TRANSITIVE EXTENSIONS OF CERTAIN PERMUTATION GROUPS OF RANK 3

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

We denote a permutation group H on a set Γ by (H, Γ). (H, Γ) is called *a permutation group of rank* 3 if (H, Γ) is transitive and (H_a, Γ), $a \in \Gamma$, has exactly three orbits, where H_a is the stabiliger of a point a, namely, $\{\alpha \in H \mid a^\alpha = a\}$

In this note the following theorems will be proved.

THEOREM 1. (I). *If (H, Γ) is a permutation group of rank* 3 *such that the lengths of orbits of (H_a, Γ), $a \in \Gamma$, are* 1, 1 *and the order of H_a, then a pair of H and H_a is one of the following:*

(1) *H is the dihedral group of order* 8 *and H_a is a subgroup of order* 2 *which is not the center of H.*

(2) *H is the symmetric group of degree* 4 *and H_a is a cyclic subgroup of order* 4.

(3) *H is the symmetric group of degree* 4 *and H_a is a non-normal elementary abelien subgroup of order* 4.

(4) *H is the general linear group $GL(2,3)$ of dimension* 2 *over $GF(3)$ and H_a is a subgroup which is isomorphic to the symmetric group S_3 of degree* 3.

(5) *H is the two dimensional linear fractional group $LF_2(7)$ over $GF(7)$ and H_a is a subgroup which is isomorphic to the alternating group A_4 of degree* 4.

(II). *If (G, Ω) is a transitive extension of (H, Γ), then G is either*

(1) $LF_2(7)$,

or (2) *$V \cdot GL(2,3)$ where V is the two dimensional vector space over $GF(3)$ and $GL(2,3)$ acts on V in the natural way,*

or (3) *the alternating group A_7 of degree* 7.

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THEOREM 2. *Let* (H, Γ) *be a transitive group of rank* 3 *and let* $\Delta_0 = \{0\}$, Δ_1, Δ_2 *be the orbits of* (H_0, Γ), $0 \in \Gamma$. *Let us assume that*

(i) H_0 *is faithful on* Δ_1 *and* Δ_2,

(ii) (H_0, Δ_1) *is a Frobenius group whose Frobenius kernel* Q *and Frobenius compliment* K *are abelian (accordingly* K *is cyclic), and* Q *is semi-regular on* Δ_2, *and*

(iii) $|\Delta_1| \neq |\Delta_2|$ *and* $|\Delta_1| \geqq 3$. *(We denote the number of points in a set* Σ *by* $|\Sigma|$*).*

If $(G, (\tilde{\Gamma})$ *is a transitive extension of* (H, Γ), *then* G *is the two dimensional linear fractional group* $LF_2(11)$ *over* $GF(11)$ *and* H *is a subgroup of* $LF_2(11)$ *which is isomorphic to the alternating group* A_5 *of degree* 5.

For a set X of permutations on a set Σ we put

$$F_\Sigma(X) = \{x \in \Sigma \mid x^\sigma = x \text{ for any } \sigma \in X\} \text{ and } f_\Sigma(X) = |F_\Sigma(X)|.$$

Proof of Theorem 1, (I). Since the stabiliger of a point has exactly two fixed points we have that $n(= |\Gamma|)$ is even and (H, Γ) is an imprimitive group with a complete system of sets of imprimitivity $\tilde{\Gamma} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_{\frac{n}{2}}\}$ such that $|\Gamma_i| = 2$ for $i = 1, 2, \ldots, \frac{n}{2}$. Put $\Gamma_i = \left\{i, \frac{n}{2} + i\right\}$ and let H_i be the stabilizer of i in (H, Γ). Let u_1 be the number of involutions in H_1 and let u_i, for $n \geqq i \geqq 2$, be the number of involutions in H which interchange 1 and i, and which are conjugate to elements of H_1. Then

