CONDENSOR PRINCIPLE AND THE UNIT CONTRACTION

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.

Introduction

Deny introduced in [4] the notion of functional spaces by generalizing Dirichlet spaces. In this paper, we shall give the following necessary and sufficient conditions for a functional space to be a real Dirichlet space.

Let $\mathscr{X}$ be a regular functional space with respect to a locally compact Hausdorff space X and a positive measure ξ in X. The following four conditions are equivalent.

(1) The unit contraction operates on $\mathscr{X}$.

(2) $\mathscr{X}$ satisfies the condensor principle.

(3) $\mathscr{X}$ satisfies the strong complete maximum principles.

(4) $\mathscr{X}$ is a real Dirichlet space.

Furthermore for an invariant functional space $\mathscr{X}$ on a locally compact abelian group X, we shall show the following equivalence without assuming the regularity.

$\mathscr{X}$ is special Dirichlet space if and only if $\mathscr{X}$ satisfies the condensor principle.

1. Preliminaries on regular functional spaces

Let X be a locally compact Hausdorff space and ξ be a positive measure in X which is everywhere dense in X (i.e., $\xi(\omega)>0$ for any non-empty open set ω in X). According to Deny [4], we give the definition of a functional space.

Definition 1. A functional space $\mathscr{X}=\mathscr{X}(X, \xi)$ with respect to X and ξ is a Hilbert space of real valued functions $u(x)$ which is locally summable for ξ, the following condition being satisfied: (i) For any compact subset K in X, there existes a positive number $A(K)$ such that

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$$\int_K |u(x)|\, d\xi(x) \leq A(K)\|u\|$$

for any u in $\mathscr{X}$.

Two functions which are equal locally almost everywhere for ξ represent the same element in $\mathscr{X}$. The norm in $\mathscr{X}$ is denoted by $\|u\|$, the associated scalar product by (u, v). Let C_K be the space of finite continuous functions with compact support provided with the topology of uniform convergence.

DEFINITION 2. A functional space $\mathscr{X}=\mathscr{X}(X, \xi)$ is said to be regular if $\mathscr{X}\cap C_K$ is dense both in $\mathscr{X}$ and in C_K.

By the condition (i), for any bounded measurable function f with compact support, there exists an element u_f in a functional space $\mathscr{X}$ such that

$$(u_f, u)=\int fu\, d\xi$$

for any u in $\mathscr{X}$. Such an element u_f is said to be the potential generated by f. More generally we define potentials as follows.

DEFINITION 3.[1] Let $\mathscr{X}$ be a regular functional space. The element u is called a potential if there exists a real Radon measure μ such that

$$(u, f)=\int f\, d\mu$$

for any f in $\mathscr{X}\cap C_K$. Such an element u is denoted by u_μ. Especially if μ is positive, u_μ is said to be a pure potential.

According to Beurling and Deny [2], we define the capacity of an open set is defined as follows:

$$Cap(\omega)=inf\{\|u\|^2;\ u\in\mathscr{X}, u(x)\geq 1 \ p.p.\ in\ \omega\}.$$

If there are no such functions, $Cap\,(\omega)=+\infty$.

LEMMA 1. *Let $\mathscr{X}$ be a regular functional space and f be a function in $\mathscr{X}\cap C_K$. Then for each positive number ε,*

$$Cap(\{x\in X;\ f(x)>\varepsilon\})\leq\frac{\|f\|^2}{\varepsilon^2}.$$

By the definition of the capacity, this is evident.

LEMMA 2. *For a relatively compact open set ω in X, put*

$$E_\omega=\overline{\{u_\mu\in\mathscr{X};\ S_\mu\subset\omega,\ \mu\geq 0\}}.$$ [2]

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[1] Cf. [2], p. 209.

[2] S_μ is the support of μ.

Then there exists a unique element u_γ *which minimizes*

$$I(u_\mu)=\|u_\mu\|^2-2\int d\mu$$

in E_ω *and for which*

$$Cap(\omega)=\|u_\gamma\|^2=\int d\gamma.$$

Proof. Obviously E_ω is a closed convex cone in $\mathscr{H}$. Since ω is a relatively compact set, there exists a function f in $\mathscr{H}\cap C_K$ such that $f(x)\geq 1$ in ω. Then

$$I(u_\mu)\geq\|u_\mu\|^2-2\int f\, d\mu=\|u_\mu-f\|^2-\|f\|^2.$$

Hence $I(u_\mu)$ is bounded from below in E_ω. Therefore there exists a unique pure potential u_γ such that

$$I(u_\gamma)\leq I(u_\mu)$$

for any u_μ in E_ω. Then

$$\int d\mu\leq(u_\gamma, u_\mu) \tag{1}$$

and

$$\int d\gamma=\|u_\gamma\|^2. \tag{2}$$

By (1), $u_\gamma(x)\geq 1$ *p.p.* in ω. Hence

$$\|u_\gamma\|^2\geq Cap(\omega).$$

On the other hand it is known that there exists a sequence (u_{f_n}) of pure potentials such that $u_{f_n}\to u_\gamma$ strongly in $\mathscr{H}$, where f_n is a positive bounded measurable function with support in ω.[3] For any u in $\mathscr{H}$ such that $u(x)\geq 1$ *p.p.* in ω,

$$(u_{f_n}, u)=\int f_n u\, d\xi\leq\int f_n\, d\xi.$$

Since the measure f_n converges vaguely to γ and ω is relatively compact,

$$\lim_{n\to\infty}\int f_n\, d\xi=\int d\gamma.$$

[3] Cf. [4], p. 3 and [6].

