ON A CLASS OF MARKOV PROCESSES TAKING VALUES ON LINES AND THE CENTRAL LIMIT THEOREM

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To Professor Kiyoshi Noshiro on the occasion of his 60–th birthday

§1. Introduction

We shall consider a class of Markov processes $(n(t), x(t))$ with the continuous time parameter $t \in [0, \infty)$, whose state space is $\{1, 2, \ldots, N\} \times R^1$. We shall assume that the processes are spacially homogeneous with respect to $x \in R^1$. In detail, our assumption is that the transition function

$$F_{ij}(x, t) = P(n(t) = j, x(t) \leqq x \mid n(0) = i, x(0) = 0), \quad t > 0, \ 1 \leqq i, j \leqq N, x \in R^1,$$

satisfies following conditions $(1, 1) \sim (1, 4)$.

(1, 1) $F_{ij}(x, t)$ is non-negative, and, for fixed i, j and t, it is monotone non-decreasing and right continuous in $x \in R^1$.

(1, 2) $$F_{ij}(+\infty, t) = \lim_{x \to \infty} F_{ij}(x, t) \leqq 1,$$
$$F_{ij}(-\infty, t) = \lim_{x \to \infty} F_{ij}(x, t) = 0, \qquad 1 \leqq i, j \leqq N, \quad t > 0,$$
$$\sum_{i=1}^{N} F_{ij}(+\infty, t) = 1, \qquad i = 1, 2, \ldots, N, \quad t > 0,$$

(1, 3) $$F_{ij}(x, t) = \sum_{k=1}^{N} \int_{R^1} F_{ik}(x - y, t) dF_{kj}(y, s) \quad t, s > 0,$$
$$1 \leqq i, k \leqq N, \quad x \in R^1,$$

(1, 4) $$\lim_{t \downarrow 0} F_{ij}(x, t) = \begin{cases} \delta_{ij}, & x \in [0, +\infty) \\ 0, & x \in (-\infty, 0). \end{cases}$$

The central limit theorem for processes of this type, in case of the discrete time parameter and in a special case of the continous time parameter, has been obtained by Keilson and Wishart [3]. In this paper, through introducing a system of generators of the semi-groups related to the processes, we show that the central limit theorem is valid for our cases of the continous time parameter.

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With the transition function $F_{ij}(x, t)$ satisfying (1, 1)~(1, 4) we associate its Fourier transform:

$$f_{ij}(z, t)=\int_{R^1} e^{ixz}dF_{ij}(x, t),\ \ t>0,\ \ 1\leqq i,\ j\leqq N,\ z\in R^1.$$

Then the matrix $\boldsymbol{f}(z, t)=(f_{ij}(z, t))$ satisfies the followings:

(1, 5) $\sum_{j=1}^{N} f_{ij}(0, t)=1,\ \ 1\leqq i\leqq N,\ \ \ t>0,$

(1, 6) $\boldsymbol{f}(z, t+s)=\boldsymbol{f}(z, t)\boldsymbol{f}(z, s)\ \ t, s>0,$

(1, 7) *$\boldsymbol{f}(z, t)$ converges, as t tends to zero, to the identity matrix $\boldsymbol{E}$ uniformly in $z\in R^1$ in the wide sense, i.e. each element $f_{ij}(z, t)$ of $\boldsymbol{f}(z, t)$ converges to δ_{ij} uniformly on any compact z-set.*

In §2, we shall determine the generator $\boldsymbol{A}(z)$, $z\in R^1$, of the semigroups $f(z, t)$, $z\in R^1$, (Theorem 1). In particular, if $N=1$, our expression of $\boldsymbol{A}(z)$ is no other than the so-called Lévy-Khintchine formula. $\{\boldsymbol{A}(z), z\in R^1\}$ in Theorem 1 characterizes all the processes whose transition functions satisfy (1, 1)~(1,4).

§3 will be devoted to the proof of the central limit theorem for our processes under some assumptions placed upon $\boldsymbol{A}(z)$ (Theorem 2). Our procedure in this section is essentially due to Keilson and Wishart [3].