$$\sum_{i=l}^{n} u_i = \frac{n}{2} u_1$$

is the number of involutions in H which are conjugate to elements of H_1. Since H_1 is transitive on $\Gamma - \Gamma_1$ we have that $u_{\frac{n}{2}+1} = \frac{n}{2} - 1$ and $u_i = 0$ or 1 simultaneously for all i other than 1 and $\frac{n}{2} + 1$. Hence we have that $u_1 = 1$ or 3. Assume that $u_1 = 1$ and let e be the involution of H_1. Then the cycle structure of e is $(1)\left(\frac{n}{2} + 1\right)(\Gamma_2)(\Gamma_3)\ldots(\Gamma_{\frac{n}{2}})$ where $(\Gamma_i) = \left(i, \frac{n}{2} + i\right)$. Let σ be an element of H which carries 1 into 2. Then $e^\sigma = (\Gamma_1)(2)\left(\frac{n}{2} + 2\right)(\Gamma_3)\ldots(\Gamma_{\frac{n}{2}})$. Hence $F_\Gamma(e\, e^\sigma) = n - 2$. Hence $n = 2$ or 4. If $n = 2$, then H is the dihedral group of order 8 and H_1 is a non central subgroup of order 2 of H. If $n = 4$, then H is the symmetric group of degree 4 and H_1 is a cyclic group of order 4 (see §126, [1]). Assume that $u_1=3$, and let e_1, e_2, e_3 be involutions of H_1. Since H_1 is regular on $\Gamma-\Gamma_1$, each two-cycle (Γ_i) appears in one (and only one) of the cycle decompositions of e_1, e_2, e_3. Hence we have the following three cases; let τ be an element of H which carries 1 into 2.

Case (i). $e_1=(1)\left(\frac{n}{2}+1\right)(\Gamma_2)(\Gamma_3)\ldots(\Gamma_{\frac{n}{2}})$

and

(Γ_1), (Γ_2), ..., $(\Gamma_{\frac{n}{2}})$ do not appear in cycle decompositions of e_2 and e_3. Then $f_\Gamma(e_1e_1^\tau)=n-4$. Hence $n=6$ and H is the symmetric group of degree 4 and H_1 is an elementary abelian non-normal subgroup of order 4 of H (see §126, [1]).

Case (ii).
$$e_1=(1)\left(\frac{n}{2}+1\right)(\Gamma_2)\ldots(\Gamma_{l+1})(U_{l+2})\ldots(U_{\frac{n}{2}})$$
$$e_2=(1)\left(\frac{n}{2}+1\right)(U_2)\ldots(U_{l+1})(\Gamma_{l+2})\ldots(\Gamma_{\frac{n}{2}})$$
$$e_3=(1)\left(\frac{n}{2}+1\right)(V_2)\ldots\ldots\ldots\ldots(V_{\frac{n}{2}})$$

where (U_i) and (V_i), $i=2,3,\ldots,\frac{n}{2}$, are two-cycles which are not equal to any one of (Γ_2), (Γ_3), ..., $(\Gamma_{\frac{n}{2}})$. Then $\frac{n}{2}=2l+1$, since e_1 and e_2 are conjugate each other. e_1^τ and e_2^τ are involutions of H_2 and two-cycles (Γ_1), (Γ_3), ..., $(\Gamma_{\frac{n}{2}})$ appear in the cycle decompositions of e_1^τ and e_2^τ. Hence, if $l\geqq 3$, then at least one of $e_i^\tau e_j$, $1\leqq i,j\leqq 2$, has more than two fixed points. This is a contradiction. Therefore $l=2$. Then H, as a permutation group on $\tilde{\Gamma}$, is doubly transitive and contains a two cycle. Hence $(H,\tilde{\Gamma})$ is the symmetric group of degree 5, but this is impossible.

Case (iii).
$$e_1=(1)\left(\frac{n}{2}+1\right)(\Gamma_2)\ldots(\Gamma_{l+1})(X_{l+1})\ldots\ldots\ldots\ldots(X_{\frac{n}{2}})$$
$$e_2=(1)\left(\frac{n}{2}+1\right)(Y_2)\ldots(Y_{l+1})(\Gamma_{l+2})\ldots(\Gamma_{m+1})(Y_{m+2})\ldots(Y_{\frac{n}{2}})$$
$$e_3=(1)\left(\frac{n}{2}+1\right)(Z_2)\ldots\ldots\ldots\ldots(Z_{m+1})(\Gamma_{m+2})\ldots(\Gamma_{\frac{n}{2}})$$

where (X_i), (Y_j), (Z_k) are two-cycles which are not equal to any one of (Γ_2), (Γ_3), . . . , $(\Gamma_{\frac{n}{2}})$. Then, since e_1, e_2, e_3 are conjugate each other, $\frac{n}{2} = 3l + 1$ and $m = 2l$. If $l \geqq 4$, then at least one of $e_i^\tau e_j$, $1 \leqq i, j \leqq 3$, has more than two fixed points which is a contradiction. Hence $l = 1$, 2 or 3. If $l = 1$, then it is easily seen that H is isomorphic to $GL(2, 3)$ and H_1 is isomorphic to S_3. If $l = 2$, then $n = 14$. H acts on $\tilde{\Gamma}$ faithfully, because if H is not faithful on $\tilde{\Gamma}$ then $e = (\Gamma_1)(\Gamma_2) \ldots (\Gamma_t)$ is an element of (H, Γ), and then $e\, e_1$ has more than two fixed points. This is impossible. Hence H has a faithful doubly transitive representation of degree 7 and the order of H is $7 \cdot 6 \cdot 4$. Hence H is isomorphic to $LF_2(7)$ and H_1 is isomorphic to A_4 (see §166, [1]). If $l = 3$, then $r = 18$, $|H| = 10 \cdot 9 \cdot 4$, and H has a faithful doubly transitive representation of degree 10 (on $\tilde{\Gamma}$). Since e_i is an odd permutation on Γ, H contains a normal subgroup H of order $10 \cdot 9 \cdot 2$, which is doubly transitive on $\tilde{\Gamma}$, but this is impossible.