Hence

$$(u_\gamma, u) \geq \int d\gamma = ||u_\gamma||^2,$$

i.e., $||u|| \geq ||u_\gamma||$. Consequently

$$Cap(\omega) = ||u_\gamma||^2 = \int d\gamma.$$

LEMMA 3. *Let $\mathscr{X}$ be a regular functional space on X and ω be an open set in X. For any increasing net $(\omega_\alpha)_{\alpha \in I}$ of relatively compact open sets exhausting ω,*

$$\lim_{\alpha \in I} Cap(\omega_\alpha) = Cap(\omega).$$

Proof. Obviously $Cap(\omega_\alpha)$ increases with α. First we suppose that $Cap(\omega) < +\infty$. Then $Cap(\omega_\alpha)$ is bounded. Let u_{γ_α} be the pure potential such that $Cap(\omega_\alpha) = ||u_{\gamma_\alpha}||^2$. Suppose that $\alpha \leq \beta$. Then

$$\begin{aligned} ||u_{\gamma_\alpha} - u_{\gamma_\alpha}||^2 &= ||u_{\gamma_\alpha}||^2 - 2(u_{\gamma_\alpha}, u_{\gamma_\beta}) + ||u_{\gamma_\beta}||^2 \\ &\leq ||u_{\gamma_\beta}||^2 - ||u_{\gamma_\alpha}||^2. \end{aligned}$$

Hence (u_{γ_β}) is a fundamental net in $\mathscr{X}$. There exists an element u in $\mathscr{X}$ such that $u_{\gamma_\alpha} \to u$ strongly in $\mathscr{X}$. For any positive bounded measurable function f with compact support such that $S_f \subset \omega$, there exists α_0 in I such that

$$(u_f, u_{\gamma_\alpha}) = \int u_{\gamma_\alpha} f d\xi \geq \int f\, d\xi$$

for any $\alpha \geq \alpha_0$. Therefore

$$(u_f, u) \geq \int f\, d\xi,$$

i.e., $u(x) \geq 1$ *p.p.* in ω. Hence

$$Cap(\omega) \leq ||u||^2.$$

Consequently

$$\lim_{\alpha \in I} Cap(\omega_\alpha) = Cap(\omega).$$

In the case that $Cap(\omega) = +\infty$, it is evident that

$$\lim_{\alpha \in I} Cap(\omega_\alpha) = +\infty$$

by the above proof.

LEMMA 4. Let ω_n be an open set in X ($n=1, 2, \ldots\ldots$).
Put

$$\omega = \bigcup_{n=1} \omega_n$$

Then

$$Cap(\omega) \leq \sum_{n=1}^{\infty} Cap(\omega_n).$$

By Lemmas 2 and 3, we can prove in the same manner as Deny [5].[4)]

RPOPOSITION 1.[5)] *Let $\mathscr{X}$ be a regular functional space on X. For any u in $\mathscr{X}$, there exists a function u^* with the following properties.*

(1.1) *$u(x)=u^*(x)$ p.p. in X and $u^*(x)=0$ outside some σ-compact set.*

(1.2) *There exists a decreasing sequence (ω_n) of open sets such that*

$$\lim_{n\to\infty} Cap(\omega_n)=0$$

and $u^*(x)$ is continuous on $\mathscr{C}\omega_n$ for each n.

(1.3) *For any pure potential u in $\mathscr{X}$, u^* is μ-measurable and*

$$(u, u_\mu)=\int u^*\, d\mu.$$

By Lemmas 1, 2, 3, and 4, we can prove in the same manner as Deny [5].
We say that u^* is *the refinement of u*. Furthermore we have

LEMMA 5. *For any u in $\mathscr{X}$, u^* is μ-measurable for any u_μ in $\mathscr{X}$ such that S_μ^+ is compact and*

$$S_\mu^+ \cap S_\mu^- = \phi.$$

Proof. S_μ^+ being compact, we can take an open set ω in X such that $\omega \supset S_\mu^+$ and

$$S_\mu^- \cap \bar{\omega} = \phi.$$

Put

$$\mathscr{X}_\omega = \{\overline{u \varepsilon C_K \cap \mathscr{X};\ S_u \subset \omega}\}.$$

Then $\mathscr{X}_\omega$ is a regular functional space on ω. We take another open set $\omega^{(1)}$

4) Cf. [5], p. 136.
5) Cf. [2], p. 209.

such that

$$S_{\mu^+}\subset\omega^{(1)}\subset\overline{\omega}^{(1)}\subset\omega.$$

Let (ω_n) be the sequence in Proposition 1. Put

$$\omega_n'=\omega^{(1)}\cap\omega_n.$$

Let $Cap'(\omega_n')$ be the capacity of ω_n' relative to the functional space $\mathscr{X}_\omega$. Obviously

$$\lim_{n\to\infty} Cap'(\omega_n')=0.$$

Let u'_{γ_n} be the pure potential in $\mathscr{X}_\omega$ such that

$$Cap'(\omega_n')=||u'_{\gamma_n}||^2.$$

Then

$$\int_{\omega_n'} d\mu^+\leq(u_\mu, u'_{\gamma_n})\leq||u_\mu||\,||u'_{\gamma_n}||\to 0$$

as $n\to+\infty$. Therefore u^* is μ^+-measurable. Similarly u^* is μ^--measurable.

2. The unit contraction and Condensor principle

First we define the unit contraction on 1-dimensional Euclidean space R.

DEFINITION 5. We call the projection T of R to the closed interval [0, 1] the unit contraction on R.

Let $\mathscr{X}$ be a regular functional space with respect to X and ξ.

DEFINITION 6. We say that the unit contraction T operates on $\mathscr{X}$ if for any u in $\mathscr{X}$, Tu is in $\mathscr{X}$ and $||Tu||\leq||u||$.

