{d^r}{du^r}(\sum_{\lambda=0}^{N} p_{\alpha_r,\lambda}(\sigma)y_\lambda) \end{pmatrix}$$

$$= \sum_{0\leqslant\lambda_0<\dots<\lambda_r\leqslant N} p_{\alpha_0,\dots,\alpha_r;\,\lambda_0,\dots,\lambda_r}(\sigma) \det\begin{pmatrix} y_{\lambda_0}, & \dots, & y_{\lambda_r} \\ \frac{dy_{\lambda_0}}{du}, & \dots, & \frac{dy_{\lambda_r}}{du} \\ \vdots & & \vdots \\ \frac{d^r y_{\lambda_0}}{du^r}, & \dots, & \frac{d^r y_{\lambda_r}}{du^r} \end{pmatrix}$$

$$= \sum_{0\leqslant\lambda_0<\dots<\lambda_r\leqslant N} p_{\alpha_0,\dots,\alpha_r;\,\lambda_0,\dots,\lambda_r}(\sigma) F_{\lambda_0,\dots,\lambda_r}(u, y_0, \dots, y_N)$$

$$= \sigma(F)_{\alpha_0,\dots,\alpha_r}(u, \sum_{\lambda=0}^{N} p_{0,\lambda}(\sigma)y_\lambda, \dots, \sum_{\lambda=0}^{N} p_{N,\lambda}(\sigma)y_\lambda)$$

$$(0 \leqslant \alpha_0 < \dots < \alpha_r \leqslant N).$$

1.2. Osculating systems of partial differential equations are defined similarly. Let $\xi_1, \dots, \xi_n$ be indeterminates and D_ξ be the sum $\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial u_i}$. Put

$$(7)\quad \det\begin{pmatrix} y_{\alpha_0}, & \dots, & y_{\alpha_r} \\ D_\xi(y_{\alpha_0}), & \dots, & D_\xi(y_{\alpha_r}) \\ \vdots & & \vdots \\ D_\xi^r(y_{\alpha_0}), & \dots, & D_\xi^r(y_{\alpha_r}) \end{pmatrix}$$

$$= \sum_{l_1+\dots+l_n=r} \xi_1^{l_1} \dots \xi_n^{l_n} P_{l_1,\dots,l_n}$$

$$\left(y_{\alpha_0}, \dots, y_{\alpha_r}, \frac{\partial y_{\alpha_0}}{\partial u_1}, \dots, \frac{\partial^{\Sigma h_i} y_{\alpha_\lambda}}{\partial u_1^{h_1}, \dots, \partial u_n^{h_n}}, \dots, \frac{\partial^r y_{\alpha_r}}{\partial u_n^r}\right)$$

$$(0 \leqslant \alpha_0 < \dots < \alpha_r \leqslant N).$$

An osculating system of partial differential equations of type (N, r, s) is a system of partial differential equations

$$(8)\quad P_{l_1,\dots,l_r}\left(y_{\alpha_0},\dots,y_{\alpha_r},\frac{\partial y_{\alpha_0}}{\partial u_1},\dots,\frac{\partial^{\Sigma h_i} y_{\alpha_\lambda}}{\partial u_1^{h_1},\dots,\partial u_n^{h_n}},\dots,\frac{\partial^r y_{\alpha_r}}{\partial u_n^r}\right)$$

$$= F_{\alpha_0,\dots,\alpha_r;\, l_1,\dots,l_r}\ (u_1,\dots,u_n, y_0,\dots,y_N)$$

$$\left(l_1+\cdots+l_n=\frac{r(r+1)}{2};\quad 0\leqslant \alpha_0<\cdots<\alpha_r\leqslant N\right)$$

such that $F_{\alpha_0,\dots,\alpha_r;\, l_1,\dots,l_r}$ are homogeneous forms of degree s in $y_0,\dots,y_N$.

For the sake of simplicity instead of (8) we shall denote

$$(9)\quad \det\begin{pmatrix} y_{\alpha_0},\dots,y_{\alpha_r} \\ D_\xi(y_{\alpha_0}),\dots,D_\xi(y_{\alpha_r}) \\ \vdots \qquad\qquad \vdots \\ D_\xi^r(y_{\alpha_0}),\dots,D_\xi^r(y_{\alpha_r}) \end{pmatrix}$$

$$= F_{\alpha_0,\dots,\alpha_r}(\xi_1,\dots,\xi_n | u_1,\dots,u_n, y_0 \dots, y_N)$$

with homogeneous forms

$$F_{\alpha_0,\dots,\alpha_r}(\xi_1,\dots,\xi_n | u_1,\dots,u_n, y_0,\dots,y_N)$$

$$=\sum_{l_1+\cdots+l_n=\frac{r(r+1)}{2}} \xi_1^{l_1}\dots\xi_n^{l_n} F_{\alpha_0,\dots,\alpha_r;\, l_1,\dots,l_r}(u_1,\dots,u_n, y_0,\dots,y_N)$$

$$(0\leqslant\alpha_0<\dots<\alpha_r\leqslant N).$$

Then a solution $(\varphi_0,\dots,\varphi_N)$ of (9) is also a solution of (8) if and only if it is a solution of the specialized system of (9) with respect to any specialization of $(\xi_0,\dots,\xi_n)$.

1.3 At the end of this paragraph we shall show some typical examples of osculating systems for $N=2$.

An osculating system of type (2, 1, 2).

$$(\mathrm{i})\qquad \begin{cases} \det\begin{pmatrix} y_0, & y_1 \\ \dfrac{dy_0}{du}, & \dfrac{dy_1}{du}\end{pmatrix} = \lambda y_0 y_1 - y_2^2 \\[2ex] \det\begin{pmatrix} y_0, & y_2 \\ \dfrac{dy_0}{du}, & \dfrac{dy_2}{du}\end{pmatrix} = -\lambda y_0 y_2 y_1^2 \quad (\lambda^3 \neq 1) \\[2ex] \det\begin{pmatrix} y_1, & y_2 \\ \dfrac{dy_1}{du}, & \dfrac{dy_2}{du}\end{pmatrix} = \lambda y_1 y_2 - y_0^2 \end{cases}$$

If we put

$$\vartheta_0(\tau|u) = \sum_{m=-\infty}^{\infty} e^{3\pi\sqrt{-1}(m^2\tau+2mu)}$$

$$\vartheta_{\frac{1}{3}}(\tau|u) = \sum_{m=-\infty}^{\infty} e^{3\pi\sqrt{-1}((m+\frac{1}{3})^2\tau+2(m+\frac{1}{3})u)}$$

$$\vartheta_{\frac{2}{3}}(\tau|u) = \sum_{m=-\infty}^{\infty} e^{3\pi\sqrt{-1}((m+\frac{2}{3})^2\tau+2(m+\frac{2}{3})u)}$$

$$\lambda = \frac{\vartheta_0(\tau|0)^3\vartheta_{\frac{1}{3}}(\tau|0)^3\vartheta_{\frac{2}{3}}(\tau|0)^3}{3\vartheta_0(\tau|0)\vartheta_{\frac{1}{3}}(\tau|0)\vartheta_{\frac{2}{3}}(\tau|0)},$$

then the projective solutions of (i) are given the theta maps φ_a: $u \to (\vartheta_0(\tau|u), \vartheta_{\frac{1}{3}}(\tau|u), \vartheta_{\frac{2}{3}}(\tau|u))$. The projective solution φ_a is an analytic homomorphism of the additive group $\boldsymbol{C}$ onto the abelian variety $Y_0^3+Y_1^3+Y_2^3-3\lambda Y_0Y_1Y_2=0$ with the origin $\varphi_a(0)=(\vartheta_0(\tau|a), \vartheta_{\frac{1}{3}}(\tau|a), \vartheta_{\frac{2}{3}}(\tau|a))$. [4)]

An osculating system of type (2, 2, 3).
If we put

$$\mu = \frac{\det\begin{pmatrix} \vartheta_0(\tau|0,), \vartheta_{\frac{1}{3}}(\tau|0), & \vartheta_{\frac{2}{3}}(\tau|0) \\ \frac{d}{du}\vartheta_0(\tau|0), & \frac{d}{du}\vartheta_{\frac{1}{3}}(\tau|0), & \frac{d}{du}\vartheta_{\frac{2}{3}}(\tau|0) \\ \frac{d^2}{du^2}\vartheta_0(\tau|0), & \frac{d^2}{du^2}\vartheta_{\frac{1}{3}}(\tau|0), & \frac{d^2}{du^2}\vartheta_{\frac{2}{3}}(\tau|0) \end{pmatrix}}{\vartheta_0(\tau|0)^3\vartheta_{\frac{1}{3}}(\tau|0)^3\vartheta_{\frac{2}{3}}(\tau|0)^3},$$

then $(\vartheta_0(t|u), \vartheta_{\frac{1}{3}}(\tau|u), \vartheta_{\frac{2}{3}}(\tau|u))$ satisfies the osculating system

$$\text{(ii)}\quad \det\begin{pmatrix} y_0, & y_1, & y_2 \\ \frac{dy_0}{du}, & \frac{dy_1}{du}, & \frac{dy_2}{du} \\ \frac{d^2y_0}{du^2}, & \frac{d^2y_1}{du^2}, & \frac{d^2y_2}{du^2} \end{pmatrix} = \mu(y_0^3+y_1^3+y_2^3),$$

because the inflex points on $Y_0^3+Y_1^3+Y_2^3-3\lambda y_0y_1y_2=0$ are the following nine points

$$\begin{matrix} (0,1,-1) & (-1,0,1) & (1,-1,0) \\ (0,1,\alpha) & (\alpha,0,1) & (1,\alpha,0) \\ (0,1,\bar{\alpha}) & (\bar{\alpha},0,1) & (1,\bar{\alpha},0) \end{matrix} \qquad (\alpha^3 \equiv 1)$$

See [2] p. 440–448, [6] p. 191–198.

and the theta function $\vartheta_0(\tau|u)^3+\vartheta_{\frac{1}{3}}(\tau|u)^3+\vartheta_{\frac{2}{3}}(\tau|u)^3$ has the exact nine zero points $a_1,\ldots,a_9$ such that $(\vartheta_0(\tau|a_i),\ \vartheta_{\frac{1}{3}}(\tau|a_i),\ \vartheta_{\frac{2}{3}}(\tau|a_i))$ $(1\leqslant i\leqslant 9)$ are the above nine infex points.

