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ALGEBRAS AND DIFFERENTIAL EQUATIONS

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0. Introduction

One purpose of this paper is a purely algebraic study of (systems of) ordinary differential equations of the type

$$D \qquad \dot{X}_i = \sum_{k_1,\cdots,k_m=1}^{n} a_i^{k_1,\cdots,k_m} X_{k_1} \dots X_{k_m} \qquad i = 1, \cdots, n\,,$$

where the coefficients are taken from a fixed associative, commutative, unital ring R, such as the field $\boldsymbol{R}$ of real or $\boldsymbol{C}$ of complex numbers or a commutative, unital Banach algebra. The right hand sides of D are considered to be elements in the polynomial ring $R[X_1, \cdots, X_n]$ of associating but non-commuting variables $X_1, \cdots, X_n$. An algebraic study calls for maps between such differential equations and, in fact, morphisms are defined between differential equations having the same arity m but not necessarily the same dimension n. These morphisms are rectangular matrices with entries in R which satisfy certain relations. This leads to a category ${}_R\mathrm{Diff}_m$ whose objects are precisely the differential equations of arity m and in which the composition of the morphisms is the usual matrix multiplication.

Given a ring R, as before, and an integer $m \geq 1$, one can define the category ${}_R\mathrm{Alg}_m$ of R-algebras of arity m. Its objects are unital R-modules A equipped with a m-ary, R-multilinear multiplication—i.e., a R-module homomorphism $\mu\colon \otimes_R^m A \to A$— and whose morphisms are R-module homomorphisms commuting with the multiplications, the composition of morphisms being the set-theoretical one. These R-algebras will, in general, not satisfy any given non-trivial relational or existential requirement; in particular, neither associativity nor commutativity nor unitality is assumed.

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The main theorem of section 1 states that the category ${}_R\mathrm{Diff}_m$ is equivalent to the full subcategory of ${}_R\mathrm{Alg}_m$ whose objects are finitely generated and free as R-modules. Hence we may view algebras as generalizations of differential equations. So we come to the second purpose of this paper, namely to develop certain constructions for and to prove theorems concerning general R-algebras, which are inspired by the study of the previously described differential equations. The equivalence ${}_R\mathrm{Diff}_m \to {}_R\mathrm{Alg}_m$ constructed in section 1 is denoted by A_m, and $A_m(D)$ is called the R-algebra associated with D. The idea of associating a R-algebra with a system D of differential equations seems to have reared its head the first time in [9]. It was subsequently used in [2], [4], [7], [8], [11]; however, only the last paper mentions functoriality. Section 1 concludes with the interpretation, in this setting, of some results of [12] and [13], and with an elaboration of previously [9] touched constructions.

Section 2 addresses itself to the functor "set of solutions". It can be easily seen that there is a functor $S: {}_R\mathrm{Diff}_m \to \mathrm{Sets}$ which assigns to each differential equation D its set $S(D)$ of solutions; in our context, solution means a n-tuple of formal power series with coefficients in R which formally satisfy D. For the purpose of analysis this is enough as a classical result says that, for R a Banach algebra, the notions of formal solution, convergent solution, and differentiable solution of D are coextensive. We proceed to define, for any R-algebra A, an associated differential operator $\partial_A: A[[t]] \to A[[t]]$ which is functorial in A. Denoting $\ker \partial_A$ by $S(A)$, it turns out that $S: {}_R\mathrm{Diff}_m \to \mathrm{Sets}$ and $S \circ A_m: {}_R\mathrm{Diff}_m \to \mathrm{Sets}$ are canonically isomorphic. Next, we show that $S: {}_R\mathrm{Alg}_m \to \mathrm{Sets}$ has a left adjoint L by constructing the value of L on the one-point set $\{\phi\}$. The ${}_R\mathrm{Alg}_m$-automorphism group of $L(\{\phi\})$ turns out to be the group of units of R, provided that the field $\boldsymbol{Q}$ of rational numbers is contained in R.

In section 3 we take up the issue of polynomial first (and higher) integrals. The assignment of the polynomial ring $R[X_1, \cdots, X_n]$ to a differential equation D of arity m and dimension n can be made into a functor P from ${}_R\mathrm{Diff}_m$ to the category of polynomial rings and R-homomorphisms. For each differential equation D there is a linear, first-order partial differential operator $\delta_D: R[X_1, \cdots, X_n] \to R[X_1, \cdots, X_n]$, which gives rise to an endomorphism δ of P. The kernel of δ_D is, by definition, the ring $I_0(D)$ of polynomial first integrals of D. Because the right hand sides of D are homogeneous polynomials of the same degree m, the homogeneous components of a first integral which is a convergent power series are elements of $I_0(D)$; in other words they can be constructed from $I_0(D)$. The knowledge of first integrals is important since they allow the reduction of the dimension of D in integrating D. In order to obtain a generalization of these concepts to R-algebras, we use the multiplication μ of a given R-algebra A to equip, functorially,

$$T_*(A) = \bigoplus_{p=0}^{\infty} \otimes_R^p A$$

with a graded R-module endomorphism $d_{\mu,*}$ (see (3.10)) of degree $1 - m$. Then we form

$$T^*(A, S) = \bigoplus_{p=0}^{\infty} \mathrm{Hom}_R\,(T_p(A), S)$$

and use $d_{\mu,*}$ to form a graded R-module endomorphism $\delta^*_{\mu,S}$ of $T^*(A, S)$. If S is a R-algebra, then $T^*(A, S)$ has a canonical R-algebra structure and $\delta^*_{\mu,S}$ becomes a R-derivation. δ^* is an endomorphism of the bifunctor $T^*(-, -)$. The main result of this section is the existence of an isomorphism of functors $P \to T^*(-, R) \circ A_m$ which commutes with $\delta : P \to P$ and $\delta^*_{-,R} \circ A_m : T^*(-, R) \circ A_m \to T^*(-, R) \circ A_m$. Hence, if we put $I_0(A, S) = \ker \delta^*_{\mu,S}$, we know that $I_0(D)$ and $I_0(A_m(D), R)$ are functorially isomorphic. Also, higher integrals are defined: $I_q(A, S) = \ker (\delta^*_{\mu,S})^{q+1}$. From an analyst's view point it is less satisfactory to deal with the non-commutative polynomial ring $R[X_1, \cdots, X_n]$ instead of the commutative polynomial ring $R[X_1, \cdots, X_n]_c$. In order to accomodate the commutative case, we construct a graded ideal $C^*(A, S)$ of $T^*(A, S)$ which is stable under $\delta^*_{\mu,S}$. Hence $\delta^*_{\mu,S}$ induces on $T^*(A, S)_c = T^*(A, S)/C^*(A, S)$ another derivation δ^*_{μ,S_c} which is used to define $I_q(A, S)_c = \ker (\delta^*_{\mu,S_c})^{q+1}$. The afore mentioned isomorphism of functors $P \to T^*(-, R) \circ A_m$ induces an isomorphism of functors $P_c \to T^*(-, R)_c \circ A_m$ which again commutes with $\delta_c : P_c \to P_c$ and $\delta^*_{-,R_c} : T^*(-, R)_c \circ A_m \to T^*(-, R)_c \circ A_m$ where δ_c is induced by δ. Thus, also in the commutative case the first and higher integrals of our differential equations are just a special instance of first and higher integrals of algebras. Section 3 closes with a remark on the parameter dependence of $I^p_q(A, S) = I_q(A, S) \cap T^p(A, S)$ resp. $I^p_q(A, S)_c = I_q(A, S)_c \cap T^p(A, S)_c$. Here we fix a n-dimensional F-vector space V and a F-algebra S which is finite dimensional as a F-vector space. The totality of F-algebras of arity m on V is identified, via the structure coefficients, with a suitable Euclidean space $S_m(V)$ over F. In the Zariski-topology of $S_m(V)$, $\dim_F I_q^p(A,S)$ and $\dim_F I_q^p(A,S)_c$ turn out to be upper semi-continuous.

Section 4 collects some properties of $I_q(A,S)$ and $I_q(A,S)_c$. It is shown that $I_q(A,S)$ and $I_q(A,S)_c$ are non-trivial provided A is nilpotent and $S \neq 0$. Next, some change-of-ring theorems are established for both $I_q(A,S)$ as well as $I_q(A,S)_c$. They imply that for any F-algebra A, F being a field, and for any finite field extension F' of $F, I_q(A,S)$ and $I_q(F' \otimes_F A, F' \otimes_F S)$ (resp. $I_q(A,S)_c$ and $I_q(F' \otimes_F A, F' \otimes_F S)_c$) determine each other completely. The significance of this lies in the fact (see [12]) that every finite-dimensional F-algebra of arity $m \geq 2$ acquires, through a finite field extension F' of F either an idempotent or a nilpotent element, and that for F-algebras which possess an idempotent element there is a way to compute $I_q(A,S)$ resp. $I_q(A,S)_c$. In particular it is shown that $I_q(A,S)$ is trivial for those finite-dimensional F-algebras A, F being a field of characteristic zero, which possess an idempotent element whose left-translation map has no eigenvalue equal to $0, -1, -2, \cdots$. These statements finally imply that $I_q(A,S)$ is Zariski-generically trivial, i.e., that $I_q(A,S)$ is trivial on a non-empty intersection of countably many Zariski-open sets of $S_m(V)$. Similar results hold for $I_q(A,S)_c$.

Section 5 takes off from the following question. Given two differential equations D_1 resp. D_2 of arity m and dimension n_1 resp. n_2, over a Banach algebra R find all germs of analytic maps $\Phi: R^{n_1} \to R^{n_2}$ which satisfy $\Phi(0) = 0$ and map every solution of D_1 which is sufficiently close to 0 into a solution of D_2. Evidently, this leads to a new category, ${}_R\mathcal{D}iff'_m$, whose objects are the differential equations of arity m over R and whose morphisms are precisely these germs. ${}_R\mathcal{D}iff'_m$ contains ${}_R\mathrm{Diff}_m$ as a subcategory. A germ Φ of an analytic map belongs to ${}_R\mathcal{D}iff'_m(D_1, D_2)$ precisely when it satisfies a certain system of non-linear partial differential equations. The formal power series which solve this system form a set ${}_R\mathcal{D}iff_m(D_1, D_2)$ which serves as the morphism set of yet another category, ${}_R\mathcal{D}iff_m \cdot {}_R\mathcal{D}iff_m$ contains ${}_R\mathcal{D}iff'_m$ as a proper subcategory. As in section 3 we proceed to cast ${}_R\mathcal{D}iff_m$ into an algebraic setting. For this purpose one defines, for two R-algebras A and B of arity m, formal power series on A with values in B whose constant term vanishes. They form a R-algebra $P(A,B)$ of arity m whose multiplication is denoted by $\mu_{A,B}$.

$\delta^*_{\mu_A}$ induces a R-module endomorphism $\delta_{A/B}$ in $P(A,B)$. The substitute for the previously mentioned system of partial differential equations is $\delta_{A/B}\lambda - \mu_{A,B}(\otimes\,{}^m\lambda) = 0$; thus we are interested in the subset ${}_R\mathscr{A}lg_m(A,B)$ of $P(A,B)$ consisting of the "solutions" of this equation. One then shows that there is a category ${}_R\mathscr{A}lg_m$ whose objects are the R-algebras of arity m and whose morphism sets are precisely the sets just described. ${}_R\mathrm{Alg}_m$ is a subcategory of ${}_R\mathscr{A}lg_m$, and a formal power series belonging to ${}_R\mathscr{A}lg_m$ has as its linear term a morphism of ${}_R\mathrm{Alg}_m$. The main theorem of this section states that ${}_R\mathscr{D}iff_m$ is equivalent to the full subcategory of ${}_R\mathscr{A}lg_m$ whose objects are finitely generated and free as R-modules. The section closes with a brief remark concerning the commutative situation.

In section 6 we discuss the symmetry group ${}_RG(A)$ of an R-algebra A, that is the group of ${}_R\mathscr{A}lg_m$-automorphisms of A. It is shown to be a split extension of ${}_R\mathrm{Aut}\,(A)$, the group of ${}_R\mathrm{Alg}_m$-automorphisms of A, by another group ${}_RU(A)$. For this group we obtain a countable tower of subgroups

$$ {}_RU(A) = {}_RU(A)^{[1]} > {}_RU(A)^{[2]} > \cdots > {}_RU(A)^{[p]} > \cdots $$

each of which is normal in its predecessor and whose intersection is the unit element. The successive quotients ${}_RU(A)^{[p-1]}/{}_RU(A)^{[p]}$ are isomorphic to an additive subgroup of $T^p(A,A) = \mathrm{Hom}_R\,(\otimes\,{}^p_R A, A)$ which is contained in

$$ Q^p_\mu = \{f : f \circ d_{\mu,p+m-1} = \mu \circ d_{f,p+m-1}\} \subset T^p(A,A)\ . $$

If the field of rational numbers is contained in R—as shall be assumed for the remainder of the introduction—then this subgroup actually coincides with Q^p_μ. This is done by constructing, for every $f \in Q^p_\mu$ an element $\lambda(f) \in {}_RU(A)^{[p-1]}$ which is mapped onto f. Forming $\lambda(f)$ from f is, in a formal sense, an exponentiation. It is shown that every element of ${}_RU(A)$ can be written uniquely as a locally finite product $\lambda(f_2)\lambda(f_3)\cdots$, with $f_p \in Q^p_\mu$. Since $\mu \in Q^m_\mu$ holds, ${}_RU(A)$ is not trivial for any non-trivial R-algebra A. Furthermore we prove that for R a Banach algebra, $\lambda(f)$ is always a convergent power series. Moreover, for a differential equation D, ${}_RG(A_m(D))$ is isomorphic to the ${}_R\mathscr{D}iff'_m$-automorphism group of D if and only if there exists an integer p_0 such that $Q^p_\mu = 0$ for $p \geq p_0$; if $R = \boldsymbol{R}$ or $\boldsymbol{C}$, this implies that ${}_RG(A_m(D))$ is a simply connected, nilpotent Lie group which is a Stein manifold. For non-trivial R-algebras A, R

again being a commutative, unital Banach algebra, the "one parameter" subgroup $\lambda(r\mu)$, $r \in R$, of ${}_RU(A)$ is shown to have geometric meaning: if $A = A_m(D)$ then $\lambda(r\mu_D)$ moves each point $a \in A_m(D)$ that is sufficiently close to 0 along the trajectory of D through a. At the end of the section there is again a discussion of the commutative situation.

The commutative analog of ${}_RG'(A_m(D))$, for R a Banach algebra, makes its first appearance in [4]. The treatment in [4] is, in contrast to ours, strictly Banach-analytic. As can be expected, there is a certain overlap between this paper and [4]. For instance: $\lambda(r\mu)$ (in [4] $p^{*\xi}$) is recognized as trajectory; the relation $f \in Q^p_{\mu c} \Leftrightarrow \mu \in Q^m_{fc}$ is obtained (although our $Q^p_{\mu c}$ is replaced in [4] by a different object); the epimorphism ${}_RU(A_m(D))^{[m-1]}_c / {}_RU(A_m(D))^{[m]}_c \to Q^m_{\mu c}$ is established in case D is "nicht entartet", i.e. $Q^2_{\mu c} = \cdots = Q^{m-1}_{\mu c} = Q^{m+1}_{\mu c} = \cdots = 0$. Here, the subscript "$c$" indicates the commutative version of the entity without this subscript.

Section 7 establishes some properties of ${}_RG(A)$. First, it is shown that there are non-trivial R-algebras of arity m which are finitely generated and free as R-modules, such that $Q^p_\mu \neq 0$ for all p. Next we show that for a R-algebra of arity $m \geq 2$ which has no $\boldsymbol{Z}$-torsion and possesses a unit element,

$$ {}_RU(A) = {}_RU(A)^{[1]} = \cdots = {}_RU(A)^{[m-1]} $$

and

$$ {}_RU(A)^{[m]} = {}_RU(A)^{[m+1]} = \cdots = 0 $$

hold and that there is a canonical injective R-module homomorphism ${}_RU(A) \to A$. The commutative analogue to this result can be found in [4]. Next, some change-of-ring theorems are established for Q^p_μ. They imply that for any F-algebra A, F being a field, and for any finite field extension F' of F, Q^p_μ and $Q^p_{F' \otimes_F \mu}$ determine each other completely. The significance of this lies in the fact (see [12]) that every finite-dimensional F-algebra of arity $m \geq 2$ acquires, through a finite field extension F' of F either an idempotent or a nilpotent element, and that for F-algebras which possess an idempotent element there is a way to compute Q^p_μ. In particular it is shown that for those finite-dimensional F-algebras A of arity m, F being a field of characteristic zero, which possess an idempotent element satisfying certain conditions,

$$
\begin{gathered}
{}_F U(A) = {}_F U(A)^{[1]} = \cdots = {}_F U(A)^{[m-1]} \cong F \\
{}_F U(A)^{[m]} = {}_F U(A)^{[m+1]} = \cdots = 0
\end{gathered}
$$

hold. These statements finally imply that Zariski-generically ${}_F U(A)$ has the structure just described. Again, a brief discussion of the commutative situation closes this section.

In [4], ${}_R U(A_m(D))_c$ is determined in case D is "nicht entartet". It is shown to be isomorphic to a certain vector space and, in case $m = 2$, a canonical injective vector space homomorphism ${}_R U(A_2(D))_c \to A_2(D)$ is obtained, provided that $A_2(D)$ has a unit element. These results of [4] are special instances of some of our results.

1. The category of differential equations of arity m

In the sequel, R denotes an associative, commutative, unital ring; all R-modules are taken to be unital, and subrings inherit the unit element.

This paper deals, in part, with differential equations D of the form

$$
\dot{X}_i = \sum_{k_1, \cdots, k_m = 1}^{n} a_i^{k_1, \cdots, k_m} X_{k_1} \cdots X_{k_m} , \qquad i = 1, \cdots, n . \tag{1.1}
$$

The right side of (1.1) is regarded as an element in the polynomial ring $R[X_1, \cdots, X_n]$ of associating but non-commuting variables X_i. $m = \operatorname{ar} D$ is called the arity of $D, n = \dim D$ is called the dimension of D. It is sometimes convenient to denote the right side of (1.1) by $D_i(X_1, \cdots, X_n)$.

Given two differential equations D' and D'' over R, with $\operatorname{ar} D' = \operatorname{ar} D'$, we define a morphism $f: D' \to D''$ to be a matrix over R

$$
f = (f_j^i : i = 1, \cdots, n' ; j = 1, \cdots, n'') ,
$$

where $n' = \dim D'$ and $n'' = \dim D''$, such that

$$
\sum_{i=1}^{n'} f_j^i D_i'(X_1', \cdots, X_n') = D_j''\left(\sum_{i=1}^{n'} f_1^i X_i', \cdots, \sum_{i=1}^{n'} f_{n''}^i X_i'\right) , \qquad j = 1, \cdots, n'' . \tag{1.2}
$$

An easy verification leads to

(1.3) PROPOSITION. *The differential equations over R of arity m and their morphisms, with composition the matrix multiplication, form a category ${}_R\mathrm{Diff}_m$.*

Let A be a unital R-module. By a R-algebra structure of arity m

on A is meant a R-module homomorphism $\mu: \otimes_R^m A \to A$, and (A, μ) is called a R-algebra of arity m. By an R-algebra morphism $f: (A', \mu') \to (A'', \mu'')$ is meant a R-module homomorphism $f: A' \to A''$ such that

$$\begin{array}{ccc} \otimes_R^m A' & \xrightarrow{\mu'} & A' \\ {\scriptstyle \otimes^m f}\downarrow & & \downarrow{\scriptstyle f} \\ \otimes_R^m A'' & \xrightarrow{\mu''} & A'' \end{array}$$

commutes. Evidently, the R-algebras of arity m and their morphisms, with composition the set theoretical composition, form a category ${}_R\mathrm{Alg}_m$.

(1.4) THEOREM. *There is a full faithful functor $A_m: {}_R\mathrm{Diff}_m \to {}_R\mathrm{Alg}_m$ which is an equivalence between ${}_R\mathrm{Diff}_m$ and the full subcategory of ${}_R\mathrm{Alg}_m$ that is defined by those algebras whose underlying R-module is finitely generated and free.*

Proof. Let D be given by (1.1). Take for the underlying R-module of $A_m(D)$ the R-module R^n and define μ_D by

$$\begin{aligned} &\mu_D((r_1^1, \cdots, r_n^1) \otimes \cdots \otimes (r_1^m, \cdots, r_n^m)) \\ &\quad = \Big(\sum_{k_1, \cdots, k_m} a_1^{k_1, \cdots, k_m} r_{k_1}^1 \cdots r_{k_m}^m, \cdots, \sum_{k_1, \cdots, k_m} a_n^{k_1, \cdots, k_m} r_{k_1}^1 \cdots r_{k_m}^m \Big) . \end{aligned} \tag{1.5}$$

Put furthermore, for any morphism f in ${}_R\mathrm{Diff}_m$, $A_m(f) = f$. An easy computation shows that (1.2) is equivalent with the relations

$$\sum_{i=1}^{n'} f_j^i a_i'^{k_1, \cdots, k_m} = \sum_{\ell_1, \cdots, \ell_m} a_j''^{\ell_1, \cdots, \ell_m} f_{\ell_1}^{k_1} \cdots f_{\ell_m}^{k_m} , \qquad \text{for all } j, k_1, \cdots, k_m . \tag{1.6}$$

It is equally easy to see that (1.6) are precisely the conditions for a R-homomorphism $f: R^{n'} \to R^{n''}$ to be a R-algebra morphism from $A_m(D')$ to $A_m(D'')$. Hence A_m is a full faithful functor. In order to obtain the second part of (1.4), let A be any R-algebra whose underlying R-module is finitely generated and free. Choose for A a basis $e^1, \cdots, e^n$ and let

$$\mu(e^{k_1} \otimes \cdots \otimes e^{k_m}) = \sum_{i=1}^{n} a_i^{k_1, \cdots, k_m} e^i . \tag{1.7}$$

Let $D_m(A)$ be the differential equation (1.1) with the coefficients $a_i^{k_1, \cdots, k_m}$ taken from (1.7). Given a morphism $f: A' \to A''$ of R-algebras, express f as a matrix with respect to the chosen bases, and denote this matrix by $D_m(f)$. Obviously, D_m is a functor to ${}_R\mathrm{Diff}_m$ from the full subcategory of ${}_R\mathrm{Alg}_m$ that is defined by those algebras whose underlying R-module is finitely generated and free. It is straightforward to check that $A_m D_m \cong \mathrm{id}$ and $D_m A_m \cong \mathrm{id}$ hold. Hence D_m is an equivalence of categories.

