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THETA-FUNCTIONS AND HILBERT MODULAR FORMS

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Introduction

The purpose of this note is to show how the theta-functions attached to certain indefinite quadratic forms of signature (2, 2) can be used to produce a map from certain spaces of cusp forms of Nebentype to Hilbert modular forms. The possibility of making such a construction was suggested by Niwa [4], and the techniques are the same as his and Shintani's [6]. The construction of Hilbert modular forms from cusp forms of one variable has been discussed by many people, and I will not attempt to give a history of the subject here. However, the map produced by the theta-function is essentially the same as that of Doi and Naganuma [2], and Zagier [7]. In particular, the integral kernel $\Omega(\tau, z_1, z_2)$ of Zagier is essentially the 'holomorphic part' of the theta-function.

Professor Asai has kindly informed me that he has also considered the case of signature (2, 2) and has obtained similar results. In [9], Professor Asai has studied the case of signature (3, 1) and has shown that forms of signature (3, 1) can be used to produce a lifting of cusp forms of Neben type to modular forms on hyperbolic 3-space with respect to discrete subgroups of $SL_2(\boldsymbol{C})$. The case of signature $(n-2, 2)$ has been considered by Rallis and Schiffman [10], [11], and by Oda [12].

1. Construction of the theta-functions

Let $k = \boldsymbol{Q}(\sqrt{\Delta})$ be the real quadratic field with discriminant Δ, and let σ be the Galois automorphism of $k/\boldsymbol{Q}$. Let

$$V = \{X \in M_2(k) \text{ such that } X^{\iota} = -X^{\sigma}\}$$
$$= \left\{X = \begin{pmatrix} x_1 & x_4 \\ x_3 & -x_1^{\sigma} \end{pmatrix};\ x_1 \in k,\ x_3,\ x_4 \in \boldsymbol{Q}\right\}.$$

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Let $Q(X) = -2\det(X)$ and $(X, Y) = -\mathrm{tr}(XY^\iota)$ where ι is the usual involution of $M_2(k)$. Then V is a rational vector space and Q is a $\boldsymbol{Q}$ valued non-degenerate quadratic form on V. Let $SO(Q)$ be the special orthogonal group of Q over $\boldsymbol{Q}$, and let $G = SL_2(k)$ viewed as an algebraic group over $\boldsymbol{Q}$. Then define a rational representation $\rho: G \to SO(Q)$ by $\rho(g)X = g^\sigma X g^\iota$ for $g \in G$ and $X \in V$.

Let $V_{\boldsymbol{R}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{R} \cong \{X = (X_1, X_2) \in M_2(\boldsymbol{R}) \times M_2(\boldsymbol{R}),\ X_1^\iota = -X_2\}$, and identify $V_{\boldsymbol{R}}$ with $M_2(\boldsymbol{R})$ via the projection $X \to X_1$ on the first factor. Then if $X = \begin{pmatrix} x_1 & x_4 \\ x_3 & x_2 \end{pmatrix} \in V_{\boldsymbol{R}}$, $Q(X) = 2(x_3x_4 - x_1x_2)$.

Let $SO(Q)^0_{\boldsymbol{R}}$ be the connected component of the special orthogonal group of $V_{\boldsymbol{R}}, Q$. Identify $G_{\boldsymbol{R}} \cong SL_2(\boldsymbol{R}) \times SL_2(\boldsymbol{R})$, and extend the representation ρ to $\rho: G_{\boldsymbol{R}} \to SO(Q)^0_{\boldsymbol{R}}$ via $\rho(g)X = g_2 X g_1^\iota$ for $g = (g_1, g_2) \in G_{\boldsymbol{R}}$ and $X \in V_{\boldsymbol{R}}$.

Let $L^2(V_{\boldsymbol{R}})$ = square integrable functions on $V_{\boldsymbol{R}}$ for Lebesgue measure, and let $S(V_{\boldsymbol{R}})$ = Schwartz functions on $V_{\boldsymbol{R}}$. Then for $\sigma \in SL_2(\boldsymbol{R})$, let $r(\sigma, Q)$ be the unitary operator on $L^2(\boldsymbol{R})$ defined by:

$$r(\sigma, Q)f(X) = \begin{cases} |a|^2\, e[(ab/2)(X, X)]f(aX) & \text{if } c = 0 \\ |c|^{-2}\,|\det Q|^{1/2} \displaystyle\int_{V_{\boldsymbol{R}}} e\left[\frac{a(X, X) - 2(X, Y) + d(Y, Y)}{c}\right] f(Y)dY & \\ & \text{if } c \neq 0\,. \end{cases}$$

Here $e[t] = e^{2\pi i t}$, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For details see [6].

