SPIRAL ASYMPTOTIC VALUES OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK

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1. Introduction

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Alice Roth has made an extensive study of entire meromorphic functions with prescribed behavior along half rays emanating from the origin (6). The question arose whether analogous results could be found for functions meromorphic in the unit disk with the same behavior prescribed along an exhaustive class of spirals emanating from the origin. In this paper, I present a class of spirals which satisfactorily fills this role. However, I make no claim to the effect that only this class will suffice.

2. The main results

The circle with origin as center and radius k in the complex plane will be denoted by $C(k)$ throughout this paper. Similarly, the open disk with origin as center and radius k will be denoted by $D(k)$.

(2. 1) Definition. For each value θ, $0 \leq \theta < 2\pi$, define $S_\theta = \left\{ z = r \exp\left[i\left(\theta + \tan\frac{\pi r}{2}\right)\right], \ 0 \leq r < 1 \right\}$. Further, define $S_{-\theta} = S_{2\pi-\theta}$ for each θ, $0 \leq \theta \leq 2\pi$. These spirals will be called Study spirals (7, p. 45).

Notice that each Study spiral originates at the origin and its argument tends monotonically to $+\infty$ as r tends to 1. Also if $\theta_1 \neq \theta_2$, then S_{θ_1} and S_{θ_2} have only the origin in common.

(2. 2) Definition. Let $\mathcal{S}$ represent the class of functions meromorphic and non-constant in $D(1)$ and which tend to a definite limit, finite or infinite, on each Study spiral as r tends to 1. (Theorem 2.15 asserts that this class is rather large.) For each function $F(z)$ define the spiral limit value function $f(\theta)$, $0 \leq \theta < 2\pi$, associated with $F(z)$ as follows:

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$$f(\theta)=\lim_{r\to 1} F\left(r\exp\left[i\left(\theta+\tan\frac{\pi r}{2}\right)\right]\right).$$

(2.3) DEFINITION. For $-2\pi\leq\theta_1<\theta_2\leq 2\pi$ and $\theta_2-\theta_1\leq 2\pi$ I shall call the region bounded by $S_{\theta_1}, S_{\theta_2}$ and $C(1)$ and containing S_{θ_3} where $\theta_3=\theta_1+\frac{\theta_2-\theta_1}{2}$ a spiral wedge and I shall denote this region by $W(\theta_1, \theta_2)$. Let $\bar{W}(\theta_1, \theta_2)$ denote the union of $W(\theta_1, \theta_2)$, S_{θ_1} and S_{θ_2} and be called a closed spiral wedge.

(2.4) LEMMA 1. *Let $F(z)\in\mathscr{S}$. In each closed region $\bar{W}(\alpha_0, \beta_0)$ there is a closed region $\bar{W}(\alpha, \beta)$, where $\alpha_0<\alpha<\beta<\beta_0$, such that either $|F(z)|\leq 2$ or $|F(z)|\geq 1$ for all sufficiently large $|z|$, where $z\in\bar{W}(\alpha, \beta)$.*

Proof. Assume the conclusion does not hold. Then there exists $1>|z_1|=r_1>1/2$ such that $z_1\in W(\alpha_0, \beta_0)$ and $|F(z_1)|<1$. Since $F(z)$ is continuous at z_1, there is an entire arc on $|z|=r_1$ such that $|F(z)|<1$ for all z on this arc. Hence there are values α_1 and β_1 such that $\alpha_0<\alpha_1<\beta_1<\beta_0$ and for $z\in W(\alpha_1, \beta_1)$ and $|z|=r_1$, $|F(z)|<1$.

Also there exists $1>|z_2|=r_2>3/4$ such that $z_2\in W(\alpha_1, \beta_1)$ and $|F(z_2)|>2$. Since $1/F(z)$ is continuous at z_2, there is an entire arc of $|z|=r_2$ on which $|F(z)|>2$. So there are values α_2, β_2 such that $\alpha_1<\alpha_2<\beta_2<\beta_1$ and for all z where $z=r_2\exp\left[i\left(\theta+\tan\frac{\pi r_2}{2}\right)\right]$ and $\alpha_2<\theta<\beta_2$, $|F(z)|>2$.

Continuing on in this way for each $n=1, 2, \ldots\ldots$ the argument reads as follows:

For $|z|>2n/2n+2$, there exists $z_{2n-1}=r_{2n-1}\exp\left[i\left(\theta_{2n-1}+\tan\frac{\pi r_{2n-1}}{2}\right)\right]$ where $\alpha_{2n-2}<\theta_{2n-1}<\beta_{2n-2}$ and $1>r_{2n-1}>\frac{2n}{2n+2}$ such that $|F(z_{2n-1})|<1$. Because $F(z)$ is continuous at z_{2n-1}, there exists an entire subarc of $|z|=r_{2n-1}$ containing the points $z=r_{2n-1}\exp\left[i\left(\theta+\tan\frac{\pi r_{2n-1}}{2}\right)\right]$ with $\alpha_{2n-1}<\theta<\beta_{2n-1}$ where $\alpha_{2n-2}<\alpha_{2n-1}<\beta_{2n-1}<\beta_{2n-2}$ on which $|F(z)|<1$.

Also there exists $z_{2n}=r_{2n}\exp\left[i\left(\theta_{2n}+\tan\frac{\pi r_{2n}}{2}\right)\right]$ where $1>r_{2n}>\frac{2n+1}{2n+2}$ and $\alpha_{2n-1}<\theta_{2n}<\beta_{2n-1}$ and for which $|F(z_{2n})|>2$. Since $1/F(z)$ is continuous at z_{2n}, there is an entire subarc of $|z|=r_{2n}$ containing the points $z=r_{2n}\exp\left[i\left(\theta+\tan\frac{\pi r_{2n}}{2}\right)\right]$ with $\alpha_{2n}<\theta<\beta_{2n}$ where $\alpha_{2n-1}<\alpha_{2n}<\beta_{2n}<\beta_{2n-1}$ on which $|F(z)|>2$.

So I have defined the following sequence of open, non-empty, nested intervals (α_1, β_1), (α_2, β_2), ...where $\alpha_{k+1}, \beta_{k+1} \in (\alpha_k, \beta_k)$ $k=0, 1, 2, \ldots\ldots$ Therefore $\bigcap_{n=1}^{\infty} (\alpha_n, \beta_n)$ is not empty. Let γ be in this set. Then for this γ and for all n, I have both $\left|F\left(r_{2n-1} \exp\left[i\left(\gamma+\tan\frac{\pi r_{2n-1}}{2}\right)\right]\right)\right| < 1$ and $\left|F\left(r_{2n} \exp\left[i\left(\gamma+\tan\frac{\pi r_{2n}}{2}\right)\right]\right)\right| > 2$. Since $\lim_{n\to\infty} r_{2n-1} = \lim_{n\to\infty} r_{2n} = 1$, this implies that no limit exists for $F(z)$ as $|z|$ tends to 1 along S_γ, contrary to my assumption. So the proposition must be true.

