A LEMMA FOR NEGATIONLESS PROPOSITIONAL LOGICS AND ITS APPLICATIONS

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(To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.)

In this note, we treat the positive sentential logic **LPS** and the primitive sentential logic (the positive implicational calculus of Hilbert) **LOS**[1]. **LOS** has 'implication' as the only logical symbol and is a subsystem of **LPS**. **LPS** can be formulated as follows:

Proposition letters: $p, q, r, \ldots$; or $p_1, p_2, p_3, \ldots$.

Logical symbols: $\rightarrow$, $\vee$ and $\wedge$.

Formation rule: as usual.

Axiom schemata:

A 1. $$A \rightarrow (B \rightarrow A),$$

A 2. $$(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)),$$

A 3. $$A \rightarrow A \vee B, \quad B \rightarrow A \vee B,$$

A 4. $$(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C)),$$

A 5. $$A \rightarrow (B \rightarrow A \wedge B),$$

A 6. $$A \wedge B \rightarrow A, \quad A \wedge B \rightarrow B.$$

Inference rule: *Modus Ponens*.

To the system **LPS** or **LOS**, by adding Peirce's law:

A 7. $$((A \rightarrow B) \rightarrow A) \rightarrow A,$$

we obtain the corresponding 'classical' logic, which is denoted by **LQS**[1] or **LOQS**, respectively.

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[1] The reference notation **LOS** to the primitive sentential logic is found in Ono [7] where it is called the sentence-logical part of the primitive logic. Church [1] refers to **LOS** and **LPS** by P^+ and P^P, respectively. As for the primitive logic **LO**, the intuitionistic positive logic **LP** and the classical positive logic **LQ**, see Ono [6]. cf. also Curry [2] and Lorenzen [3] as to the positive logics.

Now, we take the ordinary valuation $v(A)$ for the formulae on the values $\{0, 1\}$:

$$v(A \to B) = \begin{cases} 0 & \text{if } v(A) \geqq v(B), \\ 1 & \text{otherwise,} \end{cases}$$

$$v(A \vee B) = \min(v(A), v(B)),$$

$$v(A \wedge B) = \max(v(A), v(B)).$$

Throughout this paper, let $p_1, \ldots, p_n$ be distinct letters and A be such a formula that no letters other than $p_1, \ldots, p_n$ occur in A. Given an n-tuple $v(p_1), \ldots, v(p_n)$ of values of $p_1, \ldots, p_n$, denote, as convention, the letters assigned the value 0 by $r_1, \ldots, r_u$, the rest by $s_1, \ldots, s_v$. Then, the following lemma holds for **LPS** and **LOS.**

LEMMA. *For the given n-tuple $v(p_1), \ldots, v(p_n)$ of values of the letters $p_1, \ldots, p_n$,*

$$r_1, \ldots, r_u, \quad s_1 \to s_2, s_2 \to s_3, \ldots, s_v \to s_1 \vdash A$$ [2]

or

$$r_1, \ldots, r_u, \quad s_1 \to s_1, s_2 \to s_3, \ldots, s_v \to s_1 \vdash A \to s_1,$$

according as $v(A)$ is 0 or 1.

Let A be a formula of **LPS** in which no letters other than $p_1, \ldots, p_n$ occur. Then, we consider the following problem:

How much logical information Γ can we deduce A from in **LPS***?*

Of course, we can always deduce A from $p_1, \ldots, p_n$ in systems of negationless propositional logics.

Now, we restrict the entity of information to such types as p or $p \to q$. Then, we understand that $\{p\}$ is more (stronger) than $\{q \to p\}$ and also $\{q, q \to p\}$ than $\{p\}$ as information (as assumptions). Can we weaken the assumptions $\{r_1, \ldots, r_u, s_1 \to s_2, \ldots, s_v \to s_1\}$ to $\{r_{n_1}, \ldots, r_{n_i}, r_{m_1} \to r_{m_2}, \ldots, r_{m_j} \to r_{m_1}, s_1 \to s_2, \ldots, s_v \to s_1\}$, where $\{r_{n_1}, \ldots, r_{n_i}, r_{m_1}, \ldots, r_{m_j}\} = \{r_1, \ldots, r_u\}$, $i \geqq 0$ and $i + j = u$, in the lemma? It is impossible, in general. For, take p as A and 0 as the value of p, then we would have $p \to p \vdash p$, which contradicts

[2] Throughout this paper, we use '$\Gamma \vdash A$' to express 'A is deducible from Γ' in a system of logic which is arbitrary or clear from context, or in a system of logic specially noticed.

to the consistency of **LPS**. As a nontrivial counter example, one may take the sentence $r_1 \wedge r_2 \wedge (s_1 \to s_2) \wedge (s_2 \to s_1)$. In this respect, $\{r_1, \ldots, r_u, s_1 \to s_2, \ldots, s_v \to s_1\}$ in the lemma is the least information which deduces A in general. That is; our lemma gives not only a sufficient condition but also a necessary condition to solve 'how much?' of the above problem.

