

# A NOTE ON HOMOTOPY INVARIANCE OF TANGENT BUNDLES

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Although tangent bundles of manifolds are not always homotopically invariant, but in some categories of the manifolds they can be homotopically invariant.

In this note, I show that tangent bundles of  $\pi$ -manifolds and almost parallelizable manifolds depend only on their homotopy types.

We denote by  $M^n$  a  $n$ -dimensional connected closed differentiable manifold, by  $\tau M^n$  the tangent bundle, and by  $\approx$  the bundle equivalence.

1. Let  $M^n$  be a  $\pi$ -manifold. For a given immersion of  $M^n$  into  $(n+1)$ -dimensional Euclidean space or a triviality of the stable tangent bundle  $\tau M^n \oplus \varepsilon$ , we can define a map of  $M^n$  into the  $n$ -dimensional sphere  $S^n$ , so called the "normal map" or the "Gauss map". This map is covered by a bundle map of  $\tau M^n$  into  $\tau S^n$ , and its degree is decisively related to the Euler characteristic  $\chi(M^n)$  and the semi-Euler characteristic of  $M^n$ . (Milnor [3], Bredon-Kosinski [1])

**THEOREM 1.** *Let  $M_1^n$  and  $M_2^n$  be  $\pi$ -manifolds of dimension  $n$ , and let  $f: M_1^n \rightarrow M_2^n$  be an arbitrary homotopy equivalence. Then, we have  $\tau M_1^n \approx f^*(\tau M_2^n)$ .*

*Proof.* Let  $F_i$  be a framing of the stable tangent bundle  $\tau M_i^n \oplus \varepsilon$  and let  $\nu_{F_i}$  be the Gauss map for  $i = 1, 2$ . Firstly we show that we can choose the framings  $F_1$  and  $F_2$  so that  $\deg. \nu_{F_1} = \deg. \nu_{F_2}$ .

If  $n$  is even the assertion is clear, since  $\deg. \nu_{F_1} = \frac{1}{2} \chi(M_1^n)$ ,  $\deg. \nu_{F_2} = \frac{1}{2} \chi(M_2^n)$  independently of the choice of the framings and  $M_1^n, M_2^n$  are of the same homotopy type.

For  $n = 1, 3, 7$ , since  $M_1^n$  and  $M_2^n$  are parallelizable, the theorem is trivial. Let  $n = 2r + 1$ ,  $n \neq 1, 3, 7$ . By Theorem 3 of [1],  $\deg. \nu_{F_i} \equiv \sum_{k=0}^r \text{rank}$

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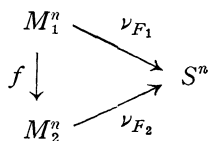
$H_k(M_i^n; Z_2) \bmod 2$  for  $i = 1, 2$ , independently of the choice of the framings. Since  $M_1^n$  and  $M_2^n$  are of the same homotopy type, we know that  $\text{deg. } \nu_{F_1} \equiv \text{deg. } \nu_{F_2} \bmod 2$  always.

Now, if we construct a new framing  $F'_1$  of  $\tau M_1^n \oplus \varepsilon$  by a continuous map  $g: M_1^n \rightarrow \text{SO}(n+1)$ , so that  $F_1(x) = g(x)F'_1(x)$ , there is a following relation between  $\text{deg. } \nu_{F'_1}$  and  $\text{deg. } \nu_{F_1}$ . ((2.2) of [1]).

$\text{deg. } \nu_{F'_1} = \text{deg. } (\pi \circ g) + \text{deg. } \nu_{F_1}$ , where  $\pi$  denotes the canonical projection of  $\text{SO}(n+1)$  onto  $S^n$ . Thus we have,

$\text{deg. } \nu_{F'_1} - \text{deg. } \nu_{F_2} = \text{deg. } (\pi \circ g) + (\text{deg. } \nu_{F_1} - \text{deg. } \nu_{F_2})$ . We note that  $\text{deg. } \nu_{F_1} - \text{deg. } \nu_{F_2}$  is even. If  $n$  is odd, since there exists a continuous map  $g$  such that  $\text{deg. } (\pi \circ g)$  can be any even integer, we can choose  $F'_1$  so that  $\text{deg. } \nu_{F'_1} = \text{deg. } \nu_{F_2}$ .

Thus we can choose the framings  $F_1$  and  $F_2$  so that  $\text{deg. } \nu_{F_1} = \text{deg. } \nu_{F_2}$ . We may assume that  $\text{deg. } f = 1$ . Then, in the following diagram,  $\text{deg. } \nu_{F_2} \circ f = \text{deg. } f \times \text{deg. } \nu_{F_2} = \text{deg. } \nu_{F_2} = \text{deg. } \nu_{F_1}$ . So that by Hopf's



Classification Theorem,  $\nu_{F_2} \circ f$  is homotopic to  $\nu_{F_1}$ , that is, the diagram is homotopy commutative. Therefore we have  $\tau M_1^n \approx \nu_{F_1}^*(\tau S^n) \approx (\nu_{F_2} \circ f)^*(\tau S^n) \approx f^*(\nu_{F_2}^*(\tau S^n)) \approx f^*(\tau M_2^n)$ .

This completes the proof of the theorem.