Proof of Theorem 1, **II.** We denote by $H_{(i)}$ the permutation group of Theorem 1, **I**, (i), and by $G_{(i)}$ a transitive extension of $H_{(i)}$. $G_{(1)}$ does not exist, because it is a doubly transitive group of degree 5 and order $5 \cdot 4 \cdot 2$, (see §166, [1]). $G_{(2)}$ does not exist and $G_{(3)} \cong LF_2(7)$, because they are doubly transitive groups of degree 7 and order $7. 6 \cdot 4$ (see §166 [1]). $G_{(4)} \cong V \cdot GL(2, 3)$, because it is a solvable doubly transitive group of degree 9 and order $9 \cdot 8 \cdot 6$ (for instance, see [3]). $G_{(5)} \cong A_7$, because it is a doubly transitive group of degree 15 and order $15 \cdot 14 \cdot 12$ (for instance, see exercises 10 (p. 162) and 4 (p. 304), [2]).

Remark. We note that the stabiligers of two points in the groups (G, Ω) of Theorem 1, (II) are not cyclic groups.

Proof of Theorem 2. Let $|\Delta_1| = n$ and put $\Delta_1 = \{1, 2, \ldots, n\}$ and let K be a stabilizer of 1 in (H_0, Δ_1). Since Q is semi-regular on Δ_2, $|\Delta_2| \equiv 0\,(n)$. We denote $|\Delta_2| = nr$ and put $\Delta_2 = \{\bar{1}, \bar{2}, \ldots, \overline{nr}\}$ where we choose the point $\bar{1}$ such that the stabilizer of $\bar{1}$ in (H_0, Δ_2), denoted by K_0, is contained in K. We also denote $|K| = q\,(\geqq 2)$.

First we claim that n is odd. We assume that n is even. Let n_0 be the number of involutions in H_0, and let $n_a, a \in \Gamma - \{0\}$, be the number of involutions in H which interchange 0 and a. Then $\{1 + n(r+1)\}\, n_0 =$

$\sum_{a\in\Gamma} n_a$ is the number of involutions in H. $n_i \leqq q$ for $1\leqq i\leqq n$, because if two involutions τ_1, τ_2 of H interchange 0 and i, then $\tau_1\tau_2$ is contained in a subgroup $K_i=\{\sigma\in H_0 \mid \sigma(i)=i\}$ of order q. $n_{\bar{i}}\leqq q/r$ for $1\leqq i\leqq nr$, because if two involutions τ_1,τ_2 of H interchange 0 and $\bar{i}$, then $\tau_1\tau_2$ is contained in a subgroup $K_i=\{\sigma\in H_0 \mid \sigma(\bar{i})=\bar{i}\}$ of order q/r. Hence $\{1+n(r+1)\}n_0\leqq n_0+nq+nrq/r=n_0+2nq$, namely, $n_0(r+1)\leqq 2q$. Since n_0 is divisible by q, we have that $r=1$. This is a contradiction.

Next we claim that q is even. We assume that q is odd. Put $\tilde{\Gamma}=\{\infty\}\cup\Gamma$. Let τ be an involution of G which ingerchanges ∞ and 0. Then $\tau^{-1}H_0\tau$ (simply denoted by $H_0^\tau)=H_0$ and $Q^\tau=Q$. Since n, the number of subgroups of H_0 of order q, is odd, there exists at least one subgroup X of H_0 of order q which is invariant by τ. Since $|\varDelta_1|\neq|\varDelta_2|$, we have that $f_{\varDelta_1}(X)=1$, namely, $\tau(i_0)=i_0$ for some $i_0\in\varDelta_1$. This means that τ is an element of a group which is isomorphic to H. Since $|H|=$ odd, this is impossible. Hence q is even.