Let $G_{\boldsymbol{R}}$ act in $L^2(V_{\boldsymbol{R}})$ via $(g \cdot f)(X) = f(\rho(g)^{-1}X)$. Then the operators $r(\sigma, Q)$ and g commute and preserve the space $S(V_{\boldsymbol{R}})$.

Let $S(V_{\boldsymbol{R}})_{2\nu} = \left\{f \in S(V_{\boldsymbol{R}}) \text{ s.t. } r(k_\theta, Q)f = e^{i\nu\theta} f,\ \forall k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\}$.

For $X \in V_{\boldsymbol{R}}$, let $R(X) = x_1^2 + x_2^2 + x_3^2 + x_4^2$; then R is a majorant of Q and $\rho(SO(2) \times SO(2)) \subset SO(Q)^0_{\boldsymbol{R}} \cap SO(R)$.

Let $\mathscr{H}_{Q,R} = \{r \in V_{\boldsymbol{C}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{C} \cong M_2(\boldsymbol{C}) \text{ s.t. } Qr = Rr, \text{ and } Q(r) = 0\}$. Then, $\mathscr{H}_{Q,R} = \boldsymbol{C}r \cup \boldsymbol{C}\bar{r}$, where $r = \begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix}$. Moreover, $R(X) + Q(X) = |(X, r)|^2$.

Now for $\nu \in \boldsymbol{Z}_{>0}$, let $f(X) = (X, r)^\nu e^{-\pi R(X)}$. Then $f \in S(V_{\boldsymbol{R}})_{2\nu}$, [6, lemma 1.2]; and if $k = (k_{\theta_1}, k_{\theta_2}) \in SO(2) \times SO(2)$, then $k \cdot f = e^{-i\nu(\theta_1 + \theta_2)} f$.

For $M \in \boldsymbol{Q}_{>0}$, let $Q_M(X) = MQ(X)$, $(\ ,\ )_M = M(\ ,\ )$, and $R_M(X) = MR(X)$. Then R_M is a majorant of Q_M, $\mathscr{H}_{Q_M, R_M} = \mathscr{H}_{Q,R}$, $R_M(X) + Q_M(X) = M^{-1}|(X, r)_M|^2$, and $f_M(X) = (X, r)_M^\nu e^{-\pi R_M(X)}$ is in $S(V_{\boldsymbol{R}})_{2\nu}$ with respect to the operators $r(\sigma, Q_M)$.

Let L be a lattice in V, and let $L_M^* = \{Y \in V \text{ s.t. } (X, Y)_M \in \boldsymbol{Z}, \forall X \in L\}$. *Assume* $L_M^* \supset L$. Then for $z = u + iv \in \mathfrak{h} =$ the upper half-plane, $g \in G_{\boldsymbol{R}}$, and $h \in L_M^*$, define the *theta-function*:

$$\theta(z, g, h) = v^{-\nu/2} \sum_{\ell \in L} \{r(\sigma_z, Q_M) f_M\}(\rho(g)^{-1}(\ell + h))$$

where

$$\sigma_z = \begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix} \in SL_2(\boldsymbol{R}) .$$

Transformation law: If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z})$, such that $\forall X, Y \in L$, $ab(X, X) \equiv cd(Y, Y) \equiv 0(2)$, and $cL_M^* \subset L$, $c(Y, Y) \equiv 0(2)$, $\forall Y \in L_M^*$, $c \neq 0$: Then

$$\theta(\gamma z, g, h) = \left(\frac{D}{d}\right) J(\gamma, z)^\nu e\left[\frac{1}{2} ab(h, h)_M\right] \theta(z, g, ah)$$

where $D = D(L) = \det((\lambda_i, \lambda_j))$ for some $\boldsymbol{Z}$ basis of L, $(-)$ is the quadratic symbol as in Shimura [5], and $J(\gamma, z) = cz + d$.