In §4, m and v which are defiend by the first and second derivatives respectively of an eigen value $\lambda(z)$ of $\boldsymbol{A}(z)$ will be expressed explicitly by $\boldsymbol{A}(z)$ and its eigenvectors. We shall further solve affirmatively the Harris' conjecture related to the expectation processes of the electron-photon cascade (Harris [2], page 194).

§2. The generators of semigroups $\boldsymbol{f}(z, t)$.

Throughout §2, §3 and the first half of §4, we shall assume that we are given a transition function $F_{ij}(x, t)$ on $\{1, 2, \dots, N\}\times R^1$ satisfiying (1, 1)~(1, 4).

In this section, we shall show the existence of the generator

$$\boldsymbol{A}(z)=\lim_{t\downarrow 0}\frac{\boldsymbol{f}(z, t)-\boldsymbol{E}}{t}\ \ \text{(Lemma 1)},$$

and determine all the possible types of the generator $\boldsymbol{A}(z)$ (Theorem 1).

LEMMA 1. The limit

$$\lim_{t\downarrow 0}\frac{\boldsymbol{f}(z, t)-\boldsymbol{E}}{t}=\boldsymbol{A}(z)$$

exists and the convergence is uniformly in $z \in R^1$ *in the wide sence.*

Proof. Let us fix $a > 0$ arbitrarily. From the property (1, 7), we can choose $v > 0$ such that, for $z \in [-a, a]$, the matrix

$$A_v(z) = \int_0^v f(z, s)ds = \left(\int_0^v f_{ij}(z, s)ds\right)$$

has its inverse $A_v^{-1}(z)$. Since

$$A_v(z)f(z, t) = \int_0^v f(z, s)\, f(z, t)ds = \int_0^v f(z, s+t)ds = \int_t^{v+t} f(z, s)ds,$$

we have

$$f(z, t) = A_v^{-1}(z)\int_t^{v+t} f(z, s)ds.$$

Therefore we can obtain that

$$\lim_{h\downarrow 0}\frac{f(z, h) - E}{h} = \lim_{h\downarrow 0} A_v^{-1}(z)\frac{1}{h}\left[\int_h^{v+h} f(z, s)ds - \int_0^v f(z, s)ds\right]$$
$$= A_v^{-1}(z)\lim_{h\downarrow 0}\frac{1}{h}\left[\int_v^{v+h} f(z, s)ds - \int_0^h f(z, s)ds\right]$$
$$= A_v^{-1}(z)[f(z, v) - E], \qquad z \in [-a, a].$$

The convergence in question is uniform since $f(z, s)$ converges to $f(z, v)$ uniformly in $z \in [-a, a]$ as s tends to v. The right side of the above equation should be independent of v because the left does not depend on v.

Thus we have proved Lamma 1.

Remark. We can express $f(z, t)$ in the form $\exp\{A(z)t\}$. In fact, $f(z, t)$ is the unique solution of the equation

$$\frac{\partial}{\partial t} f(z, t) = A(z)f(z, t)$$

with the initial condition

$$\lim_{t\downarrow 0} f(z, t) = E.$$

The following theorem gives us the possible types of $A(z)$.

THEOREM 1. *The elements* $a_{ij}(z)$, $i, j = 1, 2, \ldots, N$, *of* $A(z)$ *can be expressed in the form,*

$$a_{ij}(z)=\int_{-\infty}^{\infty}e^{izx}\Gamma_{ij}(dx) \qquad i\neq j,\quad 1\leqq i,\ j\leqq N,$$

$$a_{ii}(z)=-\sum_{j\neq i}\Gamma_{ij}(R^1)+i\nu_i z-\frac{\sigma_i^2}{2}z^2+\int_{|u|>1}(e^{izv}-1)\Pi_i(du)$$

$$+\int_{|u|\leqq 1}(e^{izu}-1-izu)\Pi_i(du), \qquad i=1,2,\dots,N$$

where Γ_{ij}*'s,* $1\leqq i\neq j\leqq N$*, are finite measures on* R^1*,* ν_i*'s,* $i=1,\dots,N$ *are real numbers,* σ_i*'s,* $i=1,\dots,N$ *are real and positive, and* Π_i*'s,* $i=1,\dots,N$ *are measures on* R^1 *such that* $\int_{|u|\leqq 1}x^2\Pi_i(dx)<\infty$ *and* $\Pi_i(u;|u|>\varepsilon)<\infty$ *for any* $\varepsilon>0$.

