CERTAIN METHOD FOR GENERATING A SERIES OF LOGICS

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Introduction. At first, we define three relations $\supseteq$, $=$, and $\supset$ in connection with a pair of logics $\boldsymbol{L}$ and $\boldsymbol{L}^*$ as follows:

$\boldsymbol{L} \supseteq \boldsymbol{L}^*$, if and only if every proposition provable in $\boldsymbol{L}^*$ is also provable in $\boldsymbol{L}$;

$\boldsymbol{L} = \boldsymbol{L}^*$, if and only if $\boldsymbol{L} \supseteq \boldsymbol{L}^*$ and $\boldsymbol{L}^* \supseteq \boldsymbol{L}$;

$\boldsymbol{L} \supset \boldsymbol{L}^*$, if and only if $\boldsymbol{L} \supseteq \boldsymbol{L}^*$ but not $\boldsymbol{L}^* \supseteq \boldsymbol{L}$.

Next, for a logic $\boldsymbol{L}$, we denote by $\boldsymbol{L}[A]$ the fortified logic of $\boldsymbol{L}$ by regarding a proposition A as a new axiom scheme.

By $\boldsymbol{LOQ}$, we denote the logic obtained by adjoining Peirce's rule,

$$P \equiv ((a \rightarrow b) \rightarrow a) \rightarrow a,$$

to the primitive logic $\boldsymbol{LO}$ (*cf.* Ono [6], [7]). According to Nagata [4], we can obtain a *descending sequence*, $\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots,$ from $\boldsymbol{LOQ}$ toward $\boldsymbol{LO}$ (*i.e.* $\boldsymbol{LOQ} = \boldsymbol{L}_1 \supset \boldsymbol{L}_2 \supset \ldots \supset \boldsymbol{L}_i \supset \ldots \supset \boldsymbol{LO}$) by the following method. A series of propositions P_i is defined recursively as follows:

$$\begin{cases} P_1 \equiv P, \\ P_{i+1} \equiv ((p_i \rightarrow P_i) \rightarrow p_i) \rightarrow p_i, \qquad (i = 1, 2, \ldots), \end{cases}$$

where p_i's are mutually distinct proposition-variables not occurring in P. For the series $P_1, P_2, \ldots,$ we can assert that

$$\boldsymbol{LOQ} = \boldsymbol{LO}[P_1] \supset \boldsymbol{LO}[P_2] \supset \ldots \supset \boldsymbol{LO}[P_i] \supset \ldots \supset \boldsymbol{LO}.$$

We have noticed that, by making use of the same method, existence of descending sequences from $\boldsymbol{K}$-series logics ($\boldsymbol{LQ}$, $\boldsymbol{LN}$, $\boldsymbol{LK}$) toward their corresponding $\boldsymbol{J}$-series logics ($\boldsymbol{LP}$, $\boldsymbol{LM}$, $\boldsymbol{LJ}$) (*cf.* Ono [6]) can be proved. We have also noticed that existence of a descending sequence from $\boldsymbol{LN}$ toward $\boldsymbol{LD} = \boldsymbol{LM}[a \vee \neg a]$ (*cf.* Curry [1]) can be proved similarly.

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Discussing with us our recent studies on the subject, Prof. T. Tugué pointed out that we would have, in a similar manner, a descending sequence toward a logic $\boldsymbol{L}$ by starting from any proposition A, not provable in $\boldsymbol{L}$, instead of starting from Peirce's rule. Guided by his valuable suggestion, we obtained the following conclusion.

For any proposition A, a series of propositions A_i is defined recursively as follows:

$$\begin{cases} A_1 \equiv A, \\ A_{i+1} \equiv ((p_i \to A_i) \to p_i) \to p_i, \qquad (i = 1, 2, \ldots), \end{cases}$$

where p_i's are mutually distinct proposition-variables not occurring in A. The proposition A is called *kernel*. Taking Peirce's rule P as the kernel A, we can produce the descending sequences described before. If we take $a \vee \neg a$ (law of the excluded middle) as the kernel A, we can produce a descending sequence from $\boldsymbol{LD}$ toward $\boldsymbol{LM}$. Along this line, we would be able to give other examples as many as we like. To show these facts, we shall use certain truth-table, called *(n, r)-evaluation.* The (n, r)-evaluation is a slight refining of the truth-table appearing in Gödel [2][1]. The refinement lies on the evaluation of negation defined as follows:

a	0	1	$\ldots$	$r-1$	r	$\ldots$	n
$\neg a$	r	r	$\ldots$	r	0	$\ldots$	0

The main purpose of this paper is to show that, for a logic $\boldsymbol{L}$ and a kernel A, we can generate a descending sequence from $\boldsymbol{L}[A]$ toward $\boldsymbol{L}$ under certain conditions. We wish to express our thanks to Profs. K. Ono and T. Tugué for their kind guidances.