An osculating system of type $(2,1,3)$.

Let $\boldsymbol{V}$ be a non-singular plane curve of order four. Then the genus of $\boldsymbol{V}$ is three and there exist a Fuchsian group Γ on the upper plane

$\boldsymbol{H}=\{\tau|Im\tau>0\}$ and a base of automorphic forms $(\varphi_0,\varphi_1,\varphi_2)$ of weight 2 with respect Γ such that the map: $\tau\rightarrow(\varphi_0(\tau),\ \varphi_1(\tau),\ \varphi_2(\tau))$ a covering map of $\boldsymbol{H}$ onto $\boldsymbol{V}$. Since it is easily observe that

$$\det\begin{pmatrix}\varphi_{\alpha_0}, & \varphi_{\alpha_1}\\ \dfrac{d\varphi_{\alpha_0}}{d\tau}, & \dfrac{d\varphi_{\alpha_0}}{d\tau}\end{pmatrix}\qquad(0\leqslant\alpha_0<\alpha_1\leqslant 3)$$

are automorphic forms of weight six, there exist cubic forms $F_{0,1}$, $F_{0,2}$, $F_{1,2}$ in y_0, y_1, y_2 with constant coefficients such that $(\varphi_0,\varphi_1,\varphi_2)$ is a solution of the osculating system of type $(2,1,3)$

$$\text{(iii)}\quad\det\begin{pmatrix}y_{\alpha_0}, & y_{\alpha_1}\\ \dfrac{dy_{\alpha_0}}{d\tau}, & \dfrac{dy_{\alpha_1}}{d\tau}\end{pmatrix}=F_{\alpha_0,\alpha_1}(y_0,y_1,y_2)\quad(0\leqslant\alpha_0<\alpha_1\leqslant N).$$

§2. Projective solutions of osculating systems of ordinary differential equations of type $(N,1,2)$.

2.1 In the following three paragraphs we shall be concerned with an osculating system of type $(N,1,2)$

$$(10)\quad\det\begin{pmatrix}y_{\alpha_0}, & y_{\alpha_1}\\ \dfrac{dy_{\alpha_0}}{du}, & \dfrac{dy_{\alpha_1}}{du}\end{pmatrix}=F_{\alpha_0,\alpha_0}(u,y_0,\ldots,y_N)\quad(0\leqslant\alpha_0<\alpha_1\leqslant N).$$

Definition 2. Let u_0 be a regular point of all the coefficients in the quadratic forms $F_{\alpha,\beta}(u,y_0,\ldots,y_N)$ in (10). $\boldsymbol{W}_{u_0}^{(F)}$ denotes the set of all the point x in the projective N-space $\boldsymbol{P}_N(\boldsymbol{C})$ such that there exists a holomorphic projective solution of (10) with the initial point x at $u=u_0$. We call $\boldsymbol{W}_{u_0}^{(F)}$ the initial variety at u_0 for the osculating system (10) of type $(N,1,2)$.

It will be shown later that the initial varieties for an osculating system of type $(N,1,2)$ are projective algebraic varieties in the projective N-space $\boldsymbol{P}_N$ which are biregular birationally equivalent each other.

Before going to the existence theorem of projective solutions, to make clear the base of the argument, we shall recollect Cauchy's existence theorem:

THEOREM (CAUCHY).[5] *Let* $f(u, y_1, \ldots, y_N)$ $(1 \leqslant \alpha \leqslant N)$ *be holomorphic functions in a neighbourhood of* $(u_0, a_1, \ldots, a_N)$. *Then there exist holomorphic functions* $\varphi_1(u, x_1, \ldots, x_N), \ldots, \varphi_N(u, x_1, \ldots, x_N)$ *in a neighbourhood of* $(u_0, a_2, \ldots, a_N)$ *such that* $(\varphi_1(u, x_1, \ldots, x_N), \ldots, \varphi_N(u, x_1, \ldots, x_N))$ *is the unique holomorphic solution of the system*

$$\frac{dy_\alpha}{du} = f_\alpha(u, y_1, \ldots, y_N) \quad (1 \leqslant \alpha \leqslant N)$$

with the initial value $(x_1, \ldots, x_N)$ *at* $u = u_0$.

THEOREM 1. *Let* u_0 *be a regular point of all the coefficients in quadratic forms* $F_{\alpha_0, \alpha_1}(u, y_0, \ldots, y_N)$ $(0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$ *and* $a = (a_0, \ldots, a_N)$ *be a point on the initial variety* $\boldsymbol{W}^{(F)}_{u_0}$ *at* u_0 *for the osculating system of type* $(N, 1, 2)$

$$\det \begin{pmatrix} y_{\alpha_0}, & y_{\alpha_1} \\ \dfrac{dy_{\alpha_0}}{du}, & \dfrac{dy_{\alpha_1}}{du} \end{pmatrix} = F_{\alpha_0, \alpha_1}(u, y_0, \ldots, y_N) \quad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N).$$

Then there exists a unique holomorphic projective solution $\varphi(u, a) = (\varphi_0(u, a), \ldots, \varphi_N(u, a))$ *of the system with the initial point a at* $u = u_0$. *Moreover* $\varphi(u, a)$ *depends analytically on the initial point, i.e. when* $a_\beta \neq 0$, *there exist holomorphic functions* $\psi_\alpha(u, x_0, \ldots, x_{\beta-1}, x_{\beta+1}, \ldots, x_N)$ *in a neighbourhood of* $(u_0, a_1/a_\beta, \ldots, a_{\beta-1}/a_\beta, a_{\beta+1}/a_\beta, \ldots, a_N/a_\beta)$ *such that, if* $(x_0, \ldots, x_{\beta-1}, 1, x_{\beta+1}, \ldots, x_N)$ *is also a point on* $\boldsymbol{W}^{(F)}_{u_0}$, *then* $(\psi_0(u, x_0, \ldots, x_{\beta-1}, x_{\beta+1}, \ldots, x_N), \ldots, \psi_{\beta-1}(u, x_0, \ldots, x_{\beta-1}, x_{\beta+1}, \ldots, x_N), 1, \psi_{\beta+1}(u, x_0, \ldots, x_{\beta-1}, x_{\beta+1}, \ldots, x_N), \ldots, \psi_N(u, x_0, \ldots, x_{\beta-1}, x_{\beta+1}, \ldots, x_N))$ *is a unique holomorphic projective solution of the osculating system with the initial point* $(x_0, \ldots, x_{\beta-1}, 1, x_{\beta+1}, \ldots, x_N)$ *at* $u = u_0$.

Proof. Since a is a point on $\boldsymbol{W}^{(F)}_{u_0}$, there exists a holomorphic projective solution $(\varphi_0, \ldots, \varphi_N)$ with the initial point a at $u = u_0$. By virtue of Lemma 2 we may assume without loss of generality that $\varphi_0(u) = a_0 \neq 0$. Put $\phi_\alpha = \varphi_\alpha/\varphi_0$ $(0 \leqslant \alpha \leqslant N)$. Then from Lemma 2 it follows

$$\det \begin{pmatrix} \phi_{\alpha_0}, & \phi_{\alpha_1} \\ \dfrac{d\phi_{\alpha_0}}{du}, & \dfrac{d\phi_{\alpha_1}}{du} \end{pmatrix} = F_{\alpha_0, \alpha_1}(u, \phi_0, \ldots, \phi_N) \quad (0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant N)$$

5) See [4] p. 29–46.

Hence

$$(*)\quad \frac{d\phi_\alpha}{du} = \det\begin{pmatrix} \phi_0, & \phi_\alpha \\ \dfrac{d\phi_0}{du}, & \dfrac{d\phi_\alpha}{du} \end{pmatrix} = F_{0,\alpha}(u, 1, \phi_1, \ldots, \phi_N) \quad (1 \leqslant \alpha \leqslant N).$$

Therefore, by virtue of Cauchy's Theorem there exist holomorphic functions $\psi_\alpha(u, x_1, \ldots, x_N)$ $(1 \leqslant \alpha \leqslant N)$ in a neighbourhood of $(u_0, a_1/a_0, \ldots, a_N/a_0)$ such that $(\psi_1(u, x_1, \ldots, x_N), \ldots, \psi_N(u, x_1, \ldots, x_N))$ is the unique holomorphic solution of the system $(*)$ with the initial value $(x_1, \ldots, x_N)$ at $u = u_0$. Since $(\varphi_0(u), \ldots, \varphi_N(u))$ and $(1, \psi_1(u, a_1/a_0, \ldots, a_N/a_0), \ldots, \psi_N(u, a_1/a_0, \ldots, a_N/a_0))$ are the same projective solutions for the osculating system in a neighbourhood of u_0, it follows the uniqueness of holomorphic projective solutions. If $(1, x_1, \ldots, x_N)$ is also a point on $W^{(F)}_{u_0}$, then by the same reason as the above $(1, \psi_1(u, x_1, \ldots, x_N), \ldots, \psi_N(u, x_1, \ldots, x_N))$ is the unique holomorphic projective solution of the osculating system with the initial point $(1, x_1, \ldots, x_N)$ at $u = u_0$. This completes the proof of Theorem.