Proof. In case that $i\neq j$, we can see by Lemma 1 that the convergence

$$a_{ij}(z)=\lim_{h\downarrow 0}\frac{f_{ij}(z,h)}{h}$$

is uniform in the wide sence in $z\in R^1$, and therefore $a_{ij}(z)$ is a positive definite function of z because $\frac{f_{ij}(z,h)}{h}$ has the same property. Then there exists a finite measure Γ_{ij} such that

$$a_{ij}(z)=\int_{-\infty}^{\infty}e^{izx}\Gamma_{ij}(dx).$$

In case that $i=j$, we note that

$$\frac{f_{ii}(z,h)-1}{h}=\frac{e^{a_{ii}(z)h}-e^{a_{ii}(0)h}}{h}+\frac{e^{a_{ii}(0)h}-1}{h}+\frac{f_{ii}(z,h)-e^{a_{ii}(z)h}}{h}.$$

Then, $\frac{f_{ii}(z,h)-e^{a_{ii}(z)h}}{h}$ tends to zero as $h\downarrow 0$ by virtue of the remark after Lemma 1. If we remark that $a_{ii}(0)=-\sum_{j=i}^{N}a_{ij}(0)=-\sum_{j=i}^{N}\Gamma_{ij}(R^1)$, $\frac{e^{a_{ii}(0)}-1}{h}$ tends to $-\sum_{j=i}^{N}\Gamma_{ij}(R^1)$ as $h\downarrow 0$. The limit of $\frac{e^{a_{ii}(z)h}-e^{a_{ii}(0)h}}{h}=e^{a_{ii}(0)h}\cdot\frac{e^{(a_{ii}(z)-a_{ii}(0))h}-1}{h}$ as $h\downarrow 0$ is the Lévy-Khintchine formula:

$$i\nu_i z-\frac{\sigma_i^2}{2}z^2+\int_{|u|>1}(e^{izu}-1)\Pi_i(du)+\int_{|u|\leqq 1}(e^{izu}-1-izu)\Pi_i(du)$$

(see for example Gnedenko and Kolmogorov [1]).

Remark: From the above discussion, we can see that Γ_{ij}, ν_i, σ_i, and Π_i are uniquely determined by the transition function $F_{ij}(x,t)$ satisfying $(1,1)\sim(1,4)$.

§3. The central limit theorem.

Let $A(z)$ be the matrix defined in Lemma 1. We shall assume

Assumption 1.

$$\int_{-\infty}^{\infty} x^2 \Gamma_{ij}(dx) < +\infty, \quad i \neq j, \ 1 \leqq i, \ j \leqq N,$$

$$\int_{-\infty}^{\infty} x^2 \Pi_i(dx) < +\infty, \quad 1 \leqq i \leqq N.$$

Assumption 2. $A(0)$ is irreducible.

The Assumption 1 is equivalent to the fact that

$$(*) \quad \int_{-\infty}^{\infty} x^2 dF_{ij}(x, 1) < \infty \quad \text{for any } i, j = 1, \dots, N.$$

In fact, Assumption 1 is equivalent to that

(**) $A(z)$ is twice differentiable with respect to z.

And (**) is equivalent to the same property of $e^{A(z)} = f(z, 1)$. This is no other than (*) (see Feller [4] page 485).

By virtue of Assumption 2, $f(0, 1) = e^{A(0)}$ is a positive stochastic matrix and therefore, by Perron-Frobenius' theorem, it has the simple eigenvalue 1 and the absolute values of the other N-1 eigenvalues are less than 1. Therefore, the equation

$$\det(A(0) - \lambda E) = \pi(0, \lambda) = 0$$

has a simple root $\lambda = 0$, and all the other roots have negative real parts. From these facts we can derive the next Lemma.