In the proof of this theorem, if  $\chi(M_1^n) = \chi(M_2^n)$  or if the semi-Euler characteristic of  $M_1^n$  is equal to that of  $M_2^n$  in mod. 2, according as  $n$  is even or odd, then the map  $f$  need not be a homotopy equivalence but the condition  $\text{deg. } f = 1$ . So, we have

**THEOREM 1'.** *Let  $M_1^n$  and  $M_2^n$  be  $\pi$ -manifolds of dimension  $n$ . If  $n$  is even and  $\chi(M_1^n) = \chi(M_2^n)$ , or if  $n$  is odd and the semi-Euler characteristic of  $M_1^n$  is equal to that of  $M_2^n$  in mod. 2, then for any continuous map  $f: M_1^n \rightarrow M_2^n$  of degree 1,  $\tau M_1^n \approx f^*(\tau M_2^n)$ .*

**COROLLARY 1.** *In the category of  $\pi$ -manifolds, whether a  $\pi$ -manifold of even dimension has an almost complex structure or not depends only on its homotopy type.*

*Proof.* This is clear.

**COROLLARY 2.** *Let  $M_1^n$  and  $M_2^n$  be  $\pi$ -manifolds of dimension  $n$ , and let  $f: M_1^n \rightarrow M_2^n$  be a homotopy equivalence. Then, for any twisted spheres  $T_1$  and  $T_2$  of*

$\Gamma^n$ , there exists a homotopy equivalence  $g: M_1^n \# T_1 \rightarrow M_2^n \# T_2$  such that  $\tau(M_1^n \# T_1) \approx g^*\tau(M_2^n \# T_2)$ .

*Proof.* Since  $M_i^n \# T_i$   $i = 1, 2$  are  $\pi$ -manifolds and are homeomorphic to  $M_i^n$ , this follows from Theorem 1.

2. Let us consider such a manifold  $M^n$  that  $M^n - (\text{a point})$  is an open  $\pi$ -manifold. If we call such a manifold to be almost  $\pi$ , a manifold  $M^n$  is almost  $\pi$  if and only if it is almost parallelizable. Because, the tangent bundle of  $M^n - (\text{a point})$  is induced from that of  $M^n - (\text{Interior of an imbedded } n\text{-disk})$ , and a manifold with boundary is  $\pi$  if and only if it is parallelizable. (Kervaire-Milnor [5]).

**THEOREM 2.** *Let  $M_1^n$  and  $M_2^n$  be almost parallelizable manifolds, and let  $f: M_1^n \rightarrow M_2^n$  be a homotopy equivalence. Then, we have that  $\tau M_1^n \approx f^*(\tau M_2^n)$ . In other words, tangent bundles of  $(n - 1)$ -parallelizable manifolds are homotopically invariant.*

*Proof.* Let  $O_i \in H^n(M_i^n; \pi_{n-1}(SO_{n+1}))$   $i = 1, 2$ . be the obstruction class for extending the triviality of  $\tau M_i^n \oplus \varepsilon$  on the  $(n - 1)$ -skeleton over the whole. If  $n \equiv 1, 2, 3, 5, 6, 7 \pmod{8}$ , by the analogous argument of Kervaire and Milnor [4], [5], we know that  $M_1^n$  and  $M_2^n$  are  $\pi$ -manifolds. So, the theorem is valid by Theorem 1. If  $n = 4k$ , since the  $k$ -th Pontrjagin classes  $P_k(M_i^n) = m O_i$  ( $m$ : an integer)  $i = 1, 2$  and the indexes of  $M_1^n$  and  $M_2^n$  are equal (we may assume that  $\text{deg. } f = 1$ .), we know that  $O_1 = f^*O_2$ . So that, the obstruction class  $f^*O_2 - O_1$  for extending the isomorphism of  $\tau M_1^n \oplus \varepsilon$  onto  $f^*(\tau M_2^n) \oplus \varepsilon$  on the  $(n - 1)$ -skeleton over the whole vanish. Thus,  $f^*(\tau M_2^n)$  is stably equivalent to  $\tau M_1^n$ . But, in this case, we can show that  $f^*(\tau M_2^n)$  is equivalent to  $\tau M_1^n$ ; If we denote by  $\alpha_i \in H^n(M_i^n; \pi_{n-1}(SO_n))$   $i = 1, 2$  the obstruction classes for extending the triviality of  $\tau M_i^n$  on the  $(n - 1)$ -skeleton over the whole, then the obstruction class for extending the isomorphism of  $\tau M_1^n$  onto  $f^*(\tau M_2^n)$  on the  $(n - 1)$ -skeleton is given by  $f^*\alpha_2 - \alpha_1$ . we can show that  $f^*\alpha_2 - \alpha_1 = 0$ . The proof is included in that of K. Shiraiwa [7].

Finally, note that if  $M^n$  is a  $(n - 1)$ -parallelizable manifold, then  $M^n$  is almost parallelizable. Because, choose a point of  $M^n$  and tie to an interior point of every  $n$ -simplex with an imbedded arc. Then, there exists a  $n$ -cell which contains the tree.

This completes the proof.

**COROLLARY 3.** *For  $(n - 1)$ -connected  $2n$ -manifolds,  $n \equiv 3, 5, 6, 7 \pmod{8}$ , their tangent bundles are homotopically invariant. For  $(n - 1)$ -connected  $(2n + 1)$ -manifolds,  $n \equiv 5, 6 \pmod{8}$ , their tangent bundles are homotopically invariant.*

*Proof.* In this case, these manifolds are almost parallelizable or stably parallelizable.

**COROLLARY 4.** *The matters corresponding to the corollaries 1, 2 are also valid for almost parallelizable manifolds.*

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