

RESIDUE RINGS OF SEMI-PRIMARY HEREDITARY RINGS*

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Introduction: Throughout this paper we assume that all rings contain an identity. We say that R is a semi-primary ring if its (Jacobson) radical N is nilpotent, and R/N is an Artinian ring. We say that R admits a splitting, and we write $R=A+B$ if A is a subring of R , if B is a two-sided ideal in R , and if $A \cap B=0$.

It has been shown in [1] that for a semi-primary ring R $l \cdot gl. \dim R = r \cdot gl. \dim R = 1 + l \cdot \text{proj. dim } N$. This common value is denoted by $gl \cdot \dim R$.

It has been shown in [2] that if R is a semi-primary hereditary ring, and I is a two-sided ideal in R , then $gl \cdot \dim R/I < \infty$.

We prove that if R is a semi-primary ring and $gl \cdot \dim R/N^2 < \infty$, then R is a residue ring of a semi-primary hereditary ring. This is a generalization of a similar result in [3]. The crucial step is a splitting theorem that we prove for a semi-primary ring R , for which $eNe=0$ for any primitive idempotent $e \in R$. This splitting theorem seems also useful in studying certain types of semi-primary subrings of a simple ring.

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§1. A Splitting Theorem.

For the rest of this section, let $R = \sum_{u=1}^t Re_u$ be a complete decomposition for the semi-primary ring R , i.e. e_1, \dots, e_t are primitive orthogonal idempotents (e.g. [4, pp 53–57]). Furthermore, assume $e_v Ne_v = 0$ for $v=1, \dots, t$. When writing e_i, e_j, \dots we always assume $1 \leq i, j, \dots \leq t$, unless otherwise stated.

Since for any $e_i, e_i Ne_i$ is the radical of $e_i Re_i$, and $e_i Re_i / e_i Ne_i$ is a division ring, we have:

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LEMMA 1: $e_i Re_i$ is a division ring for $i=1, \dots, t$. Every element $e_i r e_j \in R$ induces a homomorphism (by right multiplication) of Re_i into Re_j , and vice versa. In particular, if Re_i is isomorphic to Re_j , then $e_k N e_i \neq 0$ iff $e_k N e_j \neq 0$ for any $k, 1 \leq k < t$. Thus:

LEMMA 2: Let Re_i be isomorphic to Re_j , then $e_i N e_j = 0$.

One easily verifies that this is equivalent to:

LEMMA 2*: Every non-zero homomorphism between isomorphic components is an isomorphism.

Let Γ_0 be $\sum_{i,j} e_i Re_j$ where (i, j) ranges over all pairs such that Re_i and Re_j are isomorphic to Re_{i_0} for some fixed i_0 . Let $R_0 = \text{Hom}_R(\sum_k Re_k, \sum_k Re_k)$, where k ranges over all indices such that Re_k is isomorphic to Re_{i_0} . Let $R_1 = \text{Hom}_R(\sum_{i=1}^s B_i, \sum_{i=1}^s B_i)$, where $B_i = Re_{i_0}$ for $i=1, \dots, s$, and s is the number of components in the complete decomposition for R which are isomorphic to Re_{i_0} . Finally, let $\Gamma_1 = (e_{i_0} Re_{i_0})_s$ -the $s \times s$ matrix algebra over the division ring $e_{i_0} Re_{i_0}$. With these notations we have:

LEMMA 3: The subring Γ_0 of R is a simple ring.

Proof: It is clear that R_1 and R_0 are isomorphic. It is also clear that $\Gamma_0(\Gamma_1)$ is anti-isomorphic to $R_0(R_1)$. Thus Γ_0 and Γ_1 are isomorphic.

Let Γ be $\sum_{i,j} e_i Re_j$, where (i, j) ranges over all pairs such that Re_i is isomorphic to Re_j . Since on Γ we have a natural splitting, into subsums taken over any fixed isomorphism class of components, it follows from Lemma 3 that:

PROPOSITION 1: The subring Γ of R is a semi-simple ring.