DEFINITION 7. We say that $\mathscr{X}$ satisfies the condensor principle if for any couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively compact, there exists a potential u_μ such that

(C. 1) $0\leq u_\mu(x)\leq 1$ *p.p.* in X,

(C. 2) $u_\mu(x)=1$ *p.p.* in ω_1 and $u_\mu(x)=0$ *p.p.* in ω_0,

(C. 3) $u_\mu\in\overline{E_{\omega_1}-E_{\omega_0}}$, where E_{ω_1} and E_{ω_0} are the sets which we defined in Lemma 2.

We shall call the above potential u_μ the condensor potential with respect to ω_1 and ω_0.

LEMMA 6. *Suppose that $\mathscr{X}$ satisfies the condensor principle. For any couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively compact, put*

$$A_{1,0}=\{u\varepsilon\mathscr{X};\ u(x)\geq 1\ p.p.\ in\ \omega_1\ and\ u(x)\leq 0\ p.p.\ in.\ \omega_0\}.$$

Then there exists a unique element in $\mathscr{X}$ whose norm is minimum in $A_{1,0}$ and it is equal to the condensor potential with respect to ω_1 and ω_0.

Proof. Obviously $A_{1,0}$ is non-empty closed convex set in $\mathscr{X}$. Hence there exists a unique element $u_{1,0}$ in $A_{1,0}$ such that $||u_{1,0}||\leq||u||$ for any u in $A_{1,0}$. Let u_μ be the condensor potential with respect to ω_1 and ω_0. Since u_μ is in $A_{1,0}$, $||u_\mu||\geq||u_{1,0}||$. On ther other hand there exists a sequence $(u_{\mu_{1,n}}-u_{\mu_{0,n}})$ such that $u_{\mu_{1,n}}$ and $u_{\mu_{0,n}}$ are pure potentials,

$$S_{\mu_{1,n}}\subset\omega_1,\ S_{\mu_{0,n}}\subset\omega_0$$

and $u_{\mu_{1,n}}-u_{\mu_{0,n}}$ converges strongly to u_μ in $\mathscr{X}$ as $n\to+\infty$. For any u in $A_{1,0}$,

$$(u,\ u_{\mu_{1,n}}-u_{\mu_{0,n}})=\int u^*\,d\mu_{1,n}-\int u^*\,d\mu_{0,n}\geq(u_\mu,\ u_{\mu_{1,n}}-u_{\mu_{0,n}}),$$

because $u^*(x)\geq 1$ *p.p.p.* in ω_1 and $u^*(x)\leq 0$ *p.p.p.* in ω_0.[6] Hence

$$||u||\cdot||u_\mu||\geq(u,\ u_\mu)\geq||u_\mu||^2,$$

i.e., $||u||\geq||u_\mu||$. Consequently $u_{1,0}=u_\mu$.

LEMMA 7. *Let $\mathscr{X}$ be a regular functional space. Each element in $\overline{E_{\omega_1}-E_{\omega_0}}$ is a potential in $\mathscr{X}$.*

Proof. For any u in $\overline{E_{\omega_1}-E_{\omega_0}}$, there exists a sequence $(u_{\mu_n}-u_{\nu_n})$ of $E_{\omega_1}-E_{\omega_0}$ tending strongly to u in $\mathscr{X}$. Since

$$\bar{\omega}_0\cap\bar{\omega}_1=\phi$$

and $C_K\cap\mathscr{X}$ is dense in C_K, (μ_n) and (ν_n) are vaguely bounded. Hence we may assume that there exist positive measures μ and ν such that $\mu_n\to\mu$ and $\nu_n\to\nu$ vaguely as $n\to+\infty$. Therefore

$$(u,f)=\int f\,d(\mu-\nu)$$

for any f in $C_K\cap\mathscr{X}$. Consequently

$$u=u_{\mu-\nu}.$$

[6] Cf. [6], Lemma 2. A property is said to hold *p.p.p.* on a subset E in X if the property holds μ*-p.p.* for any pure potential u_μ in E such that $S_\mu\subset E$.

By Lemma 7, we obtain the following lemma.

LEMMA 8. *Let $\mathscr{X}$ be a regular functional space. Let $A_{1,0}$ be the same as in Lemma 6. The element u' whose norm is minimum $A_{1,0}$ is contained in $\overline{E_{\omega_1}-E_{\omega_0}}$.*

Proof. By Lemma 7, we can consider the following valuation:

$$I'(u_{\mu_1}-u_{\mu_0})=||u_{\mu_1}-u_{\mu_0}||^2-2\int d\mu_1$$

for any $u_{\mu_1}-u_{\mu_0}$ in $\overline{E_{\omega_1}-E_{\omega_0}}$. Similarly as in Lemma 2, $I'(u_{\mu_1}-u_{\mu_0})$ is bounded from below on $\overline{E_{\omega_1}-E_{\omega_0}}$. Since $\overline{E_{\omega_1}-E_{\omega_0}}$ is a non-empty closed convex set in $\mathscr{X}$, there exists a unique element $u_{\gamma_1}-u_{\gamma_0}$ in $\overline{E_{\omega_1}-E_{\omega_0}}$ such that

$$I'(u_{\gamma_1}-u_{\gamma_0})\leq I'(u_{\mu_1}-u_{\mu_0})$$

for any $u_{\mu_1}-u_{\mu_0}$ in $\overline{E_{\omega_1}-E_{\omega_0}}$. Similarly is as the proof of Lemma 2,

$$u'=u_{\gamma_1-\gamma_0}.$$

Now we remark that the regular functional space $\mathscr{X}$ satisfies the equilibrium principle if $\mathscr{X}$ satisfies the condensor principle. That is, for any relatively compact open set ω, there exists a pure potential u_μ such that

(E. 1) $\quad 0\leq u_\mu(x)\leq 1 \quad p.p.$ in X,

(E. 2) $\quad u_\mu(x)=1\ p.p.$ in ω,

(E. 3) $\quad u_\mu$ is contained in E_ω.