LEMMA 2. *There is a function $\lambda(z)$ defined on some neighbourhood $(-a, a)$ of $z=0$ which has the continuous first and second derivatives in $(-a, a)$ and satisfies that*

$$\pi(z, \lambda(z)) = 0, \qquad z \in (-a, a)$$

$$\lambda(0) = 0$$

$$\text{where } \pi(z, \lambda) = \det\,(A(z) - \lambda E).$$

Further, $\lambda'(0)$ is purely imaginary and $\lambda''(0)$ is real and non-positive.

Proof. As $\lambda = 0$ is a simple root of the equation $\Pi(0, \lambda) = 0$, we know that

$$\pi_\lambda(0, 0) = \frac{\partial \pi}{\partial \lambda}(0, 0) \neq 0.$$ If we put

$$\pi(z, \lambda) = \pi_1(z, \lambda_1, \lambda_2) + i\pi_2(z, \lambda_1, \lambda_2)$$

where $\pi_i(z, \lambda_1, \lambda_2)$, $i=1, 2$, are real valued functions and $\lambda_1=\Re e\lambda$, $\lambda_2=\Im m\lambda$, then we have

$$\pi_1(0, 0, 0)=0, \qquad \pi_2(0, 0, 0)=0.$$

On the otherhand, $\pi(z, \lambda)$ is analytic in λ. Therefore we have by Cauchy-Riemann's equation that

$$\begin{vmatrix} \frac{\partial\pi_1}{\partial\lambda_1}(0, 0, 0) & \frac{\partial\pi_2}{\partial\lambda_1}(0, 0, 0) \\ \frac{\partial\pi_1}{\partial\lambda_2}(0, 0, 0) & \frac{\partial\pi_2}{\partial\lambda_2}(0, 0, 0) \end{vmatrix} = \begin{vmatrix} \frac{\partial\pi_1}{\partial\lambda_1}(0, 0, 0) & \frac{\partial\pi_2}{\partial\lambda_1}(0, 0, 0) \\ -\frac{\partial\pi_2}{\partial\lambda_1}(0, 0, 0) & \frac{\partial\pi_1}{\partial\lambda_1}(0, 0, 0) \end{vmatrix}$$

$$=\Big(\frac{\partial\pi_1}{\partial\lambda_1}(0, 0, 0)\Big)^2+\Big(\frac{\partial\pi_2}{\partial\lambda_1}(0, 0, 0)\Big)^2 \neq 0.$$

Now the first part of Lemma 2 is the direct consequence of the implicit function theorem.

Now, it is easy to see by the form of $A(z)$ that $\pi_z(0, 0)=\frac{\partial}{\partial z}\pi(0, 0)$ is pure imaginary and $\pi_\lambda(0, 0)$ is real. Therefore $\lambda'(0)=\frac{-\pi_z(0, 0)}{\pi_\lambda(0, 0)}$ is pure imaginary. By the same method we can see that $\lambda''(0)$ is real. Let us note that $\sum_{j=1}^{N}|f_{ij}(z, 1)| \leqq 1$ where $f_{ij}(z, 1)$ is (i, j)-element of the matrix $e^{A(z)}=\boldsymbol{f}(z, 1)$. This implies, applying Frobenius' theorem again, all the eigenvalues of $A(z)$ have non-positive real parts. Especially, $\Re e\lambda(z) \leqq 0$ and $\Re e\lambda(z)$ attains its maximum 0 at $z=0$. Thus

$$\frac{d^2}{dz}\Re e\lambda(z)]_{z=0}=\Re e\lambda''(0)=\lambda''(0) \leqq 0.$$

Remark. We may assume without loss of generality that for $z\in(-a, a)$ $\lambda(z)$ is the eigenvalue of $A(z)$ the real part of which is greater than the real parts of any other eigenvalues, because the eigenvalues of $A(z)$ are continuous in z, and the eigenvalue $\lambda(0)=0$ of $A(0)$ has the maximum real part. In the following we assume the above property of $\lambda(z)$ in $z\in(-a, a)$.