2.2 It will be shown that singularities of osculating system of type $(N, 1, 2)$ are the singularities of the coefficients of the system. Therefore, if all the coefficients are holomorphic, the projective solutions are analytic maps into the projective spaces. We shall first estimate the radii of convergence for power series solutions of the following differential equations

$$(11)\quad \frac{dy_\alpha}{du} = \sum_{\lambda,\mu=1}^{N} h_{\alpha;\,\lambda,\,\mu}(u) y_\lambda y_\mu + \sum_{\lambda=1}^{N} (h_{\alpha;\,\lambda,\,0}(u) + h_{\alpha;\,0,\,\lambda}(u)) y_\lambda + h_{\alpha;\,0,\,0}(u)$$
$$(1 \leqslant \alpha \leqslant N).$$

PROPOSITION 2. Let K be a positive number not less than 1 and $(u_0, a_{1,0}, \ldots, a_{N,0})$ be a system of complex numbers such that the functions $h_{\alpha;\,\lambda,\,\mu}(u)$ $(1 \leqslant \alpha \leqslant N;\ 0 \leqslant \lambda, \mu \leqslant N)$ are holomorphic $u = u_0$ and

$$|a_{\alpha,0}| < K,$$
$$|h_{\alpha;\,\lambda,\,\mu}(u_0)| < K,$$
$$\left|\frac{1}{n!}\frac{d^n h_{\alpha;\,\lambda,\,\mu}(u_0)}{du^n}\right| < K^n \quad (1 \leqslant \alpha \leqslant N;\ 0 \leqslant \lambda, \mu \leqslant N; n = 1, 2, 3, \ldots).$$

Let r be the radius of convergence for the power series solution

$$\left(\sum_{n=0}^{\infty} a_{1,n}(u - u_0)^n, \ldots, \sum_{n=0}^{\infty} a_{N,n}(u - u_0)^n\right)$$

of (11) with constant term $(a_{1,0}, \dots, a_{N,0})$. Then it follows

$$r > \frac{1}{4(N+1)^2K^2}.$$

Proof. Putting $\varphi_\alpha(u) = \sum_{n=0}^{\infty} a_{\alpha,n}(u-u_0)^n$ and $\varphi_\alpha^{(n)} = \frac{d^n\varphi_\alpha}{du^n}$

$$(1 \leqslant \alpha \leqslant N;\ n = 0, 1, 2, \dots)$$

we have

$$a_{\alpha,n} = \frac{1}{n!}\varphi_\alpha^{(n)}(u_0)$$

$$\varphi_\alpha^{(n+1)} = \frac{d^n}{dt^n}\Big(\sum_{\lambda,\mu=0}^{N} h_{\alpha;\lambda,\mu}\varphi_\lambda\varphi_\mu\Big)$$

$$= \sum_{\lambda,\mu=0}^{N}\sum_{l=0}^{n}\sum_{p=0}^{l}\binom{n}{l}\binom{l}{p}\frac{d^{n-l}h_{\alpha;\lambda,\mu}\varphi_\lambda^{(p)}\varphi_\mu^{(l-p)}}{du^{n-l}}$$

$$a_{\alpha,n+1} = \frac{\varphi_\alpha^{(n+1)}(u_0)}{(n+1)!}$$

$$\frac{1}{n+1}\sum_{\lambda,\mu=0}^{N}\sum_{l=0}^{n}\sum_{p=0}^{l}\frac{1}{(n-l)!}\frac{d^{n-l}h_{\alpha;\lambda,\mu}(u_0)}{du^{n-l}}\frac{\varphi_\lambda^{(p)}(u_0)}{p!}\frac{\varphi_\mu^{(l-p)}(u_0)}{(l-p)!},$$

$$|a_{\alpha,n+1}| < \frac{1}{n+1}(N+1)^2\sum_{l=0}^{n}\sum_{p=0}^{l}K^{n-l+1}\operatorname*{Max}_{\lambda}|a_{\lambda,p}|\operatorname*{Max}_{\mu}|a_{\mu,l-p}|$$

$$(1 \leqslant \alpha \leqslant N;\ n = 0, 1, 2, \dots).$$

We shall prove the following inequalities by the induction on n

(*) $|a_{\alpha,n}| < 3^n(N+1)^{2n}K^{2n-1}$ $(1 \leqslant \alpha \leqslant N;\ n = 0, 1, 2, \dots)$.

This is true for $n = 0$. Assume the inequalities for $0, 1, \dots, n$. Then it follows

$$|a_{\alpha,n+1}| < \frac{(N+1)^2}{n+1}\sum_{l=0}^{n}\sum_{p=0}^{l}K^{n-l+1}\operatorname*{Max}_{\lambda}|a_{\lambda,p}|\operatorname*{Max}_{\mu}|a_{\mu,l-p}|$$

$$< \frac{(N+1)^2}{n+1}\sum_{l=0}^{n}\sum_{p=0}^{l}K^{n-l+1}K^{2l+2}3^l(N+1)^{2l}$$

$$\leqslant \frac{(N+1)^2}{n+1}\sum_{l=0}^{n}(l+1)K^{n+l+3}3^l(N+1)^{2l}$$

$$< (N+1)^2K^{2(n+1)+1}\sum_{l=0}^{\infty}3^l(N+1)^2$$

$$< 3^n(N+1)^{2(n+1)}K^{2(n+1)+1} \sum_{l=0}^{\infty} \frac{1}{3^l(N+1)^{2l}}$$

$$\leqslant 3^n(N+1)^{2(n+1)}K^{2(n+1)+1}\left(1 - \frac{1}{3(N+1)^2}\right)^{-1}$$

$$< 3^{n+1}(N+1)^{2(n+1)}K^{2(n+1)+1} .$$

This proves the inequalities (*). Hence it follows

$$\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} \leqslant 3(N+1)^2K^2 .$$

Therefore by virtue of Cauchy-Hadamard Formula[6] we have the estimation of the radius of convergence

$$r \geqslant \frac{1}{3(N+1)^2K^2} > \frac{1}{4(N+1)^2K^2} .$$

Let M be a complex analytic manifold and $M = \bigcup_{\lambda} U^{(\lambda)}$ be a covering of M by coordinate neighbourhoods $U^{(\lambda)}$ with analytic parameter $u^{(\lambda)}$. An osculating system on M of type $(N, 1, 2)$ means a collection of osculating systems

$$\det \begin{pmatrix} y_{\alpha_0}, & y_{\alpha_1} \\ \dfrac{dy_{\alpha_0}}{du^{(\lambda)}}, & \dfrac{dy_{\alpha_1}}{du^{(\lambda)}} \end{pmatrix} = F^{(\lambda)}_{\alpha_0, \alpha_1}(u^{(\lambda)}, y_0, \ldots, y_N) \quad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$$

on $U^{(\lambda)}$ such that

$$F^{(\mu)}_{\alpha_0, \alpha_1}(u^{(\mu)}, y_0, \ldots, y_N) = \frac{du^{(\lambda)}}{du^{(\mu)}} F^{(\lambda)}_{\alpha_0, \alpha_1}(u^{(\lambda)}, y_0, \ldots, y_N) \qquad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$$

on $U^{(\lambda)} \cap U^{(\mu)}$.

THEOREM 2. Let M be a complex analytic manifold of dimension one. Let $\varphi: u \to (\varphi_0(u), \ldots, \varphi_N(u))$ be a projective holomorphic solution at u_0 of an osculating system of ordinary differential equations of type $(N, 1, 2)$ and ω: $[0, 1] \to M$ be a path on M such that $\omega(0) = u_0$ and $\omega(t)$ $(0 \leqslant t < \infty)$ are regular points for all the coefficients of the osculating system. Then there exists the analytic continuation of φ along the path ω.

Proof. Let φ be a holomorphic projective solution at u_0 and ω: $t \to \omega(t)$ be a path on M such that $\omega(0) = u_0$ and $\omega(t)$ $(0 \leqslant t < \infty)$ are regular points

6) See text books on Advanced calculus.

of the all the coefficients. Let t_1 be the supremum of t such that the projective solution φ has the analytic continuation to $\omega(t)$ along ω. Then it is sufficient to show $t_1 < \infty$. Assume for a moment that $t_1 = \infty$. We shall show a contradiction. Let u be a holomorphic local parameter in an open neighbourhood U of $\omega(t_1)$. We denote by

$$\det\begin{pmatrix} y_{\alpha_0}, & y_{\alpha_1} \\ \dfrac{dy_{\alpha_0}}{du}, & \dfrac{dy_{\alpha_1}}{du}\end{pmatrix} = \sum_{\lambda,\mu=0}^{N} h_{\alpha_0,\alpha_1;\,\lambda,\mu}(u) y_\lambda y_\mu \quad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$$

the osculating system with respect to the local coordinates u. The open neighbourhood U may be regarded as metric space with the metric $|p, q| = |u(p) - u(q)|$, where $u(p)$ means the value of u at p. Since the projective solution φ is not holomorphic at $\omega(t_1)$, there exist β and t_2 such that $\varphi_\alpha / \varphi_\beta$ $(0 \leqslant \alpha \leqslant N)$ are holomorphic at $u = \omega(t)$ for $t_1 > t > t_2$ and at least one of $\varphi_\alpha / \varphi_\beta\,(0 \leqslant \alpha \leqslant N)$ are not holomorphic at $\omega(t_1)$. Let V be a compact neighbourhood of $\omega(t_1)$ contained in U such that the coefficients $h_{\alpha,\beta;\,\lambda,\mu}$ (v) are holomorphic at each point in V. Then from the Cauchy-Hadamard Formula we have a positive number ρ such that

$$\varlimsup_{n\to\infty}\left|\frac{1}{n!}\frac{d^n h_{\alpha,\beta;\,\lambda,\mu}(v)}{du^n}\right|^{\frac{1}{n}} < \frac{1}{\rho} \quad (0 \leqslant \alpha, \beta; \lambda, \mu \leqslant N)$$

for each point v in the compact set V. Hence there exists a positive number K such that

$$K > 1, \quad |h_{\alpha,\beta;\,\lambda,\mu}(v)| < K,$$

$$\left|\frac{1}{n!}\frac{d^n h_{\alpha,\beta;\,\lambda,\mu}(v)}{du^n}\right| < K^n \quad (0 \leqslant \alpha, \beta, \lambda, \mu \leqslant N;\ n = 0, 1, 2, \ldots)$$

for each point v in V. Let $p = \omega(t_3)$ be a point on the path such that $\omega(t) \in V$ for $t_1 \geqslant t \geqslant t_3$, $(t_1 \underset{\neq}{>} t_3 > t_2)$

and

$$|u(p) - u(\omega(t_1))| < [4(N+1)^2 K^2]^{-1}.$$

Then there exists β such that $\varphi_\alpha / \varphi_\beta$ $(0 \leqslant \alpha, \beta \leqslant N)$ are holomorphic at $u = \omega(t_3)$ and $|\varphi_\alpha / \varphi_\beta\,(\omega(t_3))| < 1 < K$. Hence, applying Proposition 2 to the system

$$\begin{cases} y_\beta = 1 \\ \det \begin{pmatrix} y_\alpha, & y_\beta \\ \dfrac{dy_\alpha}{du}, & \dfrac{dz_\beta}{du} \end{pmatrix} = -\dfrac{dy_\alpha}{du} = F_{\alpha,\beta}(u, y_0, \ldots. y_N) \quad (0 \leqslant \alpha \leqslant N\,;\, \alpha \neq \beta), \end{cases}$$

we observe that $\varphi_\alpha / \varphi_\beta$ $(0 \leqslant \alpha \leqslant N)$ are holomorphic in the open ball of radius $[4(N+1)^2K^2]^{-1}$ with the center $\omega(t_3)$. Therefore $\varphi_\alpha / \varphi_\beta$ $(0 \leqslant \alpha \leqslant N)$ are also holomorphic at $\omega(t_1)$, namely the map φ into $\boldsymbol{P}_N$ is holomorphic at $\omega(t_1)$. This is a contradiction to the assumption of t_1.

