ON ALGEBRAS OF DOMINANT DIMENSION ONE

BRUNO J. MUELLER

Summary. QF-3 algebras R are classified according to their second commutator algebras R' with respect to the minimal faithful module, which satisfy dom.dim. $R' \geqq 2$. The class $C(S)$ of all QF-3 algebras whose second commutator is S, contains besides S only algebras R with dom.dim. $R = 1$. $C(S)$ contains a unique (up to isomorphism) minimal algebra which can be represented as a subalgebra S_0 of S describable in terms of the structure of S, and $C(S)$ consists just of the algebras $S_0 \subset R \subset S$ (up to isomorphism). A criterion for $S_0 \neq S$ and various examples are given. Finally it is shown that the injective hull of S (as left-, right- or bimodule) is at the same time the injective hull for every $R \in C(S)$. This result sheds some light on the fact that dom.dim. $S \geqq 2$ while dom. dim. $R = 1$ for all $R \in C(S)$, $R \neq S$: We prove that no composition-factor of the R-module R' / R is isomorphic to an ideal.

The classes $C(R)$. We consider finite-dimensional algebras R with unit over a field K and unitary finitely generated R-modules. QF-3 algebras are characterized by the existence of a minimal faithful right-module X which is (unique up to isomorphism and) a direct summand in every faithful module. X is projective-injective and the sum of the isomorphism-types of dominant[1] right-ideals, hence itself a right-ideal generated by an idempotent: $X_R \cong eR_R$. The K-dual X^* of X is the minimal faithful left-module: ${}_RX^* \cong {}_RRf$. With every QF-3 algebra R one associates the second commutator R' of the minimal faithful (right-)module X, which is again a QF-3 algebra and contains R as a subalgebra, with the same unit, in a natural way: $1 \in R \subset R'$. The second commutator of the minimal faithful left-module ${}_RX^*$ is isomorphic to R', over R. Minimal faithful R'-modules are $R'f = Rf$, $eR' = eR$. (cf. Thrall [6], Morita [3], Tachikawa [5])

The following *dominant dimension* is introduced for every algebra R:

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[1] A *dominant* right-ideal is an ideal e_1R generated by a primitive idempotent e_1, which is injective.

dom.dim. $R \geqq n$ if there exists an exact sequence $0 \to R \to X_1 \to \cdots \to X_n$ of projective-injective modules X_i. It was shown in [4] that the three such dimensions obtained by using left-modules, right-modules or bimodules coincide. QF-3 algebras are characterized by dom.dim. $R \geqq 1$. The following are equivalent for any QF-3 algebra R: $R = R'$; dom.dim. $R \geqq 2$; R is the endomorphism-ring of a finitely generated fully faithful module[2]. Hence the inclusion $R \subset R'$ embedds every QF-3 algebra R into an algebra R' with dom.dim. $R' \geqq 2$, and the embedding is proper if and only if dom.dim. $R = 1$. This observation suggests the following classification:

DEFINITION. *For any algebra R with dom.dim. $R \geqq 2$, let $C(R)$ denote the class of all QF-3 algebras S such that $S' \cong R$.*

THEOREM 1. *An algebra R belongs to $C(R)$ if and only if it is isomorphic to a subalgebra R_1 of R that contains the unit 1 and suitable minimal faithful ideals eR, Rf of R.*

Proof. Morita ([3], Theorem 17.3) has shown that any R_1 satisfying those conditions is QF-3 and that $eR = eR_1$, $Rf = R_1 f$ are its minimal faithful modules. Hence

$$\text{Endo}\,(eR_{1R_1}) = eR_1e = eRe = \text{Endo}\,(eR_R)$$

and

$R = R' = \text{Endo}\,({}_{eRe}eR) = \text{Endo}\,({}_{eR_1e}eR_1) = R'_1$, proving $R_1 \in C(R)$. (We remark for later application (proof of theorem 8) that this identification of R and R'_1 is compatible with the embeddings of R_1 into R and R'_1. For $R_1 \subset R'_1 = \text{Endo}\,({}_{eR_1e}eR_1)$ by $R_1 \ni r_1 \to (x \to xr_1) \in \text{Endo}\,({}_{eR_1e}eR_1)$ and $R = \text{Endo}\,({}_{eRe}eR)$ by $R \ni r \to (x \to xr) \in \text{Endo}\,({}_{eRe}eR)$, thus $R_1 \subset R = \text{Endo}\,({}_{eRe}eR)$ again by $R_1 \ni r_1 \to (x \to xr_1) \in \text{Endo}\,({}_{eRe}eR)$.) Conversely, another result by Morita (Theorem 17.5) says that any QF-3 algebra S, as subalgebra of S', contains suitable minimal faithful ideals eS', $S'f$ of S'.