Consider a probality measure $\mu=(\mu_1, \mu_2, \ldots\ldots, \mu_N)$ on $\{1, 2, \ldots\ldots, N\}\times R^1$, where μ_i's are measues on R^1 such that $\sum_{i=1}^{N}\mu_i(R^1)=1$. Let us put

$$F_j^\mu(x, t)=\sum_{i=1}^{N}\int_{-\infty}^{\infty}\mu_i(dy)F_{ij}(x-y, t), \quad 1\leqq j\leqq N, \quad \textit{and}$$

$$\boldsymbol{F}^{\mu}(x, t)=\{F_1^{\mu}(x, t),\ F_2^{\mu}(x, t), \ldots\ldots, F_N^{\mu}(x, t)\}.$$

Then $\boldsymbol{F}^{\mu}(x, t)$ determines the distribution of a Markov process on $\{1, 2, \ldots\ldots, N\} \times R^1$ at time t with the initial distribution μ. We further put

$$\boldsymbol{f}^{\mu}(z, t)=\left\{\int_{-\infty}^{\infty} e^{ixz} dF_1^{\mu}(x, t),\ \int_{-\infty}^{\infty} e^{ixz} dF_2^{\mu}(x, t), \ldots\ldots, \int_{-\infty}^{\infty} e^{ixz} dF_N^{\mu}(x, t)\right\}.$$

Then,

$$\boldsymbol{f}^{\mu}(z, t)=\boldsymbol{f}^{\mu}(z, 0+)\boldsymbol{f}(z, t)$$

where $\boldsymbol{f}^{\mu}(z, 0+)$ is the Fourier transform of μ. Define m and v by the formula

$$(3,\ 1) \qquad m=-i\lambda'(0), \quad v=-\lambda''(0).$$

THEOREM 2. *For any initial distribution μ on $\{1, 2, \ldots\ldots, N\} \times R^1$,*

$$(3,\ 2) \qquad \lim_{t\to\infty} e^{-it^{\frac{1}{2}}mz}\boldsymbol{f}^{\mu}(t^{-\frac{1}{2}}z,\ t)=e^{\frac{-1}{2}z^2v^2}\boldsymbol{e},\ z\in(-a, a)$$

holds, where $\boldsymbol{e}=(e_1, e_2, \ldots\ldots, e_N)$ is the left eigenvector belonging to $\boldsymbol{A}(0)$ for the eigenvalue 0, and $\sum_{i=1}^{N} e_i=1$.

Proof. There is a regulaer matrix $\boldsymbol{T}(z)$ for $z\in(-a, a)$ such that

$$\boldsymbol{A}(z)=\boldsymbol{T}^{-1}(z)\boldsymbol{J}(z)\boldsymbol{T}(z)$$

where $\boldsymbol{J}(z)$ is a matrix of the Jordan's normal form whose (1, 1)-element is $\lambda(z)$. Here we can choose $\boldsymbol{T}(z)$ to be continuous in $z\in(-a, a)$. Now we have

$$(3,\ 3) \qquad \boldsymbol{f}^{\mu}(t^{-\frac{1}{2}}z,\ t)e^{-izmt^{\frac{1}{2}}}=\boldsymbol{f}^{\mu}(t^{-\frac{1}{2}}z, 0)\boldsymbol{T}^{-1}(t^{-\frac{1}{2}}z)e^{-izmt^{-\frac{1}{2}}}\exp(t\boldsymbol{J}(t^{-\frac{1}{2}}))\boldsymbol{T}(t^{-\frac{1}{2}}z).$$

In order to investigate the limit of the above expression, we first note the following

$$(3,\ 4) \qquad \lim_{t\to\infty} \boldsymbol{f}^{\mu}(t^{-\frac{1}{2}}z, 0)=\lim_{z\to 0} \boldsymbol{f}^{\mu}(z, 0)=\boldsymbol{f}^{\mu}(0,\ 0).$$

Since the first column $\boldsymbol{e}(z)$ of $\boldsymbol{T}(z)$ is the left eigenvector and the first row $\boldsymbol{i}(z)$ of $\boldsymbol{T}(z)$ is the right eigenvector of $\boldsymbol{A}(z)$ belonging to the eigenvalue $\lambda(z)$, we may assume that $\boldsymbol{e}(0)$ and $\boldsymbol{i}(0)$ are of the form

$$\boldsymbol{e}(0)=\boldsymbol{e}=(e_1, e_2, \ldots\ldots,\ e_N)$$

$$\boldsymbol{i}(0)=\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}.$$

By the continuity of each element of $\boldsymbol{T}(z)$, we have

(3, 5) $$\lim_{t\to\infty}\boldsymbol{T}(t^{\frac{-1}{2}}z)=\lim_{z\to 0}\boldsymbol{T}(z)=\begin{pmatrix} e_1, e_2, \ldots\ldots, e_N \\ * \end{pmatrix}$$