Call the differential equation (1.1) symmetric if for any permutation π of $\{1, \cdots, m\}$ and all choices of indices $i, k_1, \cdots, k_m$,

$$a_i^{k_1, \cdots, k_m} = a_i^{k_{\pi 1}, \cdots, k_{\pi m}}$$

holds. Call the R-algebra (A, μ) commutative if for any permutation π of $\{1, \cdots, m\}$ and all choices of $a_1, \cdots, a_m \in A$

$$\mu(a_1 \otimes \cdots \otimes a_m) = \mu(a_{\pi 1} \otimes \cdots \otimes a_{\pi m})$$

holds. Then we obtain from (1.1) and (1.5)

(1.8) ADDENDUM TO (1.4). *D is symmetric if and only if $A_m(D)$ is commutative.*

(1.9) COROLLARY. *${}_R\mathrm{Diff}_m$ has finite products. If R is a principal ideal domain, then ${}_R\mathrm{Diff}_m$ is finitely complete. If R is a field then ${}_R\mathrm{Diff}_m$ has coequalizers.*

Proof. Clearly, ${}_R\mathrm{Alg}_m$ is an algebraic category (see [10], p. 145 a.s.o.). Hence ${}_R\mathrm{Alg}_m$ is both complete and cocomplete ([10], p. 129, 140). Since products in ${}_R\mathrm{Alg}_m$ are cartesian products ([10], p. 129), the first claim follows from (1.4). Since equalizers in ${}_R\mathrm{Alg}_m$ are injections ([10], p. 130), the second claim follows. Since coequalizers in ${}_R\mathrm{Alg}_m$ are surjections ([10], p. 142), the last claim is verified.

It should be noted that ${}_R\mathrm{Diff}_m$ does not have finite coproducts (even for R a field), and hence fails to be finitely cocomplete.

(1.10) COROLLARY. *Let D be a differential equation of arity m and dimension n. By putting $X = \sum_{i=1}^n X_i e^i$, the differential equation reads in $A_m(D)$*

$$\dot{X} = \mu(X \otimes \cdots \otimes X) .$$

Hence, there is a bijection between constant solutions (i.e. critical points) of D and nilpotent elements of $A_m(D)$ and, in case $\boldsymbol{Q} \subset R$, a bijection between ray solutions of D and idempotent elements of $A_m(D)$[1].

[1] An element of the R-algebra A is called idempotent resp. nilpotent if $\mu(a \otimes \cdots \otimes a)$ equals a resp. 0.

The second part of (1.10) can be found, in case $R = \boldsymbol{R}$ or $\boldsymbol{C}$, in [9], p. 187, for $m = 2$, and in [2], p. 1165, for m and n arbitrary.

In ${}_R\mathrm{Diff}_m$ we have two distinguished differential equations of dimension one:

$$E_m : \dot{X}_1 = X_1^m$$

and

$$N_m : \dot{X}_1 = 0 .$$

Evidently, $A_m(N_m)$ is the null algebra of dimension one, i.e. its multiplication ν satisfies $\nu = 0$. In $A_m(E_m)$, however, the multiplication ε is given by $\varepsilon(r^1 \otimes \cdots \otimes r^m) = r^1 \cdots r^m$. With these notations, a previous result of the authors ([12], (1.1)) can be restated, in a weakened form, as follows:

(1.11) PROPOSITION. *Let F be an algebraically closed field of characteristic zero. Then, for any differential equation D, ${}_F\mathrm{Diff}_m\,(E_m \amalg N_m, D)$ contains at least two elements.*

There is another statement in [13], namely (4) Corollary, which bears restating for differential equation.

(1.12) PROPOSITION. *Let F be an algebraically closed field of characteristic zero, and let m and n be natural numbers. Identify the differential equations* (1.1) *of arity m and dimension n, with coefficients in F, via these coefficients with the points of $S = F^{n^{m+1}}$. Then there is an affine subvariety A of S, with $A \neq S$, which is defined over the prime field K of F, such that all differential equations corresponding to points of $S - A$ have precisely $n^m - 1$ ray solutions and fail to have constant solutions $\neq 0$.*

It might be appropriate to restate some well known notions for classical algebras (i.e. $m = 2$) for algebras of arity m (see also [9]).

1. *A subalgebra* (i.e. a subobject) of $A \in {}_R\mathrm{Alg}_m$ is a R-submodule A' of A such that $\mu(A' \otimes_R \cdots \otimes_R A') \subset A'$ holds, μ being the multiplication in A and $A' \otimes_R \cdots \otimes_R A'$ standing for the canonical image of $\otimes_R^m A'$ in $\otimes_R^m A$. Suppose now that R is a field and that A equals $A_m(D)$. In this case, choose a vector space basis $b^1, \cdots, b^n$ of $A_m(D)$ such that $b^1, \cdots, b^k$ forms a basis of A'. An easy computation shows that D is isomorphic to a differential equation

$$\dot{X}'_i = D'_i(X'_1, \cdots, X'_n) \qquad i = 1, \cdots, n$$

such that every monomial occuring in D'_i, $i = k+1, \cdots, n$, contains at least one of the variables $X'_{k+1}, \cdots, X'_n$. It is equally easy to see that the converse is also true.

2. An *ideal* (i.e. a kernel) of $A \in {}_R\mathrm{Alg}_m$ is a R-submodule I of A such that

$$\mu(I \otimes_R A \otimes_R \cdots \otimes_R A) \subset I ,$$
$$\mu(A \otimes_R I \otimes_R \cdots \otimes_R A) \subset I, \cdots, \mu(A \otimes_R A \otimes_R \cdots \otimes_R I) \subset I$$

hold, the notation being analogous to the one used in **1.** Again assume that R is a field and that A equals $A_m(D)$. Choose a basis just as before. Again it turns out that D is isomorphic to a differential equation

$$\dot{X}'_i = D'_i(X'_1, \cdots, X'_n) \qquad i = 1, \cdots, n \tag{1.13}$$

where each D'_i, $i = k+1, \cdots, n$ contains only the variables $X'_{k+1}, \cdots, X'_n$ (see [9], p. 188). And, again, the converse is true. A differential equation of the form (1.13) is classically called reducible. Hence, irreducibility of D is equivalent to simplicity of $A_m(D)$. At this point, we should remark that I, with the multiplication induced from A, is an R-algebra of arity m. The associated differential equation reads, in the notation of (1.13)

$$\dot{X}'_i = D'_i(X'_1, \cdots, X'_k, 0, \cdots, 0) \qquad i = 1, \cdots, k . \tag{1.14}$$

3. Given the ideal I in $A \in {}_R\mathrm{Alg}_m$, the quotient module $\overline{A} = A/I$ carries a unique R-algebra structure $\bar{\mu}$ of arity m, μ being the multiplication in A, such that the quotient map $q: A \to \overline{A}$ becomes a R-algebra homomorphism. If we put ourselves into the situation of **2** then the differential equation associated with $(\overline{A}, \bar{\mu})$ becomes

$$\dot{X}''_i = D''_i(X''_{k+1}, \cdots, X''_n) \qquad i = k+1, \cdots, n ,$$

where—by definition—

$$D''_i(X''_{k+1}, \cdots, X''_n) = D'_i(X''_1, \cdots, X''_n) .$$

4. Finally, it should be noted that the product of the two differential equations D' and D'', both of arity m, is the differential equation

$$\dot{X}_i = \sum_{k_1,\cdots,k_m=1}^{n'} a_i'^{k_1,\cdots,k_m} X_{k_1} \cdots X_{k_m} \qquad i = 1, \cdots, n'$$

$$\dot{X}_{n'+j} = \sum_{\ell_1,\cdots,\ell_m=1}^{n''} a_j''^{\ell_1,\cdots,\ell_m} X_{n'+\ell_1} \cdots X_{n'+\ell_m} \qquad j = 1, \cdots, n'' .$$

Hence, $D = D' \Pi D''$ precisely when D is completely reducible in the classical sense ([9], p. 188, Theorem 5).

5. Let (A', μ') and (A'', μ'') be R-algebras of arity m. Then we define $(A', \mu') \otimes_R (A'', \mu'')$ to be $(A' \otimes_R A'', \mu)$ where

$$\mu((a_1' \otimes a_1'') \otimes \cdots \otimes (a_m' \otimes a_m'')) = \mu'(a_1' \otimes \cdots \otimes a_m') \otimes \mu''(a_1'' \otimes \cdots \otimes a_m'') .$$

If $A' = A_m(D')$ and $A'' = A_m(D'')$ then $A' \otimes_R A''$ is finitely generated and free, and hence there is a differential equation $D' \otimes_R D''$ with $(A' \otimes_R A'', \mu) = A_m(D' \otimes_R D'')$. If A' has basis $b'^1, \cdots, b'^{n'}$ and A'' has basis $b''^1, \cdots, b''^{n''}$, with

$$\mu'(b'^{k_1} \otimes \cdots \otimes b'^{k_m}) = \sum_{i=1}^{n'} a_i'^{k_1,\cdots,k_m} b'^i$$

and

$$\mu''(b''^{\ell_1} \otimes \cdots \otimes b''^{\ell_m}) = \sum_{i=1}^{n''} a_j''^{\ell_1,\cdots,\ell_m} b''^j ,$$

then with respect to the canonical basis $b'^i \otimes b''^j$ of $A' \otimes_R A''$, $D' \otimes_R D''$ has the form

$$\dot{X}_{ij} = \sum_{k_1,\cdots,k_m=1}^{n'} \sum_{\ell_1,\cdots,\ell_m=1}^{n''} a_i'^{k_1,\cdots,k_m} a_j''^{\ell_1,\cdots,\ell_m} X_{k_1\ell_1} \cdots X_{k_m\ell_m} . \tag{1.15}$$

Conversely, each differential equation that is isomorphic to one of the form (1.15) has its associated algebra isomorphic to a tensor product.

2. The solution functor

Let D be the differential equation (1.1). By a formal solution of (1.1) is meant a n-tuple of formal power series $\mathscr{X} = (\mathscr{X}_1, \cdots, \mathscr{X}_n) \in R[[t]]^n$ with coefficients in R which formally solves (1.1). The set of formal solutions of D is denoted by $S(D)$. $S(D)$ is not empty as there is always the trivial solution 0. If R is a valued ring (with values taken in $\mathbf{R}$) then every formal solution is convergent (in the sense of Cauchy's Criterion). If R is a Banach algebra then every convergent solution is differentiable and vice versa.

(2.1) *Remark.* Let $\boldsymbol{Q}$ denote the field of rational number. If $\boldsymbol{Q} \subset R$ holds, then for every $a_0 \in R^n$ there exists one (and only one) solution $\mathscr{X}_{a_0}$ of D whose constant term equals a_0. This is obvious by the classical recursion formula for the coefficients of a formal solution. Hence, in this case, $S(D)$ is rather large.

(2.2) PROPOSITION. *There is a functor* $S: {}_R\mathrm{Diff}_m \to \mathrm{Sets}$ *that assigns to each differential equation its set of solutions.* S *preserves finite products.*

Proof. For a morphism $f: D' \to D''$, let $S(f): S(D') \to S(D'')$ be given by

$$S(f)\mathscr{X}' = \left(\sum_{i=1}^{n'} f_1^i \mathscr{X}'_i, \cdots, \sum_{i=1}^{n'} f_{n''}^i \mathscr{X}'_i\right) . \tag{2.3}$$

It follows by easy computation from (1.2) that $S(f)\mathscr{X}'$ is indeed a solution of D''. Hence S is a functor. The fact that S preserves finite products is readily verified.

Given a R-algebra (A, μ) of arity m, we can equip the R-module $A[[t]]$ of formal power series with coefficients in A with the structure of a R-algebra of arity m as follows. For

$$\mathscr{X}_i = \sum_{j=0}^{\infty} a_{ij} t^j \in A[[t]] , \qquad i = 1, \cdots, m ,$$

we put

$$\mu[[t]](\mathscr{X}_1 \otimes \cdots \otimes \mathscr{X}_m) = \sum_{j=0}^{\infty} \left(\sum_{j_1+\cdots+j_m=j} \mu(a_{1j_1} \otimes \cdots \otimes a_{mj_m})\right) t^j . \tag{2.4}$$

Clearly, $(A[[t]], \mu[[t]]) =: (A, \mu)[[t]]$ is a R-algebra of arity m. In addition, $A[[t]]$ possesses a canonical derivation $\frac{d}{dt}$ which is given by

$$\frac{d}{dt}\left(\sum_{j=0}^{\infty} a_j t^j\right) = \sum_{j=0}^{\infty} j a_j t^{j-1} .$$

If $f: A' \to A''$ is a morphism of R-algebras then we define $f[[t]]: A'[[t]] \to A''[[t]]$ by

$$f[[t]]\left(\sum_{j=0}^{\infty} a'_j t^j\right) = \sum_{j=0}^{\infty} f(a'_j) t^j .$$

It is easy to check that $f[[t]]$ is an R-algebra morphism from $(A',\mu')[[t]]$ to $(A'',\mu'')[[t]]$. Hence we have obtained an endofunctor $[[t]]$ of ${}_R\mathrm{Alg}_m$. A simple argument shows

(2.5) LEMMA. $\dfrac{d}{dt}$ *is an endomorphism of the functor* $[[t]]$.

(2.6) DEFINITION. For any R-algebra (A,μ) of arity m, define $\partial_A\colon A[[t]] \to A[[t]]$ by

$$\partial_A(\mathscr{X}) = \frac{d\mathscr{X}}{dt} - \mu[[t]](\mathscr{X}\otimes\cdots\otimes\mathscr{X})\ .$$

∂_A is called the differential operator associated with (A,μ). $\ker\partial_A = \partial_A^{-1}(0)$ is denoted by $S(A)$ and is called the set of solutions of the differential equation $\partial_A(\mathscr{X}) = 0$ associated with (A,μ).

Evidently, $0 \in S(A)$ whence $S(A)$ is not empty. (2.1) remains still in force.

An easy computation shows that, with $\mathscr{X} = (\mathscr{X}_1(t),\cdots,\mathscr{X}_n(t))$,

$$\partial_{A_m(D)}(\mathscr{X}) = \frac{d\mathscr{X}}{dt} - (D_1(\mathscr{X}),\cdots,D_n(\mathscr{X}))$$

holds.

(2.7) COROLLARY. *∂ is an endomorphism of the functor* $[[t]]$. *S is a functor from* ${}_R\mathrm{Alg}_m$ *to Sets.*

Proof. The first claim follows from (2.5). The second assertion is a consequence of the first claim and the fact that for any morphism f of R-algebras, $f(0) = 0$ holds.

(2.8) PROPOSITION. *The functors S and $S\circ A_m$ from* ${}_R\mathrm{Diff}_m$ *to Sets are canonically isomorphic.*

Proof. Let D be a differential equation of arity m and dimension n. A solution is an element $\mathscr{X} = (\mathscr{X}_1,\cdots,\mathscr{X}_n) \in R[[t]]^n \cong R^n[[t]]$. But R^n is the module underlying $A_m(D)$. If $\mathscr{X} = \sum_{j=0}^{\infty} a_j t^j \in R^n[[t]]$ and if $a_j = \sum_{k=1}^{n} r_{jk} e^k$, with e^k the unit vectors in R^n then, by various definitions,

$$\mu[[t]](\mathscr{X}\otimes\cdots\otimes\mathscr{X}) = \sum_{j=0}^{\infty}\Bigl(\sum_{j_1+\cdots+j_m=j}\mu(a_{j_1}\otimes\cdots\otimes a_{j_m})\Bigr)t^j$$

$$= \sum_{j=0}^{\infty}\left(\sum_{j_1+\cdots+j_m=j}\ \sum_{k_1,\cdots,k_m=1}^{n} r_{j_1k_1}\cdots r_{j_mk_m}\mu(e^{k_1}\otimes\cdots\otimes e^{k_m})\right)t^j$$
$$= \sum_{i=1}^{n}\sum_{j=0}^{\infty}\left(\sum_{j_1+\cdots+j_m=j}\ \sum_{k_1,\cdots,k_m=1}^{n} r_{j_1k_1}\cdots r_{j_mk_m}a_i^{k_1,\cdots,k_m}\right)t^je^i$$
$$= \sum_{i=1}^{n}\left(\sum_{k_1,\cdots,k_m=1}^{n} a_i^{k_1,\cdots,k_m}\mathscr{X}_{k_1}\cdots\mathscr{X}_{k_m}\right)e^i$$

where $\mathscr{X}_k = \sum_{j=0}^{\infty} r_{jk}t^j$. Hence it follows that $\mathscr{X}$ is a solution of D if and only if $\partial_{A_m(D)}(\mathscr{X}) = 0$; i.e. $S(D) = S(A_m(D))$. It is obvious that this identity map is natural in D.

(2.9) THEOREM. *The functor* $S\colon {}_R\mathrm{Alg}_m \to \mathrm{Sets}$ *has a left adjoint, and hence it preserves limits and monomorphisms* (= *injections*).

Proof. Since ${}_R\mathrm{Alg}_m$ is cocomplete ([10], p. 140) it suffices to show that for the one-point set $\{\Phi\}$, the functor $\mathrm{Sets}(\{\Phi\}, S\text{-})\colon {}_R\mathrm{Alg}_m \to \mathrm{Sets}$ is representable, as

$$\mathrm{Sets}\,(X, S-) = \mathrm{Sets}\left(\coprod_X \{\Phi\}, S-\right) \cong \prod_X \mathrm{Sets}\,(\{\Phi\}, S-)$$
$$\cong \coprod_X {}_R\mathrm{Alg}_m\,(L(\{\Phi\}), -) \cong {}_R\mathrm{Alg}_m\left(\prod_X L(\{\Phi\}), -\right),$$

where all isomorphisms are natural; i.e. the left adjoint of S will be $X \to \coprod_X L(\{\Phi\})$. In order to construct $L(\{\Phi\})$, let (F, μ_F) be the free object in ${}_R\mathrm{Alg}_m$ which is generated by the set $N = \{0, 1, 2, \cdots\}$; its existence is well known ([10], p. 134). The canonical image of $j \in N$ in F shall be denoted by z_j. Let I be the ideal in F that is generated by the set

$$(2.10)\qquad jz_j - \sum_{j_1+\cdots+j_m=j-1} \mu_F(z_{j_1}\otimes\cdots\otimes z_{j_m}) \qquad j = 1, 2, \cdots$$

and denote the quotient algebra $(F, \mu_F)/I$ by $(L(\{\Phi\}), \bar{\mu})$. If $z + I \in L(\{\Phi\})$ is denoted by $\bar{z}$, then—we claim—

$$\bar{\mathscr{Z}} = \sum_{j=0}^{\infty} \bar{z}_j t^j \in S(L(\{\Phi\}))$$

holds:

$$\frac{d\bar{\mathscr{Z}}}{dt} = \sum_{j=1}^{\infty} j\bar{z}_j t^{j-1} = \sum_{j=1}^{\infty} \overline{jz_j}\, t^{j-1} = \sum_{j=1}^{\infty}\left(\sum_{j_1+\cdots+j_m=j-1} \overline{\mu_F(z_{j_1}\otimes\cdots\otimes z_{j_m})}\right)t^{j-1}$$
$$= \sum_{j=0}^{\infty}\left(\sum_{j_1+\cdots+j_m=j} \bar{\mu}(\bar{z}_{j_1}\otimes\cdots\otimes \bar{z}_{j_m})\right)t^j = \bar{\mu}[[t]](\bar{\mathscr{Z}}\otimes\cdots\otimes\bar{\mathscr{Z}}).$$

In other words, we have found a solution of the differential equation associated with $L(\{\Phi\})$. Now, let $\mathscr{X} \in \mathrm{Sets}\,(\{\Phi\}, SA)$. Then $\mathscr{X} = \sum_{j=0}^{\infty} a_j t^j$. There is a unique R-algebra homomorphism $f_{\mathscr{X}}: F \to A$ which sends each z_j to the corresponding a_j. Since $\mathscr{X}$ is a solution of the differential equation associated with A, we have

$$(2.11) \qquad ja_j - \sum_{j_1+\cdots+j_m=j-1} \mu(a_{j_1} \otimes \cdots \otimes a_{j_m}) = 0 \qquad j = 1, 2, \cdots$$

Since all expressions (2.10) are in I, $f_{\mathscr{X}}$ induces a R-algebra homomorphism $\ell_{\mathscr{X}}: L(\{\Phi\}) \to A$. Evidently $\ell_{\mathscr{X}}[[t]]\bar{\mathscr{Z}} = \mathscr{X}$, and distinct solutions $\mathscr{X}$ give rise to distinct R-algebra homomorphisms ℓ. Conversely, if $\ell: L(\{\Phi\}) \to A$ is a R-algebra homomorphism then $\ell[[t]]\bar{\mathscr{Z}}$ is a solution of the differential equation associated with A. Thus

$${}_R\mathrm{Alg}_m\,(L(\{\Phi\}), A) \ni \ell \to \ell[[t]]\bar{\mathscr{Z}} \in \mathrm{Sets}\,(\{\Phi\}, SA)$$

is a bijection. Evidently, it is natural in A. The preservation properties of S are now standard ([10], p. 110), but can easily be checked independently.

(2.12) PROPOSITION. *As a R-module, $L(\{\Phi\})$ is countably but not finitely generated. Moreover, the $\bar{z}_i$ are linearly independent; in particular, $\bar{\mathscr{Z}} \neq 0$.*

Proof. Obviously, $L = L(\{\Phi\})$ is countably generated as a R-module. If L were finitely generated then so would be every homomorphic image of L, and for such a homomorphism $\ell: L \to A$, the coefficients of $\ell[[t]]\bar{\mathscr{Z}}$ would be in $\ell(L)$. So we have to find an R-algebra A for which there is a solution whose coefficients generate a submodule of A which is not finitely generated. Take for the R-module underlying A the $\bigoplus_{p=0}^{\infty} R$, and denote the unit vectors by e^i, $i = 0, 1, \cdots$. Define $\mu: \bigotimes_R^m A \to A$ by

$$\mu(e^{i_1} \otimes \cdots \otimes e^{i_m}) = \begin{cases} (i_m + 1)e^{i_m+1} & \text{for } i_1 = \cdots = i_{m-1} = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Then an easy verification shows that $\sum_{j=0}^{\infty} e^j t^j$ is indeed a solution of the differential equation associated with A which has the desired property. The linear independence of the $\bar{z}_i$ is clear as the homomorphism to go with the given solution takes $\bar{z}_i$ to e^i.

(2.13) PROPOSITION. *Suppose that R is an integral domain with $\boldsymbol{Q} \subset R$.*

Then ${}_R\mathrm{Alg}_m - \mathrm{Aut}\,(L(\{\Phi\}))$ *is isomorphic with the group of units of* R.