DEFINITION. *For any algebra R with dom.dim. $R \geqq 2$, let $C_0(R)$ denote the set of all subalgebras R_1 of R containing the unit 1 and suitable minimal faithful ideals eR, Rf of R.*

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[2] A module ${}_AX$ is *fully faithful* if it contains every indecomposable injective or projective module as a direct summand (X is a generator-cogenerator).

COROLLARY 2. *$C(R)$ and $C_0(R)$ contain the same isomorphism-types of algebras. $R_1 \in C_0(R)$, $R_1 \subset R_2 \subset R$ implies $R_2 \in C_0(R)$.*

Because of these facts it is of particular interest to characterize the minimal algebras in $C_0(R)$.

DEFINITION. *For any QF-3 algebra R, a pair of idempotents e, f will be called properly chosen if*

(1) *eR, Rf are minimal faithful modules,*

(2) $ef = fe$ (this implies that ef is again an idempotent),

(3) *the number k in a decomposition $ef = e_1 + \cdots + e_k$ into indecomposable orthogonal idempotents e_i is minimal (compared to all other pairs e', f' satisfying* (1) *and* (2). For fixed ef, k is obviously the same for each such decomposition).

The set of primitive idempotents of any algebra R falls into finitely many isomorphism-classes $E_1, \ldots, E_n$ where two primitive idempotents e_i, e_j are called isomorphic if they generate isomorphic right- (equivalent left-) ideals. Every decomposition of the unit 1 into primitive orthogonal idempotents can be written as $1 = \sum_{i=1}^{n} \sum_{j_i=1}^{n_i} e_{ij_i}$ where $e_{ij_i} \in E_i$ and the numbers n_i are the same for any such decomposition. Given two decompositions

$$1 = \sum_{i=1}^{n} \sum_{j_i=1}^{n_i} e_{ij_i} = \sum_{i=1}^{n} \sum_{j_i=1}^{n_i} e^*_{ij_i},$$

there exists an inner automorphism of R, generated by an invertible element $x \in R$, that maps e_{ij_i} onto $e^*_{ij_i}$: $x e_{ij_i} x^{-1} = e^*_{ij_i}$.

LEMMA 3. *A pair of idempotents e, f is properly chosen if and only if it is of the form $e = \sum_{i \in I} e_{i1}$, $f = \sum_{i \in J} e_{in_i}$, where $1 = \sum_{i=1}^{n} \sum_{j_i=1}^{n_i} e_{ij_i}$ is a decomposition into primitive orthogonal idempotents and the sets $I, J \subset \{1, \ldots, n\}$ characterize those classes E_i that generate dominant right-, left-ideals.*

Proof. Suppose that e, f are properly chosen. $ef = fe$ implies that $e - ef$, $f - ef$, ef, $1 - e - f + ef$ constitute a decomposition of 1 into orthogonal idempotents which can be refined to a decomposition into primitive orthogonal idempotents $1 = \sum_{i=1}^{n} \sum_{j_i=1}^{n_i} e_{ij_i}$. A suitable adjustment of the second index j_i gives $e = \sum_{i \in I} e_{i1}$ and $f = \sum_{i \in J} e_{ik_i}$, hence $ef = \sum_{\substack{i \in I \cap J \\ k_i = 1}} e_{i1}$. The minimality-requirement (3) implies $k_i \neq 1$ whenever possible, that is for $n_i > 1$.

Therefore the minimal k in (3) is the number of elements $i \in I \subset J$ with $n_i = 1$, and a further adjustment of the second index leads to $f = \sum_{i \in J} e_{in_i}$. Conversely any pair e, f of this type satisfies (1) and (2): $ef = \sum_{\substack{i \in I \cap J \\ n_i = 1}} e_{i1} = fe$, and this decomposition has the minimal number of summands, so (3) holds and e, f are properly chosen.

THEOREM 4. *Any two minimal subalgebras in* $C_0(R)$ *are isomorphic under an inner automorphism of* R. *A subalgebra* R_0 *of* R *is minimal in* $C_0(R)$ *if and only if it is of the form* $R_0 = K + eR + Rf + RfeR$ *where* e, f *are properly chosen idempotents of* R.