(3, 6) $$\lim_{t\to\infty}\boldsymbol{T}^{-1}(t^{\frac{-1}{2}}z)=\lim_{z\to 0}\boldsymbol{T}^{-1}(z)=\begin{pmatrix} 1 & \\ 1 & * \\ \vdots & \\ 1 & \end{pmatrix}$$

The absolute values of the (i, j)-elements of $\exp\{t\boldsymbol{J}(z)\}$ for $i, j\geqq 2$ and $j-i=l\geqq 0$ are less than $\frac{1}{l!}te^{-\alpha(z)t}$, where $-\alpha(z)$ denotes the maximum real part of the eigenvalues of $\boldsymbol{A}(z)$ except $\lambda(z)$.

On the other hand the (i, j)-elements of $\exp\{t\boldsymbol{J}(z)\}$ for $j-i<0$ or for $i=1$, $j\neq i$ are equal to 0. Since $\sup_{z\in(-a,a)} e^{-\alpha(z)}<1$, the (i, j)-elements of $\exp\{t\boldsymbol{J}(z)\}$ for $(i. j)\neq(1, 1)$ converge to 0 as $t\to\infty$. For the $(1, 1)$-element, we get

$$\lim_{t\to\infty}e^{-izmt^{\frac{1}{2}}}e^{t\lambda(t^{\frac{-1}{2}}z)}=e^{\frac{-1}{2}z^2v^2},$$

with m and v defined by (3, 1). Therefore

(3, 7) $$\lim_{t\to\infty}e^{-izmt^{\frac{1}{2}}}\exp\{t\boldsymbol{J}(t^{\frac{-1}{2}}z)\}=e^{\frac{-1}{2}z^2v^2}\left(\begin{array}{c|c} 1 & 0\ldots\ldots 0 \\ \hline 0 & \\ \vdots & 0 \\ 0 & \end{array}\right)$$

holds. Consequently we get (3, 1) by (3, 3)~(3, 7) and the proof is complete.

COROLLARY. *In case* $v=-\lambda''(0)>0$, *we have*

$$\lim_{t\to\infty}\boldsymbol{F}^{\mu}(xt^{\frac{1}{2}}+tm, t)=\int_0^{\frac{x}{v}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy\cdot\boldsymbol{e}$$

for any initial distribution μ.

§4. The expression of m and v, and the expectation process of the eletron-photon cascade.

Let $\boldsymbol{e}(z)$ and $\boldsymbol{i}(z)$ be the left and right eigenvector of $\boldsymbol{A}(z)$ belonging to $\lambda(z)$ respectively, as in the proof of Theorem 2. We can assume that they are twice differentiable in the neighbourhood of the origin and $\boldsymbol{e}(0)=\boldsymbol{e}$, $\boldsymbol{i}(0)=\boldsymbol{1}$.

Since $\lambda(0)=0$, taking dreivatives of the equations

(4, 1) $\boldsymbol{e}(z)\boldsymbol{A}(z)=\lambda(z)\boldsymbol{e}(z)$ and

(4, 2) $\boldsymbol{A}(z)\boldsymbol{i}(z)=\lambda(z)\boldsymbol{i}(z)$

we have

(4, 3) $\boldsymbol{e}'(0)\boldsymbol{A}(0)+\boldsymbol{e}\boldsymbol{A}'(0)=\lambda'(0)\boldsymbol{e}$ and

(4, 4) $\boldsymbol{A}(0)\boldsymbol{i}'(0)+\boldsymbol{A}'(0)\mathbf{1}=\lambda'(0)\mathbf{1}.$