This suggests that one may gain a normal form of the propositions in the positive logic. Indeed, we can obtain a *normal form theorem* in **LQS**, by making use of this fact and the fact that **LQS** is complete in the following sense: every formula A in **LQS** is provable in it if A takes identically the value 0.

As another corollary to the lemma for **LPS** (or **LOS**), we obtain a direct proof[3] of the *completeness* of **LQS** (resp. **LOQS**), by making use of the following[4]: $p \to A,\ (p \to q) \to A \vdash A$ in **LOQS** (*a fortiori*, in **LQS**). This is another proof of Curry's one who established the completeness theorem for **LQS** (in his notation, HC) by reformulating the system in Gentzen's style formalism and using the cut-elimination theorem (see Curry [2], p. 224 also p. 182).

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The proof of the lemma.

We prove the lemma for **LPS** by the induction on the number of logical symbols occurring in A (i.e. on the length of A). It will be shown in such a manner that the proof as to the case for **LOS** is automatically contained as a part. However, the latter can be carried out more easily, since the implication $\to$ is the only logical symbol in **LOS**. For brevity, we may simply write Γ the assumptions $\{r_1, \ldots, r_u, s_1 \to s_2, \ldots, s_v \to s_1\}$ below.

BASIS. There are no logical symbols in A, i.e. A is p_i. Then by the definition of Γ, we see trivially

$$\Gamma \vdash p_i \quad \text{or} \quad \Gamma \vdash p_i \to s_1,$$

according as $v(p_i)$ is 0 or 1.

INDUCTION STEP. There are three cases according to $\to$, $\vee$ or $\wedge$ be the outermost logical symol of A.

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3) After giving the proof, the author found, in Church [1], an exercise which asks to show the completeness of **LOQS** (in his notation, P_B^I) with hint. He uses a selected letter different from $p_1, \ldots, p_n$ and establishes an analogue of our lemma in **LOQS**, that is, by making use of Peirce's law. In this respect, our proof seems more cleancut.

4) See Ono [6], Foctenote 15, p. 341, and also cf. Church [1], **12.8**, p. 86.

Case 1. Let A be $B \to C$.

Subcase 1.1: $v(B \to C)=0$. In this case, $v(C)=0$ or $v(B)=1$. When $v(C)=0$, we have $\Gamma \vdash C$ as the hypothesis of the induction. Then we easily see $\Gamma \vdash B \to C$. When $v(B)=1$, we have

$$\Gamma \vdash B \to s_1$$

as the hypothesis of the induction. Now, let us assume B. Then we have s_1. Hence, we have successively $s_2, \ldots, s_v$, since $s_1 \to s_2, \ldots, s_{v-1} \to s_v$ are in Γ. So, we have $\Gamma, B \vdash p_1, \ldots, p_n$. As is easily seen, $p_1, \ldots, p_n \vdash C$ holds in the positive logics. Thus, it follows that $\Gamma, B \vdash C$ holds. The latter implies $\Gamma \vdash B \to C$ by the deduction theorem.

Subcase 1.2: $v(B \to C)=1$. In this case, $v(B)=0$ and $v(C)=1$. We wish to show $\Gamma \vdash (B \to C) \to s_1$. Assume $B \to C$. By the hypothesis of the induction, we have

$$\Gamma \vdash B \quad \text{and} \quad \Gamma \vdash C \to s_1.$$

Using these succesively, we see $\Gamma, B \to C \vdash s_1$. Therefore, it holds that

$$\Gamma \vdash (B \to C) \to s_1.$$

Case 2. Let A be $B \vee C$.

Subcase 2.1: $v(B \vee C)=0$. In this case, $v(B)=0$ or $v(C)=0$. By the hypothesis of the induction, $\Gamma \vdash B$ or $\Gamma \vdash C$, respectively. In each case, holds $\Gamma \vdash B \vee C$.

Subcase 2.2: $v(B \vee C)=1$. In this case, $v(B)=1$ and $v(C)=1$. Hence, we have simultaneously $\Gamma \vdash B \to s_1$ and $\Gamma \vdash C \to s_1$ as the hypothesis of the induction. Then it holds that

$$\Gamma \vdash B \vee C \to s_1,$$

since $B \vee C \to s_1$ is deducible from the formulae $B \to s_1$, $C \to s_1$.