(2. 5) DEFINITION. A spiral wedge $W(\alpha, \beta)$ is called a wedge of convergence for $F(z) \in \mathscr{S}$ if either a finite constant c exists so that $\lim_{r\to 1}\left\{F\left(r \exp\left[i\left(\theta+\tan\frac{\pi r}{2}\right)\right]\right)-c\right\}=0$ uniformly for $\alpha \leq \theta \leq \beta$ or if $\lim_{r\to 1} \frac{1}{F\left(r \exp\left[i\left(\theta+\tan\frac{\pi r}{2}\right)\right]\right)} = 0$ uniformly for $\alpha \leq \theta \leq \beta$.

The range of $f(z)$, denoted by $R(f)$, is the set of those values assumed by $f(z)$ at points in $D(1)$ arbitrarily near $C(1)$.

(2. 6) LEMMA 2. *Let* $F(z) \in \mathscr{S}$ *and define* $F_w(z)$ *to be* $F(z)$ *restricted to the closed spiral wedge* $\bar{W}(\alpha, \beta)$. *If* $\mathscr{C}\, R(F_w)$ (*i.e., the complement of the range of* F *restricted to this same wedge*) *contains more than two elements, then* $W(\alpha, \beta)$ *is a wedge of convergence for* $F(z)$.

Proof. The region $W(\alpha, \beta)$ is a simply connected region which possesses $C(1)$ as a single prime end. Using Carathéodory's classical theorem on prime ends (3), there is a conformal mapping from $W(\alpha, \beta)$ onto $D'(1) = |\zeta| < 1$ such that the points of S_α and S_β are mapped conformally onto $C'(1) = |\zeta| = 1$ with the exception of $\zeta = 1$. The prime end $C(1)$ corresponds to the point $\zeta = 1$. Let $\zeta = \zeta_{(z)}$ represent this mapping and let $z = z(\zeta)$ be its inverse.

Consider the composition $w = F^*(\zeta) = F(z(\zeta))$ defined for all $|\zeta| \leq 1$ with the exception of $\zeta = 1$. $\mathscr{C} R(F^*(\zeta))$ contains more than two elements by hypothesis. That $F^*(\zeta)$ represents a normal function in $D'(1)$ follows from a corollary in Noshiro's book (5, p. 89) which states that any meromorphic function defined in $D'(1)$ omitting three values is normal. (If $F(z)$ does not omit three values in $W(\alpha, \beta)$, it does for all $|z| > R$ for some $R < 1$ and $z \in W(\alpha, \beta)$.) Then map this region onto $D'(1)$ in the same way.)

I now refer the reader to a result of O. Lehto and K.I. Virtanen (4, p. 53, Theorem 2) which states that a normal meromorphic function in a Jordan region G having an asymptotic value α at a boundary point P along a Jordan curve lying in the closure of G possesses an angular limit α at the point P. I note here that the proof of this theorem yields the result that if the asymptotic path Γ lies on the boundary of G, the function not only possesses the limit α in every angle A, but it also tends uniformly to α in the part of G lying between A and the path Γ.

Let $\Gamma_\beta, \Gamma_\alpha$ be the images of S_β, S_α respectively on $C'(1)$. Then assuming $F(z)$ has the asymptotic values α' and β' along the spirals S_α and S_β respectively $F^*(\zeta)$ has the asymptotic values α' and β' along the paths Γ_α and Γ_β respectively. Then by the above result of Lehto and Virtanan, $F^*(\zeta)$ has both the angular limit α' and β' at $\zeta=1$. Thus α' must equal β' and by the note I made above $F^*(\zeta)$ must tend uniformly to α' as ζ tends to 1 in all of $\overline{D'(1)}-\{1\}$.

Hence for $\varepsilon>0$ there exists a $\delta>0$ such that for all ζ satisfying $0<|\zeta-1|<\delta$ and $\zeta\in\overline{D'(1)}$, $|F^*(\zeta)-\alpha'|<\varepsilon$. Consider $|\zeta-1|=\delta$. This circle cuts Γ_α and Γ_β and the part of this circle in $D'(1)$ together with the initial parts of Γ_α and Γ_β, up to the points of intersection, forms a closed Jordan curve in $\overline{D'(1)}$. The image of this closed Jordan curve under $z=z(\zeta)$ is a closed Jordan curve in $D(1)$. This image has a positive distance d from $C(1)$. Then for $z\in\overline{W}(\alpha,\beta)$ and $|z|>1-d$, $|F(z)-\alpha'|=|F(z(\zeta))-\alpha'|$ (where $0<|\zeta-1|<\delta$) $=|F^*(\zeta)-\alpha'|<\varepsilon$. Therefore $W(\alpha,\beta)$ is a wedge of convergence for $F(z)$ as was to be proved.

(2. 7) Definition. For $f(z)\in\mathscr{S}$, if there exists a θ such that for every $\varepsilon>0$ $F(z)$ assumes all values (including ∞) with the possible exception of at most two infinitely often in $W(\theta-\varepsilon,\theta+\varepsilon)$, S_θ will be called a Julia spiral.

(2. 8) Definition. Let $D(F)$ represent the set of those points $1/2\exp[i(\theta+\tan\pi/4)]$ on $C(1/2)$ which are cut by Julia spirals of the function $F(z)\in\mathscr{S}$.

(2. 9) Theorem. *If* $F(z)\in\mathscr{S}$, *then* $D(F)$ *is a nowhere dense closed set. Let the function* $g(\theta)$ *associate the spiral limit value* $f(\theta)$ *with each point* $1/2\exp[i(\theta+\tan\pi/4)]$ *of* $C(1/2)$. *Then this correspondence is a function of Baire class* 0 *or* 1, *whose intervals of constancy lie dense on* $C(1/2)$; *this function is constant at least on each arc of* $C(1/2)$ *in* $\mathscr{C}D(F)$. *In every closed spiral wedge which contains no point of the set* $D(F)$, $F(z)$ *tends uniformly towards the corresponding constant spiral limit value.*

Proof. In each closed disk $\overline{D(k)}$ where $0<k<1$, $F(z)$ can have at most finitely many poles. If this were not true it would have a limit point of poles in $D(1)$, and so not be meromorphic there. So there is an increasing sequence of positive numbers $\{r_n\}$ with $\lim_{n\to\infty} r_n=1$ such that, for every n, $F(z)$ has no poles on $|z|=r_n$. Thus $f(\theta)$ is the limit function of the sequence of continuous functions $f_n(\theta)=F\left(r_n \exp\left[i\left(\theta+\tan\frac{\pi r_n}{2}\right)\right]\right)$ $0\leq\theta<2\pi$.