Proof. If $\boldsymbol{Q} \subset R$ holds then $L = L(\{\Phi\})$ is just the free R-algebra over the one-point set, as can be seen from (2.1) or directly. If the free generator of L is denoted by z, then each algebra endomorphism ℓ of L is determined by $\ell(z)$. $\ell(z)$ is of the form $rz + p(z)$, where $r \in R$ and $p(z)$ is a polynomial in the non-associating variable z over R which has neither a constant nor a linear term; hence, if $p(z) \neq 0$, $\deg p \geq 2$. Suppose now that ℓ has a left-inverse ℓ'. Then

$$\begin{aligned} z &= \ell'(\ell(z)) = r\ell'(z) + \ell'(p(z)) \\ &= r\ell'(z) + p(\ell'(z)) \,. \end{aligned}$$

If d is the precise degree of $\ell'(z)$, then the degree of $p(\ell'(z))$ is $2d$. Hence we have a contradiction. Therefore $\ell(z) = rz$, and ℓ is an automorphism precisely when r is a unit.

It should be noted that the automorphism ℓ which takes z to rz maps the solution $\bar{\mathscr{Z}}$ to the solution $\sum_{j=0}^{\infty} r^{j(m-1)+1}\bar{z}_j t^j$.

We conclude this section with a statement concerning nilpotent algebras.

(2.14) DEFINITION. Let (A, μ) be a R-algebra of arity m. Then (A, μ) is said to be nilpotent of exponent $\leq e + 1$ if all e-times iterated compositions $\bigotimes_R^{e(m-1)+1} A \to A$ that can be built from μ are zero. An element $a \in A$ is called nilpotent of exponent $\leq e + 1$ if the subalgebra of A that is generated by a is nilpotent of exponent $\leq e + 1$. A is said to be a nil algebra if every element a of A is nilpotent.

(2.15) PROPOSITION. *Suppose that the* R*-algebra* A *has no* $\boldsymbol{Z}$*-torsion and that* $a_0 \in A$ *is nilpotent. If* $\mathscr{X} = a_0 + \cdots$ *is a solution of the differential equation associated with* A *then* $\mathscr{X} \in A[t]$. *In particular, if* A *is a nil algebra without* $\boldsymbol{Z}$*-torsion then* $S(A) \subset A[t]$.

Proof. Let $\mathscr{X} = \sum_{j=0}^{\infty} a_j t^j$. An easy induction argument, applied to (2.12), shows that for every $j = 0, 1, \cdots, j!\, a_j$ is in the subalgebra generated by a_0. If a_0 is nilpotent of exponent $\leq e$ then $j!\, a_j = 0$ for $j \geq e$. Since A has no $\boldsymbol{Z}$-torsion, $\mathscr{X}$ is a polynomial of degree $< e$.

(2.16) DEFINITION. Let (A, μ) be a R-algebra of arity m. Then (A, μ) is said to be associative if all twice iterated compositions $\bigotimes_R^{2m-1} A \to A$

that can be built from μ are equal to each other[2]. Similarly, one defines the notion of a power associative algebra.

(2.17) PROPOSITION. *Suppose that the R-algebra A is power associative and has no **Z**-torsion. If* $\mathscr{X} = a_0 + \cdots \in A[t]$ *is a solution of the differential equation associated with A then* a_0 *is nilpotent. In particular, if A is power associative, and* $\boldsymbol{Q} \subset R$ *holds, then* $S(A) \subset A[t]$ *implies that A is a nil algebra.*

Proof. By (2.11), $a_e = 0$ means that for an appropriate element $0 \neq n_e \in \boldsymbol{Z}$, n_e times the e-times iterated product of a_0 with itself vanishes. But this implies that a_0 is nilpotent as there is no $\boldsymbol{Z}$-torsion in A. Furthermore, $\boldsymbol{Z}$-divisibility of A implies, due to (2.11), that for every $a_0 \in A$ there exists a solution $\mathscr{X}$ whose constant term is a_0.

3. First and higher integrals

Let D be a differential equation (1.1) of arity m and dimension n. We associate with it the linear partial differential operator

$$\delta_D : R[X_1, \cdots, X_n] \to R[X_1, \cdots, X_n]$$

that is given, on the monomials, by

$$(3.1) \quad \delta_D(X_{k_1} \cdots X_{k_p}) = \sum_{i=1}^{p} X_{k_1} \cdots X_{k_{i-1}} \cdot D_{k_i}(X_1, \cdots, X_n) \cdot X_{k_{i+1}} \cdots X_{k_p} .$$

An easy argument shows

(3.2) LEMMA. δ_D *is a graded R-derivation of degree* $m-1$. *In particular,* $I(D) = \ker \delta_D$ *is a graded subalgebra of* $R[X_1, \cdots, X_n]$ *which contains R.*

(3.3) DEFINITION. The elements of $I_q(D) = \ker (\delta_D)^{q+1}$, $q = 0, 1, \cdots$, are called the $(q+1)^{st}$ (polynomial) integrals of D.

Obviously, $I_0(D) = I(D)$. Evidently one has

(3.4) COROLLARY. (i) $I_0(D) \subseteq I_1(D) \subseteq \cdots \subseteq I_q(D) \subseteq \cdots$,

(ii) $I_q(D)$ *is a graded* $I(D)$*-module,*

i(ii) $I_\omega(D) = \bigcup_{q=0}^{\infty} I_q(D)$ *is a filtered graded* $I(D)$*-module.*

There is a contravariant functor P from ${}_R\mathrm{Diff}_m$ to the category of

[2] A formally more satisfying description of these iterated compositions can be found in [5], p. 1-3.

polynomial rings over R in associating but non-commuting variables which assigns to each differential equation D of dimension n the ring $P(D) = R[X_1, \cdots, X_n]$, and to each morphism $f: D' \to D''$ the homomorphism $f^*: R[X_1'', \cdots, X_{n''}''] \to R[X_1', \cdots, X_{n'}']$ which is the substitution homomorphism induced by

$$X_j'' = \sum_{i=1}^{n'} f_j^i X_i' .$$

One verifies easily

(3.5) LEMMA. *δ is an endomorphism of the functor P.*

From (3.5) one obtains, denoting by ${}_R A$ the category of graded associative, unital R-algebras of arity 2,

(3.6) THEOREM. *There is a contravariant functor $I: {}_R\mathrm{Diff}_m \to {}_R A$ which assigns to each differential equation D its algebra of first integrals and to each morphism f of differential equations the algebra homomorphism $f*$ induced by f. There are also contravariant functors $I_q: {}_R\mathrm{Diff}_m \to \mathrm{gr\,Mod}$, $q = 1, \cdots$, to the category of graded modules, and $I_\omega: {}_R\mathrm{Diff}_m \to \mathrm{fil\,gr\,Mod}$ to the category of filtered, graded modules which assigns to each differential equation D the $I(D)$-module $I_q(D)$, $q = 1, \cdots, \omega$, and to each morphism f the (relative) module homomorphism f^* induced by f.*

(3.7) LEMMA. *Let $\mathscr{X} = \mathscr{X}(t)$ be a solution of D. Then for any $T \in R[X_1, \cdots, X_n]$,*

$$\frac{d}{dt} T(\mathscr{X}(t)) = (\delta_D T)(\mathscr{X}(t)) .$$

Proof. Easy verification.

Denote by C the ideal in $R[X_1, \cdots, X_n]$ which is generated by the polynomials $X_i X_j - X_j X_i, i, j = 1, \cdots, n$. The quotient ring $R[X_1, \cdots, X_n]_c = R[X_1, \cdots, X_n]/C$ is the polynomial ring in associating and commuting variables $X_1, \cdots, X_n$.

(3.8) PROPOSITION. *If T is an element of $(\delta_D^{q+1})^{-1}(C)$ then for every solution $\mathscr{X}(t)$ of D, $\left(\frac{d}{dt}\right)^{q+1} T(\mathscr{X}(t)) = 0$. Conversely, if $\boldsymbol{Q} \subset R$ holds then the polynomial T is in $(\delta_D^{q+1})^{-1}(C)$ if for every solution $\mathscr{X}(t)$ of D, $\left(\frac{d}{dt}\right)^{q+1} T(\mathscr{X}(t)) = 0$ holds.*

Proof. The first part of (3.8) is obvious from (3.7) and the fact that components of $\mathscr{X}(t)$ lie in a commutative ring. Conversely, (2.1) shows that for every $a \in R^n$ there is a solution $\mathscr{X}_a(t)$ whose constant term equals a. By (3.7),

$$0 = \left(\frac{d}{dt}\right)^{q+1} T(\mathscr{X}_a(t)) = (\delta_D^{q+1}T)(\mathscr{X}_a(t)) = (\delta_D^{q+1}T)(a) + \text{higher terms}\ .$$

Hence $(\delta_D^{q+1}T)(a) = 0$ for all $a \in R^n$. Since $\boldsymbol{Q}$ is contained in R, this implies that the image in $R[X_1, \cdots, X_n]_c$ of $\delta_D^{q+1}T$ vanishes—which proves our assertion.

It should be noted that $\delta_D(C) \subset C$ holds. Hence δ_D induces a derivation, denoted by δ_{Dc}, in $R[X_1, \cdots, X_n]_c$. In this latter case, there is an analog to (3.8) in which then $(\delta_D^{q+1})^{-1}(C)$ is replaced by $I_q(D)_c = \ker(\delta_{Dc})^{q+1}$.

At this point we ought to remark that the notion of first integrals for a system of ordinary (as well as partial) differential equations is old and well known ([6], p. 54) in the case $R = \boldsymbol{R}$ or $\boldsymbol{C}$. In essence, the knowledge of a first integral of D permits the reduction of $\dim D$ by 1. The reverse relationship is also classical. There one associates with a quasi-linear partial differential equation $\delta f = 0$ the system of characteristic differential equations ([3], p. 29) in such a manner that for the partial differential equation $\delta_D f = 0$, given by (3.1), the associated characteristic equation is precisely D, as given by (1.1). And again, knowledge of the solutions of the characteristic equation leads to solutions of $\delta f = 0$.

We shall now extend the functors I_q to all of ${}_R\mathrm{Alg}_m$. For that purpose, let (A, μ) be a R-algebra of arity m. Put

$$T_*(A) = \bigoplus_{p=0}^{\infty} T_p(A)\ , \tag{3.9}$$

$$\text{with } T_0(A) = R,\ T_1(A) = A,\ T_p(A) = \otimes_R^p A .$$

Obviously, $T_*(A)$ is a graded R-module and is functorial in A. Next, we define a graded R-module endomorphism $d_{\mu,*}\colon T_*(A) \to T_*(A)$ of degree $1 - m$ by $d_{\mu,p}\colon T_p(A) \to T_{p-m+1}(A)$ as follows:

$$d_{\mu,p} = 0 \qquad \text{for } p < m \tag{3.10}$$

$$d_{\mu,p} = \sum_{i=1}^{p-m+1} \otimes^{i-1} \mathrm{id}_A \otimes \mu \otimes \otimes^{p-m-i+1} \mathrm{id}_A \qquad \text{for } p \geq m\ .$$

In particular,

$$d_{\mu,m} = \mu \, .$$

Evidently, d_* is an endomorphism of the functor T_*. Hence we have obtained a functor (T_*, d_*) from ${}_R\mathrm{Alg}_m$ to the category whose objects are pairs consisting of graded R-modules M and endomorphisms d of M of degree $1-m$, and whose morphisms are graded R-module homomorphisms of degree zero which commute with the endomorphisms of the objects involved.

Let S be an associative, not necessarily commutative R-algebra of arity 2 and put

$$\begin{aligned} T^*(A,S) &= \bigoplus_{p=0}^{\infty} \mathrm{Hom}_R\,(T_p(A), S) \\ \delta^*_{\mu,S} &= \bigoplus_{p=0}^{\infty} \mathrm{Hom}_R\,(d_{\mu,p}, S) \, . \end{aligned} \tag{3.11}$$

Clearly, this establishes a bifunctor, in A and S, with values in that category which differs from the previous one by having endomorphisms of degree $m-1$.

Finally, for $f_i \in T^{p_i}(A,S)$, $i = 1, 2$, define $f_1 f_2 \in T^{p_1+p_2}(A,S)$ by

$$f_1 f_2 = \sigma \circ (f_1 \otimes f_2) \tag{3.12}$$

where $\sigma \colon S \otimes_R S \to S$ is the multiplication in S.

(3.13) THEOREM. *With the multiplication defined by* (3.12), $T^*(A,S)$ *becomes a graded, associative, unital R-algebra* (*of arity* 2) *which is commutative whenever S is.* $\delta^*_{\mu,S}$ *is a R-derivation of* $T^*(A,S)$. *Moreover, the algebra structure of* $T^*(A,S)$ *is functorial in A and S, and* δ^* *is an endomorphism of the bifunctor* T^*.

Proof. The only assertion that needs verification is that $\delta^*_{\mu,S}$ is a R-derivation. Let $f_i \in T^{p_i}(A,S)$, $i = 1, 2$. Then

$$\begin{aligned} \delta^{p_1+p_2+m-1}_{\mu,S}(f_1 f_2) &= \sigma \circ (f_1 \otimes f_2) \circ d_{\mu, p_1+p_2+m-1} \\ &= \sigma \circ (f_1 \otimes f_2) \circ \sum_{i=1}^{p_1+p_2} \otimes^{i-1} \mathrm{id}_A \otimes \mu \otimes \otimes^{p_1+p_2-i} \mathrm{id}_A \\ &= \sigma \circ (f_1 \otimes f_2) \circ \left(\sum_{i=1}^{p_1} \otimes^{i-1} \mathrm{id}_A \otimes \mu \otimes \otimes^{p_1-i} \mathrm{id}_A \right) \otimes \otimes^{p_2} \mathrm{id}_A \\ &\quad + \sigma \circ (f_1 \otimes f_2) \circ \left(\otimes^{p_1} \mathrm{id}_A \otimes \sum_{i=1}^{p_2} \otimes^{i-1} \mathrm{id}_A \otimes \mu \otimes \otimes^{p_2-i} \mathrm{id}_A \right) \end{aligned}$$

$$
\begin{aligned}
&= \sigma \circ (f_1 \otimes f_2) \circ (d_{\mu,p_1+m-1} \otimes \textstyle\bigotimes^{p_2} \mathrm{id}_A) \\
&\quad + \sigma \circ (f_1 \otimes f_2) \circ (\textstyle\bigotimes^{p_1} \mathrm{id}_A \otimes d_{\mu,p_2+m-1}) \\
&= \sigma \circ ((f_1 \circ d_{\mu,p_1+m-1}) \otimes f_2) + \sigma \circ (f_1 \otimes (f_2 \circ d_{\mu,p_2+m-1})) \\
&= (\delta_{\mu,S}^{p_1+m-1} f_1) f_2 + f_1 (\delta_{\mu,S}^{p_2+m-1} f_2) \,.
\end{aligned}
$$

(3.14) THEOREM. *There is an isomorphism of functors* $P \to T^*(-, R) \circ A_m$ *which commutes with the endomorphisms* $\delta : P \to P$ *and* $\delta^*_{-,R} \circ A_m : T^*(-, R) \circ A_m \to T^*(-, R) \circ A_m$.

Proof. Let D be a differential equation of dimension n. Then $P(D) = R[X_1, \cdots, X_n]$. We define now a R-module homomorphism $\alpha_D : P(D) \to T^*(A_m(D), R)$ as follows. Denote the unit vectors in $A_m(D)$ by $e^1, \cdots, e^n$. Then the canonical basis of $T_p(A)$ is $e^{j_1} \otimes \cdots \otimes e^{j_p}$, $j_1, \cdots, j_p = 1, \cdots, n$. Put

$$\alpha_D(X_{k_1} \cdots X_{k_p})(e^{j_1} \otimes \cdots \otimes e^{j_p}) = \delta_{k_1 j_1} \cdots \delta_{k_p j_p}$$

where δ_{kj} is the standard Kronecker symbol. Clearly, α_D is an isomorphism of graded R-algebras. Hence, the fact that α is a morphism of functors needs only to be verified in degree one—which is trivial. The remaining commutativity statement is easily checked in degree one:

$$
\begin{aligned}
(\alpha_D \delta_D X_i)(e^{k_1} \otimes \cdots \otimes e^{k_m}) &= (\alpha_D D_i(X_1, \cdots, X_m))(e^{k_1} \otimes \cdots \otimes e^{k_m}) = a_i^{k_1, \cdots, k_m} \,, \\
(\delta_{\mu_D, R} \alpha_D X_i)(e^{k_1} \otimes \cdots \otimes e^{k_m}) &= (\alpha_D X_i)(\mu(e^{k_1} \otimes \cdots \otimes e^{k_m})) = a_i^{k_1, \cdots, k_m} \,.
\end{aligned}
$$

From here, an easy induction argument on the degree of monomials shows that

$$\alpha_D \delta_D (X_{j_1} \cdots X_{j_p}) = \delta_{\mu_D, R} \alpha_D (X_{j_1} \cdots X_{j_p})$$

holds, which in turn implies the asserted commutativity.

(3.15) DEFINITION. Let A be a R-algebra of arity m. Then the elements of $I_q(A, S) = \ker (\delta^*_{\mu,S})^{q+1}$, $q = 0, 1, \cdots,$ are called the $(q+1)^{st}$ integrals of A with values in S. $I_0(A, S)$ is also denoted by $I(A, S)$, and $\bigcup_q I_q(A, S)$ is abbreviated by $I_\omega(A, S)$.

(3.16) COROLLARY. *Mutatis mutandis, the statements of* (3.4) *and* (3.6) *remain valid.*

In order to obtain an analog to (3.8), we form the R-module $W_*(A) = \bigoplus_{p=0}^{\infty} (T_p(A)[[t]])$, the direct sum of the formal power series modules

$T_p(A)[[t]]$. The operator $\dfrac{d}{dt}$, defined in the obvious manner, is a graded R-endomorphism of $W_*(A)$ of degree 0. The tensor product $\otimes : T_{p_1}(A) \times T_{p_2}(A) \to T_{p_1+p_2}(A)$ induces a tensor product, $\otimes$, on the level of formal power series thus making $W_*(A)$ into a graded, associative, unital R-algebra.

We note that for

$$\mathscr{X}_i(t) = \sum_{j=0}^{\infty} a_{ij} t^j \in A[[t]] = T_1(A)[[t]] , \qquad i = 1, \cdots, p ,$$

we obtain

$$\mathscr{X}_1(t) \otimes \cdots \otimes \mathscr{X}_p(t) = \sum_{k=0}^{\infty} \Big(\sum_{j_1+\cdots+j_p=k} a_{1j_1} \otimes \cdots \otimes a_{pj_p} \Big) t^k .$$

An easy verification shows

$$\begin{aligned} &\frac{d}{dt}(\mathscr{X}_1(t) \otimes \cdots \otimes \mathscr{X}_p(t)) \\ (3.17) \qquad &= \sum_{i=1}^{p} \mathscr{X}_1(t) \otimes \cdots \otimes \mathscr{X}_{i-1}(t) \otimes \frac{d\mathscr{X}_i(t)}{dt} \otimes \mathscr{X}_{i+1}(t) \otimes \cdots \otimes \mathscr{X}_p(t) . \end{aligned}$$

Next, every $f \in T^p(A, S)$ is extended to an element $\hat{f}$ of $\mathrm{Hom}_R\,(T_p(A)[[t]], S[[t]])$ by the formula

$$\hat{f}\Big(\sum_{k=0}^{\infty} a_k t^k\Big) = \sum_{k=0}^{\infty} f(a_k) t^k .$$

In particular,

$$(3.18) \qquad \hat{f}(\mathscr{X}_1(t) \otimes \cdots \otimes \mathscr{X}_p(t)) = \sum_{k=0}^{\infty} \Big(\sum_{j_1+\cdots+j_p=k} f(a_{1j_1} \otimes \cdots \otimes a_{pj_p} \Big) t^k$$

holds. Here we obtain

(3.19) LEMMA. *Let A be a R-algebra of arity m and let $\mathscr{X}(t)$ be a solution of the differential equation associated with A. Then for any $f \in T^p(A, S)$*

$$\frac{d}{dt}\hat{f}(\otimes^p \mathscr{X}(t)) = (\widehat{\delta^{p+m-1}_{\mu,S} f})(\otimes^{p+m-1} \mathscr{X}(t)) .$$

Proof. Denote $\delta^{p+m-1}_{\mu,S}$ by δ. Then

$$\widehat{\delta f}(\otimes^{p+m-1}\mathscr{X}(t)) = \text{(due to (3.18))}$$

$$= \sum_{k=0}^{\infty}\Big(\sum_{j_1+\cdots+j_{p+m-1}=k} \delta f(a_{j_1}\otimes\cdots\otimes a_{j_{p+m-1}})\Big)t^k$$

$$= \sum_{k=0}^{\infty}\Big(\sum_{j_1+\cdots+j_{p+m-1}=k}$$

$$\cdot\Big(\sum_{i=1}^{p} f(a_{j_1}\otimes\cdots\otimes a_{j_{i-1}}\otimes\mu(a_{j_i}\otimes\cdots\otimes a_{j_{i+m-1}})\otimes a_{j_{i+m}}\otimes\cdots\otimes a_{j_{p+m-1}})\Big)\Big)t^k$$

$$= \sum_{j_1,\cdots,j_{p+m-1}}$$

$$\cdot\Big(\sum_{i=1}^{p} f(a_{j_1}\otimes\cdots\otimes a_{j_{i-1}}\otimes\mu(a_{j_i}\otimes\cdots\otimes a_{j_{i+m-1}})\otimes a_{j_{i+m}}\otimes\cdots\otimes a_{j_{p+m-1}})\Big)$$

$$\cdot\, t^{j_1+\cdots+j_{p+m-1}}$$

$$= \sum_{i=1}^{p}\hat f(\otimes^{i-1}\mathscr{X}(t)\otimes\mu[[t]](\otimes^{m}\mathscr{X}(t))\otimes\otimes^{p-i}\mathscr{X}(t))$$

$$= \sum_{i=1}^{p}\hat f\Big(\otimes^{i-1}\mathscr{X}(t)\otimes\frac{d\mathscr{X}(t)}{dt}\otimes\otimes^{p-i}\mathscr{X}(t)\Big) = \text{(due to (3.17))}$$

$$= \hat f\Big(\frac{d}{dt}\otimes^{p}\mathscr{X}(t)\Big) = \frac{d}{dt}\hat f(\otimes^{p}\mathscr{X}(t))\ .$$

Denote by $C^p(A,S)$ the R-submodule of $T^p(A,S)$ consisting of those elements which vanish on all elements of $T_p(A)$ of the form

$$\text{(3.20)}\qquad \sum_{j_1+\cdots+j_p=k} a_{j_1}\otimes\cdots\otimes a_{j_p}\ ,\qquad a_0,a_1,\cdots,\in A, k=0,1,\cdots$$

and put $C^*(A,S)=\bigoplus_{p=2}^{\infty}C^p(A,S)$.