Case 3. Let A be $B \wedge C$.

Subcase 3.1: $v(B \wedge C)=0$. In this case, $v(B)=0$ and $v(C)=0$. As the hypothesis of the induction, we have $\Gamma \vdash B$ and $\Gamma \vdash C$ simultaneously. Then, it follows immediately that $\Gamma \vdash B \wedge C$.

Subcase 3.2: $v(B \wedge C)=1$. In this case, holds at least one of $v(B)=1$, $v(C)=1$.

Then by the hypothesis of the induction, we see

$$\Gamma \vdash B \rightarrow s_1 \quad \text{or} \quad \Gamma \vdash C \rightarrow s_1.$$

On the other hand, $B \wedge C \vdash B$ and $B \wedge C \vdash C$. Therefore, we have $\Gamma, B \wedge C \vdash s_1$ in any case, and hence $\Gamma \vdash B \wedge C \rightarrow s_1$.

Thus, the proof of the lemma is established.

A normal form of the negationless propositions.

We give a principal normal form of the formulae A in **LQS**. For the purpose, let the letters occurring in A be exactly $p_1, \ldots, p_n$.

In the first place, by the lemma, we see

*) $$\bigvee_{v(A)=0}(r_1 \wedge \ldots \wedge r_u \wedge (s_1 \rightarrow s_2) \wedge \ldots \wedge (s_v \rightarrow s_1)) \rightarrow A$$

is provable in **LPS.**

where $\bigvee_{v(A)=0}$ means the disjunction of all members $r_1 \wedge \ldots \wedge r_u \wedge (s_1 \rightarrow s_2) \wedge \ldots \wedge (s_v \rightarrow s_1)$ depending on the n-tuples of values of $p_1, \ldots, p_n$ for which A takes the value 0. In fact, let A take the value 0 for a given n-tuple $v(p_1), \ldots, v(p_n)$ of values of the letters $p_1, \ldots, p_n$. Then, by the lemma, $r_1, \ldots, r_u, s_1 \rightarrow s_2, \ldots, s_v \rightarrow s_1 \vdash A$, i.e. $r_1 \wedge \ldots \wedge r_u \wedge (s_1 \rightarrow s_2) \wedge \ldots \wedge (s_v \rightarrow s_1) \vdash A$ holds. It follows that $(r_1 \wedge \ldots \wedge r_u \wedge (s_1 \rightarrow s_2) \wedge \ldots \wedge (s_v \rightarrow s_1)) \rightarrow A$ is provable in **LPS.** Hence, we have *).

Coversely, $A \rightarrow \bigvee_{v(A)=0}(r_1 \wedge \ldots \wedge r_u \wedge (s_1 \rightarrow s_2) \wedge \ldots \wedge (s_v \rightarrow s_1))$ takes identically the value 0. For, if A takes the value 0 for any given n-tuple $v(p_1), \ldots, v(p_n)$, then the corresponding disjundctive member $r_1 \wedge \ldots \wedge r_u \wedge (s_1 \rightarrow s_2) \wedge \ldots \wedge (s_v \rightarrow s_1)$ also takes the value 0 by the convention for $\{r_1, \ldots, r_u\}$ and $\{s_1, \ldots, s_v\}$. Hence, by making use of the completeness of **LQS**, this is provable in **LQS.**

Now, we say that A is equivalent to B in **LQS**, if both the formulae $A \rightarrow B$, $B \rightarrow A$ are theorems of **LQS.** We have obtained the following:

THEOREM. *Let A be any formula in* **LQS** *and $p_1, \ldots, p_n$ be the letters occurring in A. Then A is equivalent to the formula*

**) $$\bigvee_{v(A)=0}(r_1 \wedge \ldots \wedge r_n \wedge (s_1 \rightarrow s_2) \wedge \ldots \wedge (s_v \rightarrow s_1))$$

in **LQS.**

Thus, any proposition A in the positive logics is rewritten in the form **) (which can be uniquely determined to within the order of its disjunctive members) in **LQS**, by using only the letters occurring in A. Therefore, the latter is competent for the principal normal form of the negationless propositions.

By the above result, we can say **LQS** is also *notationally complete* in the folowing sense: each of the 2^{2^n-1} possible positive propositional functions of n-variables $p_1, \ldots, p_n$ can be represented by a formula in these letters[5].

Notice: Of course, we cannot express the propositions by so-called *disjunctive* (or *conjunctive*, either) *principal normal form in* **LQS.** On the other hand, our normal form theorem is not true for the classical sentential logic **LKS.** In this sense, the theorem gives a characterization of **LQS.**

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5) See e.g. Kleene [5], p. 135.