Proof. Let R_0 be a minimal algebra in $C_0(R)$. By definition of $C_0(R)$ there exist minimal faithful ideals eR, Rf of R, contained in R_0; further the unit 1 of R lies in R_0. Hence $R_0 \supset K + eR + Rf + RfeR$, and as this is an algebra in $C_0(R)$ too, $R_0 = K + eR + Rf + RfeR$ because of the minimality of R_0.

We shall show that e, f can be replaced by a properly chosen pair. Refine $1 = e + (1 - e)$ and $1 = f + (1 - f)$ to decompositions into primitive orthogonal idempotents $1 = e_1 + \cdots + e_m = f_1 + \cdots + f_m$ of R_0. We get an inner automorphism of R_0: $xf_ix^{-1} = e_i$; $x, x^{-1} \in R_0$. Set $f' = xfx^{-1} \in R_0$, then $ef' = f'e$. Observe that

$$Rf \ni rf \to rfx^{-1} = rx^{-1}f' \in Rf'$$

is a R-isomorphism, thus Rf' is a minimal faithful module for R. Further $R_0 \supset R_0f' = R_0xfx^{-1} = R_0fx^{-1} = Rfx^{-1} = Rf'$; hence $K + eR + Rf' + Rf'eR \subset R_0$ and consequently $K + eR + Rf' + Rf'eR = R_0$.

e, f' may still not satisfy (3). But as before, the orthogonal idempotents $e - ef'$, $f' - ef'$, ef', $1 - e - f' - ef'$ can be refined in R to $1 = \sum_{i=1}^{n} \sum_{j_i=1}^{n_i} e_{ij_i}$ with $e = \sum_{i \in I} e_{i1}$, $f' = \sum_{i \in J} e_{ik_i}$. The second index can be adjusted such that $k_i = 1$ or $= n_i$, and $k_i \neq n_i$ for $i \in I \subset J$ at most. There exists an inner automorphism of R interchanging e_{i1} and e_{in_i} for $i \in I \subset J$, $k_i \neq n_i$ and leaving all other e_{ij_i} fixed: $e_{in_i} = ze_{i1}z^{-1}$. Replacing f' by $f'' = \sum_{i \in J} e_{in_i} = zf'z^{-1}$ we get ${}_RRf'' \cong {}_RRf'$ and $f''e = ef''$ so that e, f'' are properly chosen. Finally $Re_{in_i} = Rze_{i1}z^{-1} = Re_{i1}z^{-1} \subset Rf'eR$ for $k_i \neq n_i$; hence $K + eR + Rf'' +$

$Rf''eR \subset R_0$ and therefore $K + eR + Rf'' + Rf''eR = R_0$, proving that every minimal algebra in $C_0(R)$ is of the form stated in the theorem.

From Lemma 3 it is obvious that whenever e, f and e^*, f^* are two properly chosen pairs of idempotents of R, then there exists an inner automorphism of R mapping e onto e^* and f onto f^*. That completes the proof of the theorem.

DEFINITION. *For any QF-3 algebra R with dom.dim. $R \geqq 2$ and any particular minimal subalgebra R_0 in $C_0(R)$, let $C(R; R_0)$ denote the set of all algebras R_1 with $R_0 \subset R_1 \subset R$.*

COROLLARY 5. *$C(R; R_0)$ and $C(R)$ contain the same isomorphism-types of algebras.*

Proof. Any $S \in C(R)$ is isomorphic to some $R_1 \in C_0(R)$ which contains a minimal subalgebra R_{10}. R_{10} is isomorphic to R_0 by an inner automorphism of R which carries R_1 into an algebra R_2 in $C(R; R_0)$.

REMARKS. We collect a few additional (obvious) facts about $C(R)$.

(i) The (up to isomorphism unique) minimal algebra R_0 in $C(R)$ is characterized by the fact that its vector-space-dimension over K is minimal among the algebras in $C(R)$.

(ii) R is characterized in $C(R)$ by having maximal K-dimension.

(iii) While dom.dim. $R \geqq 2$, we have dom.dim. $S = 1$ for all $S \in C(R)$ that are not isomorphic to R.

(iv) If a QF-3 algebra is a ring-direct sum $R = R_1 \oplus R_2$, then so is $R' = R'_1 \oplus R'_2$. On the other hand if $R' = S_1 \oplus S_2$, then R need not decompose accordingly.