§3. Initial varieties

3.1 We shall show that the initial varieties for osculating system are projective varieties which are biregular birationally equivalent each other. Let L be the field over $\boldsymbol{C}$ generated by the coefficients in $F_{\alpha,\beta}(0 \leqslant \alpha, \beta \leqslant N)$ and all their derivatives and D be the derivation of the field $L(Y_1/Y_0, \ldots, Y_N/Y_0)$ such that the restriction of D on L coincides with the derivation $\dfrac{d}{du}$ and

$$D\Big(\frac{Y_\alpha}{Y_0}\Big) = F_{\alpha,0}\Big(u, 1, \frac{Y_1}{Y_0}, \ldots, \frac{Y_N}{Y_0}\Big) \quad (1 \leqslant \alpha \leqslant N).$$

Put

$$S_{\alpha,\beta;n} = S_{\alpha,\beta;n}(u, Y_0, \ldots, Y_N) = Y_0^n Y_\beta^{n+1} D^n\Big(\frac{Y_\alpha}{Y_\beta}\Big),$$

$$T_{\alpha,\beta;n} = T_{\alpha,\beta;Y}(u, Y_0, \ldots, Y_N) = Y_0^n Y_\beta^{n+2} D^n\Big(F_{\alpha,\beta}\Big(u, \frac{Y_0}{Y_\beta}, \ldots, \frac{Y_N}{Y_\beta}\Big)\Big),$$
$$(0 \leqslant \alpha, \beta \leqslant N;\ n = 0, 1, 2, \ldots).$$

LEMMA 1. *$S_{\alpha,\beta;n}$, $T_{\alpha,\beta;n}$ $(0 \leqslant \alpha, \beta \leqslant N\,;\ n = 0, 1, 2, \ldots)$ are homogeneous elements in the polynomial algebra $L[Y_0, \ldots, Y_N]$ such that*

$$deg\ S_{\alpha,\beta;n} = 2n + 1,$$
$$deg\ T_{\alpha,\beta;n} = 2n + 2,$$

$$(12) \quad S_{\alpha,\beta;n+1} = Y_\beta\Big[\sum_{\lambda=0}^{N} \frac{\partial S_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + Y_0 \frac{\partial S_{\alpha,\beta;n}}{\partial u}\Big] - (n+1)F_{\beta,0}S_{\alpha,\beta;n},$$

$$(13) \quad T_{\alpha,\beta;n+1} = Y_\beta\Big[\sum_{\lambda=0}^{N} \frac{\partial T_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + Y_0 \frac{\partial T_{\alpha,\beta;n}}{\partial u}\Big] - (n+2)F_{\beta,0}T_{\alpha,\beta;n}$$
$$(0 \leqslant \alpha, \beta \leqslant N\,;\ n = 0, 1, 2, \ldots).$$

Proof. Since $F_{\alpha,\beta}(0 \leqslant \alpha, \beta \leqslant N)$ are quadratic forms, $S_{\alpha,\beta;0} = Y_\alpha$ and $T_{\alpha,\beta;0} = F_{\alpha,\beta}(0 \leqslant \alpha, \beta \leqslant N)$ are forms in $L[Y_0, \ldots, Y_N]$ of degree one and two, respectively. Therefore it is sufficient to prove (12) and (13): We assume that $S_{\alpha,\beta;n}$ and $T_{\alpha,\beta;n}$ are homogeneous polynomials in $Y_0, \ldots, Y_N$ of degree $2n+1$ and $2n+2$, respectively. Then from the definitions it follows

$$Y_0^{-(2n+2)}\Big(\sum_{\lambda=0}^{N} \frac{\partial S_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + Y_0 \frac{S_{\alpha,\beta;n}}{\partial u}\Big)$$
$$= \Big[\sum_{\lambda=0}^{N} \frac{\partial S_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + Y_0 \frac{\partial S_{\alpha,\beta;n}}{\partial u}\Big]_{(u, Y_0, \ldots, Y_N) = (u, 1, Y_1/Y_0, \ldots, Y_N/Y_0)}$$
$$= D\Big(\Big(\frac{Y_\beta}{Y_0}\Big)^{n+1} D^n\Big(\frac{Y_\alpha}{Y_\beta}\Big)\Big)$$
$$= (n+1)\Big(\frac{Y_\beta}{Y_0}\Big)^n D\Big(\frac{Y_\beta}{Y_0}\Big) D^n\Big(\frac{Y_\alpha}{Y_\beta}\Big)^{n+1} + \Big(\frac{Y_\beta}{Y_0}\Big)^{n+1} D^{n+1}\Big(\frac{Y_\alpha}{Y_\beta}\Big),$$

$$D^{n+1}\Big(\frac{Y_\alpha}{Y_\beta}\Big) = \Big(\frac{Y_\beta}{Y_0}\Big)^{-(n+1)} \Big[\sum_{\lambda=0}^{N} \frac{\partial S_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + \frac{\partial S_{\alpha,\beta;n}}{\partial u}\Big]_{\substack{(u, Y_0, \ldots Y_N) \\ = (u, 1, Y_1/Y_0, \ldots, Y_N/Y_0)}}$$
$$- (n+1)\Big(\frac{Y_\beta}{Y_0}\Big)^n D\Big(\frac{Y_\beta}{Y_0}\Big) D^n\Big(\frac{Y_\alpha}{Y_\beta}\Big),$$

$$Y_0^{-(2n+3)}\Big[\sum_{\lambda=0}^{N} \frac{\partial T_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + \frac{\partial T_{\alpha,\beta;n}}{\partial u}\Big]$$
$$= \Big[\sum_{\lambda=0}^{N} \frac{\partial T_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + \frac{\partial T_{\alpha,\beta;n}}{\partial u}\Big]_{(u, Y_0, Y_1, \ldots, Y_N) = (u, 1, Y_1/Y_0, \ldots, Y_N/Y_0)}$$
$$= D\Big(\Big(\frac{Y_\beta}{Y_0}\Big)^{n+2} D^n\Big(F_{\alpha,\beta}\Big(u, \frac{Y_0}{Y}, \ldots, \frac{Y_N}{Y}\Big)\Big)\Big)$$
$$= (n+2)\Big(\frac{Y_\beta}{Y_0}\Big)^{n+1} D\Big(\frac{Y_\beta}{Y_0}\Big) D^n\Big(F_{\alpha,\beta}\Big(u, \frac{Y_0}{Y_\beta}, \ldots, \frac{Y_N}{Y_\beta}\Big)\Big)$$
$$+ \Big(\frac{Y}{Y_0}\Big)^{n+2} D^{n+1}\Big(F_{\alpha,\beta}\Big(\frac{Y_0}{Y_\beta}, \ldots, \frac{Y_N}{Y_\beta}\Big)\Big),$$

$$D^{n+1}\Big(F_{\alpha,\beta}\Big(\frac{Y_0}{Y_\beta}, \ldots, \frac{Y_N}{Y_\beta}\Big)\Big)$$
$$= \Big(\frac{Y}{Y_0}\Big)^{-(n+2)} \Big[\sum_{\lambda=0}^{n} \frac{\partial T_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + \frac{\partial T_{\alpha,\beta;n}}{\partial u}\Big]_{(u, Y_0, \ldots, Y_N) = (u, 1, Y_1/Y_0, \ldots, Y_N/Y_0)}$$
$$- (n+2)\Big(\frac{Y_\beta}{Y_0}\Big)^{n+1} D\Big(\frac{Y_\beta}{Y_0}\Big) D^n\Big(F_{\alpha,\beta}\Big(u, \frac{Y_0}{Y_\beta}, \ldots, \frac{Y_N}{Y_\beta}\Big)\Big).$$

Hence we have

$$S_{\alpha,\beta;n+1} = Y_0^{n+1} Y_\beta^{n+2} D^{n+1}\Big(\frac{Y_\alpha}{Y_\beta}\Big)$$

$$= Y_0^{2n+2} Y_\beta \Big[\sum_{\lambda=0}^{N} \frac{\partial S_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + \frac{\partial S_{\alpha,\beta;n}}{\partial u} \Big]_{(u,Y,Y_1,\dots,Y_N)=\left(u,1,\frac{Y_1}{Y_0},\dots,\frac{Y_N}{Y_0}\right)}$$

$$- (n+1)\left(\frac{Y_\beta}{Y_0}\right)^n D\left(\frac{Y_\beta}{Y_0}\right) D^n\left(\frac{Y_\alpha}{Y_\beta}\right)$$

$$= Y_\beta \Big(\sum_{\lambda=0}^{N} \frac{\partial S_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + Y_0 \frac{\partial S_{\alpha,\beta;n}}{\partial u} \Big) - (n+1) F_{\beta,0} S_{\alpha,\beta;n} \,,$$

$$T_{\alpha,\beta;n+1} = Y_0^{n+1} Y_\beta^{n+3} D^{n+1}\Big(F_{\alpha,\beta}\Big(\frac{Y_0}{Y_\beta},\dots,\frac{Y_N}{Y_\beta}\Big)\Big)$$

$$= Y_0^{2n+3} Y_\beta \Big\{ \Big[\sum_{\lambda=0}^{N} \frac{\partial T_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + \frac{\partial T_{\alpha,\beta;n}}{\partial u} \Big]_{(u,Y_0,\dots,Y_N)=(u,1,Y_1/Y_0,\dots,Y_N/Y_0)}$$

$$+ (n+2)\left(\frac{Y_\beta}{Y_0}\right)^{n+1} D\left(\frac{Y_\beta}{Y_0}\right) D^n\Big(F_{\alpha,\beta}\Big(u,\frac{Y_0}{Y_\beta},\dots,\frac{Y_N}{Y_\beta}\Big)\Big)\Big\}$$

$$= Y_\beta \Big(\sum_{\lambda=0}^{N} \frac{\partial T_{\alpha,\beta;n}}{\partial Y_\lambda} F_{\lambda,0} + Y_0 \frac{\partial T_{\alpha,\beta;n}}{\partial u} \Big) - (n+2) F_{\beta,0} T_{\alpha,\beta;n}$$

$$(0 \leqslant \alpha, \beta \leqslant N\,;\ n = 0, 1, 2, \dots).$$

This proves (12) and (13). Further more it follows

$$\deg S_{\alpha,\beta;n+1} = 2 + \deg S_{\alpha,\beta;n} = 2(n+1) + 1\,,$$
$$\deg T_{\alpha,\beta;n+1} = 2 + \deg T_{\alpha,\beta;n} = 2(n+1) + 2\,.$$

We shall prove that the initial variety $\boldsymbol{W}_{u_0}^{(F)}$ is a projective algebraic variety. A projective algebraic variety $\boldsymbol{W}$ in $\boldsymbol{P}_N$ is called the projective algebraic variety corresponding to a homogeneous ideal $\boldsymbol{a}$ in $\boldsymbol{C}[Y_0, \dots . Y_N]$ if $\boldsymbol{W}$ coincides with the set of all the point x in $\boldsymbol{P}_N$ such that $f(x) = 0$ for every f in $\boldsymbol{a}$.