In order to make the next statement more readable we denote for $f\in T^p(A,S), \hat f(\otimes^p\mathscr{X}(t))$ by $\hat f(\mathscr{X}(t))$. This notation then gives meaning to $h(\mathscr{X}(t))$ where $h\in T^*(A,S)$ is not necessarily a homogeneous element.

(3.21) PROPOSITION. *If f is an element of $((\delta^*_{\mu,S})^{q+1})^{-1}C^*(A,S)$ then for any solution $\mathscr{X}(t)$ of the differential equation associated with A, $\left(\frac{d}{dt}\right)^{q+1}\hat f(\mathscr{X}(t))=0$. Conversely, if $\boldsymbol{Q}\subset R$ holds then f belongs to $((\delta^*_{\mu,S})^{q+1})^{-1}C^*(A,S)$ if for every solution $\mathscr{X}(t)$ of the differential equation associated with A, $\left(\frac{d}{dt}\right)^{q+1}\hat f(\mathscr{X}(t))=0$ holds.*

Proof. The first part of the assertion follows from (3.19) and the definition of $C^*(A,S)$. Conversely, the analog to (2.1) shows that for every $a\in A$ there exists a solution $\mathscr{X}_a(t)$ whose constant term equals a.

By (3.19),

$$0 = \left(\frac{d}{dt}\right)^{q+1} \hat{f}(\mathscr{X}_a(t)) = \widehat{(\delta^*_{\mu,S})^{q+1}f}(\mathscr{X}_a(t))$$
$$= (\delta^*_{\mu,S})^{q+1}f(a) + \text{higher terms} .$$

Hence $(\delta^*_{\mu,S})^{q+1}f(a) = 0$ for all $a \in A$. For the homogeneous component $((\delta^*_{\mu,S})^{q+1}f)^p$ of degree p this means

$$(3.22) \qquad ((\delta^*_{\mu,S})^{q+1}f)^p(\otimes^p a_0) = 0 , \qquad \text{for all } a_0 \in A .$$

i.e. it vanishes on the elements (3.20) corresponding to $k = 0$. By substituting in (3.22) $a_0 + \lambda a_1$ for a_0, with sufficiently many $\lambda \in \boldsymbol{Q}$, one sees that $((\delta^*_{\mu,S})^{q+1}f)^p$ vanishes on the elements (3.20) corresponding to $k = 1$. Similarly, the substitution of $a_0 + \lambda a_1 + \cdots + \lambda^k a_k$ shows that $((\delta^*_{\mu,S})^{q+1}f)^p$ vanishes on all elements (3.20), whence $((\delta^*_{\mu,S})^{q+1}f)^p$ lies in $C^p(A, S)$.

(3.23) LEMMA. *$C^*(A, S) = \bigoplus_{p=0}^{\infty} C^p(A, S)$ is a graded ideal of $T^*(A, S)$ satisfying $\delta^*_{\mu,S}C^*(A, S) \subset C^*(A, S)$.*

Proof. Let $f \in T^p(A, S)$ and $g \in C^q(A, S)$. Then

$$fg\left(\sum_{j_1+\cdots+j_{p+q}=k} a_{j_1} \otimes \cdots \otimes a_{j_{p+q}}\right)$$
$$= \sum_{j_1,\cdots,j_p} f(a_{j_1} \otimes \cdots \otimes a_{j_p}) g\left(\sum_{j_{p+1}+\cdots=k-j_1-\cdots-j_p} a_{j_{p+1}} \otimes \cdots \otimes a_{j_{p+q}}\right) = 0 .$$

Similarly one shows that $C^*(A, S)$ is a right ideal. Finally,

$$d_{\mu,p}\left(\sum_{j_1+\cdots+j_p=k} a_{j_1} \otimes \cdots \otimes a_{j_p}\right)$$
$$= \sum_{j_1+\cdots+j_p=k} \sum_{i=1} a_{j_1} \otimes \cdots \otimes a_{j_{i-1}} \otimes \mu(a_{j_i} \otimes \cdots \otimes a_{j_{i+m-1}}) \otimes a_{j_{i+m}} \otimes \cdots \otimes a_{j_p} .$$

Clearly, this last sum is generated by the elements (3.20).

(3.23) implies that $\delta^*_{\mu,S}$ induces a derivation, denoted by $\delta^*_{\mu,Sc}$, on $T^*(A, S)/C^*(A, S) = T^*(A, S)_c$.

(3.24) THEOREM. *The isomorphism α of functors given in the proof of (3.14) maps the ideal $C \subset P(D)$ onto the ideal $C^*(A_m(D), R)$ and hence gives rise to a commutative diagram*

$$\begin{array}{ccc} R[X_1, \cdots, X_n]_c & \xrightarrow{\cong} & T^*(A_m(D), R)_c \\ {\scriptstyle \delta_D}\downarrow & & \downarrow{\scriptstyle \delta^*_{\mu_D}, R_c} \\ R[X_1, \cdots, X_n]_c & \xrightarrow{\cong} & T^*(A_m(D), R)_c \end{array}$$

Proof. A simple computation shows that α_D maps the generators $X_iX_j - X_jX_i$ into $C^*(A_m(D), R)$. Hence we get induced maps and the commutative diagram.

In order to show that the horizontal maps are isomorphisms, we begin with a number theoretic fact whose proof will be given below. Let $n_1 > 0, \cdots, n_q > 0$ be integers. Then (see LEMMA below) there are mutually distinct integers $z_1 > 0, \cdots, z_q > 0$ such that for any integers $n'_1 \geq 0, \cdots, n'_q \geq 0$, $n_1z_1 + \cdots + n_qz_q = n'_1z_1 + \cdots + n'_qz_q$ implies $n_1 = n'_1$, $\cdots, n_q = n'_q$. Put $k = n_1z_1 + \cdots + n_qz_q$ and $p = n_1 + \cdots + n_q$, and take the element (3.20) corresponding to k and p, with $a_{z_1}, \cdots, a_{z_q}$ mutually distinct, and all the other $a_j = 0$. If we now rename a_{z_i} by a_i, then this element will be $\sum a_{j_1} \otimes \cdots \otimes a_{j_p}$ where $(j_1, \cdots, j_p)$ runs through all permutations of the set

$$\{\underbrace{1, \cdots, 1}_{n_1}, \underbrace{2, \cdots, 2}_{n_2}, \cdots, \underbrace{q, \cdots, q}_{n_q}\} .$$

Now suppose that

$$f = \sum r^{k_1, \cdots, k_p} X_{k_1} \cdots X_{k_p} \in R[X_1, \cdots, X_n]$$

is mapped by α into $C^p(A_m(D), R)$. Then its image under α has to vanish on the elements that we just constructed. So, if we choose for the a_j mutually distinct unit vectors e^{ℓ_j}, we obtain

$$\sum r^{j_1, \cdots, j_p} = 0$$

where $(j_1, \cdots, j_p)$ runs through all permutations of the set

$$\{\underbrace{\ell_1, \cdots, \ell_1}_{n_1}, \cdots, \underbrace{\ell_q, \cdots, \ell_q}_{n_q}\} .$$

This, however, implies that the subsum of f corresponding to this summation lies in C. Since every term of f belongs to precisely one such subsum, our assertion is proved.

LEMMA. *Let $n_1 > 0, \cdots, n_q > 0$ be integers. Then there are mutually distinct integers $z_1 > 0, \cdots, z_q > 0$ such that for any integers $n'_1 \geq 0, \cdots,$*

$n'_q \geq 0$,

$$n_1 z_1 + \cdots + n_q z_q = n'_1 z_1 + \cdots + n'_q z_q$$

implies $n'_1 = n_1, \cdots, n'_q = n_q$.

Proof. Put $M = \sum_{i=1}^q n_i$ and choose $k > (2q+1)M$. Put

$$z_i = k^{q-1} + \delta_{1i} k^{q-2} + \cdots + \delta_{q-1,i} \,, \qquad i = 1, \cdots, q \,,$$

with δ_{ji} the Kronecker symbols. Then the equation in the LEMMA reads

$$k^{q-1} \sum_{i=1}^q n'_i + k^{q-2} \sum_{i=1}^q \delta_{1i} n'_i + \cdots + \sum_{i=1}^q \delta_{q-1,i} n'_i$$
$$= k^{q-1} \sum_{i=1}^q n_i + k^{q-2} \sum_{i=1}^q \delta_{1i} n_i + \cdots + \sum_{i=1}^q \delta_{q-1,i} n_i \,.$$

Hence

$$0 \leq n'_i \leq k^{1-q}(k^{q-1}M + k^{q-2}M + \cdots + M) \leq M + k^{-1}(q-1)M \leq M + 1 \,.$$

Therefore,

$$\left| \sum_{i=1}^q (n_i - n'_i) \right| \leq (q+1)M + q \leq (2q+1)M$$

and

$$\left| \sum_{i=1}^q \delta_{ji}(n_i - n'_i) \right| \leq 2M + 1 \leq (2q+1)M \,.$$

Therefore, the uniqueness of the k-adic expansion of integers implies

$$\sum_{i=1}^q (n'_i - n_i) = 0$$
$$\sum_{i=1}^q \delta_{ji}(n'_i - n_i) = 0 \qquad j = 1, \cdots, q-1 \,.$$

Since the determinant of the coefficient matrix equals 1, our assertion is verified.

Analysts may prefer working with $T^*(A, S)_c$ rather than with $T^*(A, S)$. Therefore it is advisable to have

(3.25) DEFINITION. Let A be a R-algebra of arity m. Then the elements of $I_q(A, S)_c = \ker (T^*(A, S)_c \xrightarrow{(\delta^*_\mu, S_c)^{q+1}} T^*(A, S)_c)$, $q = 0, 1, \cdots$ are called the $(q+1)^{st}$ commutative integrals of A with values in S. $I_0(A, S)_c$

is also denoted by $I(A, S)_c$, and $\bigcup_q I_q(A, S)_c$ is abbreviated by $I_\omega(A, S)_c$.

Again (3.4) and (3.6) remain valid.

In (3.11) and the subsequent discussion, S may be replaced by any $\bar{m}$-ary R-algebra $\bar{S}$. Then $T^*(A, \bar{S})$ becomes a graded $\bar{m}$-ary R-algebra by generalizing definition (3.12) in the obvious manner. With respect to this structure, $\delta^*_{\mu,\bar{S}}$ is a R-derivation in the sense that

$$\delta^*_{\mu,\bar{S}}(f_1 \cdots f_{\bar{m}}) = \sum_{i=1}^{\bar{m}} f_1 \cdots f_{i-1} \cdot \delta^*_{\mu,\bar{S}}(f_i) \cdot f_{i+1} \cdots f_{\bar{m}}$$

holds. $C^*(A, \bar{S})$ can be formed as before, and turns out to be a graded ideal of $T^*(A, \bar{S})$ (in the sense of section 1) that is stable under $\delta^*_{\mu,\bar{S}}$. Hence $T^*(A, \bar{S})_c$ and $\delta^*_{\mu,\bar{S}c}$ become available. Clearly, the R-module structure of $I_q(A, \bar{S})$ and $I_q(A, \bar{S})_c$ depends only on A and the R-module structure of $\bar{S}$.

We close this section with a brief remark pertaining to the parameter dependence of $I_q(A, S)$ resp. $I_q(A, S)_c$. Let F be a field and let V be a F-vector space of finite dimension n. If we fix a F-basis of V, then a F-algebra structure A of arity m with underlying vector space V is determined by its structure coefficients which are viewed as elements of $F^{n^{m+1}}$. Thus, we have identified the set of all such F-algebra structures with $F^{n^{m+1}}$. For any such F-algebra A, put

$$I_q^p(A, S) = I_q(A, S) \cap T^p(A, S) \text{ resp. } I_q^p(A, S)_c = I_q(A, S)_c \cap T^p(A, S)_c .$$

If $\dim_F S$ is finite then both

$$\dim_F I_q^p(A, S) \quad \text{and} \quad \dim_F I_q^p(A, S)_c$$

are finite and may be viewed as numerical functions on $F^{n^{m+1}}$. With this understanding we obtain

(3.26) PROPOSITION. *For fixed p, q, and S (with $\dim_F S < \infty$) both*

$$\dim_F I_q^p(A, S) \quad \textit{and} \quad \dim_F I_q^p(A, S)_c$$

are upper semicontinuous on $F^{n^{m+1}}$ with respect to the Zariski-topology.

Proof. Let $b_1, \cdots, b_n$ be the chosen F-basis of A and let $s_1, \cdots, s_N$ be a F-basis of S. Denote by $f_{i_1,\cdots,i_p,k}$ the element of $T^p(A, S)$ which is given by

$$f_{i_1,\cdots,i_p,k}(b_{j_1} \otimes \cdots \otimes b_{j_p}) = \delta_{i_1 j_1} \cdots \delta_{i_p j_p} \cdot s_k .$$

These elements form a canonical F-basis of $T^p(A,S)$. Hence, any $f \in T^p(A,S)$ can be written as

$$f = \sum x_{i_1,\cdots,i_p,k} f_{i_1,\cdots,i_p,k} \ .$$

From this and (3.10) it follows easily that $\delta^{p+m-1}_{\mu,S} f$ has as its coefficients with respect to the canonical basis linear combinations of the $x_{i_1,\cdots,i_p,k}$ whose coefficients are F-linear combinations of the structure coefficients of A. Hence f belongs to $I^p_q(A,S)$ precisely when the $x_{i_1,\cdots,i_p,k}$ are a solution of a certain linear homogeneous system whose coefficients are polynomials in the structure coefficients of A. The co-rank of this linear homogeneous system is upper semicontinuous on $F^{n^{m+1}}$ with respect to the Zariski-topology. Hence our assertion concerning $\dim_F I^p_q(A,S)$ is proved. A similar argument shows the validity of the second claim.

4. Properties of the functor I_*

We shall call $I_q(A,S)$ resp. $I_q(A,S)_c$ trivial if $I^p_q(A,S) = 0$ resp. $I^p_q(A,S)_c = 0$ for all $p > 0$. This is equivalent with $I_q(A,S) = S$ resp. $I_q(A,S)_c = S$.

(4.1) PROPOSITION. *If A is nilpotent of exponent $e + 1$ and $T^*(A,S) \neq 0$ resp. $T^*(A,S)_c \neq 0$ then every $I_q(A,S)$ and $I_q(A,S)_c$ is non-trivial.*

Proof. Denote $d_{\mu,p-q(m-1)} \circ \cdots \circ d_{\mu,p} \colon \bigotimes^p A \to \bigotimes^{p-(q+1)(m-1)} A$ by $d_{p,q}$. Then the image of $d_{p,q}$ will be contained in the submodule of the codomain which is generated by tensor products having as factors n_0 zero-fold products of A (i.e. elements of A), n_1 one-fold products of A (i.e. elements of $\mu(\bigotimes^m A)$), $\cdots, n_{q+1}(q+1)$-fold products of A. For these integers n_i the following relations are easily established, for $q \geq 0$ and $m \geq 2$:

$$p - (q+1)(m-1) = \sum_{i=1}^{q+1} n_i \geq 1 \ ,$$

$$q + 1 = \sum_{i=0}^{q+1} i n_i \ , \qquad n_i \geq 0 \text{ and } n_i \neq 0 \qquad \text{for some } i \geq 1 \ .$$

It is then easy to verify that $d_{p,q} = 0$ whenever

$$1 + (q+1)(m-1) \leq p \leq (q+1)e^{-1} + (q+1)(m-1)$$

holds. Thus $I_q(A,S)$ and $I_q(A,S)_c$ are non-trivial for $q \geq e - 1$, and therefore for all values of q.

(4.2) PROPOSITION. *If $A \to A''$ is a surjective morphism of R-algebras then the induced maps $I_q(A'', S) \to I_q(A, S)$ and $I_q(A'', S)_c \to I_q(A, S)_c$ are injections. In particular, for any R-algebra A there are canonical injections*

$$I_q(A/\mu(\otimes_R^m A), S) \to I_q(A, S)$$

and

$$I_q(A/\mu(\otimes_R^m A), S)_c \to I_q(A, S)_c .$$

Proof. The last assertions follow from the first ones and the fact that $\mu(\otimes_R^m A)$ is an ideal of A. Now, if $\varphi: A \to A''$ is a surjection so is the induced map $T_p(\varphi): T_p(A) \to T_p(A'')$. Hence $T^*(\varphi): T^*(A'', S) \to T^*(A, S)$ is an injection, as is $I_q(A'', S) \to I_q(A, S)$. Finally, let $\bar{f}'' \in I_q(A'', S)_c$ be in the kernel of $I_q(A'', S)_c \to I_q(A, S)_c$. Lift $\bar{f}''$ back to $T_q(A'', S)$ as f'', and denote the image of f'' in $T_q(A, S)$ by f. Then f is in $C^*(A, S)$. One concludes easily from (3.20) that this implies $f'' \in C^*(A'', S)$, and thus $\bar{f}''$ vanishes.

(4.3) LEMMA. *If S is an integral domain then so are $T^*(A, S)$ and, if S has no $\mathbf{Z}$-torsion, $T^*(A, S)_c$.*

Proof. This is obvious for $T^*(A, S)$, due to (3.12). As for the commutative case, let $f_i \in T^{p_i}(A, S)$, $i = 1, 2$, such that $f_1 f_2 \in C^*(A, S)$ holds. This implies that

$$f_1(\otimes^{p_1} a) f_2(\otimes^{p_2} a) = 0 , \qquad \text{for all } a \in A .$$

Putting

$$T_1 = \{a : f_1(\otimes^{p_1} a) = 0\} \quad \text{and} \quad T_2 = \{a : f_2(\otimes^{p_2} a) = 0\} ,$$

we obtain $A = T_1 \cup T_2$. The R-submodules of A that are either contained in T_1 or contained in T_2 are ordered by inclusion. A simple application of Zorn's lemma shows that there is a maximal such submodule, say N. We claim that $N = A$. Otherwise, assume that $N \subset T_1$ holds and choose $a_0 \notin N$. For $0 \neq n_0 \in N$ there are infinitely many pairs (k_i, ℓ_i) of integers with mutually distinct ratios such that either $k_i a_0 + \ell_i n_0 \in T_1$, for all i, or $k_i a_0 + \ell_i n_0 \in T_2$, for all i, holds. In the first case we have

$$0 = f_1(\otimes^{p_1}(k_i a_0 + \ell_i n_0)) = \sum_{j=0}^{p_1} k_i^j \ell_i^{p_1 - j} \cdot f_1\Big(\sum_j\Big)$$

where $\sum_j$ stands for a certain sum of tensor products involving a_0 and

n_0. Since S has no $\boldsymbol{Z}$-torsion this implies that, for every $j, f_1(\sum_j) = 0$. Consequently, the R-submodule of A generated by a_0 and n_0 lies in T_1. Hence we have the alternative that the R-submodule of A generated by a_0 and n_0 is contained either in T_1 or in T_2. Now, if for every $n_0 \in N$ the R-submodule generated by a_0 and n_0 were contained in T_1 then we would have a contradiction to the assumed maximality of N. Hence there is an element $0 \neq r_0 a_0 + n_0$, with $r_0 \in R$ and $n_0 \in N$, which is not in T_1 and hence not in N. Therefore, for any $n \in N$, the R-submodule generated by n and $r_0 a_0 + n_0$ is contained in T_2; hence, again, we have a contradiction. Thus $N = A = T_1$. The argument following formula (3.21) then shows that f_1 belongs to $C^*(A, S)$.

(4.4) THEOREM. *Let S be an integral domain containing $\boldsymbol{Q}$. Then $I_0(A, S)_c$ is integrally closed in $T^*(A, S)_c$.*

Proof. Let $f \in T^*(A, S)_c$ be integral over $I_0(A, S)_c$ and let $P \in I_0(A, S)_c[X]$ be a monic polynomial of least degree such that $P(f) = 0$. Then

$$0 = \delta^*_{\mu, Sc} P(f) = P'(f) \delta^*_{\mu, Sc}(f) .$$

$P'(f)$ does not vanish as P' is not the zero polynomial and has lesser degree than P. Since $T^*(A, S)_c$ is an integral domain, due to (4.3), this means that $\delta^*_{\mu, Sc}(f) = 0$, i.e., $f \in I_0(A, S)_c$.

(4.5) COROLLARY. *Let R be an integral domain containing $\boldsymbol{Q}$ and let A be a R-algebra which is free and of finite dimension n as a R-module. Then the transcendence degree of $I_0(A, R)_c$ over R is $\leq n$. If the transcendence degree equals n then $I_0(A, R)_c = T^*(A, R)_c$. If, in addition, R is factorial and if the transcendence degree equals* 1, *then $I_0(A, R)_c$ is isomorphic to the polynomial ring $R[X]$.*

Proof. By (3.24), $T^*(A, R)_c$ is isomorphic with $R[X_1, \cdots, X_n]_c$. Hence the first assertion is obvious, and the second one follows from (4.4). Now assume that the transcendence degree of $I_0(A, R)_c$ equals 1. Let $p > 0$ be minimal with respect to $I_0^p(A, R)_c \neq 0$ and choose $0 \neq t \in I_0^p(A, R)_c$. Evidently, t is a transcendence basis of $I_0(A, R)_c$ over R. Let $0 \neq u \in I_0^q(A, R)_c$. Then there is a polynomial

$$P(X, Y) = \sum \{r_i X^i Y^j : ip + jq = N\} \in R[X, Y]_c$$

with $P(t, u) = 0$. Due to (4.3) we may assume that $r_0 \neq 0$ and $r_{N/p} \neq 0$

hold. This, however, implies that every prime divisor of t also divides u, and vice versa. Thus an easy argument, involving the degrees of t and u, and the assumption that t is of minimal degree, shows that u is a power of t. Since $I_0(A, R)_c$ is graded, this finishes the proof.

Now we shall discuss some change-of-ring theorems.

(4.6) LEMMA. *Let $R \to R'$ be a unital ring homomorphism and let A be a R-algebra of arity m. Then the canonical m-ary structure $R' \otimes_R \mu$ on $R' \otimes_R A$ furnishes a commutative diagram*

$$\begin{array}{ccc} T_*(R' \otimes_R A) & \xrightarrow{\cong} & R' \otimes_R T_*(A) \\ {\scriptstyle d_{R' \otimes_R \mu, *}} \downarrow & & \downarrow {\scriptstyle R' \otimes_R d_{\mu,*}} \\ T^*(R' \otimes_R A) & \xrightarrow{\cong} & R' \otimes_R T_*(A) \end{array}$$

where the horizontal arrows are isomorphisms of degree zero. The diagram is functorial in A.