(v) For any QF-3 algebra R a minimal algebra R_0 in $C(R')$ can be constructed directly as $R_0 = K + eR + Rf + RfeR$ where e, f is any properly chosen pair of idempotents in R.

This may not be quite obvious: Since there exists a minimal subalgebra $R_0 = K + eR' + R'f + R'feR' \subset R$ with suitable properly chosen idempotents e, f of R', we get $R_0 = K + eR + Rf + RfeR$ and $e, f \in R$. $R'f = Rf$, $eR' = eR$ are minimal faithful ideals for R as well as for R'. A decomposition $ef = e_1 + \cdots + e_k$ into primitive orthogonal idempotents in R' always lies in R, hence constitutes such a decomposition with respect to R; and vice versa. Suppose k be not minimal for R; then the isomorphism-type of at least one e_i, say e_1, appears more than once in a decomposition of 1 in R, and we have $e_1R_R \cong e'_1R_R$, $e_1e'_1 = 0$. We get an inner automorphism of R that interchanges e_1 and e'_1 and leads to a R'-isomorphism $e_1R' \cong e'_1R'$, contrary to the assumption that e, f be properly chosen in R'. Thus e, f automatically are properly chosen with respect to R. —Any other properly chosen pair e^*, f^* in R can be mapped onto e, f by an inner automorphism of R and leads to an algebra $K + e^*R + Rf^* + Rf^*e^*R$ isomorphic to R_0.

We want to derive a criterion for $R = R_0$. For properly chosen idempotents e, f in R we set $ef = d$, $e - ef = e'$, $f - ef = f'$, $1 - e - f + ef = \varepsilon$. Then evaluation of $(d + e' + f' + \varepsilon) R (d + e' + f' + \varepsilon) = R = R_0 = K + e'R + Rf' + RdR$ yields the necessary and sufficient condition $f'Re' + f'R\varepsilon + \varepsilon Re' + \varepsilon R\varepsilon = f'RdRe' + f'RdR\varepsilon + \varepsilon RdRe' + \varepsilon RdR\varepsilon + K\varepsilon$, which may be split into the four conditions $f'Re' = f'RdRe'$, $f'R\varepsilon = f'RdR\varepsilon$, $\varepsilon Re' = \varepsilon RdRe'$, $\varepsilon R\varepsilon = \varepsilon RdR\varepsilon + K\varepsilon$. By construction of $d = ef$, the isomorphism-types of the idempotents in d are different from those in e', f' and ε; hence there doesn't exist any epimorphism of dR_R onto a direct summand of $e'R$, $f'R$ or εR; consequently the image of every homomorphism of dR into these modules lies in $e'N$, $f'N$, εN (N being the radical of R) and we get $e'Rd = e'Nd$, $f'Rd = f'Nd$, $\varepsilon Rd = \varepsilon Nd$. Correspondingly $dRe' = dNe'$, $dRf' = dNf'$, $dR\varepsilon = dN\varepsilon$ hold; and the above four conditions imply $f'Re' = f'NdNe' = f'N^2e'$, $f'R\varepsilon = f'NdN\varepsilon = f'N^2\varepsilon$, $\varepsilon Re' = \varepsilon NdNe' = \varepsilon N^2e'$, $\varepsilon R\varepsilon = \varepsilon NdN\varepsilon + K\varepsilon = \varepsilon N^2\varepsilon + K\varepsilon$. Then $e'R$, $f'R$ cannot contain isomorphic direct summands since that would lead to a map $e'R \to f'R$ the image of which wouldn't even be contained in $f'N$, hence to an element in $f'Re'$, not in $f'Ne'$. Similarly εR, $f'R$ and $e'R$, εR cannot have isomorphic direct summands. Finally εR cannot decompose directly, since $\varepsilon = \varepsilon_1 + \varepsilon_2$ (orthogonal idempotents) and $\varepsilon R\varepsilon = \varepsilon N^2\varepsilon + K\varepsilon$ yields $\varepsilon_1 = x + k\varepsilon$, $x \in N^2$; hence either $k = 0$, $\varepsilon_1 \in N^2$, $\varepsilon_1 = 0$ or $k \neq 0$, $0 = x\varepsilon_2 + k\varepsilon_2$, $\varepsilon_2 \in N^2$, $\varepsilon_2 = 0$.