I claim next that every spiral wedge contains a spiral wedge of convergence. Lemma 1 (2. 4) allows me to conclude that each spiral wedge contains a closed spiral wedge $\bar{W}(\alpha,\beta)$ in which either $|F(z)|\leq 2$ or $1/|F(z)|\leq 1$ for sufficiently large values of $|z|$ (i.e., $R<|z|<1$). For $z\in\bar{W}(\alpha,\beta)$ and $|z|\leq R$, $F(z)$ cannot assume any value more than a finite number of times, since $F(z)$ is not constant. For $z\in\bar{W}(\alpha,\beta)$ and $R<|z|<1$ $|F(z)|\leq 2$ (or $1/|F(z)|\leq 1$), and so the complement of the range of $F(z)$ (or the complement of the range of $1/F(z)$) restricted to $\bar{W}(\alpha,\beta)$ contains many more than two elements. Therefore, by Lemma 2 (2.6), $W(\alpha,\beta)$ is a wedge of convergence for $F(z)$ (or $1/F(z)$).

Thus it follows that wedges of convergence are everywhere dense in $D(1)$ and the intersection of the union of all wedges of convergence with $C(1/2)$ is a set everywhere dense on $C(1/2)$. The function g is constant on the intersection of each wedge of convergence with $C(1/2)$ and so the intervals of constancy of g are dense on $C(1/2)$.

Lemma 2 also allows me to conclude that if S_θ is contained in no wedge of convergence of $F(z)$, then $F(z)$ assumes every value (including ∞) with at most two exceptions infinitely often in $W(\theta-\varepsilon,\theta+\varepsilon)$ for each $\varepsilon>0$. Hence S_θ is a Julia spiral.

Conversely, no Julia spiral is contained in a wedge of convergence. If this were not true and W was a wedge of convergence containing a Julia spiral, W would have to contain an infinite number of a-points (z_0 is an a-point of $F(z)$ if $F(z_0)=a$) for all complex numbers a with at most two exceptions. Since W is a wedge of convergence there exists a constant α such that for every $\varepsilon>0$ there is a $0<R(\varepsilon)<1$ such that if $z\in\bar{W}$ and $|z|>R(\varepsilon)$, $|F(z)-\alpha|<\varepsilon$. Let a be a complex number which is not one of the two exceptional points of $F(z)$ and $|a|>|\alpha|+\varepsilon$. Then $F(z)$ must have an infinite number of a-points in $\bar{W}\cap\overline{D(R(\varepsilon))}$ and this implies $F(z)\equiv a$. Since this contradicts the fact that $F(z)\in\mathscr{S}$, I may conclude that no Julia spiral is contained in a wedge of convergence. So the set of Julia spirals is identical with the set of Study spirals that are contained in no wedge of convergence.

The set $D(F)$ is closed since it must contain all its limit points. Also, since $D(F)$ is the complement of an everywhere dense set, it is nowhere dense on

$C(1/2)$ and the theorem is proved.

It is natural to ask whether the functions of class $\mathcal{S}$ are characterized completely by the properties I have just shown for their spiral limit value function. In other words, suppose a spiral limit value function is given having the properties stated in the preceding theorem (2.9). Can a function of class $\mathcal{S}$ then be constructed that has this given function as its associated spiral limit value function? Not only is the answer to this question in the affirmative, but the desired function can even be constructed so as to be holomorphic. It is this construction that is the concern of the remainder of this section. The following two theorems are essential to constructions I will make later on. I will list them here and refer to them as needed.

(2. 10) THEOREM (Mergelyan). *If E is a closed bounded set not separating the plane, and if $f(z)$ is continuous on E and analytic in the interior points of E, then $f(z)$ can be uniformly approximated on E as closely as desired by a polynomial in z.* (8, p. 367)

(2. 11) THEOREM (Walsh). *Let the function $f(z)$ be given on the closed limited (i.e., bounded) point set C, and let distinct points $z_1, z_2, \ldots\ldots, z_\nu$ be given on C. If the function $f(z)$ can be uniformly approximated on C as closely as desired by a polynomial in z, then the function can be uniformly approximated on C as closely as desired by a polynomial $p(z)$ satisfying the auxiliary conditions*

$$p(z_k)=f(z_k) \qquad k=1, 2, \ldots\ldots, \nu .$$

(8, p. 310).

(2. 12) DEFINITION. A sequence of distinct Jordan curves $\{J_n\}_1^\infty$ shall be called increasing if J_n lies in the interior of J_{n+1} for $n=1, 2, \ldots\ldots$ I shall say that such a sequence converges to the circle $C(k)$ if the sequence is contained in $D(k)$ and for any $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)$ such that for every $n>n_0$, J_n lies in the region $k-\varepsilon<|z|<k$. If $\{J_n\}_1^\infty$ is an increasing sequence I shall define D_n to be the closed region bounded by and including J_n for $n=1, 2, \ldots\ldots$ and $D_0=\phi$. Then I define $A_n=\overline{D_{n+1}-D_n}$ for $n=0, 1, 2, \ldots\ldots$

(2. 13) THEOREM. *Let $\{G_n\}_1^\infty$ be a sequence of open, disjoint, connected sets in $D(1)$ such that G_n contains the closed (i.e., closed in $D(1)$) set M_n for each $n=1, 2,\ldots\ldots$ Let $f_n(z)$ be defined and holomorphic for $z\in G_n$ for each $n=1, 2, \ldots\ldots$ Given $\varepsilon>0$, the following conditions are sufficient for the existence of a function $F(z)$ holomorphic in $D(1)$ and satisfying*

$$|F(z)-f_n(z)|<\varepsilon \qquad \text{for } z\in M_n, \quad n=1,2,\ldots\ldots$$

1. *There exists a sequence of closed sets* $\{B_n\}_1^\infty$ *having no cluster point* in* $D(1)$ *such that for each* n, M_n *is contained in the interior of* B_n *and* B_n *is contained in* G_n.

2. *There exists an increasing sequence of distinct Jordan curves* $\{J_n\}_1^\infty$ *in* $D(1)$ *converging to* $C(1)$ *such that if* b_n *is the boundary of* B_n,

a) $J_m\cap b_n$ *is a finite point set for each* m *and* n.

b) $A_m\cap b_n$ *is a finite number of arcs, each with a finite length, for each* m *and* n.

c) $D_n\cup(A_n\cap\{\bigcup_{k=1}^{\infty} b_k\})$ *does not separate the plane for* $n=0, 1, \ldots\ldots$

Proof. Since M_n is contained in the interior of B_n for each n, $b_n\cap M_n=\phi$ for each n. For $k\neq n$, $B_k\cap B_n=\phi$ since $G_k\cap G_n=\phi$ and so $b_k\cap M_n=\phi$. Therefore the sets $\bigcup_{k=1}^{\infty} b_k$ and $\bigcup_{k=1}^{\infty} M_k$ have no point in common. Moreover, both of these sets are closed in $D(1)$ since each term is closed and neither sequence has a cluster point in $D(1)$.

Let $p_n=\left(\bigcup_{k=1}^{\infty} b_k\right)\cap A_n$ for $n=0, 1, \ldots\ldots$. p_n is closed and so has a positive distance S_n from $\bigcup_{k=1}^{\infty} M_k$, $n=0, 1, 2, \ldots\ldots$. Since $\{B_n\}$ has no cluster point in $D(1)$, at most a finite number of terms of this sequence can have a non-empty intersection with A_n for $n=0, 1, 2, \ldots\ldots$. Therefore, since $A_n\cap b_m$ is a finite number of arcs each with a finite length for each m, p_n has a finite length which I shall designate by l_n for $n=0, 1, 2, \ldots\ldots$.