THEOREM 3. *Let $\boldsymbol{a}_{u_0}^{(F)}$ be the homogeneous ideal in $\boldsymbol{C}[Y_0, \dots, Y_N]$ generated by homogeneous forms*

$$S_{\alpha,\beta;n+1}(u_0, Y_0, \dots, Y_N) - Y_0 T_{\alpha,\beta;n}(u_0, Y_0, \dots, Y_N)$$
$$(0 \leqslant \alpha, \beta \leqslant N\,;\ n = 0, 1, 2, \dots).$$

Then the initial variety $\boldsymbol{W}_{u_0}^{(F)}$ at u_0 for the osculating system (1) of type $(N, 1, 2)$

$$\det \begin{pmatrix} y_{\alpha_0}\,, & y_{\alpha_1} \\ \dfrac{dy_{\alpha_0}}{du}\,, & \dfrac{dy_{\alpha_1}}{du} \end{pmatrix} = - F_{\alpha,\beta}(u, y_0, \dots, y_N) \quad (0 \leqslant \alpha_0, \alpha_1 \leqslant N)$$

is the projective algebraic variety corresponding to the homogeneous ideals $\boldsymbol{a}_{u_0}^{(F)}$.

Proof. Since u_0 is a regular point of all the coefficients in $F_{\alpha,\beta}$ $(0 \leqslant \alpha, \beta \leqslant N)$, by virtue of (12) and (13) the coefficients in $S_{\alpha,\beta;n}$, $T_{\alpha,\beta;n}$ $(0 \leqslant \alpha, \beta \leqslant N; n = 0, 1, 2, \ldots)$ are holomorphic at u_0. Hence $S_{\alpha,\beta;n+1}(u_0, Y_0, \ldots, Y_N) - Y_0 T_{\alpha,\beta;n}(u_0, Y_0, \ldots, Y_N)$ $(0 \leqslant \alpha, \beta \leqslant N;\ n = 0, 1, 2, \ldots)$ are homogeneous elements in $\boldsymbol{C}[Y_0, \ldots, Y_N]$. Let $\boldsymbol{V}$ be the projective variety corresponding to $\boldsymbol{a}^{(F)}$. We shall show $\boldsymbol{W}_{u_0}^{(F)} \subset \boldsymbol{V}$. Let x be a point on $\boldsymbol{W}_{u_0}^{(F)}$ and $\varphi(u, x)$ be the holomorphic projective solution at u_0 such that $\varphi(u_0, x) = x$. By virtue of Proposition 1 we may assume without loss of generality that $x_\beta \neq 0$ $(0 \leqslant \beta \leqslant N)$, where $x = (x_0, \ldots, x_N)$. Then from Theorem 1 it follows

$$\det \begin{pmatrix} 1, \dfrac{\varphi_\alpha}{\varphi_\beta} \\ 0, \dfrac{d\frac{\varphi_\alpha}{\varphi_\beta}}{du} \end{pmatrix} = \frac{d\frac{\varphi_\alpha}{\varphi_\beta}}{du} = F_{\alpha,\beta}\Big(u, \frac{\varphi_0}{\varphi_\beta}, \ldots, \frac{\varphi_0}{\varphi_\beta}\Big) \quad (0 \leqslant \alpha, \beta \leqslant N).$$

Since

$$\left(\frac{d\frac{\varphi_\alpha}{\varphi_0}}{du}\right)_{u=u_0} = -F_{\alpha,0}\Big(u_0, 1, \frac{\varphi_1}{\varphi_0}, \ldots, \frac{\varphi_N}{\varphi_0}\Big) = D\Big(\frac{Y_\alpha}{Y_0}\Big)_{(u_0, \varphi_1/\varphi_0, \ldots, \varphi_N/\varphi_0)}$$
$$(0 \leqslant \alpha \leqslant N),$$

there exists a specialization of $L[Y_1/Y_0, \ldots, Y_N/Y_0]$ onto $L[\varphi_1/\varphi_0, \ldots, \varphi_N/\varphi_0]$ such that $(Y_1/Y_0, \ldots, Y_N/Y_0) \to (\varphi_1/\varphi_0, \ldots, \varphi_N/\varphi_0)$ and the derivation D corresponds to the derivation $\dfrac{d}{du}$. Therefore, applying this specialization on $S_{\alpha,\beta;n+1} - Y_0 T_{\alpha,\beta;n}$, we have

$$\varphi_0^{-(2n+3)}[S_{\alpha,\beta;n+1}(\varphi_0, \ldots \varphi_N) - \varphi_0 T_{\alpha,\beta;n}(\varphi_0, \ldots, \varphi_N)]$$

$$= \Big(\frac{\varphi_\beta}{\varphi_0}\Big)^{n+2} \frac{d^{n+1}\frac{\varphi_\alpha}{\varphi_\beta}}{du^{n+1}} - \Big(\frac{\varphi_\beta}{\varphi_0}\Big)^{n+2} \frac{d^n}{du^n} F_{\alpha,\beta}\Big(\frac{\varphi_0}{\varphi_\beta}, \ldots, \frac{\varphi_N}{\varphi_\beta}\Big)$$

$$= \Big(\frac{\varphi_\beta}{\varphi_0}\Big)^{n+1} \frac{d^{n+1}\frac{\varphi_\alpha}{\varphi_\beta}}{du^{n+1}} - \frac{d^n}{du^n}\Big(\frac{d\frac{\varphi_\alpha}{\varphi_\beta}}{du}\Big) = 0.$$

On the other hand $\varphi_\alpha(u_0) = x_\alpha \neq 0$ $(0 \leqslant \alpha \leqslant N)$, hence we have

$$S_{\alpha,\beta;n+1}(x_0, \ldots, x_N) - x_0 T_{\alpha,\beta;n}(x_0, \ldots, x_N) = 0$$
$$(0 \leqslant \alpha, \beta \leqslant N;\ n = 0, 1, 2, \ldots).$$

This means $x \in \boldsymbol{V}$, namely $\boldsymbol{W}_{u_0}^{(F)} \subset \boldsymbol{V}$. We shall next prove $\boldsymbol{W}_{u_0}^{(F)} \supset \boldsymbol{V}$. For a given point $x = (x_0, \ldots, x_N)$ on $\boldsymbol{V}$ we shall construct a holomorphic projective solution φ such that $\varphi(u_0) = x$. By virtue of Lemma 2 we may assume without loss of generality that $x_\alpha \neq 0$ $(0 \leqslant \alpha \leqslant N)$. From the Cauchy Existence Theorem there exists a unique holomorphic solution $(\varphi_{0,\beta}(u), \ldots, \varphi_{N,\beta}(u))$ at u_0 of the differential equations

$$\frac{dz_{\alpha,\beta}}{du} = F_{\alpha,\beta}(u, z_{0,\beta}, \ldots, z_{N,\beta}) \quad (0 \leqslant \alpha \leqslant N)$$

with the initial condition $(\varphi_{0,\beta}(u_0), \ldots, \varphi_{N,\beta}(u_0)) = (x_0 / x_\beta, \ldots, x_N / x_\beta)$. Therefore it is sufficient to show that

$$\varphi_{\alpha,\beta} = \varphi_{\alpha,0} / \varphi_{\beta,0} \quad (0 \leqslant \alpha, \beta \leqslant N),$$

because

$$\det\begin{pmatrix} \varphi_{\alpha_0}, & \varphi_{\alpha_1} \\ \dfrac{d\varphi_{\alpha_0}}{du}, & \dfrac{d\varphi_{\alpha_1}}{du} \end{pmatrix} = -F_{\alpha_0,\alpha_1}(u, \varphi_0, \ldots, \varphi_N) \quad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N).$$

The specialization

$$\left(\frac{Y_1}{Y_0}, \ldots, \frac{Y_N}{Y_0}, D\right) \to \left(\varphi_{1,0}, \ldots, \varphi_{N,0}, \frac{d}{du}\right)$$

implies the relation

$$\frac{d^{n+1}\left(\dfrac{\varphi_{\alpha,0}}{\varphi_{\beta,0}}\right)}{du^{n+1}} = D^{n+1}\left(\frac{Y_\alpha}{Y_\beta}\right)_{(\varphi_{1,0}, \ldots, \varphi_{N,0})}$$

$$= \varphi_{\beta,0}^{-(n+2)} S_{\alpha,\beta;n+1}(1, \varphi_{1,0}, \ldots, \varphi_{N,0})$$

and the specialization

$$\left(\frac{Y_0}{Y_\beta}, \ldots, \frac{Y_N}{Y_\beta}, D\right) \to \left(\varphi_{0,\beta}, \ldots, \varphi_{N,\beta}, \frac{d}{du}\right)$$

implies another relation

$$\frac{d^{n+1}\varphi_{\alpha,\beta}}{du^{n+1}} = \frac{d^n F_{\alpha,\beta}(\varphi_{0,\beta}, \ldots, \varphi_{N,\beta})}{du^n}$$

$$= D^n\Big(F_{\alpha,\beta}\Big(\frac{Y_0}{Y_\beta}, \ldots, \frac{Y_N}{Y_\beta}\Big)\Big)_{(\varphi_{0,\beta}, \ldots, \varphi_{N,\beta})}$$

$$= \varphi_{0,\beta}^{-n} T_{\alpha,\beta;n}(\varphi_{0,\beta}, \ldots, \varphi_{N,\beta}) \quad (0 \leqslant \alpha, \beta \leqslant N;\ n = 0, 1, 2, \ldots).$$