Proof. The canonical structure, $R' \otimes_R \mu$, is given by

$$R' \otimes_R \mu((r'_1 \otimes a_1) \otimes \cdots \otimes (r'_m \otimes a_m)) = r'_1 \cdots r'_m \otimes \mu(a_1 \otimes \cdots \otimes a_m)$$

μ being the multiplication on A. The horizontal isomorphism $R' \otimes_R T_*(A) \to T_*(R' \otimes_R A)$ is given by (see [1], p. 489)

$$r' \otimes (a_1 \otimes \cdots \otimes a_p) \mapsto r'((1 \otimes a_1) \otimes \cdots \otimes (1 \otimes a_p)) .$$

These definitions insure functoriality. The commutativity of the diagram follows by easy verification.

(4.7) LEMMA. *Let S' be a R'-algebra of arity 2, and let $R \to R'$ be a unital ring homomorphism. Then the isomorphism in* (4.6) *induces isomorphisms of graded algebras*

$$T^*(R' \otimes {}_R A, S') \xrightarrow{\cong} T^*(A, {}_R S')$$

and

$$T^*(R' \otimes {}_R A, S')_c \xrightarrow{\cong} T^*(A, {}_R S')_c ,$$

where ${}_R S'$ is the canonical R-algebra structure of arity 2 *on S'. They are functorial in A and S' and render the following diagrams commutative*

$$\begin{array}{ccc} T^*(R' \otimes {}_R A, S') & \xrightarrow{\cong} & T^*(A, {}_R S') \\ {\scriptstyle \delta^*_{R' \otimes_R \mu, S'}} \downarrow & & \downarrow {\scriptstyle \delta^*_{\mu, {}_R S'}} \\ T^*(R' \otimes {}_R A, S') & \xrightarrow{\cong} & T^*(A, {}_R S') \end{array}$$

resp.

$$\begin{array}{ccc} T^*(R' \otimes {}_R A, S')_c & \xrightarrow{\cong} & T^*(A, {}_R S')_c \\ {\scriptstyle \delta^*_{R' \otimes_R \mu, S' c}} \downarrow & & \downarrow {\scriptstyle \delta^*_{\mu, {}_R S' c}} \\ T^*(R' \otimes {}_R A, S')_c & \xrightarrow{\cong} & T^*(A, {}_R S')_c \end{array}$$

Proof. The first isomorphism sends $f' \in T^p(R' \otimes {}_R A, S')$ into the element $f \in T^p(A, {}_R S')$ that is given by

$$f(a_1 \otimes \cdots \otimes a_p) = f'((1 \otimes a_1) \otimes \cdots \otimes (1 \otimes a_p)) \,. \tag{4.8}$$

An easy verification shows the required commutativity. From (4.8) it follows immediately that the first isomorphism maps $C^*(R' \otimes {}_R A, S')$ onto $C^*(A, {}_R S')$. Hence the remainder of (4.7) is established.

Again, let $R \to R'$ be a unital ring homomorphism. Then there are, for any R-algebra of arity 2, canonical R-module homomorphisms

$$\begin{aligned} &T^*(A, S) \longrightarrow T^*(A, {}_R(R' \otimes {}_R S)) \xrightarrow{\cong} T^*(R' \otimes {}_R A, R' \otimes {}_R S) \\ &T^*(A, S)_c \longrightarrow T^*(A, {}_R(R' \otimes {}_R S))_c \xrightarrow{\cong} T^*(R' \otimes {}_R A, R' \otimes {}_R S)_c \end{aligned} \tag{4.9}$$

which are functorial in A and S. Here, the two last isomorphisms are the ones described in (4.7). The first homomorphism in the top line of (4.9) is the composition with the map $S \ni s \mapsto 1 \otimes s \in {}_R(R' \otimes {}_R S)$; since it maps $C^*(A, S)$ into $C^*(A, {}_R(R' \otimes {}_R S))$, it induces the first homomorphism of the bottom line of (4.9). With these homomorphisms we obtain

(4.10) PROPOSITION. *Let $R \to R'$ be an injective, unital ring homomorphism, let A be a R-algebra of arity m, and let S be a R-algebra of arity two which is flat as a R-module. Then there are isomorphisms of R-algebras, resp. R-modules both of which are functorial in A and S,*

$$I_q(A, S) \cong I_q(R' \otimes {}_R A, R' \otimes {}_R S) \cap \operatorname{im}\,(T^*(A, S) \to T^*(R' \otimes {}_R A, R' \otimes {}_R S))$$

and

$$I_q(A, S)_c \cong I_q(R' \otimes {}_R A, R' \otimes {}_R S)_c \cap \operatorname{im}\,(T^*(A, S)_c \to T^*(R' \otimes {}_R A, R' \otimes {}_R S)_c) \,.$$

Proof. Since S is a flat R-module, the top homomorphism (4.9) is injective, and the first set of isomorphisms follows from (4.7) by easy diagrammatics. The second set of isomorphisms is established similarly as one verifies easily that $T^*(A, S) \to T^*(A, {}_R(R' \otimes_R S))$ maps $C^*(A, S)$ onto $C^*(A, {}_R(R' \otimes_R S)) \cap \operatorname{im}(T^*(A, S))$.

(4.11) PROPOSITION. *Let $R \to R'$ be a unital ring homomorphism and let A be a R-algebra of arity m. Suppose that either R' or A is a finitely generated projective R-module. Then there are isomorphisms of R-algebras resp. R-modules, both of which are functorial in A and S,*

$$R' \otimes_R I_q(A, S) \cong I_q(R' \otimes_R A, R' \otimes_R S)$$

and

$$R' \otimes_R I_q(A, S)_c \cong I_q(R' \otimes_R A, R' \otimes_R S)_c \ .$$

Proof. There is a canonical R'-module homomorphism

$$\omega : R' \otimes_R T^*(A, S) \to T^*(R' \otimes_R A, R' \otimes_R S)$$

which is given by

$$\omega(r' \otimes f)((r_1 \otimes a_1) \otimes \cdots \otimes (r_p \otimes a_p)) = r'r_1 \cdots r_p \otimes f(a_1 \otimes \cdots \otimes a_p) \ .$$

It follows from [1], p. 489 and 282, that under the above assumptions ω is an isomorphism. An easy verification shows that the following diagram commutes

$$\begin{array}{ccc} R' \otimes_R T^*(A, S) & \xrightarrow{\omega} & T^*(R' \otimes_R A, R' \otimes_R S) \\ {\scriptstyle R' \otimes \delta^*_{\mu,S}} \Big\downarrow & & \Big\downarrow {\scriptstyle \delta^*_{R' \otimes_R \mu, R' \otimes_R S}} \\ R' \otimes_R T^*(A, S) & \xrightarrow{\omega} & T^*(R' \otimes_R A, R' \otimes_R S) \ . \end{array}$$

Thus we have the first batch of isomorphisms. Since ω maps $R' \otimes_R C^*(A, S)$ onto $C^*(R' \otimes_R A, R' \otimes_R S)$—as is checked easily—the second batch of isomorphisms is established similarly.

(4.12) COROLLARY. *Let $R \to R'$ be a unital injective ring homomorphism, let A be a R-algebra of arity m, and let S be a R-algebra of arity 2 which is flat as a R-module. Suppose that either R' or A is a finitely generated projective R-module. Then*

(i) $I_q(A, S)$ (resp. $I_q(A, S)_c$) *is trivial if and only if* $I_q(R' \otimes_R A, R' \otimes_R S)$ (resp. $I_q(R' \otimes_R A, R' \otimes_R S)_c$) *is*

(ii) $I_q(A, S) = T^*(A, S)$ (resp. $I_q(A, S)_c = T^*(A, S)_c$) *if and only if*

$I_q(R' \otimes_R A,\ R' \otimes_R S) = T^*(R' \otimes_R A,\ R' \otimes_R S)$ (resp. $I_q(R' \otimes_R A,\ R' \otimes_R S)_c = T^*(R' \otimes_R A, R' \otimes_R S)_c$).

Proof. (4.10) and (4.11).

(4.13) THEOREM. *Let F be a field of characteristic zero and let A be a F-algebra of arity m with $\dim_F A < \infty$. Suppose that A possesses a non-trivial idempotent e such that the F-endomorphism τ of A, which is given by*

$$\tau(a) = \mu(e \otimes \cdots \otimes e \otimes a)\ ,$$

satisfies

$$\det(\ell \cdot \mathrm{id}_A + \tau) \neq 0 \qquad \text{for } \ell = 0, 1, \cdots .$$

Then $I_q(A, S)$ is trivial.

Proof. We shall prove that $d_{\mu,p}, p \geq m$, is a surjection; this implies that $\delta^*_{\mu,S}$ is injective, and hence that $I_q(A, S)$ is trivial. In order to obtain the required surjectivity we show, by induction on q, that $\otimes^{p-q-m+1} e \otimes T_q(A)$ is in the image of $d_{\mu,p}$. This is clear for $q = 0$ as

$$d_{\mu,p}(\otimes^p e) = (p - m + 1) \otimes^{p-m+1} e\ ,$$

and $p - m + 1 \neq 0$ since F has characteristic zero. Next, one verifies easily that for $p - q \geq m - 1$

$$\begin{aligned} &d_{\mu,p}(\otimes^{p-q} e \otimes a_1 \otimes \cdots \otimes a_q) \\ &\quad = (p - q - m + 1) \otimes^{p-q-m+1} e \otimes a_1 \otimes \cdots \otimes a_q \\ &\qquad + \otimes^{p-q-m+1} e \otimes \tau(a_1) \otimes a_2 \otimes \cdots \otimes a_q + \otimes^{p-q-m+2} e \otimes b\ , \end{aligned} \tag{4.14}$$

with $b \in T_{q-1}(A)$ holds. It is well known that there is a F-basis

$$b_{i,j}\ , \qquad i = 1, \cdots, n_j; j = 1, \cdots, t$$

of A such that

$$\begin{aligned} \tau(b_{i,j}) &= b_{i+1,j}\ , \qquad i = 1, \cdots, n_j - 1 \\ \tau(b_{n_j,j}) &= \sum_{i=1}^{n_j} r_{i,j} b_{i,j} \end{aligned}$$

with suitable scalars $r_{i,j} \in F$. If we apply (4.14) to $a_1 = b_{i,j}$ then the induction hypothesis leads to

$$\begin{aligned} &(p - q - m + 1) \otimes^{p-q-m+1} e \otimes b_{i,j} \otimes a_2 \otimes \cdots \otimes a_q \\ &\qquad + \otimes^{p-q-m+1} e \otimes \tau(b_{i,j}) \otimes a_2 \otimes \cdots \otimes a_q \in \mathrm{im}\, d_{\mu,p}\ . \end{aligned} \tag{4.15}$$

The element occurring in (4.15) can be written in the form

$$\otimes^{p-q-m+1} e \otimes ((p-q-m+1)\,\mathrm{id}_A + \tau)(b_{i,j}) \otimes a_2 \otimes \cdots \otimes a_q \,.$$

Since, by assumption, $(p-q-m+1)\,\mathrm{id}_A + \tau$ is invertible and since the elements $b_{i,j}$ form a F-basis of A, we finally obtain

$$\otimes^{p-q-m+1} e \otimes T_q(A) \subset \mathrm{im}\, d_{\mu,p} \,.$$

(4.16) COROLLARY. *Let F be a field of characteristic zero, and let S be a F-algebra of arity 2 with $\dim_F S < \infty$. Then $I_q(A,S)$ is Zariski-generically trivial.*

Proof. The proof of (3.26) shows that $I_q(A,S)$ is trivial on a countable intersection of Zariski-open sets. This intersection is not empty as the hypotheses of (4.13) are satisfied for any R-algebra which possesses a unit element.

It follows easily from the first part of the proof of (4.13)—up to and including (4.14)—that the following assertion is valid.

(4.17) PROPOSITION. *Suppose that $\boldsymbol{Q} \subset R$ holds. Then for any R-algebra A of arity m which possesses a unit element*[3]*, $I_q(A,S)$ is trivial.*

We turn now to the commutative analog of (4.13). Here we have

(4.18) THEOREM. *Let F be a field of characteristic zero and let A be a F-algebra of arity m with $\dim_F A < \infty$. Suppose that A possesses an idempotent element e such that the F-endomorphism T of A, which is given by*

$$T(a) = \sum_{i=1}^{m} \mu(\otimes^{i-1} e \otimes a \otimes \otimes^{m-i} e) \,,$$

satisfies the following conditions

$$\det\left(\sum_{i=1}^{k} \otimes^{i-1}\mathrm{id}_A \otimes T \otimes \otimes^{k-i}\mathrm{id}_A + (p-k-m+1)\otimes^k \mathrm{id}_A\right) \neq 0 \tag{4.19}$$

$$m+1 \leq p \,,\ 1 \leq k \leq p-m+1 \,.$$

Then $I_q(A,S)_c$ is trivial. The stated conditions are satisfied if no finite sum of eigenvalues of T is a negative integer or zero. In fact, (4.19)

[3] The element u of the R-algebra A of airty m is called *a* unit element of A if, for all $a \in A$ and $i = 1, \cdots, m$,

$$\mu(\otimes^{i-1} u \otimes a \otimes \otimes^{m-i} u) = a$$

holds.

need only be satisfied for all sufficiently large values of p.

Proof. We shall prove that $d_{\mu,p}$, $p \geq m+1$, maps the F-vector subspace $C_p(A)$ of $T_p(A)$ that is generated by the elements (3.20) onto $C_{p-m+1}(A)$. Then

$$\delta^p_{\mu,S} f \in C^p(A, S)$$

implies that

$$0 = \delta^p_{\mu,S} f(C_p(A)) = f(d_{\mu,p} C_p(A)) = f(C_{p-m+1}(A))$$

holds, and hence the canonical image of f in $T^*(A, S)_c$ vanishes; thus $\delta^*_{\mu,Sc}$ is injective and our assertion follows. The fact that $d_{\mu,p}$ maps $C_p(A)$ onto $C_{p-m+1}(A)$ is established by an induction argument similar to the one used in the proof of (4.13). In order to set up the induction we return to the proof of (3.24). There, certain elements $\sum a_{j_1} \otimes \cdots \otimes a_{j_p}$ of $C_p(A)$ were constructed. We denote this element by $c_p(a_2 \otimes \cdots \otimes a_q)$ if, in that notation, $n_1 = p - q + 1$, $n_2 = \cdots = n_q = 1$ and $a_1 = e$ hold. $\otimes^p e$ is denoted by $c_p(\phi)$, and corresponds to $q = 1$. First we observe that

$$d_{\mu,p} c_p(\phi) = (p - m + 1) c_{p-m+1}(\phi)$$

holds, which provides the starting point for the induction argument. Next, an easy computation shows that

$$\begin{aligned} d_{\mu,p} c_p(a_2 \otimes \cdots \otimes a_q) &- \sum_{i=2}^{q} c_{p-m+1}(a_2 \otimes \cdots \otimes a_{i-1} \otimes Ta_i \otimes a_{i+1} \otimes \cdots \otimes a_q) \\ &- (p - q - m + 2) c_{p-m+1}(a_2 \otimes \cdots \otimes a_q) \end{aligned}$$

is a linear combination of terms of the form $c_{p-m+1}(b_2 \otimes \cdots \otimes b_r)$ with $r < q$. Hence, by induction hypothesis, this expression is in $d_{\mu,p} C_p(A)$. Since

$$\begin{aligned} \sum_{i=2}^{q} & c_{p-m+1}(a_2 \otimes \cdots \otimes a_{i-1} \otimes Ta_i \otimes a_{i+1} \otimes \cdots \otimes a_q) \\ &+ (p - q - m + 2) c_{p-m+1}(a_2 \otimes \cdots \otimes a_q) \\ &= c_{p-m+1}\Bigg(\Bigg(\sum_{i=1}^{q-1} \otimes^{i-1} \mathrm{id}_A \otimes T \otimes \otimes^{q-1-i} \mathrm{id}_A \\ &\qquad + (p - q - m + 2) \otimes^{q-1} \mathrm{id}_A\Bigg)(a_2 \otimes \cdots \otimes a_q)\Bigg), \end{aligned}$$

the conditions (4.19) show that $c_{p-m+1}(b_2 \otimes \cdots \otimes b_q)$ lies in $d_{\mu,p} C_p(A)$, for any choice of $b_2, \cdots, b_q$. This finishes the induction proof as the elements $c_{p-m+1}(b_2 \otimes \cdots \otimes b_q)$, $q = 2, \cdots, p - m + 1$, generate $C_{p-m+1}(A)$. The last assertion follows by computing the eigenvalues of the F-endomorphisms (4.19).

(4.20) COROLLARY. *Let F be a field of characteristic zero, and let S be a F-algebra of arity* 2 *with* $\dim_F S < \infty$. *Then $I_q(A,S)_c$ is Zariski-generically trivial.*

Proof. Same as for (4.16).

We close this section with a claim whose proof is obtained by going through the proof of (4.18).

(4.21) PROPOSITION. *Suppose that $\boldsymbol{Q} \subset R$ holds. Then for any R-algebra A of arity m which possesses a unit element, $I_q(A,S)_c$ is trivial.*

5. The categories ${}_R\mathscr{D}iff'_m$ and ${}_R\mathscr{A}lg_m$

Let D' and D'' be differential equation of arity m over a commutative unital Banach algebra R, and denote by n' resp. n'' their dimension. Recall that any formal solution is then a convergent solution in the sense that the power series converges for sufficiently small values of the variable. We are interested in the germs of analytic maps $F: R^{n'} \to R^{n''}$ which have constant term zero and map every solution of D' with sufficiently small constant term into a solution of D''. If the components of F are denoted by $F_1, \cdots, F_{n''}$ then, with $X'(t) = (X'_1(t), \cdots, X'_{n'}(t))$ a solution of D' for which $F \circ X'(t)$ exists,

$$\frac{d}{dt} F_j(X'_1(t), \cdots, X'_{n'}(t)) = \sum_{i=1}^{n'} \frac{\partial F_j}{\partial X'_i}(X'(t)) \cdot D'_i(X'(t)) = D''_j(F(X'(t))) \; , \qquad j = 1, \cdots, n'' \; .$$

By (2.1), this is equivalent with

$$\sum_{i=1}^{n'} \frac{\partial F_j}{\partial X'_i}(X') \cdot D'_i(X') = D''_j(F(X')) \; , \qquad j = 1, \cdots, n'' \; . \tag{5.1}$$

If f is a morphism from D' to D'', then the convergent power series

$$F(X') = \left(\sum_{i=1}^{n'} f_1^i X'_i, \cdots, \sum_{i=1}^{n'} f_{n''}^i X'_i\right)$$

satisfies (5.1) as (1.2) shows. Conversely, every F which satisfies (5.1) and consists of the linear term only, arises from such a morphism f. An easy verification leads to

(5.2) PROPOSITION. *Let R be a commutative unital Banach algebra. Then the differential equations over R of arity m and the germs of analytic maps which have constant term zero and satisfy* (5.1), *with composition of such maps the set-theoretical one, form a category ${}_R\mathscr{D}iff'_m$ which contains ${}_R\mathrm{Diff}_m$ as a subcategory. The functor $S\colon {}_R\mathrm{Diff}_m \to \mathrm{Sets}$ extends canonically to ${}_R\mathscr{D}iff'_m$.*

As in the case of first and higher integrals we wish to find a purely algebraic setting for ${}_R\mathscr{D}iff'_m$. Again, it will be done in a non-commutative setting. Since we will have to deal now with several R-algebras of arity m at the same time we shall denote now the multiplication in A by μ_A, the one in B by μ_B, etc.

Given $A, B \in {}_R\mathrm{Alg}_m$ we put

$$P(A,B) = \prod_{p=1}^{\infty} \mathrm{Hom}_R\,(T_p(A), B)\ . \tag{5.3}$$

The p^{th} component of the element $\lambda \in P(A,B)$ shall be denoted by λ^p. $P(A,B)$ is made into a R-algebra of arity m by putting

$$(\mu_{A,B}(\lambda_1 \otimes \cdots \otimes \lambda_m))^p = \sum_{j_1+\cdots+j_m=p} \mu_B \circ (\lambda_1^{j_1} \otimes \cdots \otimes \lambda_m^{j_m})\ . \tag{5.4}$$

If $f\colon A' \to A$ and $g\colon B \to B'$ are morphisms of R-algebras, we put

$$(P(f,g)(\lambda))^p = g \circ \lambda^p \circ T_p(f)\ .$$

An easy verification shows

(5.5) LEMMA. *P is a bifunctor from ${}_R\mathrm{Alg}_m^{op} \times {}_R\mathrm{Alg}_m$ to ${}_R\mathrm{Alg}_m$.*

$P(A,B)$ should be interpreted as the algebra of formal power series on A with values in B whose constant term is zero. We can apply the constructions of section 3 to the algebra $P(A,B)$; the graded endomorphism $d_{\mu_{A,B},*}$ of $T_*(P(A,B))$ shall be denoted by $d_{(A,B),*}$.

Next we define a map

$$\circ\colon P(A,B) \times P(B,C) \to P(A,C)$$

by defining $\lambda \circ \kappa = \circ(\kappa, \lambda)$ through

$$(\lambda \circ \kappa)^p = \sum_q \sum_{j_1+\cdots+j_q=p} \lambda^q \circ (\kappa^{j_1} \otimes \cdots \otimes \kappa^{j_q})\ . \tag{5.6}$$

This latter sum is clearly finite as $j_i \geq 1$ forces $q \leq p$.

(5.7) PROPOSITION. *The R-algebras of arity m together with $P(A,B)$ as the morphisms from A to B, with the composition defined by* (5.6), *form a category ${}_R\mathscr{A}_m$ with internal hom-functor which contains ${}_R\mathrm{Alg}_m$ as a subcategory.*

Proof. It is easy to see that the identity morphism $\mathrm{id}_A: A \to A$ is given by $(\mathrm{id}_A)^p = 0$, for $p > 1$, while $(\mathrm{id}_A)^1$ is the identity map $A \to A$. As for associativity of the composition, this can be verified by a straightforward, but somewhat messy computation. The imbedding of ${}_R\mathrm{Alg}_m$ into ${}_R\mathscr{A}_m$ is achieved by assigning to $f \in {}_R\mathrm{Alg}_m(A,B)$ the element $\bar{f} \in {}_R\mathscr{A}_m(A,B)$ that is given by $\bar{f}^p = 0$, for $p > 1$, and $\bar{f}^1 = f$.