For each $S_n>0$ and $l_n\geq 0$ I define a positive number ε_n so small that

$$\sum_{n=0}^{\infty}\frac{\varepsilon_n\cdot l_n}{S_n}<2\pi\varepsilon \tag{1}$$

and

$$\sum_{n=0}^{\infty}\varepsilon_n\cdot l_n<1. \tag{2}$$

Since $b_n\subset G_n$ for each n, $f_n(z)$ is continuous on b_n for each n. Moreover, since the terms of $\{b_n\}$ are pairwise disjoint and the sequence has no cluster point in $D(1)$, a continuous function $f(z)$ can be defined on $\bigcup_{k=1}^{\infty} b_k$ by the correspondence $f(z)=f_n(z)$ for z on b_n $n=1, 2, \ldots\ldots$.

* A point is a cluster point of a sequence $\{B_n\}$ if every one of its neighborhoods contains a point from an infinite number of terms of the sequence.

Let $\varepsilon(r)$ be a positive continuous function for $0 \leq r < 1$ such that $\lim_{r\to 1} \varepsilon(r)=0$. I shall now construct a function $F_1(z)$ holomorphic in $D(1)$ such that for $z \in \bigcup_{k=1}^{\infty} b_k$ $|F_1(z)-f(z)| < \varepsilon(|z|)$.

To this end let ε'_0, ε'_1, ... be positive numbers such that $\varepsilon'_n > \varepsilon'_{n+1}$ for all n and $\varepsilon'_n < \varepsilon(r)$ for $r \leq \max_{z\in J_{n+1}} |z|$ for $n=0, 1, 2, \ldots\ldots$ Let $\eta_n = \varepsilon'_n - \varepsilon'_{n+1}$ so that η_0, η_1, ... are positive numbers and $\sum_{\nu=n}^{\infty} \eta_\nu = \varepsilon'_n \quad n=0, 1, 2, \ldots\ldots$.

As has already been pointed out, at most a finite number of terms in the sequence $\{B_n\}$ have a non-empty intersection with $J_{\nu+1}$ for each ν. Since $b_k \cap J_{\nu+1}$ is a finite point set for each k, $\left(\bigcup_{k=1}^{\infty} b_k\right) \cap J_{\nu+1}$ contains at most a finite number of points which I shall designate as $z_1^{(\nu)}$, $z_2^{(\nu)}$, ..., $z_{n(\nu)}^{(\nu)}$. By Condition 2c in the hypothesis, for $\nu=0, 1, 2, \ldots\ldots$, $D_\nu \cup p_\nu$ is a closed bounded set that does not separate the plane.

Now, by induction, for $\nu=0$, $D_0 \cup p_0$ satisfies the condition of the theorems in Sections (2. 10) and (2. 11) and so I may conclude that there is a polynomial $\pi_0(z)$ such that $|f(z)-\pi_0(z)| < \eta_0$ for $z \in D_0 \cup p_0$ and $f(z_k^0)=\pi(z_k^0)$ for $k=1, 2, \ldots\ldots$, $n(0)$.

Assume that $\nu > 0$ and polynomials $\pi_0(z)$, ... , $\pi_{\nu-1}(z)$ have been defined so that $\pi_{\nu-1}(z_k^{\nu-1}) = f(z_k^{\nu-1}) - \pi_0(z_k^{\nu-1}) - \cdots\cdots - \pi_{\nu-2}(z_k^{\nu-1})$ for $k=1, 2, \ldots\ldots$, $n(\nu-1)$. Then the theorems in Sections (2. 10) and (2. 11) assert the existence of a polynomial $\pi_\nu(z)$ so that $|\pi_\nu(z)| < \eta_\nu$ for $z \in D_\nu$, $|f(z)-[\pi_0(z)+\pi_1(z)+\cdots\cdots+\pi_\nu(z)]| < \eta_\nu$ for $z \in p_\nu$, and $\pi_\nu(z_k^\nu) = f(z_k^\nu) - \sum_{n=0}^{\nu-1} \pi_n(z_k^\nu)$, $k=1, \ldots\ldots, n(\nu)$. Now define $F_1(z) = \sum_{\nu=0}^{\infty} \pi_\nu(z)$.

For $z \in D_k$, since $D_k \subset D_{k+n}$ for all $n \geq 1$, $|\pi_\nu(z)| < \eta_\nu$ for $\nu=k, k+1, \ldots\ldots$. Also there are constants $\mathcal{M}_\nu > 0$ such that $|\pi_\nu(z)| < \mathcal{M}_\nu$ for $\nu=1, 2, \ldots\ldots, k-1$ because $\pi_\nu(z)$ is a polynomial and D_k is compact. Since $\sum_{\nu=k}^{\infty} \eta_\nu$ converges, by the Weierstrass M-test, $\sum_{\nu=0}^{\infty} \pi_\nu(z)$ is uniformly convergent on D_k. But every compact subset of $D(1)$ is contained in D_k for some k. Therefore $F_1(z) = \sum_{\nu=0}^{\infty} \pi_\nu(z)$ converges uniformly on every compact subset of $D(1)$ and so represents a holomorphic function in $D(1)$.

Moreover, for $z \in p_k$, $|f(z)-F_1(z)| = |f(z) - \sum_{\nu=0}^{\infty} \pi_\nu(z)| \leq |f(z) - \sum_{\nu=0}^{k} \pi_\nu(z)| + \sum_{\nu=k+1}^{\infty} |\pi_\nu(z)| < \eta_k + \sum_{\nu=k+1}^{\infty} \eta_\nu$ since $p_k \subset D_{k+n}$ for $n \geq 1$. But $\sum_{\nu=k}^{\infty} \eta_\nu < \varepsilon'_k < \varepsilon(r)$ for $r \leq \max_{z\in J_{k+1}} |z|$ and so in particular for all $z \in p_k$. Since k is arbitrary, for any $z \in \bigcup_{k=0}^{\infty} p_k = \bigcup_{n=1}^{\infty} b_n$, $|f(z) - F_1(z)| < \varepsilon(|z|)$ and the construction is finished.

Using this function $F_1(z)$ and the construction employed by Roth (6, pp. 107–109), the desired function $F(z)$ results.

(2. 14) Corollary. *If each of the regions G_n in* (2. 13) *can be mapped conformally onto a subset of $|z'| < 1$ by a function that has a continuous extension onto the boundary of G_n and also maps $\bar{G}_n \cap C(1)$ onto the single point $z' = 1$, then there exists a function $F(z)$ holomorphic in $D(1)$ such that $|F(z) - f_n(z)| < \varepsilon$ for $z \in M_n$ ($n = 1, 2, \ldots\ldots$) and also $[F(z) - f_n(z)]$ tends uniformly to* 0 *as $|z|$ tends to* 1 ($z \in M_n$; $n = 1, 2, \ldots\ldots$).