In particular, if $N_0 \in Z_{>0}$ such that $N_0 L_M^* \subset L$, and $N_0(X, X) \equiv 0(2)$, $\forall X \in L_M^*$, $N = 4N_0$. Then,

$$\forall \gamma \in \Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}),\ c \equiv b \equiv 0(N),\ a \equiv d \equiv 1(N)\right\} ,$$
$$\theta(\gamma z, g, h) = J(\gamma, z)^\nu \theta(z, g, h) .$$

Moreover, let $\Gamma_L = \{g \in SL_2(k) \text{ s.t. } \rho(g)L = L\}$. Then Γ_L preserves L_M^*, and $\forall g' \in \Gamma_L$,

$$\theta(z, g'g, h) = \theta(z, g, \rho(g')^{-1}h) .$$

Remark. These transformation laws follow easily from Propositions 1.6 and 1.7 of Shintani [6], and hold for analogous functions constructed from any $f \in S(V_{\boldsymbol{R}})_{2\nu}$. For the particular f chosen above, they could be proved just as in Siegel [8] and Shimura [5]. In fact,

$$r(\sigma_z, Q)f(X) = ve[\tfrac{1}{2}u(X, X)]v^{\nu/2}(X, r)^\nu e^{-\pi v R(X)} .$$

So that,

$$\theta(z, g, h) = v \sum_{\ell \in L} (\rho(g)^{-1}(\ell + h), r)^\nu e^{i\pi(uQ + ivR)(\rho(g)^{-1}(\ell + h))} .$$

It should be noted that $\theta(z, g, h)$ is not holomorphic in z.

2. The inner product with the Poincaré series

Since M will be fixed throughout this section, it will be dropped as a subscript e.g. $(\ ,\ ) = (\ ,\ )_M$.

Let $N = 4N_0$ as before.

Let $S_\nu(\Gamma(N))$ be the space of cusp forms of weight ν for $\Gamma(N)$. Then for $\varphi \in S_\nu(\Gamma(N))$, the following integral is well defined:

$$\Psi(g, h) = \int_{\mathscr{F}_N} \varphi(z)\overline{\theta(z, g, h)}v^{\nu-2}dudv$$

where $\mathscr{F}_N$ is a fundamental domain for $\Gamma(N)$.

Now assume that $\nu > 2$, and let $\Gamma_\infty = \{\gamma \in \Gamma(N) \text{ s.t. } \gamma\infty = \infty\}$. Let $\mathscr{R}$ = a set of representatives for $\Gamma_\infty\backslash\Gamma(N)$, and let

$$\varphi_n(z) = \frac{1}{N}\sum_{\gamma\in\mathscr{R}} J(\gamma, z)^{-\nu}e\left[\frac{n}{N}\gamma z\right]$$

be the n-th Poincaré series for $\Gamma(N)$ of weight ν. Let

$$\Psi_n(g, h) = \int_{\mathscr{F}_N} \varphi_n(z)\overline{\theta(z, g, h)}v^{\nu-2}dudv\ .$$

PROPOSITION 1. *If $\nu \geq 7$, $n > 0$, then:*

$$\Psi_n(g, h) = \pi^{-\nu}\Gamma(\nu)M \sum_{\substack{\ell\in L \\ (\ell+h, \ell+h)=2n/N}} (\rho(g)^{-1}(\ell + h), r)^{-\nu}\ .$$

Proof.

$$\begin{aligned}
\Psi_n(g, h) &= \int_{\mathscr{F}_N} \varphi_n(z)\overline{\theta(z, g, h)}v^{\nu-2}dudv \\
&= \frac{1}{N}\int_{\mathscr{F}_N}\Big(\sum_{\gamma\in\mathscr{R}} J(\gamma, z)^{-\nu}e\left[\frac{n}{N}\gamma z\right]\Big)\overline{\theta(z, g, h)}v^{\nu-2}dudv \\
&= \frac{1}{N}\sum_{\gamma\in\mathscr{R}}\int_{\mathscr{F}_N} J(\gamma, z)^{-\nu}e\left[\frac{n}{N}\gamma z\right]\overline{\theta(z, g, h)}v^{\nu-2}dudv \\
&= \frac{1}{N}\sum_{\gamma\in\mathscr{R}}\int_{\gamma\mathscr{F}_N} J(\gamma, \gamma^{-1}z)^{-\nu}e\left[\frac{n}{N}z\right]\overline{\theta(\gamma^{-1}z, g, h)}v(\gamma^{-1}z)^{\nu}v^{-2}dudv \\
&= \frac{1}{N}\sum_{\gamma\in\mathscr{R}}\int_{\gamma\mathscr{F}_N} e\left[\frac{n}{N}z\right]\overline{\theta(z, g, h)}v^{\nu-2}dudv \\
&= \frac{1}{N}\int_{\mathscr{F}_\infty} e\left[\frac{n}{N}z\right]\overline{\theta(z, g, h)}v^{\nu-2}dudv
\end{aligned}$$