Next we define a R-module homomorphism

$$\delta_{A/B}: P(A,B) \to P(A,B)$$

by

$$(\delta_{A/B}\lambda)^{p+m-1} = \lambda^p \circ d_{\mu_A, p+m-1} \tag{5.8}$$

Here one verifies

(5.9) LEMMA.

$$\delta_{A/B}(\mu_{A,B}(\lambda_1 \otimes \cdots \otimes \lambda_m)) = \mu_{A,B}\left(\sum_{i=1}^{m} \lambda_1 \otimes \cdots \otimes \lambda_{i-1} \otimes \delta_{A/B}\lambda_i \otimes \lambda_{i+1} \otimes \cdots \otimes \lambda_m\right).$$

Finally we need the R-module homomorphism

$$\eta: \otimes_R^q P(A,B) \to P(A, \otimes_R^q B)$$

that is given by

$$\eta(\lambda_1 \otimes \cdots \otimes \lambda_q)^p = \sum_{j_1 + \cdots + j_q = p} \lambda_1^{j_1} \otimes \cdots \otimes \lambda_q^{j_q}. \tag{5.10}$$

Evidently

$$\mu_{A,B}(\lambda_1 \otimes \cdots \otimes \lambda_m) = P(A, \mu_B) \circ \eta(\lambda_1 \otimes \cdots \otimes \lambda_m)$$

and

$$(\lambda \circ \kappa)^p = \sum_q \lambda^q \circ \eta(\otimes^q \kappa)^p$$

hold.

(5.11) Definition. Let A, B be R-algebras of arity m and put

$$_R\mathscr{A}lg_m(A,B) = \{\lambda : \delta_{A/B}\lambda = \mu_{A,B}(\bigotimes^m \lambda)\} \subset P(A,B) .$$

(5.12) Proposition. *For any R-algebras A, B of arity m*

(i) $_R\mathrm{Alg}_m\,(A,B) \subset {_R\mathscr{A}lg_m}\,(A,B)$ *(in the sense of* (5.7))

(ii) *if* $\lambda \in {_R\mathscr{A}lg_m}\,(A,B)$ *then* $\lambda^1 \in {_R\mathrm{Alg}_m}\,(A,B)$.

Proof. (i) $f \in {_R\mathrm{Alg}_m}$ is equivalent with $f \circ \mu_A = \mu_B \circ \bigotimes^m f$. Due to (5.4), (5.8), (5.10)

$$(\delta_{A/B}\bar{f})^m = f \circ d_{\mu_A,m} = f \circ \mu_A = \mu_B \circ (\bigotimes^m \bar{f})^m = (\mu_{A,B}(\bigotimes^m \bar{f}))^m .$$

Thus

$$\delta_{A/B}\bar{f} = \mu_{A,B}(\bigotimes^m \bar{f}) .$$

(ii) Due to (5.4), (5.8), (5.10)

$$\lambda^1 \circ \mu_A = \lambda^1 \circ d_{\mu_A,m} = (\delta_{A/B}\lambda)^m = (\mu_{A,B}(\bigotimes^m \lambda))^m = \mu_B(\bigotimes^m \lambda^1) .$$

(5.13) Lemma. *Let $\kappa \in {_R\mathscr{A}_m}(A,B)$ and $\lambda \in {_R\mathscr{A}_m}(B,C)$. Then for $q \leq p$,*

$$(\delta_{B/C}\lambda)^q \circ \eta(\bigotimes^q \kappa)^p = \lambda^{q-m+1} \circ (d_{(A,B),q}(\bigotimes^q \kappa))^p .$$

Proof.

$$\begin{aligned}
(\delta_{B/C}\lambda)^q \circ \eta(\bigotimes^q \kappa)^p &= \lambda^{q-m+1} \circ d_{\mu_B,q} \circ \sum_{j_1+\cdots+j_q=p} \kappa^{j_1} \otimes \cdots \otimes \kappa^{j_q} \\
&= \lambda^{q-m+1} \circ \left(\sum_{i=1}^{q-m+1} \bigotimes^{i-1} \mathrm{id}_B \otimes \mu_B \otimes \bigotimes^{q-m-i+1} \mathrm{id}_B\right) \circ \sum_{j_1+\cdots+j_q=p} \kappa^{j_1} \otimes \cdots \otimes \kappa^{j_q} \\
&= \lambda^{q-m+1} \circ \sum_{j_1+\cdots+j_q=p} \sum_{i=1}^{q-m+1} \kappa^{j_1} \otimes \cdot\cdot \otimes \kappa^{j_{i-1}} \otimes \mu_B \circ (\kappa^{j_1} \otimes \cdot\cdot \otimes \kappa^{j_{i+m-1}}) \otimes \kappa^{j_{i+m}} \otimes \cdot\cdot \\
&= \lambda^{q-m+1} \circ \sum_{j_1+\cdots+j_q=p} \sum_{i=1}^{q-m+1} \kappa^{j_1} \otimes \cdot\cdot \otimes \kappa^{j_{i-1}} \otimes \mu_{A,B}(\bigotimes^m \kappa)^{j_i+\cdots+j_{i+m-1}} \otimes \kappa^{j_{i+m}} \otimes \cdot\cdot \\
&= \lambda^{q-m+1} \circ (d_{(A,B),q}(\bigotimes^q \kappa))^p .
\end{aligned}$$

(5.14) Theorem. *The R-algebras of arity m with the morphisms from A to B precisely the elements of $_R\mathscr{A}lg_m(A,B)$ form a subcategory $_R\mathscr{A}lg_m$ of $_R\mathscr{A}_m$.*

Proof. It is very easy to see that the identity morphism id_A belongs to $_R\mathscr{A}lg_m(A,A)$. Take $\kappa \in {_R\mathscr{A}lg_m}(A,B)$ and $\lambda \in {_R\mathscr{A}lg_m}(B,C)$. Then

$$
\begin{aligned}
((\delta_{A/C}(\lambda\circ\kappa))^p &= (\lambda\circ\kappa)^{p-m+1}\circ d_{\mu_{A},p} = \sum_q \sum_{j_1+\cdots+j_q=p-m+1} \lambda^q\circ(\kappa^{j_1}\otimes\cdots\otimes\kappa^{j_q})\circ d_{\mu_{A},p}\\
&= \sum_q \sum_{j_1+\cdots+j_q=p-m+1} \lambda^q\circ((\delta_{A/B}\kappa)^{j_1+m-1}\otimes\kappa^{j_2}\otimes\cdots\otimes\kappa^{j_q}\\
&\quad + \kappa^{j_1}\otimes(\delta_{A/B}\kappa)^{j_2+m-1}\otimes\kappa^{j_3}\otimes\cdots+\cdots) = (\text{due to } (5.11))\\
&= \sum_q \sum_{j_1+\cdots+j_q=p-m+1} \lambda^q\circ(\mu_{A,B}(\otimes^m\kappa)^{j_1+m-1}\otimes\kappa^{j_2}\otimes\cdots\otimes\kappa^{j_q}+\cdots)\\
&= \sum_q \lambda^q\circ(d_{(A,B),q+m-1}(\otimes^{q+m-1}\kappa))^p = (\text{due to } (5.13))\\
&= \sum_q (\delta_{B/C}\lambda)^{q-m+1}\circ\eta(\otimes^{q-m+1}\kappa)^p\\
&= \sum_q (\delta_{B/C}\lambda)^{q}\circ\eta(\otimes^{q}\kappa)^p = \sum_q (\mu_{B,C}(\otimes^m\lambda))^q\circ\eta(\otimes^q\kappa)^p\\
&= \sum_q \sum_{q_1+\cdots+q_r=q} \mu_C\circ(\lambda^{q_1}\otimes\cdots\otimes\lambda^{q_r})\circ\eta(\otimes^q\kappa)^p\\
&= \sum_q \sum_{q_1+\cdots+q_r=q} \sum_{i_1+\cdots+i_q=p} \mu_C\circ(\lambda^{q_1}\otimes\cdots\otimes\lambda^{q_r})\\
&\quad \circ(\kappa^{i_1}\otimes\cdots\otimes\kappa^{i_{q_1}}\otimes\cdots\otimes\kappa^{i_q})\\
&= \sum_q \sum_{q_1+\cdots+q_r=q} \sum_{i_1+\cdots+i_q=p} \mu_C\circ((\lambda^{q_1}\circ(\kappa^{i_1}\otimes\cdots\otimes\kappa^{i_{q_1}}))\\
&\quad \otimes(\lambda^{q_2}\circ(\kappa^{i_{q_1+1}}\otimes\cdots))\otimes\cdots)\\
&= \sum_{j_1+\cdots+j_m=p} \mu_C\circ((\lambda\circ\kappa)^{j_1}\otimes\cdots\otimes(\lambda\circ\kappa)^{j_m})\\
&= \mu_{A,C}(\otimes^m(\lambda\circ\kappa))^p\ ,
\end{aligned}
$$

which finishes the proof.

We now return to (5.1). The germs of analytic maps considered there shall be regarded as elements F in the formal power series module $R^{n''}[[X'_1,\cdots,X'_{n'}]]$ of associating but non-commuting variables. However, (5.1) remains meaningful for formal power series F with $F^0=0$. If we use the formal power series F with $F^0=0$ which satisfy (5.1) rather than the germs of analytic maps with $F^0=0$ which satisfy (5.1) then we obtain, similar to (5.2), but now for any base ring R, a category ${}_R\mathscr{D}\mathit{iff}_m$.

(5.15) THEOREM. *There is a full faithful functor $\mathscr{A}_m: {}_R\mathscr{D}\mathit{iff}_m\to{}_R\mathscr{A}\mathit{lg}_m$ which is an equivalence between ${}_R\mathscr{D}\mathit{iff}_m$ and the full subcategory of ${}_R\mathscr{A}\mathit{lg}_m$ that is defined by those algebras whose underlying R-module is finitely generated and free.*

Proof. We put $\mathscr{A}_m(D)=A_m(D)$ and define $\mathscr{A}_m(F)$ by the following formula

$$\mathscr{A}_m(F)^p=\sum_{i=1}^{n''}\alpha_{D'}(F_i^p)e''^i$$

where the e''^i are the unit vectors in $A_m(D'')$, $\alpha_{D'}$ is as in (3.14), and the F_i^p are the components of F^p. Clearly, $\mathscr{A}_m(F)$ is in $P(A_m(D'), A_m(D''))$. Evidently, $\mathscr{A}_m$ maps identities to identities. Suppose we are given $F: D' \to D$ and $G: D \to D''$. Let

$$G^q(X) = \sum_{k_1,\cdots,k_q=1}^{n} c^{k_1,\cdots,k_q} X_{k_1} \cdots X_{k_q} .$$

Then

$$(G \circ F)^p = \sum_{q \leq p} \sum_{k_1,\cdots,k_q=1}^{n} \sum_{j_1+\cdots+j_q=p} c^{k_1,\cdots,k_q} F_{k_1}^{j_1} \cdots F_{k_q}^{j_q}$$

and thus

$$\begin{aligned} \mathscr{A}_m(G \circ F)^p &= \sum_{i=1}^{n''} e''^i \alpha_{D'}((G \circ F)_i^p) \\ &= \sum_{q \leq p} \sum_{k_1,\cdots,k_q=1}^{n} \sum_{j_1+\cdots+j_q=p} c^{k_1,\cdots,k_q} \alpha_{D'}(F_{k_1}^{j_1}) \cdots \alpha_{D'}(F_{k_q}^{j_q}) . \end{aligned}$$

On the other hand

$$\begin{aligned} (\mathscr{A}_m(G) \circ \mathscr{A}_m(F))^p &= \sum_{q \leq p} \sum_{j_1+\cdots+j_q=p} \mathscr{A}_m(G)^q \circ (\mathscr{A}_m(F)^{j_1} \otimes \cdots \otimes \mathscr{A}_m(F)^{j_q} \\ &= \sum_{q \leq p} \sum_{j_1+\cdots+j_q=p} \sum_{i=1}^{n''} \alpha_D(G_i^q) e''^i \circ \left(\sum_{k=1}^{n} \alpha_{D'}(F_k^{j_1})\, e^k \otimes \cdots \otimes \sum_{k=1}^{n} \alpha_{D'}(F_k^{j_q}) e^k \right) \\ &= \sum_{q \leq p} \sum_{j_1+\cdots+j_q=p} \sum_{i=1}^{n''} \alpha_D(G_i^q) e''^i \circ \sum_{k_1,\cdots,k_q=1}^{n} \alpha_{D'}(F_{k_1}^{j_1}) \cdots \alpha_{D'}(F_{k_q}^{j_q}) e^{k_1} \otimes \cdots \otimes e^{k_q} \\ &= \sum_{q \leq p} \sum_{j_1+\cdots+j_q=p} \sum_{k_1,\cdots,k_q=1}^{n} c^{k_1,\cdots,k_q} \alpha_{D'}(F_{k_1}^{j_1}) \cdots \alpha_{D'}(F_{k_q}^{j_q}) . \end{aligned}$$

This makes $\mathscr{A}_m$ at least a functor into ${}_R\mathscr{A}_m$. Next, for $F: D' \to D''$, put $\delta = \delta_{\mathscr{A}_m(D')/\mathscr{A}_m(D'')}$ to obtain by (3.14)

$$\begin{aligned} (\delta \mathscr{A}_m(F))^p &= (\mathscr{A}_m(F))^{p-m+1} \circ d_{\mu_{D'},p} = \sum_{i=1}^{n''} \alpha_{D'}(F_i^{p-m+1}) \circ d_{\mu_{D'},p} e''^i \\ &= \sum_{i=1}^{n''} \delta_{\mu_{D'},R}^p \alpha_{D'}(F_i^{p-m+1}) e''^i = \sum_{i=1}^{n''} \alpha_{D'}(\delta_{D'} F_i^p) e''^i , \end{aligned}$$

which by (3.1) is the linear combination of the e''^i with coefficients the left sides of (5.1′) for $p - m + 1$. Finally, put $\mu = \mu_{\mathscr{A}_m(D'),\mathscr{A}_m(D'')}$ and $\mu'' = \mu_{D''}$ to obtain

$$\mu(\otimes^m \mathscr{A}_m(F))^p = \sum_{j_1+\cdots+j_m=p} \mu'' \circ (\mathscr{A}_m(F)^{j_1} \otimes \cdots \otimes \mathscr{A}_m(F)^{j_m}$$

$$= \sum_{j_1+\cdots+j_m=p} \mu'' \circ \left(\sum_{i=1}^{n''} \alpha_{D'}(F_i^{j_1}) e''^i \otimes \cdots \otimes \sum_{i=1}^{n''} \alpha_{D'}(F_i^{j_m}) e''^i \right)$$

$$= \sum_{j_1+\cdots+j_m=p} \mu'' \circ \sum_{i_1,\cdots,i_m=1}^{n''} \alpha_{D'}(F_{i_1}^{j_1}) \cdots \alpha_{D'}(F_{i_m}^{j_m}) e^{i_1} \otimes \cdots \otimes e^{i_m}$$

$$= \sum_{i=1}^{n''} \sum_{i_1,\cdots,i_m=1}^{n''} \alpha_{D'} \left(\sum_{j_1+\cdots+j_m=p} a_i''^{i_1,\cdots,i_m} F_{i_1}^{j_1} \cdots F_{i_m}^{j_m} \right) e''^i$$

which is the linear combination of the e''^i with coefficients the right sides of (5.1) for $p - m + 1$. Hence $F \in {}_R\mathscr{D}iff_m(D', D'')$ is equivalent with $\mathscr{A}_m(F) \in {}_R\mathscr{A}lg_m(\mathscr{A}_m(D'), \mathscr{A}_m(D''))$. Thus, $\mathscr{A}_m$ is full and faithful. The assertion that $\mathscr{A}_m$ is an equivalence with the described subcategory is obvious.

Next we should like to remark, without giving the routine proof, that any $\lambda \in {}_R\mathscr{A}lg_m(A, B)$ with $\lambda^p = 0$, for $p \geq p_0$, induces a map $S(\lambda)$: $S(A) \to S(B)$ by putting

$$S(\lambda)(\mathscr{X}(t)) = \sum_{p=1}^{p_0} \widehat{\lambda^p}(\otimes^p \mathscr{X}(t)) \ .$$

Hence, the functor $S: {}_R\mathrm{Alg}_m \to \mathrm{Sets}$ can be extended to the subcategory of ${}_R\mathscr{A}lg_m$ whose morphisms are "finite" in the above sense.

(5.3) and (5.4) still make sense in case the R-algebras A and B have arities, say m' and m, which are possibly distinct from each other. In this event, (5.5) is replaced by a bifunctor $P: {}_A\mathrm{Alg}_{m'}^{op} \times {}_R\mathrm{Alg}_m \to {}_R\mathrm{Alg}_m$. For three R-algebras A, B and C of respective arity m', m, and m'', (5.6) remains meaningful. This allows us, just as in (5.7), to form a category, ${}_R\mathscr{A}$, which contains $\coprod_m {}_R\mathrm{Alg}_m$ as a subcategory. The definition (5.8) of $\delta_{A/B}$ also makes sense in this setting, and (5.9) remains valid. (5.10) and (5.11) remain meaningful, and the validity of (5.12), (5.13), and (5.14) can be established just as in the previous proofs.

We close this section by pointing out that there is a commutative version to the R-algebra $P(A, B)$; it is obtained by dividing out by the ideal $\prod_{p=1}^{\infty} C^p(A, B)$—see the remarks on page 3.18. The resulting R-algebras, $P(A, B)_c$, can be used to form categories ${}_R\mathscr{A}_{mc}$ and ${}_R\mathscr{A}lg_{mc}$ into which many of the arguments of this section carry over. In particular, the multiplication $\mu_{A,B}$ on $P(A, B)$ induces a multiplication $\mu_{A,Bc}$ on $P(A, B)_c$, $\delta_{A/B}: P(A, B) \to P(A, B)$ induces $\delta_{A/Bc}: P(A, B)_c \to P(A, B)_c$, and

$${}_R\mathscr{A}lg_{mc}(A, B) = \{\lambda : \delta_{A/Bc}\lambda = \mu_{A,Bc}(\otimes^m \lambda)\} \subset P(A, B)_c \ .$$

6. The symmetry group of an algebra

Given a differential equation D of arity m over a Banach algebra R, it is important to obtain information on the group of ${}_R\mathscr{D}iff'_m$-automorphisms of D (see [4]). This group shall be called the symmetry group of D and is denoted by ${}_RG'(D)$. If we replace ${}_R\mathscr{D}iff'_m$ by ${}_R\mathscr{D}iff_m$ then the corresponding group will be denoted by ${}_RG(D)$. In view of (5.15) it makes sense to investigate ${}_RG(A)$, the group of ${}_R\mathscr{A}lg_m$-automorphisms of an R-algebra A of arity m; again ${}_RG(A)$ will be called the symmetry group of A. The group of ${}_R\mathrm{Alg}_m$-automorphisms of A, an algebraic group, will be denoted by ${}_R\mathrm{Aut}(A)$.

(6.1) PROPOSITION. *${}_R\mathrm{Aut}(A)$ is a subgroup of ${}_RG(A)$.*

(6.2) PROPOSITION. *The set of elements $\lambda \in {}_R\mathscr{A}lg_m(A, A)$ with $\lambda^1 = \mathrm{id}_A$ forms a normal subgroup ${}_RU(A)$ of ${}_RG(A)$, and ${}_RG(A)$ is a split extension of ${}_R\mathrm{Aut}(A)$ by ${}_RU(A)$.*

Proof. Evidently, the identity morphism on A is in ${}_RU(A)$. Due to (5.6), $(\lambda \circ \kappa)^1 = \lambda^1 \circ \kappa^1$; hence ${}_RU(A)$ is closed under forming product. A simple computation shows that inverses exist in ${}_RU(A)$. Normality of ${}_RU(A)$ follows from the relation on first components that was just stated. Since ${}_R\mathrm{Aut}(A) \cap {}_RU(A) = 1$, the quotient map ${}_RG(A) \to {}_RG(A)/{}_RU(A)$ is an injection on ${}_R\mathrm{Aut}(A)$. It follows from (5.6) that $\lambda \in {}_RG(A)$ is equivalent with $\lambda^1 \in {}_R\mathrm{Aut}(A)$. Hence ${}_RG(A) = {}_R\mathrm{Aut}(A) \cdot {}_RU(A)$. Thus the quotient map induces an isomorphism ${}_R\mathrm{Aut}(A) \cong {}_RG(A)/U(A)$. The splitting map is obvious.

The commutative analog of ${}_RU(A_m(D)) \cap {}_RG'(D)$ has been determined in [4] in case R is a Banach algebra and D is "nicht entartet".

Due to (6.2), a further discussion of the structure of ${}_RG(A)$ has to focus on ${}_RU(A)$.

Denote by ${}_RU(A)^{[p]}$ the set of all $\lambda \in {}_RU(A)$ with $\lambda^2 = \cdots = \lambda^p = 0$. Put ${}_RU(A)^{[1]} = {}_RU(A)$, and write the multiplication, that is composition, in ${}_RU(A)$ as addition. Then we obtain, denoting—as we did prior to section 5—the multiplication in A by μ,

(6.3) THEOREM. *${}_RU(A)^{[p]}$ is a normal subgroup of ${}_RU(A)^{[p-1]}$, and the quotient group ${}_RU(A)^{[p-1]}/{}_RU(A)^{[p]}$ is isomorphic to an additive subgroup of*

$$(6.4) \qquad Q_\mu^p = \left\{ f : f \circ d_{\mu, p+m-1} = \mu \circ \sum_{i=1}^m \otimes^{i-1} \mathrm{id}_A \otimes f \otimes \otimes^{m-i} \mathrm{id}_A \right\} \subset T^p(A, A) .$$

If for every $r \in R$, either r or $-r$ has a $(p-1)^{th}$ root, then this additive subgroup is in fact a R-submodule of $T^p(A, A)$.