Definition. For integers n and r such that $1 \leq r \leq n$, any evaluation having the following truth-value properties is called *(n, r)-evaluation:*

$$a \to b = \begin{cases} 0 & \text{if } a \geq b, \\ b & \text{if } a < b, \end{cases}$$

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[1] In Gödel [2], it is discussed that there is "eine monoton abnehmende Folge von Systemen" between $\boldsymbol{LK}$ and $\boldsymbol{LJ}$ by considering the formula $F_n \equiv \bigvee_{1 \leq i < k \leq n}(a_i \equiv a_k)$ with respect to a many-valued evaluation. This fact enables us to do the same discussion between $\boldsymbol{LK}$ and a logic, in which every provable proposition is (n, r)-true (*cf.* Definition) for any n and some r. Moreover, in Umezawa [8]—[10] and Nishimura [5], there are detailed discussions on intermediate logics between $\boldsymbol{LK}$ and $\boldsymbol{LJ}$.

$$a \vee b = \mathrm{Min}(a, b),$$
$$a \wedge b = \mathrm{Max}(a, b),$$
$$\neg a = \begin{cases} r & \text{if } a < r, \\ 0 & \text{if } a \geq r; \end{cases}$$

where the truth-values of propositions a, b, denoted simply by a, b, respectively, runs over the set $\{0, 1, \ldots, n\}$. If we take the logical constant $\curlywedge$ (contradiction) whose truth-value is defined by r, and define $\neg a$ by $a \to \curlywedge$, then the above truth-value property for negation is obtained. For the predicate logics, we take a domain of k individual objects $\{\xi_1, \xi_2, \ldots, \xi_k\}$ and define the truth-value of

$$(\xi)\, a(\xi) \quad \text{by} \quad a(\xi_1) \wedge a(\xi_2) \wedge \ldots \wedge a(\xi_k),$$
$$(\exists \xi)\, a(\xi) \quad \text{by} \quad a(\xi_1) \vee a(\xi_2) \vee \ldots \vee a(\xi_k).$$

Any proposition whose truth-value is always 0 is called *(n, r)-true.*

For any n and r, all the axiom schemes[2] of ***LM*** are (n, r)-true, and all the inference rules of ***LM*** deduce (n, r)-true conclusions, whenever their assumptions are all (n, r)-true. $\curlywedge \to a$ is (n, n)-true, $a \vee \neg a$ is $(n, 1)$-true, and $(\curlywedge \to a) \vee b \vee \neg b$ (*cf.* Example) is both $(n, 1)$- and (n, n)-true for all n, r.

Before stating the theorem, the following two lemmas are remarkable.

LEMMA 1. *If the kernel $A \leq n - j$ $(0 \leq j \leq n-1)$ for the (n, r)-evaluation, then, $A_i \leq n - j - i + 1$ $(1 \leq i \leq n - j + 1)$.*

LEMMA 2. *If the kernel A takes the truth-value $n - j$ $(0 \leq j \leq n-1)$ for the (n, r)-evaluation, then, A_i takes $n - j - i + 1$ $(1 \leq i \leq n - j + 1)$.*

THEOREM. *Let **L** be a logic such that $\boldsymbol{LK} \supset \boldsymbol{L} \supseteq \boldsymbol{LO}$. Assume that there exists a function $r = r(n)$ $(r = 1, 2, \ldots, n)$ satisfying the following conditions.*

(1) *For all n, every **L**-provable[3] proposition is (n, r)-true.*

(2) *There exists a non-negative integer j such that, for all $n (n \geq 2)$ larger than j, a proposition A can never takes the truth-value larger than $n - j$, but can take certainly $n - j$ by the (n, r)-evaluation.*

*Then, there is a descending sequence from **L**[A] toward **L**, i.e.,*

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[2] *cf.* *H* system of Curry [1].

[3] In this paper, for a logic ***L***, a proposition A is called to be ***L-provable*** when A is provable in ***L***.

$$\boldsymbol{L}[A] = \boldsymbol{L}[A_1] \supset \boldsymbol{L}[A_2] \supset \ldots \supset \boldsymbol{L}[A_i] \supset \ldots \supset \boldsymbol{L}.$$

Proof. (i) The cases $j \neq 0$ or $n \geq 3$. If we take $(n+j-1, r)$-evaluation in place of (n, r)-evaluation, then, $A \leq n-j$ turns out to be $A \leq n-1$. By Lemma 1, $A_i \leq n-i$ holds; hence, $A_n = 0$ always holds. Since all the $\boldsymbol{L}$-provable propositions are $(n+j-1, r)$-true by assumption, all the $\boldsymbol{L}[A_n]$-provable propositions are always $(n+j-1, r)$-true. By Lemma 2, however, $A_i = n-i$ holds; hence, $A_{n-1} = 1$ holds. Therefore, A_{n-1} is not $\boldsymbol{L}[A_n]$-provable. Namely, $\boldsymbol{L}[A_{n-1}] \supset \boldsymbol{L}[A_n] \supset \boldsymbol{L}$.