Such element u_μ is called an equilibrium potential of ω.

LEMMA 9. *Let $\mathscr{X}$ be the regular functional space which satisfies the condensor principle. For any couple of open sets ω_1 and ω_0 with disjoing closures, ω_1 being relatively compact, let u_μ be the condensor potential with respect to ω_1 and ω_0. Then*

$$\int d\mu\geq 0.$$

Proof. We take a relatively compact open set ω such that $\omega\supset\overline{\omega_1}$. Let u_ν be the equilibrium potential of ω. Since by Lemma 5,

$$u_\nu^*(x)=1 \quad p.p.p. \text{ in } \omega,$$
$$0\leq u_\nu^*(x)\leq 1 \quad p.p.p. \quad \text{in } X,$$

we have

$$(u_\mu, u_\nu)=\int u_\nu{}^*d\mu^+-\int u_\nu{}^*d\mu^-\leq\int d\mu^+-\int_\omega d\mu^-.$$

On the other hand since we have

$$u_\mu{}^*(x)\geq 0 \quad p.p.p. \quad \text{in } X,$$

$$(u_\mu, u_\nu)=\int u_\mu{}^*dv\geq 0.$$

Hence

$$\int d\mu^+\geq\int_\omega d\mu^-.$$

ω being arbitrary, we obtain that the total mass of μ is non-negative.

LEMMA 10. *Let $\mathscr{H}$ be the same as above. Let F_1 be a compact and F_0 be a closed set such that*

$$F_1\cap F_0=\phi.$$

Then there exists a potential u_μ in $\mathscr{H}$ such that

(C'. 1) $\qquad 0\leq u_\mu{}^*(x)\leq 1 \quad p.p. \quad X,$

(C' 2) $\qquad u_\mu{}^*(x)=1 \quad p.p.p. \text{ in } F_1, \; u_\mu{}^*(x)=0 \quad p.p.p. \text{ in } F_0,$

(C' 3) $\qquad S_{\mu}+\subset F_1, S_{\mu}-\subset F_0,$

(C' 4) $\qquad \int d\mu\geq 0.$

Proof. We take two decreasing nets $(\omega_{1,\alpha})_{\alpha\varepsilon I}$ and $(\omega_{0,\alpha})_{\alpha\varepsilon I}$ of open sets converging to F_1, F_0 such that $\omega_{1,\alpha}$ is relatively compact for any $\alpha\varepsilon I$,

$$\omega_{1,\alpha}\supset F_1, \; \omega_{0,\alpha}\supset F_0$$

and for any, $\alpha<\beta$,

$$\overline{\omega_{1,\alpha}}\subset\omega_{1,\beta}, \; \overline{\omega_{0,\alpha}}\subset\omega_{0,\beta}.$$

Let u_{μ_α} be the condensor potential with respect to $\omega_{1,\alpha}$ and $\omega_{0,\alpha}$. Since $u^*_{\mu_\alpha}(x)$ is bounded in X, by Lemma 5,

$$(u_{\mu_\alpha}, u_{\mu_\beta})=\int u_{\mu_\alpha}^*d\mu_\beta^+-\int u_{\mu_\beta}^*d\mu_\beta^-=||u_{\mu_\beta}||^2$$

for any $\alpha\leq\beta$. Hence $||u_{\mu_\alpha}||\geq||u_{\mu_\beta}||$ for any $\alpha\leq\beta$, i.e., $\{||u_{\mu_\alpha}||\}$ is convergent. Furthermore we have

$$||u_{\mu_\alpha}-u_{\mu_\beta}||^2=||u_{\mu_\alpha}||^2-2(u_{\mu_\alpha}, u_{\mu_\beta})+||u_{\mu_\beta}||^2=||u_{\mu_\alpha}||^2-||u_{u_\beta}||^2.$$

Therefore there exists an element u in $\mathscr{X}$ such that $u_{\mu_\alpha}\to u$ strongly in $\mathscr{X}$. Obviously the sets $(\mu_\alpha^+)_{\alpha\in I}$ and $(\mu_\alpha^-)_{\alpha\in I}$ are vaguely bounded, and hence we may assume that there exist two positive measures μ_1 and μ_0 such that $(\mu_\alpha^-)_{\alpha\in I}$ and (μ_α^-) converge vaguely to μ_1 and, μ_0, respectively. By the definition of a potential in $\mathscr{X}$,

$$u=u_{\mu_1-\mu_0}.$$

We shall show that this element u is the required element. Evidently

$$S_{\mu_1}\subset F_1, S_{\mu_0}\subset F_0.$$

Since we have

$$u_{\mu_\alpha}^*=1 \text{ } p.p.p. \text{ in } \omega_{1,\alpha} \text{ and } u_{\mu_\alpha}^*=0 \text{ } p.p.p. \text{ in } \omega_{0,\alpha},$$

$$u^*=1 \text{ } p.p.p. \text{ in } F_1 \text{ and } u^*=0 \text{ } p.p.p. \text{ in } F_0.$$

It is evident that u satisfies the condition (C'. 1). Finally we prove that u satisfies the condition (C'. 4). $S_{\mu_\alpha^\pm}$ being in a fixed compact set,

$$\lim_{\alpha\in I}\int d\mu_\alpha^+=\int d\mu_1.$$

On the other hand

$$\lim_{\alpha\in I}\int d\mu_\alpha^-\geq\int d\mu_0.$$

By Lemma 9, we obtain the inequality

$$\int d\mu_1\geq\int d\mu_0.$$

We call such a potential u_μ the condensor potential with respect to F_1 and F_0. Now we consider the strong complete maximum principle.