Proof. It is easy to see from (5.10) that ${}_RU(A)^{[p]}$ is a subgroup of ${}_RU(A)$. Moreover, for $\kappa \in {}_RU(A)^{[p-1]}$, we obtain from (5.10)

$$(-\kappa)^p = -(\kappa^p)$$

and thus, for $\lambda \in {}_RU(A)^{[p]}$ and $1 < p' \leq p$,

$$\begin{aligned} ((-\kappa + \lambda) + \kappa)^{p'} &= \sum_q \sum_{j_1 + \cdots + j_q = p'} (-\kappa + \lambda)^q \circ (\kappa^{j_1} \otimes \cdots \otimes \kappa^{j_q}) \\ &= \mathrm{id} \circ \kappa^{p'} + (-\kappa + \lambda)^{p'} \circ \mathrm{id} \\ &= \kappa^{p'} + \sum_q \sum_{j_1 + \cdots + j_q = p'} (-\kappa)^q \circ (\lambda^{j_1} \otimes \cdots \otimes \lambda^{j_q}) \\ &= \kappa^{p'} + (-\kappa)^{p'} \circ \mathrm{id} + \mathrm{id} \circ \lambda^{p'} = 0 . \end{aligned}$$

Hence ${}_RU(A)^{[p]}$ is a normal subgroup of ${}_RU(A)^{[p-1]}$. Another easy computation shows that

$$(\lambda + \kappa)^p = \kappa^p \quad \text{and} \quad (\kappa_1 + \kappa_2)^p = \kappa_1^p + \kappa_2^p .$$

Hence the map

$${}_RU(A)^{[p-1]} \ni \kappa \to \kappa^p \in T^p(A, A)$$

induces a homomorphiom

$$(6.5) \qquad {}_RU(A)^{[p-1]} / {}_RU(A)^{[p]} \to T^p(A, A)$$

which is clearly an injection. The image of (6.5) is an additive subgroup of $T^p(A, A)$. Let $r \in R$ and $\kappa \in {}_RU(A)^{[p-1]}$, and define $r \cdot \kappa$ by $(r \cdot \kappa)^{p'} = r^{p'-1} \cdot \kappa^{p'}$. Then a simple computation shows that $r \cdot \kappa$ is again in ${}_RU(A)^{[p-1]}$. Thus, under the additional assumptions, the image of (6.5) is a R-submodule of $T^p(A, A)$. In any case, if we take the p^{th} component of the defining relation (5.11) for κ, then we get by (5.4) and (5.8)

$$\kappa^p \circ d_{\mu, p+m-1} = \delta_\mu^{p+m-1} \kappa^p = \mu \circ \sum_{i=1}^m \otimes^{i-1} \mathrm{id}_A \otimes \kappa^p \otimes \otimes^{m-i} \mathrm{id}_A .$$

Hence κ^p lies in Q_μ^p.

Let us return for a moment to (6.4). Given $f \in Q_\mu^p$, we can equip the R-module A with the multiplication f, thereby obtaining a R-algebra (A, f)

of arity p; the graded R-homomorphism (3.10) corresponding to (A, f) is $d_{f,*}$. Then we obtain

(6.6) COROLLARY. (i) $Q^p_\mu = \{f : f \circ d_{\mu,p+m-1} = \mu \circ d_{f,p+m-1}\} \subset T^p(A, A)$.
(ii) $f \in Q^p_\mu$ *if and only if* $\mu \in Q^m_f$.
(iii) $\mu \in Q^m_\mu$.

For R a Banach algebra and $A = A_m(D)$ an analog to (6.6) can be found in [4].

Let B be a R-algebra of arity m. A R-module homomorphism $\delta : B \to B$ is called a R-derivation if

$$\delta_{\mu_B}(b_1 \otimes \cdots \otimes b_m) = \sum_{i=1}^{m} \mu_B(b_1 \otimes \cdots \otimes b_{i-1} \otimes \delta b_i \otimes b_{i+1} \otimes \cdots \otimes b_m) , \quad b_1, \cdots, b_m \in B$$

holds.

(6.7) LEMMA. *Let (B, μ_B) be a R-algebra of arity m without $\boldsymbol{Z}$-torsion and let $\delta : B \to B$ be a $\boldsymbol{Z}$-derivation with respect to μ_B. Let furthermore $\bar{f} : \otimes^p_R B \to B$ satisfy the following conditions*

(i) $\delta(\bar{f}(b_1 \otimes \cdots \otimes b_p)) = \bar{f}\left(\sum_{i=1}^{p} b_1 \otimes \cdots \otimes b_{i-1} \otimes \delta b_i \otimes b_{i+1} \otimes \cdots \otimes b_p\right)$.

(ii) $\bar{f} \circ d_{\mu_B, p+m-1} = \mu_B \circ d_{\bar{f}, p+m-1}$.

Suppose that $a_0 \in B$ satisfies $\delta a_0 = \mu_B(\otimes^m a_0)$ and that $a_1, a_2, \cdots \in B$ are subject to

$$j a_j = \sum_{j_1 + \cdots + j_p = j-1} \bar{f}(a_{j_1} \otimes \cdots \otimes a_{j_p}) , \qquad j = 1, 2, \cdots$$

Then

$$\delta a_j = \mu_B\left(\sum_{j_1 + \cdots + j_m = j} a_{j_1} \otimes \cdots \otimes a_{j_m}\right) .$$

Proof. The claim is correct for $j = 0$ by assumption. We proceed by induction on j. Assume that the formula is correct for $i \le j - 1$. Then

$$j \delta a_j = \delta\left(\sum_{j_1 + \cdots + j_p = j-1} \bar{f}(a_{j_1} \otimes \cdots \otimes a_{j_p})\right)$$
$$= \sum_{j_1 + \cdots + j_p = j-1} \bar{f}\left(\sum_{i=1}^{p} a_{j_1} \otimes \cdots \otimes a_{j_{i-1}} \otimes \mu_B\left(\sum_{k_1 + \cdots + k_m = j_i} a_{k_1} \otimes \cdots \otimes a_{k_m}\right) \otimes a_{j_{i+1}} \otimes \cdots\right)$$

$$
\begin{aligned}
&= \sum_{\ell_1+\cdots+\ell_{m+p-1}=j-1} \bar{f} \circ d_{\mu_B, m+p-1}(a_{\ell_1} \otimes \cdots \otimes a_{\ell_{m+p-1}}) \\
&= \sum_{\ell_1+\cdots+\ell_{m+p-1}=j-1} \mu_B \circ d_{\bar{f}, m+p-1}(a_{\ell_1} \otimes \cdots \otimes a_{\ell_{m+p-1}}) \\
&= \mu_B\Bigg(\sum_{\ell_1+\cdots+\ell_{m+p-1}=j-1} \sum_{i=1}^{m} a_{\ell_1} \otimes \cdots \otimes a_{\ell_{i-1}} \\
&\qquad\qquad \otimes \bar{f}(a_{\ell_i} \otimes \cdots \otimes a_{\ell_{i+p-1}}) \otimes a_{\ell_{i+p}} \otimes \cdots \Bigg) \\
&= \mu_B\Bigg(\sum_{j_1+\cdots+j_m=j-1} \sum_{i=1}^{m} \sum_{\ell_1+\cdots+\ell_{1+p-1}=j_i} a_{j_1} \otimes \cdots \otimes a_{j_{i-1}} \\
&\qquad\qquad \otimes \bar{f}(a_{\ell_i} \otimes \cdots \otimes a_{\ell_{i+p-1}}) \otimes a_{j_{i+1}} \otimes \cdots \Bigg) \\
&= \mu_B\Bigg(\sum_{j_1+\cdots+j_m=j-1} \sum_{i=1}^{m} a_{j_1} \otimes \cdots \otimes a_{j_{i-1}} \otimes (j_i+1) a_{j_{i+1}} \otimes a_{j_{i+1}} \otimes \cdots \Bigg) \\
&= j\mu_B\Bigg(\sum_{j_1+\cdots+j_m=j} a_{j_1} \otimes \cdots \otimes a_{j_m} \Bigg) .
\end{aligned}
$$

Since B has no $\boldsymbol{Z}$-torsion, the induction step is completed.

For R a Banach algebra, $A = A_m(D)$, D "nicht entarted", and $p = m$ the commutative version of the following result appears in [4].

(6.8) THEOREM. *Suppose that $\boldsymbol{Q} \subset R$ holds. Then for any R-algebra A and any p, the map ${}_RU(A)^{[p-1]}/{}_RU(A)^{[p]} \to Q_\mu^p$ established in (6.3) is an isomorphism.*

Proof. We apply (6.7) to $(B, \mu_B) = (P(A, A), \mu_{A,A})$, $\delta = \delta_{A/A}$, $a_0 = \mathrm{id}_A \in P(A, A)$, and $\bar{f} = \mu_{f,f}$, where f is a given element of Q_μ^p and $\mu_{f,f}$ is obtained just as $\mu_{A,A}$ using f rather than μ. We have to verify the conditions (i) and (ii) of (6.7). Using (3.10), (5.4) and (5.8) we get

$$
\begin{aligned}
&(\delta_{A/A}(\mu_{f,f}(\lambda_1 \otimes \cdots \otimes \lambda_p)))^{q+m-1} = \mu_{f,f}(\lambda_1 \otimes \cdots \otimes \lambda_p)^q \circ d_{\mu, q+m-1} \\
&\quad = \sum_{j_1+\cdots+j_p=q} f \circ (\lambda_1^{j_1} \otimes \cdots \otimes \lambda_p^{j_p}) \circ \sum_{i=1}^{q} \otimes^{i-1} \mathrm{id}_A \otimes \mu \otimes \otimes^{q-i} \mathrm{id}_A \\
&\quad = \sum_{j_1+\cdots+j_p=q} f \circ \sum_{i=1}^{p} \lambda_1^{j_1} \otimes \cdots \otimes \lambda_{i-1}^{j_{i-1}} \otimes (\lambda_i^{j_i} \circ d_{\mu, j_i+m-1}) \otimes \lambda_{i+1}^{j_{i+1}} \otimes \cdots \\
&\quad = \sum_{j_1+\cdots+j_p=q} f \circ \sum_{i=1}^{p} \lambda_i^{j_1} \otimes \cdots \otimes \lambda_{i-1}^{j_{i-1}} \otimes (\delta_{A/A}\lambda_i^{j_i})^{j_i+m-1} \otimes \lambda_{i+1}^{j_{i+1}} \otimes \cdots \\
&\quad = \mu_{f,f}\Bigg(\sum_{i=1}^{p} \lambda_1 \otimes \cdots \otimes \lambda_{i-1} \otimes \delta_{A/A}\lambda_i \otimes \lambda_{i+1} \otimes \cdots \otimes \lambda_p \Bigg)^q ,
\end{aligned}
$$

which establishes (i). Furthermore we have

$$\begin{aligned}
&\mu_{f,f}(d_{(A,A),p+m-1}(\lambda_1 \otimes \cdots \otimes \lambda_{p+m-1}))^q \\
&\quad = \mu_{f,f} \sum_{i=1}^{p} \lambda_1 \otimes \cdots \otimes \lambda_{i-1} \otimes \mu_{A,A}(\lambda_i \otimes \cdots \otimes \lambda_{i+m-1}) \\
&\quad\quad \otimes \lambda_{i+m} \otimes \cdots \otimes \lambda_{p+m-1})^q \\
&\quad = \sum_{j_1+\cdots+j_{p+m-1}=q} f \circ \sum_{i=1}^{p} \lambda_1^{j_1} \otimes \cdots \otimes \lambda_{i-1}^{j_{i-1}} \otimes \mu(\lambda_i^{j_i} \otimes \cdots \otimes \lambda_{i+m-1}^{j_{i+m-1}}) \\
&\quad\quad \otimes \lambda_{i+m}^{j_{i+m}} \otimes \cdots \otimes \lambda_{p+m-1}^{j_{p+m-1}} .
\end{aligned}$$

Since f is an element of Q_μ^p, the "inside" sum satisfies

$$\begin{aligned}
&f \circ \sum_{i=1}^{p} \lambda_1^{j_1} \otimes \cdots \otimes \lambda_{i-1}^{j_{i-1}} \otimes \mu(\lambda_i^{j_i} \otimes \cdots \otimes \lambda_{i+m-1}^{j_{i+m-1}}) \otimes \lambda_{i+m}^{j_{i+m}} \otimes \cdots \otimes \lambda_{p+m-1}^{j_{p+m-1}} \\
&\quad = \mu \circ \sum_{i=1}^{m} \lambda_1^{j_1} \otimes \cdots \otimes \lambda_{i-1}^{j_{i-1}} \otimes f(\lambda_i^{j_i} \otimes \cdots \otimes \lambda_{i+p-1}^{j_{i+p-1}}) \otimes \lambda_{i+p}^{j_{i+p}} \otimes \cdots \otimes \lambda_{p+m-1}^{j_{p+m-1}} ,
\end{aligned}$$

as can be checked on elements immediately. So, if we reverse the preceding argument, we obtain a verification of (ii). Since $\boldsymbol{Q} \subset R$ holds, we can define, inductively, the a_j's to satisfy the required relations in (6.7). Then, however, by (6.7),

$$\delta_{A/A} a_j = \mu_{A,A}\Big(\sum_{j_1+\cdots+j_m=j} a_{j_1} \otimes \cdots \otimes a_{j_m}\Big) . \tag{6.9}$$

a_0, by definition, has the following components

$$a_0^1 = \mathrm{id}_A , \quad a_0^q = 0 \qquad \text{for } q \neq 1 .$$

A simple verification shows that

$$a_1^p = f , \quad a_1^q = 0 \qquad \text{for } q \neq p$$

A routine induction argument shows that

$$a_j^q = 0 \qquad \text{for } q \neq j(p-1)+1 .$$

Define $\lambda \in P(A, A)$ by

$$\left.\begin{aligned} \lambda^q &= 0 \qquad \text{for } q \neq j(p-1)+1 \\ \lambda^{j(p-1)+1} &= a_j^{j(p-1)+1} \end{aligned}\right. \qquad j = 0, 1, \cdots$$

Evidently, $\lambda^1 = \mathrm{id}_A$ and $\lambda^p = f$. (6.9) is then equivalent with $\lambda \in {}_R\mathscr{A}lg_m(A, A)$; and thus λ belongs to ${}_RU(A)^{[p-1]}$ and has f as its image in Q_μ^p.

Given $f \in Q_\mu^p$, the element $\lambda \in {}_RU(A)^{[p-1]}$ which was constructed in the proof of (6.8) shall be denoted by $\lambda(f)$.

(6.10) COROLLARY. *Suppose that* $\boldsymbol{Q} \subset R$ *holds. For* $f \in Q^p_\mu$, *the element* $\lambda(f)$ *satisfies the following conditions*:

$$\lambda(f)^q = 0 \qquad \textit{for } q \not\equiv 1 \bmod (p-1)$$

$$\lambda(f)^{j(p-1)+1} = \frac{1}{j!} d_{f,p} \circ \cdots \circ d_{f,j(p-1)+1} \qquad \textit{for } j = 1, 2, \cdots$$

Proof. Apply (6.7) to $(B, \mu_B) = (P(A, A), \mu_{f,f})$, $\delta = \delta_{f,f}$, $a_0 = \mathrm{id}_A$, and $\bar{f} = \mu_{f,f}$. Then, obviously, (i) and (ii) of (6.7) are satisfied. Hence

$$\delta_{f/f} a_{j-1} = \mu_{f,f}\Bigl(\sum_{j_1 + \cdots + j_p = j-1} a_{j_1} \otimes \cdots \otimes a_{j_p}\Bigr) = j a_j \,.$$

Therefore,

$$j\lambda(f)^{j(k-1)+1} = j a_j^{j(p-1)+1} = (\delta_{f/f} a_{j-1})^{j(p-1)+1} = a_{j-1}^{(j-1)(p-1)+1} \circ d_{f,j(p-1)+1} \,.$$

Thus, a straightforward induction argument delivers our formula.

(6.11) COROLLARY. *Suppose that* $\boldsymbol{Q} \subset R$ *holds. Then for any non-trivial* R-*algebra* A *of arity* m, $Q^m_\mu \neq 0$ *and hence both* ${}_RU(A)^{[m-1]}$ *and* ${}_RU(A)$ *are non-trivial.*

Proof. By (6.6), $0 \neq \mu \in Q^m_\mu$. By (6.8), μ is the image of some element $\neq 1$ in ${}_RU(A)^{[m-1]}$, which is contained in ${}_RU(A)$.

(6.12) COROLLARY. *Suppose that* $\boldsymbol{Q} \subset R$ *holds. Let* N *be a trivial* R-*algebra of arity* m *with* $T^p(N, N) \neq 0$ *for some* $p \geq 2$. *Then both* ${}_RU(A)^{[p-1]}$ *and* ${}_RU(A)$ *are non-trivial. In particular, this is true if* R *is any field of characteristic zero.*

Proof. $T^p(N, N) = Q^p_0$.

Before formulating the next statement we ought to recall that, rather at the beginning of this section, the product—that is composition— in ${}_RU(A)$ was written as (not necessarily commutative) addition. With this understanding we obtain

(6.13) COROLLARY. *Suppose that* $\boldsymbol{Q} \subset R$ *holds. Then, for every* $f \in {}_RU(A)$ *there exists uniquely a sequence of elements* $f_p \in Q^p_\mu$, $p = 2, 3, \cdots$ *such that* $f = \lambda(f_2) + \lambda(f_3) + \cdots$; *this sum is locally finite.*

Proof. Clearly, $-\lambda(f^2) + f$ has vanishing second component, and $f_2 = f^2$ is an element of Q^2_μ. Suppose we have constructed $f_2, \cdots, f_q$ such that $f_p \in Q^p_\mu$, $p = 2, \cdots, q$, and that $-\lambda(f_q) - \cdots - \lambda(f_2) + f$ is in

${}_RU(A)^{[q+1]}$. Then we take $f_{q+1} = (-\lambda(f_q) - \cdots - \lambda(f_2) + f)^{q+1}$. These inductively constructed elements will satisfy our claim. Uniqueness of the f_p's is obvious.

For a differential equation D of arity m we denote by Q_D^p the R-module Q_μ^p, where μ is the multiplication of the R-algebra $A_m(D)$. Furthermore, for $\lambda \in P(A_m(D), A_m(D))$ we denote by $\alpha_D^{-1}(\lambda)$ the formal power series

$$\sum_{q=1}^{\infty} \alpha_D^{-1}(\lambda^q) ,$$

where α_D^{-1} is applied to each component of λ^q separately. With these notations we have

(6.14) Proposition. *Let R be a commutative unital Banach algebra. Let furthermore D be a differential equation of arity m and dimension n over R. Then for any $f \in Q_D^p$, $\alpha_D^{-1}(\lambda(f))$ is in ${}_R\mathscr{D}iff'_m(D, D)$ and satisfies $\lambda(f) = \mathscr{A}_m(\alpha_D^{-1}(\lambda(f)))$. Moreover, ${}_RG'(D) = {}_RG(D)$ if and only if either of the following conditions is satisfied*

(i) *there exists an integer p_0 such that*

$${}_RU(A_m(D))^{[p_0]} = {}_RU(A_m(D))^{[p_0+1]} = \cdots$$

(ii) *there exists an integer p_0 such that*

$$Q_D^{p_0} = Q_D^{p_0+1} = \cdots = 0 .$$

In this case, ${}_RG'(D)$ has the additional properties stated in (7.16) for ${}_FU(A)$.

Proof. Due to the definition of $\mathscr{A}_m$—see proof of (5.15)—the asserted equality is clear. Hence it remains to be shown that $\alpha_D^{-1}(\lambda(f))$ is a convergent power series. Since the roles of f and μ are interchangeable, it suffices to deal with μ rather than f. Due to (3.14) and (6.10) we obtain, putting $A = A_m(D)$,

$$\begin{aligned}\alpha_D^{-1}(\lambda(\mu)^{j(m-1)+1}) &= \frac{1}{j!}\alpha_D^{-1}(d_{\mu_D,m} \circ \cdots \circ d_{\mu_D,j(m-1)+1}) \\ &= \frac{1}{j!}\alpha_D^{-1}(\delta_{A/A}^{j(m-1)} \circ \cdots \circ \delta_{A/A}^{2m-1}(d_{\mu_D,m})) \\ &= \frac{1}{j!}\delta_D^{j-1}(\alpha_D^{-1}(d_{\mu_D,m})) .\end{aligned}$$

A straightforward computation shows that

$$\alpha_D^{-1}(d_{\mu_D,m}) = (D_1(X), \cdots, D_n(X))$$

holds. Put

$$M = \max\{|a_i^{k_1,\cdots,k_m}|\}$$

and let $P \in R[X_1, \cdots, X_n]$ be a homogeneous polynomial of degree g whose coefficients are bounded, in norm, by M'. Then a simple computation shows that the coefficients of $\delta_D P$ are bounded, in norm, by gMM'. Hence the coefficients of $\delta_D^{j-1}P$ are bounded by

$$g(g+m-1)\cdots(g+(j-2)(m-1))M'M^{j-1} .$$

This, in turn, shows that the coefficients of $\alpha_D^{-1}(\lambda(\mu)^{j(m-1)+1})$ are bounded by

$$\frac{1}{j!}m(2m-1)\cdots((j-1)(m-1)+1)M^j \leq (mM)^j .$$

Since $\alpha_D^{-1}(\lambda(\mu)^{j(m-1)+1})$ has at most $n^{j(m-1)+1}$ monomial terms, this implies convergence. The remaining assertion follows from (6.13).

The example

$$\dot{X}_i = X_i^m , \qquad i = 1, \cdots, n$$

shows that $\alpha_D^{-1}(\lambda(f))$ cannot be expected to be an entire function.

(6.15) PROPOSITION. *Let R be a commutative unital Banach algebra. Let furthermore D be a differential equation of arity m over R whose associated R-algebra $A_m(D)$ has multiplication μ_D. Then, for every $a \in A_m(D)$ that is sufficiently close to 0, there exists a $\varepsilon > 0$ such that for all $s \in R$ with $|s| < \varepsilon$,*

$$\alpha_D^{-1}(\lambda(s\mu_D)) \cdot \mathscr{X}_a(t) = \mathscr{X}_{\mathscr{X}_a(s)}(t)$$

holds, where $\mathscr{X}_a(t)$ denotes the unique solution of D with $\mathscr{X}_a(0) = a$.

Proof. Evidently, the left side of the formula constitutes a solution of D, due to (6.14) and the definition of ${}_R\mathscr{D}iff'_m$. Hence we only have to determine its constant term. A straight forward computation shows that this constant term equals

$$a + \sum_{j=1}^{\infty} \lambda(\mu_D)^{j(m-1)+1}(\bigotimes{}^{j(m-1)+1} a)s^j . \tag{6.16}$$

Denote the coefficient of s^j in (6.16) by a_j and put $b_j = \lambda(\mu_D)^{j(m-1)+1}$. Then $a_j = b_j(\otimes^{j(m-1)+1} a)$. Since $\lambda(\mu_D)$ is in ${}_RU(A_m(D))$, (5.8) and (5.11) imply

$$b_{j-1} \circ d_{\mu_D, j(m-1)+1} = \sum_{j_1+\cdots+j_m=j-1} \mu_D \circ (b_{j_1} \otimes \cdots \otimes b_{j_m}) \, , \qquad j = 1, 2, \cdots \, .$$

(6.10) shows that

$$jb_j = b_{j-1} \circ d_{\mu_D, j(m-1)+1} \, .$$

Hence

$$ja_j = \sum_{j_1+\cdots+j_m=j-1} \mu_D(a_{j_1} \otimes \cdots \otimes a_{j_m}) \, .$$

This, finally, shows that (6.16) equals $\mathscr{X}_a(s)$.