where $\mathscr{F}_\infty$ is a fundamental domain for Γ_∞. Take $\mathscr{F}_\infty = \{z \in \mathfrak{H} \text{ s.t. } 0 \leq \operatorname{Re} z \leq N\}$,

$$\begin{aligned}\Psi_n(g,h) &= \frac{1}{N}\int_0^\infty\int_0^N e\Big[\frac{n}{N}z\Big]v^{-\nu/2}\sum_{\ell\in L} ve\Big[-\frac{u}{2}(\ell+h,\ell+h)\Big] \\ &\qquad\times \overline{f(v^{1/2}\rho(g)^{-1}(\ell+h))}v^{-2}dudv \\ &= \frac{1}{N}\int_0^\infty e^{-2\pi nv/N}v^{\nu/2-1}\sum_{\ell\in L}\int_0^N e\Big[\frac{n}{N}u-\frac{u}{2}(\ell+h,\ell+h)\Big]du \\ &\qquad\times \overline{f[v^{1/2}\rho(g)^{-1}(\ell+h))}dv \\ &= \int_0^\infty e^{-2\pi nv/N}v^{\nu/2-1}\sum_{\substack{\ell\in L\\(\ell+h,\ell+h)=2n/N}} v^{\nu/2}(\rho(g)^{-1}(\ell+h),\overline{r})^{\nu}e^{-\pi vR(\rho(g)^{-1}(\ell+h))}dv\,.\end{aligned}$$

If $\nu\geq 7$, the sum and integral in the last expression can be switched,

$$\begin{aligned}\Psi_n(g,h) &= \sum_{\substack{\ell\in L\\(\ell+h,\ell+h)=2n/N}}\int_0^\infty v^{\nu-1}e^{-2\pi nv/N}(\rho(g)^{-1}(\ell+h),\overline{r})^{\nu}e^{-\pi vR(\rho(g)^{-1}(\ell+h))}dv \\ &= \pi^{-\nu}\Gamma(\nu)\sum_{\substack{\ell\in L\\(\ell+h,\ell+h)=2n/N}}(\rho(g)^{-1}(\ell+h),\overline{r})^{\nu}\Big(\frac{2n}{N}+R(\rho(g)^{-1}(\ell+h))\Big)^{-\nu}\,.\end{aligned}$$

But now,

$$\begin{aligned}2n/N+R(\rho(g)^{-1}(\ell+h)) &= (Q+R)(\rho(g)^{-1}(\ell+h)) \\ &= M^{-1}\,|(\rho(g)^{-1}(\ell+h),r)|^2\,,\end{aligned}$$

by the property of r remarked in section 1. Substituting this into the last expression yields the desired result.

Now, as observed in section 1, if $k=(k_{\theta_1},k_{\theta_2})\in SO(2)\times SO(2)$, then $k\cdot f=e^{-i\nu(\theta_1+\theta_2)}f$. Consequently,

$$\theta(z,gk,h)=e^{-i\nu(\theta_1+\theta_2)}\theta(z,g,h)$$

and so,

$$\Psi(gk,h)=e^{i\nu(\theta_1+\theta_2)}\Psi(g,h)\,.$$

Then for $(z_1,z_2)\in\mathfrak{h}\times\mathfrak{h}$, and $\sigma_{z_1,z_2}=(\sigma_{z_1},\sigma_{z_2})$, the function

$$\psi(z_1,z_2,h)=(v_1v_2)^{-\nu/2}\Psi(\sigma_{z_1,z_2},h)$$

satisfies

$$\psi(gz_1,g^{\sigma}z_2,h)=J(g,z_1)^{\nu}J(g,z_2)^{\nu}\psi(z_1,z_2,\rho(g)^{-1}h)$$

for all $g\in\Gamma_L$.