(ii) The case $j = 0$ and $n = 2$. By assumption, there exists r such that A_1 can take 2 by $(2, r)$-evaluation. However, $A_2 \leq 1$ by Lemma 1. Hence, A_1 is not $\boldsymbol{L}[A_2]$-provable. Therefore, $\boldsymbol{L}[A_1] \supset \boldsymbol{L}[A_2]$. *q. e. d.*

EXAMPLE. In the following table, descending sequences from $\boldsymbol{L}[A]$ toward $\boldsymbol{L}$ are exhibited by showing their kernels A and the numbers r appearing in the assumption of the theorem. We can further substitute, in the table, $\boldsymbol{LM}[\neg a \vee \neg\neg a]$ or $\boldsymbol{LM}[(a \to b) \vee (b \to a)]$ etc. for $\boldsymbol{LM}$. In the following table, $\boldsymbol{A} \cap \boldsymbol{B}$ denotes the logics in which any proposition is provable, if and only if it is both $\boldsymbol{A}$- and $\boldsymbol{B}$-provable.

	$\boldsymbol{L}[A]$	$\boldsymbol{L}$	A	r
1	$\boldsymbol{LOQ}$	$\boldsymbol{LO}$	$P \equiv ((a \to b) \to a) \to a$	$1 \leq r \leq n$
2	$\boldsymbol{LQ}$	$\boldsymbol{LP}$	P	$1 \leq r \leq n$
3	$\boldsymbol{LN}$	$\boldsymbol{LM}$	P	$1 \leq r \leq n$
4	$\boldsymbol{LK}$	$\boldsymbol{LJ}$	P	$r = n$
5	$\boldsymbol{LN}$	$\boldsymbol{LD}$	P	$r = 1$
6	$\boldsymbol{LD}$	$\boldsymbol{LM}$	$a \vee \neg a$	*e. g.* $r = n$
7	$\boldsymbol{LJ}$	$\boldsymbol{LM}$	$\curlywedge \to a$	$1 \leq r \leq n-1$
8	$\boldsymbol{LN}_i \equiv \boldsymbol{LM}[P_i]$	$\boldsymbol{LD}_i \equiv \boldsymbol{LM}[B_i]\ (B \equiv a \vee \neg a)$	P_i	$r = 1$
9	$\boldsymbol{LN}_i$	$\boldsymbol{LJ}_i \equiv \boldsymbol{LM}[C_{i+1}]\ (C \equiv \curlywedge \to a)$	P_i	$r = n$
10	$\boldsymbol{LJ} \cap \boldsymbol{LN}$	$\boldsymbol{LM}$	$(\curlywedge \to a) \vee b \vee (b \to c)$	$1 \leq r \leq n-1$
11	$\boldsymbol{LN}$	$\boldsymbol{LJ} \cap \boldsymbol{LN}$	P	$r = n$
12	$\boldsymbol{LJ} \cap \boldsymbol{LD}$	$\boldsymbol{LM}$	$(\curlywedge \to a) \vee b \vee \neg b$	*e. g.* $r = n-1$

13	$\boldsymbol{LD}$	$\boldsymbol{LJ} \cap \boldsymbol{LD}$	$a \vee \neg a$	$r = n$
14	$\boldsymbol{LJ}$	$\boldsymbol{LJ} \cap \boldsymbol{LD}$	$\curlywedge \to a$	$r = 1$
15	$\boldsymbol{LJ} \cap \boldsymbol{LN}$	$\boldsymbol{LJ} \cap \boldsymbol{LD}$	$(\curlywedge \to a) \vee b \vee (b \to c)$	$r = 1$

(As for the correlations of logics in the lines 10—15 under $\boldsymbol{L}[A]$ and $\boldsymbol{L}$, see Miura [3].)

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Added in proof April 28, 1967.

After this paper had been admitted, we found the fact that a series of propositions P_i, appearring in Nagata [4] and also in this paper, has been introduced by A.S. Troelstra in the slightly different form. (See [11] cited below.) A result of Nagata [4] has been already used in Troelstra [11] in order to verify one of his theorems. Moreover, there are some arguments in [11] connected with ours in the present paper.

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