The underlying additive group of R admits a decomposition $R = \sum_{i,j=1}^t e_i Re_j$. Let $R_1 = \sum_{i,j} e_i Re_j$ where (i, j) ranges over all pairs such that Re_i is not isomorphic to Re_j . We have $R = \Gamma + R_1$, and it is clear that $R_1 \subset N$. Our next step is to show that $R_1 = N$. We will be done once we show that R_1 is a two-sided ideal in R . Since $r = \sum_{i,j=1}^t e_i r e_j$ for any $r \in R$, and since R_1 is closed under addition, it suffices to show that $e_i r e_j \in R_1$, implies $e_i r e_j s e_k \in R_1$ and $e_l v e_i r e_j \in R_1$ for all $1 \leq i, j, k, l \leq t$ and $r, s, v \in R$. But $e_i r e_j s e_k \notin R_1$ only if Re_i is isomorphic to Re_k , whence by Lemma 2* this element induces an isomorphism of Re_i onto Re_k , and this is impossible since $e_i r e_j \in R_1 \subset N$. A similar argument shows that $e_l v e_i r e_j \in R_1$.

This proves:

THEOREM 1. THE SPLITTING THEOREM: *Let R be a semi-primary ring, and let $R = \sum_{u=1}^t Re_u$ be a complete decomposition for R . If $e_i Ne_i = 0$ for $i = 1, \dots, t$, then R admits a splitting $R = \Gamma + N$. $\Gamma = \sum_{i,j} e_i Re_j$ where (i, j) ranges over all pairs such that Re_i is isomorphic to Re_j . $N = \sum_{i,j} e_i Re_j$ where (i, j) ranges over all pairs such that Re_i is not isomorphic to Re_j .*

With the assumptions and notations of Theorem 1, using Lemma 1 one can easily prove that the center of R is a direct product of fields. The center of R is a field only if 0 and 1 are the unique central idempotents in R . One can also show that if $R = \Gamma_1 + N$ is another splitting for R , then there exists an invertible element s in R such that the automorphism $r \rightarrow srs^{-1}$ takes Γ onto Γ_1 .

The splitting theorem enables us to view N as a Γ - Γ bimodule. Define $\Omega(\Gamma, N) = \sum_{i=0}^{\infty} N^{(i)}$, where $N^{(0)} = \Gamma$ and $N^{(i)} = N^{(i-1)} \otimes_{\Gamma} N$. Letting $n_1 \otimes \dots \otimes n_i \otimes n^1 \otimes \dots \otimes n^j = n_1 \otimes \dots \otimes n_i \otimes n^1 \otimes \dots \otimes n^j$ and extending \otimes distributively, $\Omega(\Gamma, N)$ becomes a ring (identifying $N^{(i)} \otimes_{\Gamma} \Gamma$, $\Gamma \otimes_{\Gamma} N^{(i)}$ and $N^{(i)}$ for $i \geq 0$). Letting $f(n_1 \otimes \dots \otimes n_k) = n_1 \dots \cdot n_k$ and extending f linearly, f is a ring epimorphism from $\Omega(\Gamma, N)$ onto R . If for some m , $N^{(m)} = 0$ then $M = \sum_{i=1}^{m-1} N^{(i)}$ is a nilpotent two-sided ideal and $\Omega(\Gamma, N)/M$ is semi-simple. Thus $\Omega(\Gamma, N)$ is a semi-primary ring with radical M . Furthermore, $M = \Omega(\Gamma, N) \otimes_{\Gamma} N$, and since N is Γ -projective, M is $\Omega(\Gamma, N)$ -projective. By [1], this implies that $\Omega(\Gamma, N)$ is an hereditary ring.

If E_0, \dots, E_k are primitive idempotents in R , then (E_0, \dots, E_k) is an R -connected sequence of length k if $E_i NE_{i+1} \neq 0$ for $i = 0, \dots, k-1$. It is obvious that $N^{(m)} = 0$ if there are no R -connected sequences of length m .

2. Applications. We first deal with the case $gl.\dim R/N^2 < \infty$. Thus let R be a semi-primary ring and $gl.\dim R/N^2 < \infty$. Let $\tilde{R} = R/N^2, \tilde{N} = N/N^2$. With the notations of section 1 we have that $\tilde{R} = \sum_{u=1}^t \tilde{R} \tilde{e}_u$ is a complete decomposition for \tilde{R} , where \tilde{e}_i is the canonical image of e_i in \tilde{R} for $i = 1, \dots, t$. By a result in [3] concerning semi-primary rings for which the square of the radical is zero, we conclude that \tilde{R} -connected sequences are bounded in length. This implies:

LEMMA 4: *R -connected sequences are bounded in length.*

Proof: We show that if $e_i Ne_j \neq 0$ then there exists an \tilde{R} -connected sequence

of the form $(\tilde{e}_i, \dots, \tilde{e}_j)$. If $(\tilde{e}_i, \tilde{e}_j)$ is \tilde{R} -connected we are done. Otherwise, $e_i N e_j \in N^2$ and there readily follows the existence of a primitive idempotent e_k such that $e_i N e_k N e_j \neq 0$. If $(\tilde{e}_i, \tilde{e}_k, \tilde{e}_j)$ is \tilde{R} -connected we are done. Otherwise, either $e_i N e_k \in N^2$ or $e_k N e_j \in N^2$. Let $e_i N e_k \in N^2$, then we can find a primitive idempotent e_t such that $0 \neq e_i N e_t N e_k N e_j \in N^3$. Since N is nilpotent, this procedure must end and the result follows.

In particular, we must have $e_i N e_i = 0$ for $i = 1, \dots, t$, thus by Theorem 1, $R = \Gamma + N$. The ring $\Omega(\Gamma, N)$ as constructed at the end of § 1 is a semi-primary hereditary ring in this case. Combining this with the result in [2] concerning residue rings of semi-primary hereditary rings we have:

THEOREM 2. *Let R be a semi-primary ring, then the following are equivalent:*

- (a) *R is a residue ring of a semi-primary hereditary ring.*
- (b) *All residue rings of R have finite global dimension.*
- (c) *$gl \cdot dim R/N^2 < \infty$.*

Remark that under each of these equivalent conditions $eNe = 0$ for any primitive idempotent $e \in R$.

In particular, if R is a semi-primary hereditary ring, its center is a direct product of fields. The center of R is a field only if 0 and 1 are the unique central idempotents in R .

For the rest, let D be a division ring and let D_n denote the $n \times n$ matrix algebra over D . Let R be a semi-primary subring of D_n , such that $R = \sum_{i=1}^n R e_i$ is a complete decomposition for R . Without loss of generality we may assume that e_i is the matrix whose $(\alpha, \beta)^{th}$ component is $(e_i)_{\alpha\beta} = \delta_{i\alpha} \delta_{i\beta}$ for all $i, \alpha, \beta = 1, \dots, n$. We can (naturally) identify $e_i D_n e_i$ with D , and $e_i R e_i$ with a subring of D , for $i = 1, \dots, n$. In particular $e_i N e_i = 0$ for $i = 1, \dots, n$, and by Theorem 1 $R = \Gamma + N$. We want to show now that $\Omega(\Gamma, N)$ is a semi-primary hereditary ring. This follows from the fact that any element $e_i r e_j \in R$ induces an isomorphism from $D_n e_i$ onto $D_n e_j$. Thus in particular $e_i r e_j \neq 0$ and $e_k s e_i \neq 0$ imply $e_k s e_i r e_j \neq 0$, or $e_i N e_j \neq 0$ and $e_j N e_k \neq 0$ imply $e_i N e_j N e_k \neq 0$. Since N is nilpotent this implies that R -connected sequences are bounded in length. Thus we proved:

THEOREM 3. *Let R be a semi-primary subring of D_n , containing n orthogonal idempotents, then $gl \cdot dim R/I < \infty$ for any two-sided ideal I in R .*

Let R be a semi-primary subring of D_n . Let $C(R)$ be the subset of D_n

consisting of elements $V \in D_n$ for which $Vr = rV$ for all $r \in R$. Set $C(D_n)$ to be the center of D_n . One can show that $C(R) = C(D_n)$ implies that (a) 0 and 1 are the unique central idempotents in R and (b) R contains n orthogonal idempotents. If D is a field one easily verifies that (a) and (b) imply $C(R) = C(D_n)$.

REFERENCES

- [1] M. Auslander-On the Dimension of Modules and Algebras III, Global Dimension, Nagoya Math. J., **9**(1955), pp. 67-77.
- [2] S. Eilenberg, H. Nagao and T. Nakoyama-On the Dimension of Modules and Algebras IV, Dimension of Residue Rings of Hereditary Rings, Nagoya Math. J., **10** (1956), pp 87-95
- [3] J.P. Jans and T. Nakayama-On the Dimension of Modules and Algebras VIII, Algebras with Finite Dimensional Residue Algebras, Nagoya Math. J., **11** (1957), pp 67-76.
- [4] N. Jacobson-Structure of Rings, Amer. Math. Soc. Col. Pub., vol. **37** (1956).

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