Summarizing: We have shown that $R = R_0$ implies that R is selfbasic and that ε is either primitive or zero. Therefore $\varepsilon R\varepsilon$ is local and has radical $\varepsilon N\varepsilon$; and the condition $\varepsilon R\varepsilon = \varepsilon NdN\varepsilon + K\varepsilon$ gives $\varepsilon NdN\varepsilon = \varepsilon N\varepsilon$ and $\varepsilon R\varepsilon / \varepsilon N\varepsilon \cong K$ if $\varepsilon \neq 0$.

Thus we have proved one direction of the following

THEOREM 6. *A QF-3 algebra R is minimal in $C(R')$ if and only if*

(1) *R is selfbasic,*

(2) *there exists at most one type of idempotents ε such that $R\varepsilon$, εR both are not dominant,*

(3) $f'Re' = f'NdNe'$; $f'R\varepsilon = f'NdN\varepsilon$, $\varepsilon Re' = \varepsilon NdNe'$, $\varepsilon N\varepsilon = \varepsilon NdN\varepsilon$, $\varepsilon R\varepsilon / \varepsilon N\varepsilon \cong K$ *(if ε exists); where d (e', f') is the sum of those idempotents e_i of a decomposition into primitive orthogonal idempotents $1 = e_1 + \cdots + e_n$ for which Re_i, e_iR are both dominant (e_iR but not Re_i is dominant; Re_i but not e_iR is dominant).*

Conversely these conditions (1) to (3) immediately lead back to the former conditions for $R = R_0$. This completes the proof.

REMARKS. (i) We are particularly interested in the case dom.dim. $R \geqq 2$. Here the conditions of the theorem characterize those R for which $C(R)$ is trivial (to say it contains the isomorphism-type of R only).

(ii) Applied to R_0 itself the theorem describes properties of the minimal algebras in the classes $C(R)$.

(iii) The conditions can be simplified in certain cases, e.g.: If $NdN = 0$ (in particular if $d = 0$, which for dom.dim. $R \geqq 2$, hence $R = \mathrm{Endo}({}_AX)$ means that A doesn't have any dominant ideals; or if $N^2 = 0$) they reduce to $f'Re' = f'R\varepsilon = \varepsilon Re' = 0$, $\varepsilon R\varepsilon = K\varepsilon$. If $e' = f' = 0$ (for dom.dim. $R \geqq 2$ this means that A is Frobenius) they reduce to $\varepsilon N\varepsilon = \varepsilon N(1-\varepsilon)N\varepsilon$, $\varepsilon R\varepsilon / \varepsilon N\varepsilon \cong K$.

EXAMPLES. The following remarks are obtained by specializing results of Harada [1] for semi-primary rings to our case of algebras; but easy direct proofs could be given as well. R denotes a QF-3 algebra, A its endomorphism-ring and R' its second commutator, both with respect to the minimal faithful module.

(i) These three statements are equivalent: R' is semi-simple; A is semi-simple; the socle of R is projective. Then, if $e_1R, \ldots, e_kR$ represent the different types of dominant ideals, the $D^{(i)} = e_iRe_i$ are division-rings and we have $A = \bigoplus_{i=1}^{k} D^{(i)}$, $R' = \bigoplus_{i=1}^{k} D^{(i)}_{n_i}$ (ring-direct sum of $n_i \times n_i$-matrix-rings over the $D^{(i)}$) where $n_i = D^{(i)}$-dim e_iR.

(ii) Equivalent: R' is simple; $A = D$ is a division-ring; there exists only one dominant type eR and the unique minimal subideal of eR is projective. Then $D = eRe$ and $R' = D_n$ where $n = D$-dim eR. The minimal subalgebra R_0 in $C(D_n)$ is $R_0 = \sum_{k=1}^{n} D c_{1k} + \sum_{i=2}^{n} D c_{in} + K(\sum_{j=2}^{n-1} c_{jj})$; observe $R_0 \neq D_n$ for $n > 1$.