On the other hand, the point x belongs to $\boldsymbol{V}$, hence it follows

$$\left[\frac{d^{n+1}\Big(\dfrac{\varphi_{\alpha,0}}{\varphi_{\beta,0}}\Big)}{du^{n+1}} - \frac{d^{n+1}\varphi_{\alpha,\beta}}{du^{n+1}}\right]_{u=u_0}$$

$$= \varphi_{\beta,0}^{-(n+2)} S_{\alpha,\beta;n+1}(1, \varphi_{1,0}, \ldots, \varphi_{N,0}) - \varphi_{0,\beta}^{-n} T_{\alpha,\beta;n}(\varphi_{0,\beta}, \ldots, \varphi_{N,\beta})]_{u=u_0}$$

$$= x^{-(n+2)} x_0^{n+2} S_{\alpha,\beta;n+1}\Big(1, \frac{x_1}{x_0}, \ldots, \frac{x_N}{x_0}\Big) - x_\beta^n x_0^{-n} T_{\alpha,\beta;n}\Big(\frac{x_0}{x_\beta}, \ldots, \frac{x_N}{x_\beta}\Big)$$

$$= x_0^{-(n+1)} x_\beta^{-(n+2)} S_{\alpha,\beta;n+1}(x_0, \ldots, x_N) - x_0 T_{\alpha,\beta;n}(x_0, \ldots, x_N) = 0$$

Since $x_0 x_\beta \neq 0$, we have

$$[S_{\alpha,\beta;n+1} - Y_0 T_{\alpha,\beta;n}]_{(x_0, \ldots, x_N)} = 0 \quad (0 \leqslant \alpha, \beta \leqslant N;\ n = 0, 1, 2, \ldots).$$

This shows

$$\left(\frac{d^n\Big(\dfrac{\varphi_{\alpha,0}}{\varphi_{\beta,0}}\Big)}{du^n}\right)_{u=u_0} = \left(\frac{d^n\varphi_{\alpha,\beta}}{du^n}\right)_{u_0=u_0} \quad (0 \leqslant \alpha, \beta \leqslant N;\ n = 0, 1, 2, \ldots).$$

Since $\varphi_{\alpha,0}/\varphi_{\beta,0}$ and $\varphi_{\alpha,\beta}$ $(0 \leqslant \alpha, \beta \leqslant N)$ are holomorphic at u_0, it follows $\varphi_{\alpha,0}/\varphi_{\beta,0} = \varphi_{\alpha,\beta}$ $(0 \leqslant \alpha, \beta \leqslant N)$. This completes the proof of Theorem.

3.2 Applying Chow's theorem[7] we shall show that projective solutions induce biregular birational transformations between the initial varieties.

THEOREM 4. *Let $M^{(F)}$ be the set of all the regular points of the coefficients in an osculating system of type* $(N, 1, 2)$

$$det \begin{pmatrix} y_{\alpha_0} & , & y_{\alpha_1} \\ \dfrac{dy_{\alpha_0}}{du} & , & \dfrac{dy_{\alpha_1}}{du} \end{pmatrix} = F_{\alpha_0,\alpha_1}(u, y_0, \ldots, y_N)$$

$$(0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$$

[7] See [1] p. 893–914.

on a complex analytic manifold M of dimension one. Assume $M^{(F)}$ is a connected open set in M. Let $(\tilde{M}^{(F)}, p)$ be the universal covering space of $M^{(F)}$ with the canonical map $p: \tilde{M}^{(F)} \to M^{(F)}$. The osculating system can be naturally considered as an osculating system on $\tilde{M}^{(F)}$. Then each point $\tilde{u}_0$ on $\tilde{M}^{(F)}$ there exists a unique holomorphic map

$$\Phi_{\tilde{u}_0} : \tilde{M}^{(F)} \times \boldsymbol{W}^{(F)}_{p(u_0)} \to \boldsymbol{P}_N$$

such that 1^0 *for a fixed point x_0 on the initial variety $\boldsymbol{W}^{(F)}_{p(\tilde{u}_0)}$ at $p(\tilde{u}_0)$ the map $\tilde{u} \to \Phi_{\tilde{u}_0}(\tilde{u}, x_0)$ is the projective solution with the initial point x_0 at $p(u_0)$,* 2^0 *for a fixed u_1 on $M^{(F)}$ the map $x \to \Phi_{\tilde{u}_0}(\tilde{u}_1, x)$ is a biregular birational map of the initial variety $\boldsymbol{W}^{(F)}_{p(\tilde{u}_0)}$ onto $\boldsymbol{W}^{(F)}_{p(\tilde{u}_1)}$,*

$$3^0 \quad \Phi_{\tilde{u}_1}(\tilde{u}, \Phi_{\tilde{u}_0}(\tilde{u}_1, x))\, \Phi_{\tilde{u}_0}(\tilde{u}, x) \quad (\tilde{u}_0, \tilde{u}_1, \tilde{u} \in M^{(F)},\ x \in \boldsymbol{W}^{(F)}_{p(u_0)}).$$

Proof. The uniqueness of $\Phi_{\tilde{u}_0}$ is a consequence of the uniqueness of the projective solution with the given initial point. It is sufficient to prove that for each point u_0 on $M^{(F)}$ there exist a positive number r and a unique holomorphic map Φ_{u_0}: $B(r, u_0) \times \boldsymbol{W}^{(F)}_{u_0} \to \boldsymbol{P}_N$ such that for a fixed x_0 in $\boldsymbol{W}^{(F)}_{u_0}$ the map $u \to \Phi_{u_0}(u, x_0)$ is a projective solution with the initial point x_0 at u_0, 2^0 for a fixed u_1 in the open ball $B(r, u_0)$ the map $x \to \Phi_{u_0}(u_1, x)$ is a biregular birational map of the initial variety $\boldsymbol{W}^{(F)}_{u_0}$ onto $\boldsymbol{W}^{(F)}_{u_1}$, 3^0 $\varphi_{u_1}(u, \varphi_{u_0}(u_1, x)) = \varphi_{u_0}(u, x)$ $(u, u_1 \in B(r, u_0),\ x \in \boldsymbol{W}^{(F)}_{u_0})$. Let $\varphi_{u_0}(u, x)$ be the holomorphic projective solution at u_0 with the initial point x. Then for every point u_1 in a small neighbourhood of u_0 $\varphi_{u_0}(u, x)$ is regarded as the projective solution at u_1 with the initial point $\varphi_{u_0}(u_1, x)$, hence by the uniqueness property it follows

$$\varphi_{u_1}(u, \varphi_{u_0}(u_1, x)) = \varphi_{u_0}(u, x).$$

By virtue of Proposition 3 for each point $u_0 \times x_0$ in $u_0 \times \boldsymbol{W}^{F}_{u_0}$ there exist a positive number $r(t_0, x_0)$ and a neighbourhood U_{x_0} of x_0 in $\boldsymbol{W}^{(F)}_{u_0}$ such that the map φ_{u_0}: $u \times x \to \varphi_{u_0}(u, x)$ of $B(r(u_0, x_0), u_0) \times U_{x_0}$ into $\boldsymbol{P}_N$ are holomorphic, where $B(r(u_0, x_0), u_0)$ is the open ball of radius $r(u_0, x_0)$ with the center u_0 with respect to the metric $|u - u_1|$. Since $\boldsymbol{W}^{(F)}_{u_0}$ is a projective variety, it is a compact subset in $\boldsymbol{P}_N$. Hence there exist positive numbers r and a such that if $|u_1 - u_0| < s$ the map $u \times x \to \varphi_{u_1}(u, x)$ is a holomorphic map of $B(r, u_1) \times \boldsymbol{W}_{u_1}^{(F)}$ into $\boldsymbol{P}_N$. For fixed x_0 on $\boldsymbol{W}_{t_0}^{(F)}$ the map $u \to \varphi_{u_0}(u, x_0)$ is obviously the projective solution at u_0 with the initial point x_0. Let u_2 be a point satisfying $|u_2 - u_0| < \mathrm{Min}\left(s, \frac{1}{2} r\right)$. Then the map $x \to \varphi_{u_0}(u_2, x)$ is a holomorphic map of the initial variety $\boldsymbol{W}_{u_0}^{(F)}$ into $\boldsymbol{W}_{u_2}^{(F)}$. We shall prove first that this map is biholomorphic. Since $u \times y \to \varphi_{u_2}(u, y)$ is holomorphic in $B(r, u_2) \times \boldsymbol{W}_{u_2}^{(F)}$ and $|u_2 - u_0| < \frac{1}{2} r$, the map $y \to \varphi_{u_2}(u_0, y)$ is a holomorphic map of $\boldsymbol{W}_{u_2}^{(F)}$ into $\boldsymbol{W}_{u_0}^{(F)}$. On the other hand $\varphi_{u_2}(u_0, \varphi_{u_0}(u_2, x)) = x$ and $\varphi_{u_0}(u_2, \varphi_{u_2}(u_0, y)) = y$, hence the map $x \to \varphi_{u_0}(u_2, x)$ is a biholomorphic map of $\boldsymbol{W}_{u_0}^{(F)}$ onto $\boldsymbol{W}_{u_1}^{(F)}$ for every u_2 satisfying $|u_2 - u_0| < \mathrm{Min}\left(s, \frac{1}{2} r\right)$, Finaly, using Chow's Theorem, we shall prove the map $x \to \varphi_{u_0}(u_2, x)$ is a birational map of $\boldsymbol{W}_{u_0}^{(F)}$ onto $\boldsymbol{W}_{u_1}^{(F)}$, Since the initial varieties $\boldsymbol{W}_{u_0}^{(F)}$ and $\boldsymbol{W}_{u_2}^{(F)}$ are projective varieties in $\boldsymbol{P}_N$ the graph Γ of the biholomorphic map $x \to \varphi_{u_0}(u_2, x)$ is a closed complex space in the product $\boldsymbol{P}_N \times \boldsymbol{P}_N$. Since $\boldsymbol{P}_N \times \boldsymbol{P}_N$ can be embedded in a large complex projective space, the graph Γ is regarded as a closed complex analytic space in a complex projective space. Hence by virtue of Chow's Theorem Γ is a projective variety and thus the map $x \to \varphi_{u_0}(u_2, x)$ is a rational map of $\boldsymbol{W}_{u_0}^{(F)}$ onto $\boldsymbol{W}_{u_2}^{(F)}$. This completes the proof of Theorem.