Proof. If the given mapping is possible, G_n can be mapped conformally onto H_n, a subset of the following region H in the ζ-plane, in such a way that $\bar{G}_n \cap C(1)$, which I shall call E_n, corresponds to the point $\zeta = -\infty$.

$$H: \quad R(\zeta) < -1 \qquad \text{and} \qquad |I(\zeta)| < 1.$$

Let $\zeta = \zeta_n(z)$ designate this mapping and let $z = z_n(\zeta)$ be the inverse of $\zeta = \zeta_n(z)$.

For N sufficiently large, the line $\zeta = -N + iv$, $-1 < v < +1$, cuts the region H_n and divides it into two subsets H_n^1 and H_n^2 where H_n^2 represents the unbounded subset. Let G_n^1 and G_n^2 be the images of H_n^1 and H_n^2 respectively under $z = z_n(\zeta)$. If $\{z_n\}$ is a sequence of points in G_n^1, $|\zeta(z_n)| < N + 1$ and so no sequence in G_n^1 can converge to a point of E_n and so not to a point of $C(1)$. Therefore there is a positive distance $d(N)$ between $\bar{G}_n^1$ and $C(1)$. If $z \in G_n$ and $|z| > 1 - d(N)$ then z must be in G_n^2 and thus $R(\zeta_n(z)) < -N$ (i.e., the real part of $\zeta_n(z) < -N$).

Now I have $\zeta_n(z)$ holomorphic in G_n for each $n = 1, 2, \ldots\ldots$; and for $z \in M_n \subset G_n$ and $|z| > 1 - d(N)$ this implies that $R(\zeta_n(z)) < -N$. So, $R(\zeta_n(z)) \to -\infty$ uniformly as $|z| \to 1$ in M_n ($n = 1, 2, \ldots\ldots$).

Therefore, by (2. 13), there exists a function $F^*(z)$ holomorphic in $D(1)$ such that $|F^*(z) - \zeta_n(z)| < \varepsilon$ for $z \in M_n$ ($n = 1, 2, \ldots\ldots$). Since $R(\zeta_n(z)) \to -\infty$ uniformly as $|z| \to 1$ in M_n, $n = 1, 2, \ldots\ldots$, $R(F^*(z)) \to -\infty$ uniformly as $|z| \to 1$ in M_n $n = 1, 2, \ldots\ldots$.

Since $F^*(z)$ is holomorphic in $D(1)$, $w(z) = e^{F^*(z)}$ is holomorphic in $D(1)$ and $w(z) \neq 0$ for $z \in D(1)$. Furthermore, $|w(z)| = e^{R(F^*(z))} \to 0$ uniformly as $|z| \to 1$ for $z \in M_n$, $n = 1, 2, \ldots\ldots$.

Since $f_n(z)$ is holomorphic in G_n for each n and $w(z) \neq 0$ for any z in G_n for each n, $f_n(z)/w(z)$ represents a holomorphic function in G_n for each $n = 1, 2, \ldots\ldots$

By (2. 13), there exists a function $F^{**}(z)$ holomorphic in $D(1)$ such that $|F^{**}(z)-f_n(z)/w(z)|<\varepsilon$ for $z\in M_n$ $n=1,2,\ldots\ldots$. Let $F(z)=F^{**}(z)\cdot w(z)$. Then $F(z)$ is holomorphic for $z\in D(1)$ and $|F(z)-f_n(z)|<\varepsilon|w(z)|$ for $z\in M_n$ $n=1, 2, \ldots\ldots$. Since $|w(z)|$ tends uniformly to 0 as $|z|$ tends to 1 for $z\in M_n$ $(n=1,2,\ldots\ldots)$, the theorem is proved.

I have now developed enough technique to proceed with the details of the answer presented on page 260.

(2. 15) THEOREM. *Let Δ be a non-empty closed subset of $C(1/2)$ which is nowhere dense on $C(1/2)$. Let $f(\varphi)$ (which may assume the value ∞) be defined for $0\leq\varphi<2\pi$ such that when $f(\varphi)$ is associated by g with each point $1/2\ exp\,[i(\varphi+tan\,\pi/4)]$ of $C(1/2)$ $0\leq\varphi<2\pi$ the correspondence is a function of Baire class 0 or 1 and g is constant at least on each circular arc disjoint from the set Δ. Then there exists a function $F(z)$ holomorphic in $D(1)$ such that:*

1) $\lim\limits_{r\to1} F\left(r\ exp\left[i\left(\varphi+tan\frac{\pi r}{2}\right)\right]\right)=f(\varphi) \qquad 0\leq\varphi<2\pi\,.$

2) *If W is a closed spiral wedge which contains no point of Δ and $f(\varphi)$ is the corresponding constant spiral limit value, $F(z)$ tends uniformly to $f(\varphi)$ as $|z|$ tends to 1, $z\in W$.*

3) *Every Study spiral cutting $C(1/2)$ at a point of Δ is a Julia spiral.*

*Proof.** Since $f(\varphi)$ is defined on $C(1/2)$ and is of Baire class 0 or 1, there exists a sequence of functions $f_1(\varphi)$, $f_2(\varphi),\ldots\ldots$ which are continuous on $C(1/2)$ and converge to $f(\varphi)$. Let $f(\varphi)=r_0(\varphi)\exp\,[i\theta_0(\varphi)]$ where $r_0(\varphi)$ and $\theta_0(\varphi)$ are continuous functions of φ. Now define $f_n(\varphi)=r_n(\varphi)\exp\,[i\theta_n(\varphi)]$ where $\theta_n(\varphi)=\theta_0(\varphi)$ if $r_n(\varphi)=0$. Since $\lim\limits_{n\to\infty} f_n(\varphi)=f(\varphi)$, there is a proper determination of $\theta_n(\varphi)$ for each n and each φ such that $\lim\limits_{n\to\infty}\theta_n(\varphi)=\theta_0(\varphi)$ and $\lim\limits_{n\to\infty} r_n(\varphi)=r_0(\varphi)$.

Let $d_r^{(n)}=\dfrac{\frac{n}{n+1}-r}{\frac{n}{n+1}-\frac{n-1}{n}}$ for $n=1, 2, \ldots\ldots$ and $\frac{n-1}{n}<r\leq\frac{n}{n+1}$. Then $0\leq d_r^{(n)}<1$, $n=1,2,\ldots\ldots\ldots$. Let $h(0)=0=f_0(\varphi)$. Then for $\frac{n-1}{n}<r\leq\frac{n}{n+1}$ set $h(r\exp\,[i(\varphi+\tan\,\pi r/2)])=\{r_n(\varphi)+d_r^{(n)}(r_{n-1}(\varphi)-r_n(\varphi))\}\exp\,[i(\theta_n(\varphi)+d_r^{(n)}\{\theta_{n-1}(\varphi)-\theta_n(\varphi)\})]$. Then $h(z)$ is continuous in $D(1)$ and it can be easily verified that $\lim\limits_{r\to1} h(r\exp\,[i(\varphi+\tan\pi r/2)])=f(\varphi)$.