DEFINITION 7.[6]) We say that a regular functional space $\mathscr{X}$ satisfies the strong complete maximum principle if the following condition is fulfiled. For a potential u_f, f being locally summable for ξ, and a pure potential u_ν in $\mathscr{X}$ and a non-negative constant c, suppose that

$$u_f^*(x)\leq u_\nu^*(x)+c$$

$p.p.p.$ on K_{f^+}. Then

$$u_f(x) \leq u_\nu(x)+c$$

p.p. in X.

In this definition, K_{f^+} is a set whose complement is of f^+-measure zero.

By the above lemmas, we obtain the following theorem.

THEOREM 1. *If a regular funtiocnal space* $\mathscr{X}$ *satisfies the condensor principles, then* $\mathscr{X}$ *satisfies the strong complete maximum principle.*

Proof. Let u_f, u_ν and c be the same as in Definition 7. Suppose that there exists a compact set K_1 in $\mathscr{C}K_{f^+}$ such that $\xi(K_1) > 0$ and

$$u_f(x) > u_\nu(x)+c$$

on K_1. Since

$$u_f^*(x)=u_f(x)\ p.p. \text{ in } X \text{ and } u_\nu^*(x)=u_\nu(x)\ \ p.p. \text{ in } X,\ u_f^*(x) > u_\nu^*(x)+c$$

p.p. on K_1. Therefore there exists a compact set K_2 in K_1 such that $\xi(K_2) > 0$ and

$$u_f^*(x) > u_\nu^*(x)+c$$

on K_2. By Proposition 1, there exists a decreasing sequence (ω_n) of open sets such that

$$\lim_{n\to\infty} Cap(\omega_n)=0,$$

$u_f^*(x)$ and $u_\nu^*(x)$ are continuous on $\mathscr{C}\omega_n$. Since $\xi(\omega_n) \searrow 0$ as $n \to +\infty$, there exists a number n such that

$$\xi(K_2 \cap \mathscr{C}\omega_n) > 0.$$

We take a compact set K such that

$$K \subset K_2 \cap \mathscr{C}\omega_n \text{ and } \xi(K) > 0.$$

Then $u_f^*(x)$ and $u_\nu^*(x)$ are continuous and $u_f^*(x) > u_\nu^*(x)+c$ on K, and hence there exists a positive number a such that

$$u_f^*(x)-u_\nu^*(x)-c > a$$

on K. Since f is locally summable for ξ, there exists an open set G such that $G \supset K$ and

$$\int_G f^+(x)d\xi(x) < \frac{1}{2}a\cdot Cap(K),$$

where

$$Cap(K) = \inf_{k \subset \omega} Cap(\omega),$$

because we have

$$\int_K f^+(x) d\xi(x) = 0 \text{ and } Cap(K) > 0.$$

Put

$$K'_{f^+} = K_{f^+} \cap \mathscr{C} G.$$

By the measurablity of f, there exists an increasing sequence (F_n) of compact sets such that $F_n \subset K'_{f^+}$ and

$$\lim_{n \to \infty} \xi(F_n \cap F) = \xi(K'_{f^+} \cap F)$$

for any compact set F. Let u_{μ_n} be the condensor potential with respect to K and F_n. Similarly as the proof of Lemma 10, there exists a potential u_μ such that $u_{\mu_n} \to u_\mu$ strongly in $\mathscr{X}$ and $S_{\mu^+} \subset K$. By Lemmas 9 and 10,

$$(u_\mu, u_\nu) = \int (u_f^*(x) - u_\nu^*(x))\, d\mu \geq (a+c)\int d\mu^+ - c\int d\mu^- \geq a\int d\mu^+ = a\|u_\mu\|^2 \geq a \cdot \mathrm{Cap}(K).$$

Let $(G_\alpha)_{\alpha \in I}$ be an increasing net of relatively compact open sets such that $G_\alpha \supset G$ and $G_\alpha \nearrow X$. Similarly as the above, we can take the condensor potential u_{μ_α} with respect to K and $K'_{f^+} \cup \mathscr{C} G_\alpha$. Since u_{μ_α} is a bounded measurable function with compact support, u_{μ_α} is f-integrable and

$$(u_{\mu_\alpha}, u_f - u_\nu) = \int u_{\mu_\alpha}(x) f^+(x) d\xi(x)$$

$$- \left(\int u_{\mu_\alpha}(x) f^-(x) d\xi(x) + \int u^*_{\mu_\alpha}(x) d\nu(x) \right)$$

$$\leq \int_G u_{\mu_\alpha}(x) f^+(x) d\xi(x) \leq \int_G f^+(x) d\xi(x) \leq \frac{1}{2} a \cdot Cap(K).$$

Now since $(u_{\mu_\alpha})_{\alpha \in I}$ converges strongly to u_μ in $\mathscr{X}$,

$$(u_\mu, u_f - u_\nu) \leq \frac{1}{2} a \cdot Cap(K).$$

This is a contradiction and the proof is completed.

3. Main theorems

First we consider the resolvent operator on a regular functional space $\mathscr{X}$

or $L^2=L^2(\xi)$.