(iii) R' is simple for every indecomposable hereditary QF-3 algebra R (Mochizuki [2]). Actually $R \in C(D_n)$ is hereditary if and only if (up to isomorphism) $T_n \subset R \subset D_n$ where T_n denotes the algebra of (upper) triangular matrices. Any such R is of the form

$$R = \begin{pmatrix} D_{n_1} & D_{n_1,n_2} & \cdots & D_{n_1,n_k} \\ 0 & D_{n_2} & \cdots & D_{n_2,n_k} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & D_{n_k} \end{pmatrix}$$

Injective hulls. LEMMA 7. *Let S be a QF-3 algebra, M' a S'-(left-) module hence a S-module, and M a S-submodule of M' such that $S'M = M'$. Suppose that all simple S'-submodules of M' are isomorphic to ideals. Then the S'-injective hull H' of M' (considered as S-module) is the S-injective hull of M.*

Proof. Consider any simple S'-submodule I' of M'. I' being isomorphic to a S'-ideal and S' being QF-3, we get a S'-monomorphism of I' into a minimal faithful ideal $S'f = Sf$ which yields an epimorphism $eS = eS' \cong S'f^* \to I'^*$. Hence $I'^*e \neq 0$ and $eI' \neq 0$. But $eM' = eS'M = eSM \subset M$, consequently $0 \neq eI' \subset I' \cap M$ and $I' \cap M \neq 0$. Furthermore the S'-injective hull $H'(I')$ of I' is isomorphic to some $S'f_1 = Sf_1$, $f = f_1 + \cdots$; hence $H'(I')$ has a unique minimal submodule I when considered as S-module. We get $I \subset I' \cap M \subset M$ and $S'I = I'$ since $S'I$ is a S'-submodule of the simple S'-module I'. The S-injective hull of I, being isomorphic to Sf_1, is isomorphic to $H'(I')$ as S-module.

Let $\bigoplus_{k=1}^{n} I'_k$ be the S'-socle of M'. As we have seen, each I'_k contains a unique simple S-submodule I_k and the S'-injective hull H' of M', being the direct sum of the S'-injective hulls $H'(I'_k)$ of the I'_k, is isomorphic to the S-injective hull of $\bigoplus I_k$ as S-module. Since $\bigoplus I_k$ is semi-simple and is contained in M, it is in the socle of M: $\text{socle}(M) = \bigoplus I_k \oplus J$. Thus the S-injective hull of M is isomorphic to the direct sum of H' and the S-injective hull $H(J)$ of J, as S-module. On the other hand $M \subset M' \subset H'$ and the fact that H' is S-injective imply that the S-injective hull of M is contained in H'; hence a K-vector-space-dimension argument yields $H(J) = 0$ and the assertion of the lemma.

THEOREM 8. *Let R be a QF–3 algebra. Then the R'-injective hull H' of R' is the R-injective hull of R when considered as R-module, where all modules are either left-, right- or bimodules.*

Proof. Applying Lemma 7 to $S=R$, $M=R$, $M'=R'$ we get the result for left-modules. A similar argument holds for right-modules.

Considering bimodules, to say modules over the enveloping algebra $R^e=R\otimes_K R^0$, we show that $(R^e)'$ can be identified with $(R')^e$ by an isomorphism which carries R^e as (natural) subalgebra of $(R^e)'$ into R^e as subalgebra of $(R')^e$ determined by R as (natural) subalgebra of R'. Observe dom.dim. $(R')^e=$ dom.dim. $R'\geqq 2$ (Mueller [4], Lemma 6). We have $1\otimes 1^0\in R^e\subset (R')^e$; and the $(R')^e$-left- resp. right-modules $R'f\otimes(eR')^0$, $eR'\otimes(R'f)^0$ where $R'f=Rf$, $eR'=eR$ are minimal faithful R'- and R-ideals, are projective-injective-faithful. We have $R'f\otimes(eR')^0=Rf\otimes(eR)^0$, $eR'\otimes(R'f)^0=eR\otimes(Rf)^0\subset R^e$; hence Theorem 1 yields $R^e\in C((R')^e)$, to say $(R')^e\cong(R^e)'$, and this isomorphism carries R^e as subalgebra of $(R')^e$ into R^e as subalgebra of $(R^e)'$, as indicated above (cf. the proof of Theorem 1). Now choose $S=R^e$, $S'=(R^e)'=(R')^e$; $M=R$, $M'=R'$. We get $S'M=(R')^eR=R'RR'=R'=M'$ and a simple $(R')^e$-submodule of R' – a simple two-sided R'-ideal – is isomorphic to a $(R')^e$-ideal since the QF–3 algebra R' can be embedded as $(R')^e$-module into a projective module. Thus Lemma 7 yields the desired result in this case too.