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* The argument used by Roth (6, p. 117) to construct a function $h(z)$ continuous in the entire plane does not hold if $f(\varphi)=\infty$ for some φ. A slight modification of the construction of the function $h(z)$ used in the first part of this proof can be used to correct this error.

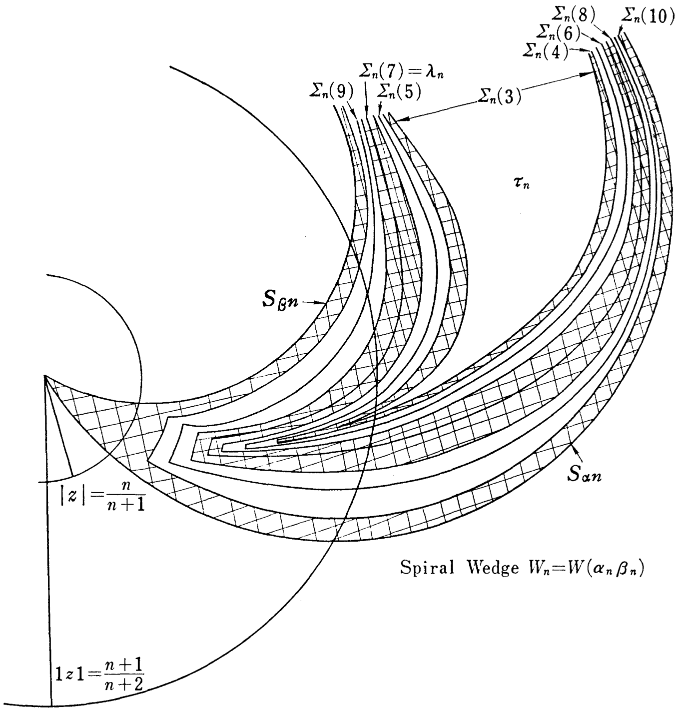
Let Γ be the set of all Study spirals that intersect the set $\varDelta$ on $C(1/2)$. Then Γ is closed and nowhere dense in $D(1)$. It follows from a theorem of Bagemihl and Seidel (1, p. 190, Corollary 1) that there exists a function $G_\Gamma(z)$ holomorphic in $D(1)$ such that if $1/2 \exp[i(\varphi+\tan\pi/4)]$ is in $\varDelta$ then $\lim_{r\to1} G_\Gamma(r\exp[i(\varphi+\tan\pi r/2)])=\lim_{r\to1} h(r\exp[i(\varphi+\tan\pi r/2)])=f(\varphi)$.

The complementary set $G=C(1/2)-\varDelta$ is an open subset of $C(1/2)$ and so consists of countably many pairwise disjoint open arcs $g_0, g_1, g_2, \ldots\ldots$. Let W_n be the spiral wedge for which $W_n\cap C(1/2)=g_n$ $n=0,1,2,\ldots\ldots$. Let $W_n=W(\alpha_n,\beta_n)$ where S_{α_n} and S_{β_n} are the Study spirals that bound W_n, $\alpha_n<\beta_n$. Let C_n (finite or infinite) be the constant value of $f(\varphi)$ associated with the arc g_n. Let $r_k^{(n)}=\frac{n}{n+1}+\frac{1}{k}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right)$ $(k=3,4,\ldots\ldots,10;\ n=0,1,2,\ldots\ldots)$. Define $\sum_n(k)=\{z\,|\,z=r\exp\left[i(\alpha_n+\tan\frac{\pi r}{2}+\sigma_k^{(n)}(r))\right],\ 1>r\geq r_k^{(n)}\}\cup\{z\,|\,z=r\exp\left[i\left(\beta_n+\tan\frac{\pi r}{2}-\sigma_k^{(n)}(r)\right)\right],\ 1>r\geq r_k^{(n)}\}\cup\{z=r_k^{(n)}\exp i\theta\,|\,\alpha_n+\tan\frac{\pi r_k^{(n)}}{2}+\sigma_k^{(n)}(r_k^{(n)})\leq\theta\leq\beta_n+\tan\frac{\pi r_k^{(n)}}{2}-\sigma_k^{(n)}(r_k^{(n)})\}$ where $\sigma_k^{(n)}(r)=(1-r)\frac{(\beta_n-\alpha_n)}{k}$ $(k=3,4,\ldots\ldots,10;\ n=0,1,2,\ldots\ldots)$. Notice that $\lim_{r\to1}\sigma_k^{(n)}(r)=0$ for all k and n. Thus $\sum_n(k)$ "opens up" in W_n as $r\to1$ and its "ends" approach S_{α_n} and S_{β_n} asymptotically from within W_n as $r\to1$. Let $\lambda_n=\sum_n(7)$. Let $B(\lambda_n)$ be the closed region bounded by $\sum_n(6)$ and $\sum_n(8)$ and containing λ_n. Let $G(\lambda_n)$ be the open region bounded by $\sum_n(5)$ and $\sum_n(9)$. Then $B(\lambda_n)$ is closed and contains λ_n in its interior and $B(\lambda_n)$ is contained in $G(\lambda_n)$ $(n=0,1,2,\ldots\ldots)$.

Let τ_n be the closed region entirely contained in W_n which is bounded by $\sum_n(3)$. Define $B(\tau_n)$ to be the closed region bounded by $\sum_n(4)$ and containing τ_n in its interior. Let $G(\tau_n)$ be the open region bounded by $\sum_n(5)$ and containing $B(\tau_n)$.

Lastly, let $B(\Gamma)$ be the closed region bounded by $\bigcup_{n=0}^{\infty}\sum_n(10)$ and containing Γ in its interior. Let $G(\Gamma)$ be the open region bounded by $\bigcup_{n=0}^{\infty}\sum_n(9)$ and containing $B(\Gamma)$ in its interior.

If I define $J_n=\left\{z\,|\,|z|=\frac{n}{n+1}\right\}$ $(n=1,2,\ldots\ldots)$ then $\{J_n\}$ is an increasing sequence of distinct Jordan curves converging to $C(1)$. Consider the open regions $G(\Gamma)$, $G(\lambda_0)$, $G(\tau_0)$, $G(\lambda_1)$, $G(\tau_1)$, ... and, for the sake of simple notation,



rename these regions $G_1, G_2, \ldots\ldots$. Each region contains a certain closed set (e.g., $G(\Gamma)\supset\Gamma$, $G(\lambda_i)\supset\lambda_i$, and $G(\tau_i)\supset\tau_i$) and I shall correspondingly label these $M_1, M_2, \ldots\ldots$. The closed set contained in G_i and containing M_i in its interior I shall call B_i (e.g., $G_i=G(\tau_i)$, $B_i=B(\tau_i)$ and $M_i=\tau_i$). It is immediate at this point that the sequences $\{G_n\}_1^\infty, \{B_n\}_1^\infty, \{M_n\}_1^\infty$ and $\{J_n\}_1^\infty$ have been constructed to satisfy the conditions stated in (2. 13).