By (4, 3), we have

(4, 5) $m=-i\lambda'(0)=-\boldsymbol{e}\boldsymbol{A}'(0)\mathbf{1}.$

Differentiating (4, 1) twice and noting (4, 4), we have

(4, 6) $v=-\lambda''(0)=-\boldsymbol{e}\boldsymbol{A}''(0)\mathbf{1}+2\boldsymbol{e}'(0)\boldsymbol{A}(0)\boldsymbol{i}'(0).$

The following Lemma 3 will illustrate the probalibistic measning of m and v. Let $\{n(t), x(t)\}$ be the Markov process on $\{1, 2, \ldots\ldots, N\}\times R^1$ whose transition function are governed by $F_{ij}(x, t)$ and whose initial distrivution is $\delta_0(dx)\boldsymbol{e}$, where δ_0 is the δ-measure concentrated at the origin.

LEMMA 3. $$m=-i\boldsymbol{e}\boldsymbol{A}'(0)\mathbf{1}=\frac{\boldsymbol{E}(x(t))}{t}, \text{ for any } t>0$$

$$v=-\boldsymbol{e}\boldsymbol{A}''(0)\mathbf{1}+2\boldsymbol{e}'(0)\boldsymbol{A}(0)\boldsymbol{i}'(0)=\lim_{t\to+\infty}\frac{\boldsymbol{E}[x(t)-\boldsymbol{E}(x(t))]^2}{t}.$$

Proof. We can see that

$$\boldsymbol{E}(x(t))=-i\boldsymbol{e}\left(\frac{\partial}{\partial z}\boldsymbol{f}(z, t)\right)_{z=0}\mathbf{1}=-i\boldsymbol{e}\left(\frac{\partial}{\partial z}e^{t\boldsymbol{A}(z)}\right)_{z=0}\mathbf{1}=-it\boldsymbol{e}\boldsymbol{A}(0)\mathbf{1}mt,$$

and

$$\boldsymbol{E}[x(t)-\boldsymbol{E}(x(t))]^2=\boldsymbol{E}(x(t)^2)-m^2t^2=-\boldsymbol{e}\left(\frac{\partial^2}{\partial z^2}e^{t\boldsymbol{A}(z)}\right)\mathbf{1}+\lambda'(0)^2t^2$$

$$=-t\boldsymbol{e}\boldsymbol{A}''(0)\mathbf{1}-2\sum_{k=2}^{\infty}\frac{t^k}{k!}\boldsymbol{e}\boldsymbol{A}'(0)\boldsymbol{A}^{k-2}(0)\boldsymbol{A}'(0)\mathbf{1}+\lambda'(0)^2t^2.$$

While, the second term of the last expression is, by virtue of (4, 3) and (4, 4), equal to $-\lambda'(0)^2t^2-2\boldsymbol{e}'(0)(\exp(t\boldsymbol{A}(0))-\boldsymbol{E}-t\boldsymbol{A}(0))\boldsymbol{l}'(0)$. Thus, we see that the second equation of Lemma 3 is valid.

In the remainder of this section, let us discuss the case where $N=2$. In this case, $\boldsymbol{A}(0)$ can be written in the form $\boldsymbol{A}(0)=\begin{pmatrix}-a & a\\ b & -b\end{pmatrix}$ and by the assumption that $\boldsymbol{A}(0)$ is irreducible, $a>0$ and $b>0$ hold. Therefore

$$\boldsymbol{A}(0)^n=(-1)^{n-1}(a+b)^{n-1}\boldsymbol{A}(0),\ n>1,$$

and, by (4, 3), (4, 4) and (4, 6), we have

$$v=-\boldsymbol{e}\boldsymbol{A}''(0)\mathbf{1}+\frac{2}{(a+b)^2}\boldsymbol{e}\boldsymbol{A}'(0)\boldsymbol{A}(0)\boldsymbol{A}'(0)\mathbf{1}, \tag{4, 7}$$

The expectation process introduced by Harris ([2] Chap. VII) related to the electron-photon cascade can be dealt with as a special case of our discussion with $N=2$.

Harris has conjectured that the central limit theorem holds for the expectation process, with m and v given by the right hand sides of the equations (4, 5) and (4, 7) respectively ([2] page 198). Theorem 2 and the above discussions show the validity of his conjecture.

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