Mochizuki [2] observed that for hereditary QF–3 algebras R (where R' is semi-simple), R' itself is the injective hull of ${}_RR$ and R_R. We see that this phenomenon is rather exceptional:

COROLLARY 9. *R' is the injective hull of R as left- and/or right-R-module if and only if R' is quasi-Frobenius. R' is the injective hull of R as R-R-bimodule if and only if R' is separable.*

Theorem 8 allows the construction of the following diagram of left-, right- or bimodules:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & R & \longrightarrow & H' & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & R' & \longrightarrow & H' & \longrightarrow & X'_2 & \longrightarrow & \cdots & \longrightarrow & X'_n \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

where all rows and columns are exact, the bottom row contains R'-homomorphisms while all other maps are R-homomorphisms; where H', X'_2, $\ldots$, X'_n are R'-injective-projective and therefore also R-injective-projective; where $2 \leqq n = \text{dom.dim.}\, R' \leqq \infty$; and where the top row cannot be extended further by R-injective-projective modules if $R \neq R'$. Hence the socle of the R-module H' / R must contain a simple module non-isomorphic to an ideal while the socle of H' / R' as R'- or R-module contains only simple modules isomorphic to ideals (cf. [4], proof of Lemma 7). Consequently since $H' / R' \cong H'/R \big/ R'/R$ as R-modules, socle (R' / R) has to contain a simple R-module non-isomorphic to an ideal. We show the following stronger fact:

THEOREM 10. *Let R be a QF-3 algebra. Then all composition-factors of R' / R as R-left-, right- or bimodule, are not isomorphic to ideals.*

Proof. We apply Lemma 7 choosing S either $= R$ or $= R^e$ (then identifying $S' = (R^e)'$ with $(R')^e$ as before) and $M' = R'$, $R \subset M \subset R'$ any S-submodule of R'. Then the S'-injective hull H' of R' is the S-injective hull of M when considered as S-module, and we get the exact sequence of S-modules $0 \longrightarrow M \xrightarrow{\alpha} H'$. Suppose it can be extended to $0 \longrightarrow M \xrightarrow{\alpha} H' \xrightarrow{\beta} X$ where X is S-injective-projective. Then we get a diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & M & \xrightarrow{\alpha} & H' & \xrightarrow{\beta} & X \\ & \downarrow & & \downarrow & & \downarrow \\ & S' \otimes_S M & \xrightarrow{1_{S'} \otimes \alpha} & S' \otimes_S H' & \xrightarrow{1_{S'} \otimes \beta} & S' \otimes_S X \\ & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ & R' & \xrightarrow{\varphi} & S' \otimes_S H' & \xrightarrow{1_{S'} \otimes \beta} & S' \otimes_S X \end{array}$$

where Ψ is the epimorphism $s' \otimes m \rightarrow s'm$ and φ is the homomorphism $R' \rightarrow H' \rightarrow S' \otimes {}_S H'$. The maps $H' \rightarrow S' \otimes {}_S H'$, $X \rightarrow S' \otimes {}_S X$ are S-isomorphisms since ${}_S H'$, ${}_S X$ are injective-projective. All squares are commutative — the one in the lower left corner because of $s' \otimes h' = 1' \otimes s'h' \in S' \otimes {}_S H'$ (use the isomorphism between $S' \otimes {}_S H'$ and H'). The bottom row is a complex $(= 0)$ since the middle row obviously is and ψ is epimorphic. Finally φ is S'-monomorphic, for a simple S'-submodule I' of Ker φ gives $I' \cap M \neq 0$ and $I' \cap M \subset \text{Ker}\, \alpha$ since $M \rightarrow S' \otimes {}_S M \rightarrow R'$ turns out to be the injection $M \rightarrow R'$; but α is monomorphic. Now diagram-chasing shows that $M \rightarrow R'$ is epimorphic which is a contradiction whenever $M \neq R'$; hence in this case an extension $0 \rightarrow M \rightarrow H' \rightarrow X$ cannot exist, meaning that socle$({}_S H' / M)$ will contain a simple module J non-isomorphic to an ideal. Since J cannot lie in H'/R', it has to be contained in the socle of R'/M.

Now suppose $R \subset M \subsetneqq R'$ and that all factors of M/R are non-isomorphic to ideals. Then there exists $M \subset M_1 \subset R'$ such that $J \cong M_1/M$ and all factors of M_1/R are non-isomorphic to ideals. Thus the theorem is proved by induction.

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McMaster University, Hamilton, Ontario, Canada