Now consider $\sum_0(9)$. It is disjoint from G_n for each $n=0, 1, \ldots\ldots$. Let σ represent that part of $\sum_0(9)$ defined by $\left\{z \,|\, z=r\exp i\left(\beta_0+\tan\frac{\pi r}{2}-\sigma_9^{(0)}(r)\right)\right]$ $1>r \geq r_9^{(0)}\Big\}$. This σ is a spiral in $D(1)$ in the z-plane. Let $G=D(1)-\sigma$. Let $D'(1)$ represent the set $|\zeta|<1$. The initial point of σ is the impression of one prime end of G, while every other point of σ is the impression of two prime ends of G. Since σ converges to $C(1)$, $C(1)$ is the impression of a single prime end P of G. Using Carathéodory's classical results (3), there exists a one-to-one conformal mapping from G onto the unit disk $D'(1)$ so that the initial point of σ and the prime end P correspond, respectively, to the points -1, 1 while the other points of σ are mapped onto $C'(1)-\{1,-1\}$. Let $\zeta=\zeta_\sigma(z)$ be the mapping from G to $D'(1)$ and $z=z_\sigma(\zeta)$ the inverse of this mapping.

Each of the regions G_n has the property that $\bar{G}_n \cap C(1)=C(1)$. Furthermore $\zeta_\sigma(z)$ restricted to G_n has a continuous extension onto $\bar{G}_n$ which maps $C(1)$ onto the single point $\zeta=1$ for each n. Therefore the continuous extension of $\zeta_\sigma(z)$ restricted to $\bar{G}_n$ satisfies the condition stated in (2. 14) for each $n=0, 1, \ldots\ldots$.

It remains only for me to define a function $f_n(z)$ holomorphic in G_n ($n=0, 1, \ldots\ldots$) in order for me to be able to use the conclusions of (2. 13) and (2. 14). To do this I must consider the regions $G_1, G_2, \ldots\ldots$ as formerly represented: $G(\Gamma), G(\lambda_0), G(\tau_0), \ldots\ldots$.

Let $G_\Gamma(z)$ be the holomorphic function defined on the region $G(\Gamma)$. If the constant C_n associated with W_n is finite, let the function $f^*(z)\equiv C_n$ be the holomorphic function defined on $G(\tau_n)$. Then define $f^{**}(z)=1/\zeta_\sigma(z)-1$ for $z\in G(\lambda_n)$. This function is then holomorphic on $G(\lambda_n)$ and tends uniformly to ∞ as $|z|$ tends to 1 in $G(\lambda_n)$.

On the other hand, if $C_n=\infty$ for W_n, define $f^*(z)=1/\zeta_\sigma(z)-1$ for $z\in G(\tau_n)$. This function is then holomorphic in $G(\tau_n)$ and tends to ∞ uniformly as $|z|$ tends to 1, $z\in G(\tau_n)$. Then define $f^{**}(z)\equiv 0$ for $z\in G(\lambda_n)$.

Thus, reverting back to the simpler notation, with each region G_n I have associated a function $f_n(z)$ holomorphic there. Given $\varepsilon>0$, (2. 14) allows me to conclude the existence of a function $F(z)$ holomorphic in $D(1)$ such that $|F(z)-f_n(z)|<\varepsilon$ for $z\in M_n$ ($n=1, 2, 3, \ldots\ldots$) and also $[F(z)-f_n(z)]$ tends uniformly to zero as $|z|$ tends to 1, $z\in M_n$ ($n=1, 2, \ldots\ldots$).

Now consider the Study spiral S_φ. For some n, S_φ is eventually in M_n. Hence there is an R, $0<R<1$ such that for $|z|=r>R$ and $z\in S_\varphi$, $z\in M_n$. If $f(\varphi)$ is finite, $f_n(z)=f(\varphi)$ for $z\in S_\varphi$ and $|z|>R$ and so $\lim_{r\to1}\left|F\left(r\exp\left[i\left(\varphi+\tan\frac{\pi r}{2}\right)\right]\right)-f(\varphi)\right|\leq\lim_{r\to1}\left|F\left(r\exp\left[i\left(\varphi+\tan\frac{\pi r}{2}\right)\right]\right)-f_n\left(r\exp\left[i\left(\varphi+\tan\frac{\pi r}{2}\right)\right]\right)\right|$ $+\lim_{r\to1}\left|f_n\left(r\exp\left[i\left(\varphi+\tan\frac{\pi r}{2}\right)\right]\right)-f(\varphi)\right|=0$. Thus, for $f(\varphi)$ finite, $\lim_{\substack{|z|\to1\\ z\in S_\varphi}}F(z)=f(\varphi)$. If, on the other hand, $f(\varphi)=\infty$, for $z\in S_\varphi$ and $|z|>R$ it is true that $|F(z)-f_n(z)|<\varepsilon$. This implies that $|f_n(z)|-\varepsilon<|F(z)|$ and since $\lim_{\substack{|z|\to1\\ z\in S_\varphi}}|f_n(z)|=\infty$ it follows that $\lim_{\substack{|z|\to1\\ z\in S_\varphi}}|F(z)|=\infty$ or $\lim_{\substack{|z|\to1\\ z\in S_\varphi}}F(z)=f(\varphi)$.

Now suppose W is a closed spiral wedge which contains no point of Δ. Then $W \cap C(1/2)$ is contained entirely in g_m for some m. It follows then that W is eventually contained in M_n for some n. That is, there exists $0 < R < 1$ such that for $z \in W$ and $|z| > R$, $z \in M_n$.

If the corresponding constant spiral limit value C_m is finite, $f_n(z) = C_m$ for all $z \in M_n$, so $\lim_{\substack{|z| \to 1 \\ z \in W}} |F(z) - C_m| = \lim_{\substack{|z| \to 1 \\ z \in M_n}} |F(z) - f_n(z)|$ and since $[F(z) - f_n(z)]$ tends to zero uniformly as $|z|$ tends to 1 for $z \in M_n$, it follows that $F(z)$ tends uniformly to the corresponding constant limit value as $|z| \to 1$ for $z \in W$.

If $C_m = \infty$, $f_n(z)$ tends to ∞ uniformly as $|z| \to 1$ and z remains in M_n. If $z \in W$, and $|z| > R$, z must be in M_n. If $z \in W$ and $|z| > R$, it follows that $|F(z) - f_n(z)| < \varepsilon$. Then $|f_n(z)| - \varepsilon < |F(z)|$ and $|F(z)|$ must tend to ∞ uniformly as $|z| \to 1$ for $z \in W$. Thus, in this case also, $F(z)$ tends uniformly to the corresponding constant limit value as $|z|$ tends to 1 for $z \in W$.