LEMMA 11.[7] *Let f be in L^2 or in $\mathscr{X}$. For each positive number λ, there exists a unique element $R_\lambda f$ in $\mathscr{X}$ which minimizes the following quadratic form:*

$$F(u)=\|u\|^2+\int|u(x)-f(x)|^2 d\xi(x)$$

in the set

$$A_f=\{u\varepsilon\mathscr{X};\ u-f\varepsilon L^2\}.$$

$R_\lambda f$ is also the only element u in $\mathscr{X}$ such that $u-f$ is in L^2 and

$$\lambda(u,v)+\int(u-f)v\,d\xi=0$$

for any v in $L^2\cap\mathscr{X}$.

This is obtained by Beurling and Deny [2] for the case when $\mathscr{X}$ is a Dirichlet space. For the case when $\mathscr{X}$ is a regular functional space, this is proved in the same way. We call such an operator R_λ the resolvent operator. Before we prove the main theorem, we prepare the following lemma.

LEMMA 12. *Let $\mathscr{X}$ be a regular functional space on X. Suppose that $\mathscr{X}$ satisfies the strong complete maximum principle. Then for any positive bounded function f with compact support,*

$$0\leq R_\lambda f(x)\leq M$$

p.p. in X, where

$$M=\operatorname*{ess.sup}_{x\varepsilon X} f(x).$$

Proof. First we shall prove that

$$R_\lambda f(x)\geq 0$$

p.p. in X. By the second part of Lemma 11, $R_\lambda f$ is the potential generated by $f-R_\lambda f$ in $\mathscr{X}$. Since the potential u_f generated by f is in $\mathscr{X}$, there exists a potential $u_{R_\lambda f}$ generated by $R_\lambda f$ in $\mathscr{X}$. Then

$$u_f-\lambda R_\lambda f=u_{R_\lambda f}\,.$$

Hence

$$u_f^*(x)-\lambda(R_\lambda f)^*(x)=u_{R_\lambda f}^*(x)$$

[7] Cf. [2], p. 211.

p.p.p. in X. Since

$$R_\lambda f(x)=(R_\lambda f)^*(x)$$

p.p. in X, we have

$$u_{R_\lambda f}=u_{(R_\lambda f)*}.$$

Since

$$u_f^*(x)\geq u^*_{(R_\lambda f)^+}(x)$$

p.p.p. on $K_{(R_\lambda f)*+}$, by Theorem 1,

$$u_f(x)\geq u_{(R_\lambda f)^*}(x)$$

p.p. in X. Therefore $R_\lambda f\geq 0$ *p.p.* in X.

Next we shall show that

$$R_\lambda f(x)\leq M$$

p.p. in X. There exists a function g in C_K such that $g(x)\geq f(x)$ *p.p.* in X and $g(x)\leq M$. Since by the above arguement, R_λ is a positive operator,

$$R_\lambda f(x)\leq R_\lambda g(x)$$

p.p. in X. Similarly as above,

$$(R_\lambda g)^*(x)=u^*_{g-(R_\lambda g)^+}(x)$$

p.p.p. in X. Similarly as in the first part of this lemma,

$$M\geq g(x)\geq (R_\lambda g)^*(x)$$

p.p.p. in $K_{((g-R_\lambda g)^*)^+}$. Hence

$$M\geq u_{(g^*-(R_\lambda g)^*)}(x)$$

p.p.p. in $K_{(g-(R_\lambda g)^*)^+}$. By the strong complete maximum principle,

$$M\geq u_{g-(R_\lambda g)^*}(x)$$

p.p. in X. Consequently

$$R_\lambda f\leq R_\lambda g\leq M$$

p.p. in X. This completes the proof.

Now we shall show the following main theorem.

THEOREM 2. *Let $\mathscr{X}$ be a regular functional space with respect to X and ξ.*

Then the following four conditions are equivalent.

(1) *The unit contraction operates on* $\mathscr{X}$.

(2) $\mathscr{X}$ *satisfies the condensor principle.*

(3) $\mathscr{X}$ *satisfies the strong complete maximum principle.*

(4) $\mathscr{X}$ *is a real Dirichlet space with respect to* X *and* ξ.[8)]

Proof. First we shall prove the implication (1) $\Rightarrow$ (2). For any couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively compact, let $A_{1,0}$, E_{ω_1} and E_{ω_0} be the same as defined before. Let $u_{1,0}$ be a unique element in $\mathscr{X}$ whose norm is minimum in $A_{1,0}$. Since the unit contraction T operates on $\mathscr{X}$, $Tu_{1,0}$ is in $A_{1,0}$ and

$$\|Tu_{1,0}\| \leq \|u_{1,0}\|.$$

Therefore $Tu_{1,0}=u_{1,0}$. By Lemma 8, $u_{1,0}$ belongs to $\overline{E_{\omega 1}-E_{\omega_0}}$ and hence it is the condensor potential with respect to ω_1 and ω_0.

The implication (2) $\Rightarrow$ (3) was proved in Theorem 1.

Next we shall show the implication (3) $\Rightarrow$ (4). For a positive number λ, let R_λ be a resolvent operator. For any f, g in $C_K \cap \mathscr{X}$,

$$(R_\lambda f, R_\lambda g)=\frac{1}{\lambda}\int(f-R_\lambda f)R_\lambda g\, d\xi=\frac{1}{\lambda}\int(g-R_\lambda g)R_\lambda f\, d\xi,$$

Hence

$$(R_\lambda f,\ g)=(R_\lambda g,\ f)$$

and

$$\int R_\lambda f g\, d\xi=\int R_\lambda g f\, d\xi.$$

Hence by Lemma 12, there exists a positive symmetric measure σ_λ on $X\times X$ such that

$$\int R_\lambda f(x)g(x)\, d\xi(x)=\iint f(x)g(y)\, d\sigma_\lambda(x, y)$$

for any f, g in C_K and σ_λ is sub-markovian, *i. e.*, the projection of σ_λ on X is less than or equal to ξ. Let m_λ be the density of the projection of σ_λ on X. By the second part of Lemma 11, for any f, g in $C_K \cap \mathscr{X}$,