PROPOSITION 2. *If* $\nu\geq 7$, $\psi(z_1,z_2,h)$ *is a holomorphic automorphic form of weight* ν *on* $\mathfrak{h}\times\mathfrak{h}$ *with respect to*

$$\Gamma_{L,h}=\{g\in\Gamma_L\text{ s.t. }\rho(g)^{-1}h\equiv h \bmod L\}\,.$$

In particular,

$$\begin{aligned}\psi_n(z_1, z_2, h) &= (v_1v_2)^{-\nu/2}\Psi_n(\sigma_{z_1,z_2}, h)\\ &= M^{1-\nu}\pi^{-\nu}\Gamma(\nu) \sum_{\substack{\ell\in L\\ (\ell+h,\ell+h)=2n/N}} (-x_3z_1z_2 + x_1z_1 + x_1^\sigma z_2 + x_4)^{-\nu}\end{aligned}$$

where

$$\ell + h = \begin{pmatrix} x_1 & x_4 \\ x_3 & -x_1^\sigma \end{pmatrix}, \quad x_1 \in k, \quad x_3, x_4 \in Q .$$

Recall that $(,) = (,)_M$.

Proof. The only point to be proved is that $\psi(z_1, z_2, h)$ is holomorphic; and, since the Poincare series $\psi_n(z)$ span $S_\nu(\Gamma(N))$, it will be sufficient to prove that the $\psi_n(z_1, z_2, h)$ are holomorphic. Since

$$\rho(g) \in SO(Q) , \qquad (\rho(g)^{-1}(\ell + h), r) = (\ell + h, \rho(g)r) .$$

On the other hand,

$$\rho(\sigma_{z_1,z_2})r = \sigma_{z_2} r \sigma_{z_1}^\iota = (v_1v_2)^{-1/2}\begin{pmatrix} -z_1 & z_1z_2 \\ -1 & z_1 \end{pmatrix} .$$

Then if $\ell + h$ is as above,

$$(\ell + h, \rho(\sigma_{z_1,z_2})r) = (v_1v_2)^{-1/2}M(-x_3z_1z_2 + x_1z_1 + x_1^\sigma z_2 + x_4) .$$

Substituting this into the formula for Ψ_n given in proposition 1, and multiplying the result by $(v_1v_2)^{-\nu/2}$ yields the desired expression for ψ_n. Finally observe that, since

$$M^{-1}|(\rho(\sigma_{z_1,z_2})^{-1}(\ell + h), r)|^2 = (Q + R)(\rho(\sigma_{z_1,z_2})^{-1}(\ell + h)) ,$$

and $Q(\ell + h) = 2n/N > 0$, and R is positive definite, the expression $-x_3z_1z_2 + x_1z_1 + x_1^\sigma z_2 + x_4$ never vanishes on $\mathfrak{h} \times \mathfrak{h}$. Thus ψ_n is holomorphic as claimed.

3. An example

Take $M = 1$, so that $Q_M(X) = Q(X) = -2\det(X)$. For $N \in Z_{>0}$, let

$$L = \left\{\begin{pmatrix} x_1 & x_4 \\ x_3 & -x_1^\sigma \end{pmatrix} \text{ s.t. } x_1 \in \mathcal{O}_k,\ x_3 \in NZ,\ x_4 \in Z\right\} .$$

$$L^* = \left\{\begin{pmatrix} y_1 & y_4 \\ y_3 & -y_1^\sigma \end{pmatrix} \text{ s.t. } y_1 \in \mathfrak{D}^{-1},\ y_3 \in Z,\ y_4 \in \frac{1}{N}Z\right\} .$$

Then $(\ ,\ )$ is even integral on $L, N'(\ ,\ )$ is even integral on L^*, where N' is the least common multiple of N and Δ.

$$D(L) = N^2\Delta \quad \text{and} \quad L^*/L = \mathfrak{D}^{-1}/\mathcal{O}_k \oplus Z/NZ \oplus \frac{1}{N}Z/Z\ .$$