DEFINITION 3 For each closed path ω on $M^{(F)}$ starting from u_0 there exists a biregular birational map σ_ω of $\boldsymbol{W}_{u_0}^{(F)}$ such that the analytic continuation of the projective solution $\varphi_{u_0}(u, x)$ along ω coincides with the projective solution $\varphi_{u_0}(u, \sigma_\omega x)$. The biregular birational transformation σ_ω depends only on the homotopy class of the path ω. Therefore the map $\omega \to \sigma_\omega$ induces a representation of the fundamental group $\pi_1(M^{(F)}, u_0)$ of $M^{(F)}$ by biregular birational transformations of the initial variety $\boldsymbol{W}_{u_0}^{(F)}$. We shall such the representation *the monodoromy group* of the osculating system of type $(N, 1, 2)$.

3.3 Let us characterize osculating systems of type $(N, 1, 2)$ with the initial variety $\boldsymbol{P}_N$.

DEFINITION 4. A *Riccati system* means an osculating system

$$\det\begin{pmatrix} y_{\alpha_0}, & y_{\alpha_1} \\ \frac{dy_{\alpha_0}}{du}, & \frac{dy_{\alpha_1}}{du} \end{pmatrix} = y_{\alpha_0} L_{\alpha_1}(u, y_0, \ldots, y_N) - y_{\alpha_1} L_{\alpha_0}(u, y_0, \ldots, y_N)$$

$$(0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$$

such that $L_\alpha(u, y_0, \ldots, y_N)$ $(0 \leqslant \alpha \leqslant N)$ are linear forms in $y_0, \ldots, y_N$.

PROPOSITION 3. *An osculation system of type $(N, 1, 2)$ is a Riccati system if and only if the initial varieties for the system are the whole projective space $\boldsymbol{P}_N$.*

Proof. Assume first that the system is a Riccati system, i.e. $F_{\alpha,\beta} = y_\alpha L_\beta - y_\beta L_\alpha$ with linear forms L_α $(0 \leqslant \alpha \leqslant N)$. Let $(\varphi_0, \ldots, \varphi_N)$ be a solution of the linear system

$$\frac{dy_\alpha}{du} = L_\alpha(u, y_0, \ldots, y_N) \quad (0 \leqslant \alpha \leqslant N).$$

We may assume without loss of generality that φ_α $(0 \leqslant \alpha \leqslant N)$ are not constant zero. Then it follows

$$\det\begin{pmatrix} \frac{\varphi_\alpha}{\varphi_\beta}, & \frac{\varphi_\beta}{\varphi_\beta} \\ \frac{d\frac{\varphi_\alpha}{\varphi_\beta}}{du}, & \frac{d\frac{\varphi_\beta}{\varphi_\beta}}{du} \end{pmatrix} = \frac{d\frac{\varphi_\alpha}{\varphi_\beta}}{du} = \varphi_\beta^{-2}(\varphi_\alpha L_\beta - \varphi_\beta L_\alpha)$$

$$= F_{\alpha,\beta}\left(u, \frac{\varphi_0}{\varphi_\beta}, \ldots, \frac{\varphi_N}{\varphi_\beta}\right), \quad (0 \leqslant \alpha < \beta \leqslant N)$$

This means that $\varphi\colon u \to (\varphi_0(u), \ldots, \varphi_N(u))$ is a projective solution of the Riccati system. For a linear system we may choose arbitrary initial values at the regular points, hence the initial varieties of Riccati systems are the whole projective N-space $\boldsymbol{P}_N$. We shall next assume that the initial variety $\boldsymbol{W}_{u_0}^{(F)}$ at a regular point u_0 coincides with $\boldsymbol{P}_N$. Let K be the field over Q generated by the values of the coefficients in $F_{\alpha,\beta}$ $(0 \leqslant \alpha, \beta \leqslant N)$ at $u = u_0$. Let $(\xi_0, \ldots, \xi_N)$ be a system of complex numbers such that $\dim_K K(\xi_0, \ldots, \xi_N) = N+1$ and $\varphi = (\varphi_0, \ldots, \varphi_N)$ be the unique holomorphic projective solution of the osculating system such that $\varphi(u_0) = (\xi_0, \ldots, \xi_N)$. Putting $(u, y_0, \ldots, y_N) = (u_0, \xi_0, \ldots, \xi_N)$ in the relation (2), we have

$$\xi_\gamma F_{\alpha,\beta}(u_0, \xi_0, \ldots, \xi_N) + \xi_\alpha F_{\beta,\gamma}(u_0, \xi_0, \ldots, \xi_N) + \xi_\beta F_{\gamma,\alpha}(u_0, \xi_0, \ldots, \xi_N) = 0$$

$$(0 \leqslant \alpha, \beta, \gamma \leqslant N),$$

Since u_0 can be replaced by an arbitrary regular point and $\dim_K K(\xi_0, \ldots, \xi_N) = N+1$, it follows

$$Y_\gamma F_{\alpha,\beta}(u, Y_0, \ldots, Y_N) + Y_\alpha F_{\beta,\gamma}(u, Y_0, \ldots, Y_N) + Y_\beta F_{\gamma,\alpha}(u, Y_0, \ldots, Y_N) = 0 \quad (0 \leqslant \alpha, \beta, \gamma \leqslant N)$$

with indeterminates $Y_0, \ldots, Y_N$. We may put

$$F_{\alpha,0}(u, Y_0, \ldots, Y_N) = H_\alpha(u, Y_1, \ldots, Y_N) - Y_0 L_\alpha(u, Y_0, \ldots, Y_N)$$

with quadratic forms $H_\alpha(u, Y_1, \ldots, Y_N)$ in $Y_1, \ldots, Y_N$ and linear forms $L_\alpha(u, Y_0, \ldots, Y_N)$ $(1 \leqslant \alpha \leqslant N)$.

Then we have

$$\begin{aligned} Y_0 F_{\alpha,\beta}(u, Y_0, \ldots, Y_N) &= Y_\beta F_{\alpha,0}(u, Y_0, \ldots, Y_N) - Y_\alpha F_{\beta,0}(u, Y_0, \ldots, Y_N) \\ &= Y_\beta H_\alpha(u, Y_1, \ldots, Y_N) - Y_\alpha H_\beta(u, Y_1, \ldots, Y_N) \\ &\quad + Y_0 \{Y_\alpha L_\beta(u, Y_0, \ldots, Y_N) - Y_\beta L_\alpha(u, Y_0, \ldots, Y_N)\}. \end{aligned}$$

Hence it follows

$$\begin{aligned} &Y_\beta H_\alpha(u, Y_1, \ldots, Y_N) = Y_\alpha H_\beta(u, Y_1, \ldots, Y_N) \\ &F_{\alpha,\beta}(u, Y_0, \ldots, Y_N) = Y_\beta L_\alpha(u, Y_0, \ldots, Y_N) - Y_\beta L_\alpha(u, Y_0, \ldots, Y_N) \\ &\qquad (1 \leqslant \alpha, \beta \leqslant N). \end{aligned}$$

Moreover $H_\alpha(u, Y_1, \ldots, Y_N)$ is divisible by Y_α and $Y_\alpha^{-1} H_\alpha(u, Y_1, \ldots, Y_N) = Y_\beta^{-1} H_\beta(u, Y_1, \ldots, Y_N)$ $(1 \leqslant \alpha, \beta \leqslant N)$. Therefore we may put

$$H_\alpha(u, Y_1, \ldots, Y_N) = -Y_\alpha L_0(u, Y_0, \ldots, Y_N) \quad (1 \leqslant \alpha \leqslant N)$$

with a linear form $L_0(u, Y_0, \ldots, Y_N)$. This proves that

$$F_{\alpha,\beta}(u, Y_0, \ldots, Y_N) = Y_\alpha L_\beta(u, Y_0, \ldots, Y_N) - Y_\beta L_\alpha(u, Y_0, \ldots, Y_N) \quad (0 \leqslant \alpha < \beta \leqslant N),$$

namely the osculating system is a Riccati system.

§4. Osculating systems of type $(N, 1, 2)$ with constant coefficients

We shall show that projective solutions for osculating system of type $(N, 1, 2)$ with constant coefficients are given by mean of analytic homomorphism of the additive group $\boldsymbol{C}$ into commutative algebraic transformation groups of the initial varieties.

THEOREM 5. *Let* $W_0^{(F)}$ *be the initial variety at* the origin $u=0$ *for an osculating system of type* $(N,1,2)$ *with constant coeffcients:*

$$det \begin{pmatrix} y_{\alpha_0} & , & y_{\alpha_1} \\ \frac{dy_{\alpha_0}}{du} & , & \frac{dy_{\alpha_1}}{du} \end{pmatrix} = F_{\alpha_0,\alpha_1}(z_0, \ldots, z_N) \quad (0 \leqslant \alpha_0 < \alpha_1 \leqslant N)$$

and $\varphi(u,x)$ *be the holomorphic projective solution at* $u=0$ *with the initial point* x. *Then there exist a commutative algebraic transformation group* G *of the projective variety* $W_0^{(F)}$ *and an analytic homomorphism* ρ *of the additive group* C[8] *of complex numbers into* G *such that*

$$\varphi(u,x) = \rho(u)x \quad (u \in C, \ x \in W_0^{(F)}).$$

Proof. By virtue of Theorem 2 and 3 the projective solutions $\varphi(u,x)$ $(x \in W_0^{(F)})$ are holomorphic on the whole complex plane C and the initial varieties $W_{u_1}^{(F)}$coincides with the initial variety $W_0^{(F)}$ at the origin. Hence by virtue of Theorem 3 the map: $x \to \varphi(u,x)$ is a biregular birational map of the initial variety $W_0^{(F)}$ onto itself. Since a translation: $u \to u+u_0$ of the independent variable leaves the quadratic system invariant, the map: $u \to \varphi(u+u_0,x)$ is a holomorphic projective solution with the initial point $\varphi(u_0,x)$ at $u=0$. Hence from Theorem 1 we have

$$\varphi(u,\varphi(v,x)) = \varphi(v,\varphi(u,x)) = \varphi(u+v,x)$$

and

$$\varphi(-u,\varphi(u,x)) = \varphi(u,\varphi(-u,x)) = \varphi(0,x) = x.$$

This shows that the map: $u \to \varphi(u,x)$ is a one-parameter group with the origin at x. Let V be an irreducible component of $W_0^{(F)}$ and $\xi = (\xi_0, \ldots, \xi_N)$ be a generic point of V over the field C of complex numbers. Since the map $\rho(u)$: $x \to \varphi(u,x)$ is a biregular birational transformation of V onto V, there exist a system $(R_0(x), \ldots, R_s(x))$ of homogeneous forms of the same degree in $x_0, \ldots, x_N$ with coefficients in C and a system $(a_{0,0}(u), \ldots, a_{N,s}(u))$ of holomorphic functions in a neighbourhood of $u=0$ such that

[8] C means sometimes the additive group of complex numbers and sometimes the field of complex numbers.