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8) A real Dirichlet space with respect to X and ξ is a Dirichlet space with respect to X and ξ which consists of real functions. For Dirichlet spaces, see [2], p. 209.

$$(R_\lambda f,\ g)=\frac{1}{\lambda}\int(f-R_\lambda f)g\,d\xi$$
$$=\frac{1}{\lambda}\left\{\int(1-m_\lambda)fg\,d\xi+\frac{1}{2}\iint(f(x)-f(y))(g(x)-g(y))\,d\sigma_\lambda(x,y)\right\}$$

Now by the first part of Lemma 11, for any positive number λ,

$$\|R_\lambda f\|\leq\|f\|.$$

And by the second part of Lemma 11,

$$(R_\lambda f,\ R_\lambda f-f)=-\int|R_\lambda f-f|^2 d\xi.$$

Therefore $R_\lambda f\to f$ strongly in L^2, and hence $R_\lambda f\to f$ weakly in $\mathscr{X}$ as $\lambda\to 0$. Since

$$\lim_{\lambda\to 0}\|R_\lambda f\|\geq\|f\|\geq\|R_\lambda f\|$$

for any $\lambda>0$, $R_\lambda f\to f$ strongly in $\mathscr{X}$ as $\lambda\to 0$. Next we shall prove the following assertion: for a function f in C_K, suppose that

$$H_\lambda(f)=\frac{1}{\lambda}\left\{\int(1-m_\lambda)|f|^2\,d\xi+\frac{1}{2}\iint|f(x)-f(y)|^2\,d\sigma_\lambda(x,\ y)\right\}$$

is bounded with respect to λ. Then f is in $\mathscr{X}$ and $H_\lambda(f)\to\|f\|^2$ as $\lambda\to 0$. In fact,

$$H_\lambda(f)=\frac{1}{\lambda}\int(1-R_\lambda f)\,f\,d\xi\geq\frac{1}{\lambda}\int(f-R_\lambda f)\,R_\lambda f\,d\xi=\|R_\lambda f\,\|^2.$$

Hence $(R_\lambda f)$ is bounded with respect to λ, and we may assume that there exists an element u in $\mathscr{X}$ such that $R_\lambda f\to u$ weakly in $\mathscr{X}$ as $\lambda\to 0$. On the other hand by the second part of Lemma 11, $R_\lambda f\to f(x)$ *p.p.* in X. Consequently $u(x)=f(x)$ *p.p.* in X, *i, e.*, f is in $\mathscr{X}$ and $H_\lambda(f)\to\|f\|^2$ as $\lambda\to 0$. Thus we obtain:

For any f in $C_K\cap\mathscr{X}$ and any normal contraction T[9] on R, Tf is in $\mathscr{X}$ and $\|Tf\|\leq\|f\|$. Because Tf is in C_K and

$$H_\lambda(Tf)\leq H_\lambda(f)$$

for any λ.

Furthermore for any u in $\mathscr{X}$, there exists a sequence (f_n) in $C_K\cap\mathscr{X}$ converging to u. By the results that Tf_n is in $\mathscr{X}$, $\|Tf_n\|\leq\|f_n\|$ and $Tf_n(x)$ converges to $Tu(x)$ *p.p.* in X, Tu is in $\mathscr{X}$ and $\|Tu\|\leq\|u\|$. Consequently $\mathscr{X}$

is a real Dirichlet space.

The implication (4) $\Rightarrow$ (1) is evident. This completes the proof.

By the above main theorem, we obtain the following another characterization of a real Dirichlet space.

THEOREM 3. *A regular functional space* $\mathscr{X}$ *is a real Dirichlet space if and only if there exists number* $M \neq 0$ *such that* u_M *is in* $\mathscr{X}$ *and* $\|u_M\| \leq \|u\|$ *for any* u *in* $\mathscr{X}$, *where*

$$u_M(x) = \inf(u(x), M)$$

if $M > 0$,

$$u_M(x) = \sup(u(x), M)$$

if $M < 0$.

Proof. Suppose that there exists a number $M \neq 0$ such that u_M is in $\mathscr{X}$ and $\|u_M\| \leq \|u\|$. It is sufficient to prove the thoerem for the case $M > 0$. Put

$$u_1(x) = \inf(u(x). 1)$$

for any u in $\mathscr{X}$. Then

$$u_1(x) = M^{-1} \inf(Mu(x), M),$$

and hence u_1 is in $\mathscr{X}$ and $\|u_1\| \leq \|u\|$. On the other hand for a sequence (a_n) of negative numbers tending to 0.,

$$u_{a_n}(x) = \sup(u(x),\ a_n) = \frac{a_n}{M} \inf\left(\frac{M}{a_n} u(x), M\right).$$

Hence u_{a_n} is in $\mathscr{X}$ and $\|u_{a_n}\| \leq \|u\|$. We may assume that there exists an element u' such that $u_{a_n} \to u'$ weakly in $\mathscr{X}$. Since $u_{a_n}(x)$ converges to $u'(x)$ *p.p.* in X, u^+ is in $\mathscr{X}$ and

$$\|u\| \geq \varliminf_{n \to \infty} \|u_{a_n}\| \geq \|u^+\|$$

Let T be the unit contraction on R. Then $Tu = u_1{}^+$. Consequently T operates on $\mathscr{X}$. By Theorem 2, $\mathscr{X}$ is a real Dirichlet space.

The converse is evident. This completes the proof.