Moreover,

$$\Gamma_L \supseteq \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL_2(\mathcal{O}_k) \text{ s.t. } \operatorname{tr}(\gamma^\sigma \alpha y_1) \in NZ,\ \forall y_1 \in \mathcal{O}_k,\ \gamma\gamma^\sigma \in NZ\right\}$$
$$\supseteq \tilde{\Gamma}_0(N) = \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL_2(\mathcal{O}_k) \text{ s.t. } \gamma \in N\mathcal{O}_k\right\}\ .$$

Now for $r \in Z/NZ$, let $h_r = \begin{pmatrix}0 & 0\\ r & 0\end{pmatrix} \in L^*$. Then $(h_r, h_r) = 0$, and if $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \tilde{\Gamma}_0(N)$, then $\rho(g)^{-1}h_r \equiv h_{\alpha\alpha^\sigma r} \bmod L$.

Let χ be a character of $(Z/NZ)^x$, and set

$$\theta(z, g, \chi) = \sum_{\substack{r \in Z/NZ\\ (r,N)=1}} \chi(r)\theta(z, g, h_r)\ .$$

Then, $\forall\gamma \in \Gamma_0(N')$,

$$\theta(\gamma z, g, \chi) = \chi(d)\left(\frac{\Delta}{d}\right)J(\gamma, z)^\nu\theta(z, g, \chi)$$

Thus by the procedure of section 2, $\theta(z, g, \chi)$ yields a map

$$S_\nu\left(\Gamma_0(N'), \chi\cdot\left(\frac{\Delta}{*}\right)\right) \longrightarrow S_\nu(\tilde{\Gamma}_0(N), \tilde{\chi})\ ,$$

where $\tilde{\chi}(\delta) = \chi(\delta\delta^\sigma)$.

In particular, taking $N = 1$, and ν even yields a map

$$S_\nu\left(\Gamma_0(\Delta), \left(\frac{\Delta}{*}\right)\right) \longrightarrow S_\nu(SL_2(\mathcal{O}_k))\ .$$

4. The 'Mellin transform'

Let $\psi(z_1, z_2) \in S_\nu(SL_2(\mathcal{O}_k))$ with ν even. Then ψ has a Fourier expansion of the form:

$$\psi(z_1, z_2) = \sum_{\substack{\xi \in \mathfrak{D}^{-1}\\ \xi \gg 0, \bmod U_k^2}} c(\xi) \sum_{n=-\infty}^{\infty} e[\xi\varepsilon_0^{2n}z_1 + \xi^\sigma\varepsilon_0^{-2n}z_2]\ ,$$

where $\mathfrak{D}^{-1}$ is the inverse different of k, and ε_0 is a fundamental unit.

The 'Mellin transform' of ψ is given by:

$$\begin{aligned} D^*(s, \psi) &= \int_0^\infty \int_{-\log \varepsilon_0}^{\log \varepsilon_0} \psi(ire^w, ire^{-w}) r^{2s-1} dw\, dr \\ &= \frac{1}{2}(2\pi)^{-2s}\Gamma(s)^2 \sum_{\substack{\xi \in \mathfrak{D}^{-1} \\ \xi \gg 0, \bmod U_k^2}} c(\xi)(\xi\xi^\sigma)^{-s} . \end{aligned}$$

Now suppose that $\varphi \in S_\nu(\Gamma_0(\Delta), (\Delta/*))$ with ν even, and consider its image under the map given at the end of section 3:

$$\psi(z_1, z_2) = \int_{\mathcal{F}_{\Gamma_0(\Delta)}} \varphi(z)\overline{\theta(z, g, 1)} v^{\nu-2} du\, dv .$$

Then $\psi(z_1, z_2) \in S_\nu(SL_2(\mathcal{O}_k))$. Set $\psi_1(z_1, z_2) = (z_1 z_2)^{-\nu}\psi(-1/z_1, -1/z_2)$, and consider the Mellin transform $D^*(s, \psi_1)$ as above.