(i) $\varphi(u, \xi) = (\sum_{l=0}^{s} a_{0,l}(u)R_l(\xi), \ldots, \sum_{l=0}^{s} a_{N,l}(u)R_l(\xi))$ and (ii) $R_1(\xi), \ldots, R_s(\xi)$ are lineary independent over $\boldsymbol{C}$. We denote by T the projective variety in the projective Ns-space $\boldsymbol{P}_{(N+1)(s+1)-1}$ such that the point $(a_{0,0}(u), \ldots, a_{N,s}(u))$ is a generic point of $\boldsymbol{T}$ over $\boldsymbol{C}$. For the sake of simplicity we mean by the same symble $\rho(u)$ the point $(a_{0,0}(u), \ldots, a_{N,s}(u))$ on $\boldsymbol{T}$ and denote by $\boldsymbol{C}(\rho(u))$ the field generated by the quotients $a_{\alpha,l}(u) / a_{\beta,h}(u)$ $(0 \leqslant \alpha, \beta \leqslant N;$ $0 \leqslant l, h \leqslant s)$ over $\boldsymbol{C}$. Since ξ is a generic point of V over C and $R_0(\xi), \ldots, R_s(\xi)$ are linearly independent over $\boldsymbol{C}$, there exist $\boldsymbol{C}$-rational points $a^{(0)}, \ldots, a^{(s)}$ on $\boldsymbol{V}$ such that

$$\det \begin{pmatrix} R_0(a^{(0)}), \ldots, R_0(a^{(s)}) \\ \vdots \\ R_s(a^{(0)}), \ldots, R_s(a^{(s)}) \end{pmatrix} \neq 0.$$

Therefore from the linear equations

$$\varphi_\alpha(u, a^{(j)}) = \sum_{l=0}^{s} a_{\alpha,l}(u)R_l(a^{(j)}) \quad (0 \leqslant \alpha \leqslant N;\ 0 \leqslant j \leqslant s)$$

it follows that $a_{\alpha,l}(u)$ $(0 \leqslant \alpha \leqslant N;\ 0 \leqslant l \leqslant s)$ are linear combinations of $\varphi_\alpha(u, a^{(j)})$ $(0 \leqslant \alpha \leqslant N;\ 0 \leqslant j \leqslant s)$ with coefficients in $\boldsymbol{C}$. This means that $\boldsymbol{C}(\rho(u)) = \boldsymbol{C}(\varphi(u, a^{(0)}), \ldots, \varphi(u, a^{(s)}))$, where $\boldsymbol{C}(\varphi(u, a^{(0)}), \ldots, \varphi(u, a^{(s)}))$ is the field generated by the inhomogeneous coordinates of $\varphi(u, a^{(l)})$ $(0 \leqslant l \leqslant s)$ over $\boldsymbol{C}$. Since $\varphi(u, \varphi(v, a)) = \varphi(v, \varphi(u, a)) = \varphi(u + v, a)$, by virtue of Theorem 3 it follows that $\boldsymbol{C}(\varphi(u + v, a)) = \boldsymbol{C}(\varphi(u, a),\ \varphi(v, a))$. Hence we have

$$\begin{aligned} \boldsymbol{C}(\rho(u + v)) &= \boldsymbol{C}(\varphi(u + v, a^{(0)}), \ldots, \varphi(u + v, a^{(s)})) \\ &= \boldsymbol{C}(\varphi(u, a^{(0)}), \ldots, \varphi(u, a^{(s)}),\ \varphi(v, a^{(0)}), \ldots, \varphi(v, a^{(s)})) \\ &= \boldsymbol{C}(\rho(u), \rho(v)). \end{aligned}$$

This means that there exists a rational map α: $\boldsymbol{T} \times \boldsymbol{T} \rightarrow \boldsymbol{T}$ such that $\alpha(\rho(u), \rho(v))\ \rho(u + v)$ and α is defined over $\boldsymbol{C}$. Let us next show that there exists a rational map β such that $\beta(\rho(u)) = \rho(-u)$ and β is defined over $\boldsymbol{C}$. Since $\varphi(-u, \varphi(u, a^{(l)}) = \varphi(0, a^{(l)}) = a^{(l)}$ $(0 \leqslant l \leqslant s)$ and

$$\det \begin{pmatrix} R_0(a^{(0)}), \ldots, R_0(a^{(s)}) \\ \vdots \\ R_s(a^{(0)}), \ldots, R_s(a^{(s)}) \end{pmatrix} \neq 0.$$

we have

$$\varphi_{\alpha}(0, a^{(j)}) = \sum_{l=0}^{s} a_{\alpha,l}(-u)R_l(\varphi(u, a^{(j)}))$$

and

$$\det\begin{pmatrix} R_0(\varphi(u, a^{(0)})), \ldots, R_0(\varphi(u, a^{(s)})) \\ \vdots \\ R_s(\varphi(u, a^{(0)})), \ldots, R_s(\varphi(u, a^{(s)})) \end{pmatrix} \neq 0$$

This means that

$$\begin{aligned} &\boldsymbol{C}(\rho(-u)) = \boldsymbol{C}(\varphi(u, a^{(0)}), \ldots, \varphi(u, a^{(s)}), \varphi(0, a^{(0)}), \ldots, \varphi(0, a^{(s)})) \\ &= \boldsymbol{C}(\varphi(u, a^{(0)}), \ldots, \varphi(u, a^{(s)})) \\ &= \boldsymbol{C}(\rho(u)). \end{aligned}$$

Therefore there exists a rational map β: $\boldsymbol{T} \to \boldsymbol{T}$ such that $\beta(\rho(u)) = \rho(-u)$ and β is defined over $\boldsymbol{C}$. These rational maps α and β satisfy the conditions:

$$\alpha(\rho(u), \rho(v)) = \alpha(\rho(v), \rho(u)) = \rho(u+v),$$

$$\alpha(\rho(u), \alpha(\rho(v), \rho(w))) = \alpha(\alpha(\rho(u), \rho(v)), \rho(w)) = \rho(u+v+w),$$

$$\alpha(\beta(\rho(u)), \rho(u)) = \alpha(\rho(u), \beta(\rho(u)) = \rho(0).$$

Hence we have a commutative normal law of composition $\circ$ on T such that $\rho(u)\circ\rho(v) = \alpha(\rho(u), \rho(v))$ and $\rho(u)\circ\rho(-u) = \rho(-u)\circ\rho(u) = \rho(0)$. By virtue of the general theory on algebraic group[9] there exist a commutative algebraic group $\boldsymbol{G_V}$ and a birational equivalence ψ of T to $\boldsymbol{G}$ such that $\psi(\rho(u)\circ\rho(v))\psi(\rho(u))\circ\psi(\rho(v))$ and $\psi(\rho(u))\circ\psi(\rho(-u))) = \psi(\rho(0))$. This shows that $\boldsymbol{G_V}$ is regarded as a commutative transformation group of $\boldsymbol{V}$ such that $\rho_{\boldsymbol{V}}(u)x\varphi(u,x)$ and $\rho_{\boldsymbol{V}}(u) = \psi(\rho(u))$. Let $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_r$ be the irreducible components of the initial variety $\boldsymbol{W}_0^{(F)}$, $\tilde{\boldsymbol{G}}$ be the direct sum $\boldsymbol{G}_{V_1}\oplus\ldots\oplus\boldsymbol{G}_{V_r}$ and ρ be the direct sum $\rho_{V_1}\oplus\ldots\oplus\rho_{V_r}$. Let $\boldsymbol{G}$ be the Zariski closure of the image $\rho(\boldsymbol{C})$ in the commutative algebraic group $\tilde{\boldsymbol{G}}$. Then $\boldsymbol{G}$ is a commutative algebraic transformation group of the initial variety $\boldsymbol{W}_0^{(F)}$ such that

9) See [3] Chap. IX Algebraic groups.

$$\rho(u)x = \varphi(u, x) \quad (u \in \boldsymbol{C}, x \in \boldsymbol{W}_0^{(F)}).$$

COROLLARY 1.[10] *Let x be a point on $\boldsymbol{W}_0^{(F)}$. Then there exists a commutative algebraic group $\boldsymbol{G}_x$ and an analytic homomorphism ρ_x of the additive group $\boldsymbol{C}$ into $\boldsymbol{G}_x$ such that* (i) *$\boldsymbol{G}_x$ is a local closed subvariety in $\boldsymbol{W}_0^{(F)}$ and x is the origin of $\boldsymbol{G}_x$,* (ii) $\rho_x(u) = \varphi(u, x)$ $(u \in \boldsymbol{C})$, (iii) *the Zariski closure of $\boldsymbol{G}_x$ in $\boldsymbol{W}_0^{(F)}$ coincides with that of $\varphi(\boldsymbol{C}, x)$.*

Proof. Let $\boldsymbol{H}_x$ be the subgroup of $\boldsymbol{G}$ consisting of all element g such that $gx = x$. Then $\boldsymbol{H}_x$ is a normal algebraic subgroup of $\boldsymbol{G}$. Let $\boldsymbol{G}_x$ be the quotient group $\boldsymbol{G}/\boldsymbol{H}_x$ and π be the natural map: $\boldsymbol{G} \to \boldsymbol{G}_x$. Let ρ_x be the composite $\pi\rho$. Then, identifying $\boldsymbol{G}_x$ with the image $\boldsymbol{G}_x x$ of x by $\boldsymbol{G}_x$, we have Corollary.

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[10] This means that the Zariski closure of a projective solution for a system with constant coefficients is a Zariski closure of a commutative algebraic group.