DEFINITION 8. We say that the positive contraction on R operates on a regular functional space $\mathscr{X}$ if for any u in $\mathscr{X}$, u^+ is in $\mathscr{X}$ and $\|u^+\| \leq \|u\|$.

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9) A normal contraction T is a transformation of R into itself such that $|Ta_1 - Ta_2| \leq |a_1 - a_2|$ for any couple a_1 and a_2 in R and $T(0) = 0$. Cf. [2], p. 209.

Remark. There exists a regular functional space on which the positive contraction operates and which is not a real Dirichlet space. We can construct such an example when X is a finite space. (*Cf.* [1].)

Similarly as Theorem 2, we obtain the following theorem. First we give a definition.

DEFINITION 9.[10] We say that a regular functional space satisfies the balayage principle if the following condition is satisfied: for any pure potential u_μ and any open set ω in X, there exists a pure potential $u_{\mu'}$ such that

(B. 1) $u_\mu(x) \geqq u_{\mu'}(x)$ *p.p.* in X,

(B. 2) $u_\mu(x) = u_{\mu'}(x)$ *p.p.* in, ω,

(B. 3) $u_{\mu'} \varepsilon E_\omega$.

THEOREM 4. *A regular functional space $\mathscr{X}$ satisfies the balayage principle if and only if the positive contraction operates on $\mathscr{X}$.*

We can prove in the same way as the proof of Theorem 2.

4. Special Dirichlet spaces

Let X be a locally compact abelian group and ξ be the Haar measure on X which we denote by dx.

DEFINITION 10.[11] A functional space $\mathscr{X}$ with respact to X and ξ is called an invariant functional space if for any x in X and any u in $\mathscr{X}$,

$$U_x u \in \mathscr{X} \text{ and } \|U_x u\| = \|u\|,$$

where $U_x u$ is a function obtained from u by the translation x (*i.e.*, $U_x u(y) = u(y-x)$).

DEFINITION 11.[12] An invariant functional space $\mathscr{X}$ is called a special Dirichlet space if $\mathscr{X}$ is a real Dirichlet space.

LEMMA 13. *For any u in an invariant functional space $\mathscr{X}$ and any bounded measurable function f with compact support, $u*f$ is in $\mathscr{X}$ and*

$$(u*f, v) = \int (U_{-x}u, v) f dx$$

for any v in $\mathscr{X}$.

10) Cf. [2], p. 210.

11) After Deny's terminology, this is the functional space which is invariant by the transtion.

12) Cf. [2], p. 215.

For the proof, see [3] and [4].

Using Theorem 2, we obtain the following theorem.

THEOREM 5. *An invariant functional space $\mathscr{X}$ is a special Dirichlet space if and only if $\mathscr{X}$ satisfies the condensor principle.*[13)]

Proof. It is well-known that a special Dirichlet space satisfies the condensor principle. It is sufficient to prove the "if" part. By Lemma 13 and the condensor principle, $C_K \cap \mathscr{X}$ is total in C_K.[14)] We shall show that $C_K \cap \mathscr{X}$ is dense in $\mathscr{X}$. Put

$$\mathscr{X}' = \overline{C_K \cap \mathscr{X}},$$

Then by Theorem 2, $\mathscr{X}'$ is a special Dirichlet space on X. First we shall prove that for each u in $\mathscr{X}$ with compact support, u is in $\mathscr{X}'$. We take a net $(f_\alpha)_{\alpha \in I}$ of C_K such that

$$f_\alpha(x) \geq 0, \int f_\alpha(x)\, dx = 1$$

and $(f_\alpha)_{\alpha \in I}$ converges vaguely to the unit measure ε at 0 and (S_{f_α}) converges to $\{0\}$. Since the mapping: $x \to U_x u$ is strongly continuous for any u in $\mathscr{X}$, there exists α_0 in I such that

$$||U_x u - u|| < \delta$$

for any $x \in -S_{f_\alpha}$, $\alpha \geq \alpha_0$, for a given positive number δ. Therefore

$$||u*f_\alpha - u||^2 = ||u*f_\alpha||^2 - 2(u*f_\alpha, u) + ||u||^2 < 4||u||\delta + \delta^2.$$

$u*f_\alpha$ is in $C_K \cap \mathscr{X}$, and hence u is in $\mathscr{X}'$. Let $(F_\alpha)_{\alpha \in J}$ be a net of compact sets such that $F_\alpha \to X$. Put

$$E_{\mathscr{C}F_\alpha} = \overline{\left\{ \begin{array}{l} u_f \in \mathscr{X};\ f \text{ is a bounded measurable function with compact support} \\ S_f \subset \mathscr{C}F_\alpha \end{array} \right\}}.$$

Then $E_{\mathscr{C}F_\alpha}$ is a closed subspace of $\mathscr{X}$. For any u in $\mathscr{X}$, let u_α be the projection of u to $E_{\mathscr{C}F_\alpha}$. Then $u(x) = u_\alpha(x)$ *p.p.* in $\mathscr{C}F_\alpha$. Hence by the above result, $u - u_\alpha$ is in $\mathscr{X}'$. On the other hand obviously (u_α) converges strongly

13) Let ω be an open set in X and the notation E_ω be the same as in Lemma 2. Without the condition of regularity, we can only consider potentials generated by bounded measurable functions with compact support. Then $E_\omega = \overline{\{u_f \in \mathscr{X};\ S_f \subset \omega\}}$.

14) Cf. [6].

to 0 in $\mathscr{X}$, hence $(u-u_\alpha)$ converges strongly to u. That is, u is in $\mathscr{X}'$. Consequently $\mathscr{X}$ is a special Dirichlet space.

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