THEOREM. $D^*(s, \psi_1) = C\cdot(2\pi)^{-2s}\Gamma(s)^2\zeta(2s - \nu + 1)L(s)$

where

$$C = 2\pi(i)^\nu \left(\sum_{\substack{\varepsilon=0 \\ \text{even}}}^{\nu} \binom{\nu}{\varepsilon} \pi^{-\varepsilon}\right)$$

and

$$L(s) = \sum_{\substack{\xi \in \mathfrak{D}^{-1} \\ \xi \gg 0, \xi \bmod U_k^2}} A(\xi)(\xi\xi^\sigma)^{-s}$$

$$A(\xi) = \sum_\tau a^\tau_{\xi\xi^\sigma\Delta/(\Delta, c^2)} \cdot \frac{\Delta}{(\Delta, c^2)} \cdot \overline{c(\xi, \tau)} ,$$

where the last sum runs over a set of coset representatives

$$\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{for } \dot{\Gamma}_\infty \backslash SL_2(\mathbf{Z})/\Gamma_0(\Delta) ;$$

the a^τ_n *are the Fourier coefficients of* φ *at the cusp corresponding to* τ, *i.e.*

$$\varphi(\tau^{-1}z)J(\tau^{-1}, z)^{-\nu} = \sum_{n=1}^\infty a^\tau_n e\left[\frac{nz}{\Delta/(\Delta, c^2)}\right] .$$

And $c(\xi, \tau)$ *is given by*:

$$c(\xi, \tau) = \Delta^{-1/2}|c|^{-1} \sum_{r \in \mathcal{O}_k/c\mathcal{O}_k} e\left[\frac{ar\,r^\sigma - \operatorname{tr}(r\xi^\sigma) + d\xi\xi^\sigma}{c}\right], \qquad \text{if } \tau \neq 1_2 ,$$

$$c(\xi, \tau) = \begin{cases} 1 & \text{if } \xi \in \mathcal{O}_k , \\ 0 & \text{if } \xi \notin \mathcal{O}_k \end{cases} \qquad \text{if } \tau = 1_2 .$$

Proof. This theorem is proved by a direct computation of the integral along the same lines as the computation in Niwa [4].

Set $D(s, \psi_1) = \zeta(2s - \nu + 1)L(s)$.

Now suppose that $\Delta = q \equiv 1(4)$, and further assume that the class number of $k = 1$. If

$$\varphi \in S_\nu\left(\Gamma_0(q), \left(\frac{q}{*}\right)\right), \qquad \varphi(z) = \sum_{n=1}^{\infty} a_n e[nz] ,$$

set $L(s, \varphi) = \sum_{n=1}^{\infty} a_n n^{-s}$.

PROPOSITION. *Suppose that φ is a common eigenfunction of all the Hecke operators, and that $a_1 = 1$. Set $\varphi_1(z) = \varphi(-1/qz)\cdot q^{\nu/2}(qz)^{-\nu}$. Then if ψ and ψ_1 are as in the theorem,*

$$D^*(s, \psi_1) = C\cdot q^{1/2-\nu/2}q^s(2\pi)^{-2s}\Gamma(s)^2 L(s, \varphi)L(s, \varphi_1) .$$

This proposition shows that the map from $S_\nu(\Gamma_0(q), (q/*)) \to S_\nu(SL_2(\mathcal{O}_k))$ by the theta-function is the same, up to a constant factor, as that given by Naganuma [3].

Remarks. 1) By taking non-trivial characters χ in the construction of section 3, it is possible to produce Hilbert modular forms from automorphic forms for various congruence subgroups. For example, taking $N = \Delta$, and $\chi = (\Delta/*)$, should yield the map of Doi and Naganuma [2], on forms of Haupt-type. Taking $N =$ a multiple of Δ, and $\chi = \chi_1(\Delta/*)$, should yield the map given by H. Cohen [1].

2) It is possible to carry out all of the constructions of sections 1 and 2 with an arbitrary indefinite quaternion algebra $A_0/\boldsymbol{Q}$ in place of $M_2(\boldsymbol{Q})$. The corresponding theta-functions will give maps from automorphic forms of $\mathfrak{h}$ with respect to congruence subgroups of $SL_2(\boldsymbol{Z})$ to holomorphic automorphic forms on $\mathfrak{h} \times \mathfrak{h}$ with respect to the unit groups of orders in $A = A_0 \otimes_{\boldsymbol{Q}} k$. The functions $\psi_n(z_1, z_2)$ will then be the analogue of Zagier's functions $\omega_n(z_1, z_2)$, and should be significant in the study of cycles in the surfaces attached to A.

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