Next we claim that $q=r$. We assume that $q\neq r$. Let K_0' be a subgroup of H_0 which is conjugate to K_0 by an element of G. Then $f_{\varDelta_1}(K'_0)\neq 0$, because $(|K_0'|,n)=1$. Hence $K_0'^{\sigma_i}\leqq K$ for some i of $f_{\varDelta_1}(K_0)$, where σ_i is an element of Q such that $\sigma_i(1)=i$. Since K is cyclic, $K_0'^{\sigma_i}=K_0$. This means that if a subgroup of H_0 is conjugate to K_0 in G, then they are conjugate in H_0. Hence, by a theorem of Witt (§9, [5]), the normalizer of K_0 in G, denoted by $N(K_0)$, is doubly transitive on $F_{\tilde{\Gamma}}(K_0)$. Since $(H_0,\varDelta_2)$ $(H_0, H_0/K_0)$ and K is abelian, we have that $f_{\varDelta_2}(K_0)=f_{H/K_0}(K_0)=r$, hence $f(K_0)=r+3$. Then it is easily seen that $(N(K_0)/K_0, F_{\tilde{\Gamma}}(K_0))$ is a doubly transitive group of degree $r+3$, K/K_0 is the stabilizer of two points $\infty,0$ in this group, $F_{F_{\tilde{\Gamma}}(K_0)}(K/K_0)=\{\infty,0,1\}$, and K/K_0 is cyclic and regular on $F_{\tilde{\Gamma}}(K_0)-\{\infty,0,1\}$. Hence the group $(N(K_0)/K_0, F_{\tilde{\Gamma}}(K_0))$ should be one of the groups in Theorem 1, (II). From the remark at the end of proof of Theorem 1, $(N(K_0)/K_0, F_{\tilde{\Gamma}}(K_0))$ can not exist, because the stabilizer of two points is cyclic. Hence $q=r$.

Let τ be an involution of G. Since r is even, τ is conjugate to an element of $H-\bigcup_{\sigma\in G}H_0^\sigma$ or K. Hence $f_{\tilde{\Gamma}}(\tau)=1$ or 3. Let τ_0 be an involution of G which interchanges ∞ and 0. Since $H_0^{\tau_0}=H_0$ and $|\varDelta_1|\neq|\varDelta_2|$, $\varDelta_i^{\tau_0}=\varDelta_i$. Since $|\varDelta_1|$ is odd, τ_0 leaves a point of $\varDelta_1$, say 1, invariant. Let α_i, $i \in \Delta_1$, be an element of Q such that $\alpha_i(1) = i$. Then $\tau_0^{-1}\alpha_i\tau_0 = \alpha_{\tau_0(i)}$. Hence, since $|Q|$ is odd, $|C_Q(\tau_0)| = 1$ or 3. We have that $Q = Q_1 \times Q_2$ where $Q_1 = C_Q(\tau_0)$ and $Q_2 = \{\alpha \in Q \mid \alpha^{\tau_0} = \alpha^{-1}\}$. In fact, for any element α of Q, $\alpha\alpha^{\tau_0} \in C_Q(\tau_0)$, and hence the order of $\alpha\alpha^{\tau_0}$ is 1 or 3. Hence $\alpha = (\alpha^2\alpha^{\tau_0})(\alpha^2\alpha^{2\tau_0})$ where $\alpha^2\alpha^{\tau_0} \in Q_2$ and $\alpha^2\alpha^{2\tau_0} \in Q_1$. Let τ_1 be an involution of K. Then we know that $\tau_1^{-1}\alpha\tau_1 = \alpha^{-1}$ for all $\alpha \in Q$, and hence $Q_2 = C_Q(\tau_0\tau_1)$. Since $\tau_0\tau_1$ is an involution which interchanges $\infty, 0$, and which fixes 1, we have that $|Q_2| = |C_Q(\tau_0\tau_1)| = 1$ or 3. Hence $n = |Q| = 3$ or 9. If $n = 3$, then $q = r = 2$, and we have that $G \cong LF_2(11)$ and $H \cong A_5$ (for instance, see [4]). If $n = 9$, then $q = r = 8$, 4, or 2, and it is easy to prove non-existence of such groups.

References

[1] Burnside, W.S., Theory of groups of finite order, 2nd. ed. Cambridge Univ. Press, 1911.
[2] Carmichael, R.D., Introduction to the theory of groups of finite order, Boston, 1937.
[3] Huppert, B., Zweifach transitive, auflosbare Permutations-gruppen, Math. Zeitschr. 68 (1957). 126–150.
[4] Ito, N., A note on transitive permutation groups of degree p, Osaka Math. J. 14 (1962), 213–218.
[5] Wielandt, H., Finite permutation groups, Acad. Press, 1964.

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