Lastly consider any spiral wedge W^* which contains a point of Δ. Then W^* contains the Study spiral through that point and so cannot be a wedge of convergence. Since the arcs $g_1, g_2, \ldots\ldots$ are everywhere dense on $C(1/2)$ and since $W^* \cap C(1/2)$ is not contained entirely in any one of them, $W^* \cap C(1/2)$ contains at least a part of g_n for some n. Thus W^* contains at least a terminal part of λ_n. Then W^* contains spirals along which $|F(z)|$ is both bounded and unbounded and so $|F(z)|$ is neither bounded or uniformly unbounded in W^*, as $|z| \to 1$ and thus W^* cannot be a wedge of convergence. Since I have previously shown that the set of Julia spirals is identical with the set of Study spirals that are contained in no wedge of convergence, it follows that each Study spiral cutting $C(1/2)$ at a point of Δ is a Julia spiral and the theorem is complete.

3. An extension to a larger class

Harold Bohr has considered (2) the class of entire functions that are bounded on each half line emanating from the origin. This class, of course, includes much of the class of functions considered by Roth and it is natural to ask whether analogous results can be obtained for functions holomorphic in $D(1)$ where the class of spirals in the disk plays the role of half lines in the plane. For my purposes I again consider the class of Study spirals as rather natural analogues in the disk of the half lines in the plane, but of course make no claim as to their exclusiveness.

(3. 1) Definition. M^* shall denote the class of functions $F(z)$ holomor-

phic in $D(1)$ and bounded on each Study spiral.

It follows from this that if $F(z)$ is in M^* for each θ $(0 \leqslant \theta < 2\pi)$ there is a real positive number $L(\theta)$ such that for all $z \in S_\theta$, $|F(z)| < L(\theta)$.

I shall use i to denote an open connected subset of $C(1/2)$ and I shall distinguish between three distinct types of intervals as follows:

1) The entire circle, therefore an interval without frontier points, which I shall designate as I.

2) An interval with exactly one frontier point $A = 1/2 \exp\left[i\left(\alpha + \tan\frac{\pi}{4}\right)\right]$, therefore the entire circle except this point, which I shall designate by I_A.

3) An interval with two frontier points $A = 1/2 \exp\left[i\left(\alpha + \tan\frac{\pi}{4}\right)\right]$ and $B = 1/2 \exp\left[i\left(\beta + \tan\frac{\pi}{4}\right)\right]$. I shall designate this interval by I_{AB} where the order of the two frontier points is so chosen that as z moves from A to B on I_{AB} it travels in the counterclockwise direction.

(3. 2) Definition. I shall say that $F(z) \in M^*$ is bounded on an interval i if $F(z)$ is uniformly bounded on all Study spirals which pass through a point of i. That is, there must exist a constant $B(i)$ such that for all θ for which S_θ cuts i, $L(\theta) < B(i)$.

(3. 3) Definition. Concerning two intervals i_1 and i_2, I shall say that i_1 is a subinterval of i_2 iff not only all points of i_1 are contained in i_2 but also all frontier points of i_1 are also in i_2.

(3. 4) Definition. An interval i is called a complete interval of boundedness for a function $F(z) \in M^*$ iff 1) $F(z)$ is bounded in every subinterval of i and 2) $F(z)$ is not bounded in any interval which contains a frontier point of i.

It follows that two complete intervals of boundedness for a given function are either disjoint or identical. Also, if $F(z)$ is bounded on an interval i, there is a unique interval i^* containing i which is a complete interval of boundedness.

(3. 5) Theorem. *If $F(z) \in M^*$ and i is an arbitrarily small interval on $C(1/2)$, then i contains a subinterval on which $F(z)$ is bounded. Consequently the complete intervals of boundedness are everywhere dense on $C(1/2)$.*

Proof. Let i be bounded by $1/2 \exp[i(\alpha_0 + \tan \pi/4)]$ and $1/2 \exp[i(\beta_0 + \tan \pi/4)]$ where $\alpha_0 < \beta_0 < \alpha_0 + 2\pi$ and let $F_r(\theta) = f(r \exp[i(\theta + \tan \pi r/2)])$ for

$\alpha_0 < \theta < \beta_0$. If the theorem is not true, there exists r_1 and θ_1 such that $|F_{r_1}(\theta_1)| > 1$. Since F_{r_1} is a continuous function of θ, there exists an entire interval (α_1, β_1) containing θ_1 where $\alpha_0 < \alpha_1 < \beta_1 < \beta_0$ such that $|F_{r_1}(\theta)| > 1$ for $\theta \in (\alpha_1, \beta_1)$.

If (α_1, β_1) is not an interval of boundedness, then there exists r_2 and θ_2 such that $|F_{r_2}(\theta_2)| > 2$. Again by continuity, there exists an entire interval (α_2, β_2) containing θ_2 where $\alpha_1 < \alpha_2 < \beta_2 < \beta_1$ such that $|F_{r_2}(\theta)| > 2$ for $\theta \in (\alpha_2, \beta_2)$.

Continuing in this way, if no interval of boundedness is encountered, there exists $\theta_0 \in \bigcap_{n=1}^{\infty} (\alpha_n, \beta_n)$ such that $|F_{r_n}(\theta_0)| > n$ for each positive integer n. But this denies the existence of $L(\theta_0)$ and so proves the theorem.

It is natural to ask, as in the previous section, whether a converse can be found for this theorem. The following is the affirmative answer.

(3.6) THEOREM. *Let $i_1, i_2, \ldots\ldots$ be an arbitrary (finite or countably infinite) set of intervals on $C(1/2)$ such that no two of these intervals possess a common point and the intervals are everywhere dense on $C(1/2)$. Then there exists a function of class M^* whose complete intervals of boundedness are precisely the intervals $i_1, i_2, \ldots\ldots$.*

Proof. If I is the only interval that is given, any function constant in $D(1)$ will suffice. Otherwise the set $C(1/2) - \bigcup_{n=1}^{\infty} i_n$ is closed and not empty, and I shall call it Δ. Now apply the theorem of Section 2.15 with $f(\varphi) = 0$ for all φ $(0 \leq \varphi < 2\pi)$, and the conclusion follows immediately.

REFERENCES

[1] F. Bagemihl and W. Seidel, *Some boundary properties of analytic functions.* Math. Zeit. **61,** 186–199 (1954).

[2] H. Bohr, *Über ganze transzendente Funktionen, die auf jeder durch den Nullpunkt gehenden Geraden beschränkt sind.* Opusc. Math. A. Wiman Dedicata (1930).

[3] C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete.* Math. Annalen **73,** 323–370 (1913).

[4] O. Lehto and K.I. Virtanen, *Boundary behaviour and normal meromorphic functions.* Acta Math. **97,** 47–65 (1957).

[5] K. Noshiro, *Cluster Sets.* Springer-Verlag, Berlin, 1960.

[6] A. Roth, *Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen.* Comment. Math. Helvet. **11,** 77–125 (1938).

[7] E. Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie, Zweites Heft, Konforme Abbildung einfachzusammenhängender Bereiche.* Leipzig and Berlin, 1913.

[8] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Plane.* Amer. Math. Soc. Colloquium Publications. 20, Third Edition, 1960.

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