Thus, the "one-parmeter" subgroup $\{\lambda(s\mu_D) : s \in R\}$ of ${}_RU(A_m(D))$ permits a geometrical interpretation: given any initial value $a \in A_m(D)$ which is sufficiently close to $\mathcal{O}$, it moves a along the trajectory through a.

We come now to a brief remark concerning the parameter dependence of Q^p_μ. Using the same conventions as at the end of section 3 we obtain

(6.17) PROPOSITION. *For fixed p, $\dim_F Q^p_\mu$ is upper semicontinuous on $F^{n^{m+1}}$ with respect to the Zariski-topology.*

Proof. Let $e^1, \cdots, e^n$ be the unit vectors in F^n. For $f \in T^p(A, A)$ put

$$f(e^{k_1} \otimes \cdots \otimes e^{k_p}) = \sum_{i=1}^{n} f_i^{k_1, \cdots, k_p} e^i \, .$$

Then it follows from (6.4) that $f \in Q^p_\mu$ is equivalent with the $f_i^{k_1, \cdots, k_p}$ satisfying a certain system L_p of homogeneous linear equations whose coefficients are Z-linear combinations of the structure coefficients of the F-algebra A. The corank of L_p is therefore upper semicontinuous on $F^{n^{m+1}}$ with respect to the Zariski-topology. Hence our first assertion is established.

We close this section with stating—without proof—the changes that occur as one switches from ${}_R\mathscr{A}lg_m$ to ${}_R\mathscr{A}lg_{mc}$. If the corresponding groups are denoted by ${}_RG'(D)_c, {}_RG(D)_c, {}_RG(A)_c, {}_RU(A)_c$, and ${}_RU(A)_c^{[p]}$ then all previous statements and proofs remain in force, provided that

(i) in (6.4) and subsequently Q_μ^p is replaced by

$$Q_{\mu c}^p = \{f : \delta_{\mu,Ac}^{m+p-1} f = \delta_{f,Ac}^{m+p-1} \mu_c\} \subset T^p(A,A)_c$$

where μ_c is the canonical image of μ in $T^m(A,A)_c$

(ii) in (6.10), the pertinent formula is replaced by

$$\lambda(f)^{j(p-1)+1} = \frac{1}{j!} \delta_{f,Ac}^{(p-1)+1} \circ \cdots \circ \delta_{f,Ac}^{2p-1}(f)$$

(iii) in (6.11) and (6.12), A is assumed to be a non-trivial commutative R-algebra.

These claims are verified by simply checking the previous proofs.

7. Properties of the symmetry group

(7.1) PROPOSITION. *Suppose that* $\boldsymbol{Q} \subset R$ *holds. Then there are non-trivial R-algebras A of arity m, whose underlying R-module is finitely generated and free, such that* ${}_RU(A)^{[p-1]}/{}_RU(A)^{[p]} \neq 0$ *for all* $p \geq 2$. *In case R is a Banach algebra,* ${}_RU(A)^{[p-1]}$ *contains for all* $p \geq 2$ *global analytic maps which are not in* ${}_RU(A)^{[p]}$ *and, if* $A = A_m(D)$, ${}_R\mathscr{D}iff'_m(D,D) \subsetneqq {}_R\mathscr{D}iff_m(D,D)$.

Proof. Let $A = R^n$ and define, on A, a m-ary multiplication μ such that

$$\mu(\otimes^m A) \subseteq R^{n-1} \times \{0\} \text{ and}$$
$$0 \neq annA = \{a : \mu(\otimes^{i-1} A \otimes a \otimes \otimes^{m-i} A) = 0,\ i = 1, \cdots, m\}$$

This can be easily done. Then there are elements $0 \neq f^p \in T^p(A,A)$ with

$$f^p(\otimes^p A) \subset annA \quad \text{and} \quad f^p(\otimes^{i-1} A \otimes \mu(\otimes^m A) \otimes \otimes^{p-i} A) = 0\ , \quad i = 1, \cdots, p\ .$$

Define $\lambda \in P(A,A)$ by

$$\lambda^1 = \mathrm{id}_A\ , \quad \lambda^p = f^p\ , \quad \text{and} \quad \lambda^{p'} = 0 \quad \text{for } p' \neq p, 1\ .$$

A straightforward computation shows that λ is indeed in ${}_RU(A)^{[p-1]}$ as $(\delta_{A,A}(\lambda))^1 = \mu_A = \mu_{A,A}(\otimes^m \lambda)^1$ and $(\delta_{A/A}\lambda)^q = 0 = \mu_{A,A}(\otimes^m \lambda)^q$, $q > 1$. In case R is a Banach algebra and $A = A_m(D)$, this λ is a global analytic map $R^n \to R^n$. On the other hand, if we define $\lambda \in P(A,A)$ by

$$\lambda^1 = \mathrm{id}_A\ , \quad \lambda^p = r_p f^p\ , \qquad p = 2, 3, \cdots$$

with $f^p \neq 0$ satisfying the previous relations then a suitable choice of the $r_p \in R$ will force λ to be a formal power series in ${}_RU(A)$ which is not convergent.

Thus, ${}_R\mathscr{D}iff_m(D', D'')$ is—in general—a proper subset of ${}_R\mathscr{D}iff'_m(D', D'')$.

For R a Banach algebra and $A = A_m(D)$ the commutative version of the following result and of (7.5) can be found in [4].

(7.2) PROPOSITION. *Let A be a R-algebra of arity $m \geq 2$ which has no $\boldsymbol{Z}$-torsion. If A has a unit element then*

(i) ${}_RU(A)^{[1]} = \cdots = {}_RU(A)^{[m-1]} = {}_RU(A)$ *and*

(ii) ${}_RU(A)^{[m]} = {}_RU(A)^{[m+1]} = \cdots = 0$.

Proof. (i) By (6.3) it suffices to show that the R-modules (6.4) vanish for $p = 2, \cdots, m-1$. The defining relation for the R-module (6.4) reads on elements $a_1 \otimes \cdots \otimes a_{p+m-1}$

$$(7.3)\quad \begin{aligned} &\sum_{i=1}^{p} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{p+m-1}) \\ &= \sum_{i=1}^{m} \mu(a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+p-1}) \otimes a_{i+p} \otimes \cdots \otimes a_{p+m-1}) . \end{aligned}$$

Let u be the unit element of A. Then (7.3) renders for $u = a_1 = \cdots = a_{p+m-1}$

$$(7.4)\qquad pf(\otimes^p u) = mf(\otimes^p u) ,$$

and hence $f(\otimes^p u) = 0$. This will now be used to start an induction argument. So, assume that

$$f(a_1 \otimes \cdots \otimes a_\ell \otimes \otimes^{p-1-r} u \otimes a_{p-r+1} \otimes \cdots \otimes a_p) = 0$$

for all choices of a_i and $\ell < k, r < q$. Keeping in mind that $k + q \leq p \leq m-1$, we evaluate (7.3) on $a_1 \otimes \cdots \otimes a_k \otimes \otimes^{p-k-q+m-1} u \otimes a_{p-q+1} \otimes \cdots \otimes a_p$ and obtain for $k = 0, q > 0$

$$(p - q)f(\otimes^{p-q} u \otimes a_{p-q+1} \otimes \cdots \otimes a_p) = 0 ,$$

for $k > 0, q = 0$

$$(p - k)f(a_1 \otimes \cdots \otimes a_k \otimes \otimes^{p-k} u) = 0 ,$$

for $k > 0, q > 0$

$$(p - k - q + 2)f(a_1 \otimes \cdots \otimes a_k \otimes \otimes^{p-k-q} u \otimes a_{p-q+1} \otimes \cdots \otimes a_p)$$

$$= \mu_A(f(a_1 \otimes \cdots \otimes a_k \otimes \otimes^{p-k} u) \otimes \otimes^{m-1-q} u \otimes a_{p-q+1} \otimes \cdots \otimes a_p)$$
$$+ \mu_A(a_1 \otimes \cdots \otimes a_k \otimes \otimes^{m-1-k} u \otimes f(\otimes^{p-q} u \otimes a_{p-q+1} \otimes \cdots \otimes a_p) .$$

The first of these relations renders through induction

$$f(u \otimes a_2 \otimes \cdots \otimes a_p) = 0 ,$$

the second one furnishes

$$f(a_1 \otimes \cdots \otimes a_{p-1} \otimes u) = 0 ;$$

and these, together with the third one, finally result in

$$f(a_1 \otimes \cdots \otimes a_p) = 0 , \qquad a_1, \cdots, a_p \in A ,$$

by choosing $k = 1$, $q = p - 1$.

(ii) Let λ be in ${}_RU(A)^{[m]}$. Then, by definition, $\lambda^2 = \cdots = \lambda^m = 0$. We prove, by induction on p, that $\lambda^p = 0$ for all $p > m$. If $\lambda^q = 0$ for $2 \leq q < p$, then it follows from (5.11)—by computing the $(p + m - 1)^{st}$ component—that $f = \lambda^p$ satisfies (7.3). Hence we obtain again (7.4) and, as $p > m$, $f(\otimes^p u) = 0$. This, however, makes it possible to repeat the argument of (i). Thus $f = \lambda^p = 0$, which sets the induction in motion.

(7.5) PROPOSITION. *Suppose that $\boldsymbol{Q} \subset R$ holds. Let A be a R-algebra of arity m which possesses a unit element. Then ${}_RU(A) \cong Q^m_\mu$. Moreover, there is a canonical, injective R-homomorphism $Q^m_\mu \to A$.*

Proof. (6.8) and (7.2) render the isomorphism ${}_RU(A) \cong Q^m_\mu$. Denote, again, by u the unit element of A. Evidently, the map

$$Q^m_\mu \ni f \to f(\otimes^m u) \in A$$

is a R-homomorphism. In order to see that it is an injection we prove, by induction on $\ell + r$, that $f(\otimes^m u) = 0$ implies

$$f(a_1 \otimes \cdots \otimes a_\ell \otimes \otimes^{m-\ell-r} u \otimes a_{\ell+1} \otimes \cdots \otimes a_{\ell+r}) = 0 \tag{7.6}$$

for all ℓ, r with $\ell + r \leq m$. Assume that this is true for all ℓ, r with $\ell + r < k + q$. By evaluating (7.3) on $a_1 \otimes \cdots \otimes a_k \otimes \otimes^{2m-k-q-1} u \otimes a_{k+1} \otimes \cdots \otimes a_{k+q}$ we obtain, using the induction hypothesis

$$(m - q)f(\otimes^{m-q} u \otimes a_1 \otimes \cdots \otimes a_q) = 0 \qquad \text{if } k = 0, q \neq 0$$
$$(m - k)f(a_1 \otimes \cdots \otimes a_k \otimes \otimes^{m-k} u) = 0 \qquad \text{if } k \neq 0, q = 0$$
$$(m - q - k + 2)f(a_1 \otimes \cdots \otimes a_k \otimes \otimes^{m-k-q} u \otimes a_{k+1} \otimes \cdots \otimes a_{k+q}) = 0$$
$$\text{if } k \neq 0, q \neq 0 .$$

These relations, however, imply (7.6) as is easily seen. Thus, the last claim is established.

Next, we shall deal with change of rings. Here we have

(7.7) LEMMA. *Let $R \to S$ be a unital ring homomorphism such that S is a finitely generated projective R-module. Then for any R-algebra A of arity m there is a S-module isomorphism*

$$\omega : S \otimes_R T^p(A, A) \to T^p(S \otimes_R A, S \otimes_R A)$$

that is given by

$$\omega(s \otimes f)((s_1 \otimes a_1) \otimes \cdots \otimes (s_p \otimes a_p)) = ss_1 \cdots s_p \otimes f(a_1 \otimes \cdots \otimes a_p) .$$

Proof. [1], p. 257, 279, 282, 283.

(7.8) PROPOSITION. *Let $R \to S$ be a unital ring homomorphism such that S is a finitely generated projective R-module. Then ω induces isomorphisms*

$$S \otimes_R Q^p_\mu \cong Q^p_{S \otimes_R \mu} \quad \text{and} \quad Q^p_\mu \cong \omega(1 \otimes_R T^p(A, A)) \cap Q^p_{S \otimes_R \mu} .$$

Proof. By (7.7) and the definition of $S \otimes_R \mu$ we have

$$\begin{aligned} &\omega(s \otimes f)((s_1 \otimes a_1) \otimes \cdots \otimes (s_{i-1} \otimes a_{i-1}) \otimes (S \otimes_R \mu)((s_i \otimes a_i) \otimes \cdots) \otimes (s_{i+m} \otimes a_{i+m}) \otimes \cdots) \\ &\quad = ss_1 \cdots s_{p+m-1} \otimes f(a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu(a_i \otimes \cdots) \otimes a_{i+m} \otimes \cdots \otimes a_{p+m-1}) \end{aligned}$$

and

$$\begin{aligned} &(S \otimes_R \mu)((s_1 \otimes a_1) \otimes \cdots \otimes (s_{i-1} \otimes a_{i-1}) \otimes \omega(s \otimes f)((s_i \otimes a_i) \otimes \cdots) \otimes (s_{i+p} \otimes a_{i+p}) \otimes \cdots) \\ &\quad = ss_1 \cdots s_{p+m-1} \otimes \mu(a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots) \otimes a_{i+p} \otimes \cdots \otimes a_{p+m-1}) . \end{aligned}$$

These formulas, when added up with respect to i, show that ω maps $S \otimes_R Q^p_\mu$ into $Q^p_{S \otimes_R \mu}$. Conversely, if $\bar{f}$ is in $Q^p_{S \otimes_R \mu}$ then we have

$$\begin{aligned} &\sum_{i=1}^{p} \bar{f}((1 \otimes a_1) \otimes \cdots \otimes (1 \otimes a_{i-1}) \otimes (S \otimes_R \mu)((1 \otimes a_i) \otimes \cdots \\ &\qquad\qquad \otimes (1 \otimes a_{i+m-1})) \otimes (1 \otimes a_{i+m}) \otimes \cdots) \\ &\quad = \sum_{i=1}^{m} (S \otimes_R \mu)(1 \otimes a_1) \otimes \cdots (1 \otimes a_{i-1}) \otimes \bar{f}((1 \otimes a_i) \otimes \cdots \\ &\qquad\qquad \otimes (1 \otimes a_{i+p-1})) \otimes (1 \otimes a_{i+p}) \otimes \cdots) \end{aligned} \tag{7.9}$$

It is well known ([1], p. 238) that there are finitely many elements $\sigma_j \in S$ and $\sigma_j^* \in S^*$ such that $\mathrm{id}_S = \sum_j \sigma_j^* \sigma_j$ holds. Let

$$\bar{f}((1 \otimes a_1) \otimes \cdots \otimes (1 \otimes a_p)) = \sum_\ell t_\ell(a_1, \cdots, a_p) \otimes b_\ell(a_1, \cdots, a_p) ,$$

where $t_\ell(a_1, \cdots, a_p) \in S$ and $b_\ell(a_1, \cdots, a_p) \in A$, and put, identifying $R \otimes_R A$ with A,

$$\begin{aligned} f_j(a_1 \otimes \cdots \otimes a_p) &= (\sigma_j^* \otimes \mathrm{id}_A)(\bar{f}((1 \otimes a_1) \otimes \cdots \otimes (1 \otimes a_p))) \\ &= \sum_\ell \sigma_j^*(t_\ell(a_1, \cdots, a_p)) \otimes b_\ell(a_1, \cdots, a_p) \ . \end{aligned}$$

Then (7.9) becomes

$$\begin{aligned} &\sum_{i=1}^{p} f_j(a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{m+p-1}) \\ &\quad = \sum_{i=1}^{m} \sum_\ell \sigma_j^*(t_\ell(a_i, \cdots, a_{i+p-1})) \otimes \mu(a_1 \otimes \cdots \otimes a_{i-1} \otimes b_\ell(a_i, \cdots, a_{i+p-1}) \\ &\qquad\qquad \otimes a_{i+p} \otimes \cdots \otimes a_{m+p-1}) \\ &\quad = \sum_{i=1}^{m} \mu(a_1 \otimes \cdots \otimes a_{i-1} \otimes f_j(a_i \otimes \cdots \otimes a_{i+p-1}) \otimes a_{i+p} \otimes \cdots \otimes a_{m+p-1}) \ . \end{aligned}$$

In other words, f_j belongs to Q^p_μ. Now,

$$\begin{aligned} \bar{f}((s_1 \otimes a_1) \otimes \cdots \otimes (s_p \otimes a_p)) &= s_1 \cdots s_p \bar{f}((1 \otimes a_1) \otimes \cdots \otimes (1 \otimes a_p)) \\ &= s_1 \cdots s_p \sum_\ell \sum_j \sigma_j^*(t_\ell(a_1, \cdots, a_p))\sigma_j \otimes b_\ell(a_1, \cdots, a_p) \\ &= s_1 \cdots s_p \sum_j \sigma_j \otimes f_j(a_1 \otimes \cdots \otimes a_p) \\ &= \sum_j \omega(\sigma_j \otimes f_j)((s_1 \otimes a_1) \otimes \cdots \otimes (s_p \otimes a_p)) \ , \end{aligned}$$

and thus $\bar{f} = \sum_j \omega(\sigma_j \otimes f_j)$ as had to be shown to obtain the first isomorphism. As for the second isomorphism, we only have to show that $\omega(1 \otimes f) \in Q^p_{S \otimes_R \mu}$ implies $f \in Q^p_\mu$. However, substituting $\omega(1 \otimes f)$ for $\bar{f}$ in (7.8) leads immediately to (7.3), as had to be shown.

(7.10) THEOREM. *Let F be a field of characteristic zero and let A be a F-algebra of arity $m \geq 2$ with $\dim_F A < \infty$. Let e be a non-trivial idempotent of A such that for the F-endomorphisms τ_i of A, which are given by*

$$\tau_i(a) = \mu(\otimes^{i-1} e \otimes a \otimes \otimes^{m-i} e) \ , \qquad i = 1, \cdots, m \ ,$$

the following conditions are satisfied:

(i) *for $p = 2, \cdots, p \neq m$*

$$\ker\left(p \cdot \mathrm{id}_A - \sum_{i=1}^{m} \tau_i\right) = 0$$

(ii) *for any $L \in \mathrm{Hom}_F(A, A)$ and any $\ell = 1, 2, \cdots$*

(α) $\ell L = \tau_1 L - L\tau_1$ *implies* $L = 0$

(β) $\ell L = \tau_m L - L\tau_m$ *implies* $L = 0$

(iii) *for any* $L \in \mathrm{Hom}_F(A \otimes_F A, A)$ *and any* $\ell = 0, 1, \cdots$

$$\ell L + L \circ (\tau_1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \tau_m) = 0 \qquad \textit{implies } L = 0$$

(iv) $$\frac{\det\left(\mathrm{id}_A \cdot X - \sum_{i=1}^{m} \tau_i\right)}{(X - m)}(m) \neq 0$$

Then

(1) ${}_F U(A) = {}_F U(A)^{[1]} = \cdots = {}_F U(A)^{[m-1]}$

(2) ${}_F U(A)^{[m]} = {}_F U(A)^{[m+1]} = \cdots = 0$

(3) *there is a canonical, injective F-homomorphism* $Q_\mu^m \to A$

(4) ${}_F U(A)$ *is a one-dimensional F-vector space.*

Proof. (1) It follows immediately from (7.3) and (7.10), (i), that for $p < m$ and $f \in Q_\mu^p$, $f(\otimes^p e) = 0$.

It is well known that there is a F-basis

$$b'_{i,j}\,, \qquad i = 1, \cdots, n_j\,; j = 1, \cdots, t\,,$$

of A such that

$$\tau_m(b'_{i,j}) = b'_{i+1,j}\,, \qquad i = 1, \cdots, n_j - 1$$
$$\tau_m(b'_{n_j,j}) = \sum_{i=1}^{n_j} r_{i,j} b'_{i,j}$$

with suitable scalars $r_{i,j} \in F$. We want to prove, by induction on r, that, for $r < p$

(7.11) $$f(\otimes^{p-r} e \otimes a_1 \otimes \cdots \otimes a_r) = 0 \qquad \text{for all } a_1, \cdots, a_r \in A\,.$$

Assume that (7.11) is valid for $r < q$. By evaluating (7.3)—see proof of (7.2)—on $\otimes^{p-q+m-1} e \otimes b'_{i,j} \otimes a_2 \otimes \cdots \otimes a_q$ we obtain

(7.12) $$\begin{aligned}(p - q)&f(\otimes^{p-q} e \otimes b'_{i,j} \otimes a_2 \otimes \cdots \otimes a_q)\\ &+ f(\otimes^{p-q} e \otimes \tau_m(b'_{i,j}) \otimes a_2 \otimes \cdots \otimes a_q)\\ &= \tau_m(f(\otimes^{p-q} e \otimes b_{i,j} \otimes a_2 \otimes \cdots \otimes a_q)\,.\end{aligned}$$

Put

$$f_{i,j} = f(\otimes^{p-q} e \otimes b'_{i,j} \otimes a_2 \otimes \cdots \otimes a_q)\,.$$

Then (7.12) becomes

$$(p-q)f_{i,j} + f_{i+1,j} = \tau_m(f_{i,j}) , \qquad i = 1, \cdots, n_j - 1$$
$$(p-q)f_{n_j,j} + \sum_{i=1}^{n_j} r_{i,j} f_{i,j} = \tau_m(f_{n_j,j}) .$$

Define the F-endomorphism L of A by putting $L(b'_{i,j}) = f_{i,j}$. Then the last relations can be written as

$$(p-q)L = \tau_m L - L\tau_m .$$

By assumption, $q \neq p$ implies $L = 0$ and thus $f_{i,j} = 0$. Therefore, (7.11) is valid for $r = 0, \cdots, p-1$. Similarly, we obtain that

(7.13) $\quad f(a_1 \otimes \cdots \otimes a_r \otimes \otimes^{p-r} e) = 0 \qquad \text{for all } a_1, \cdots, a_r \in A$

is valid for $r = 0, \cdots, p-1$.

Next, choose a F-basis $'b_{k,\ell}$ of A which behaves relative to τ_1 as the basis $b'_{i,j}$ does relative to τ_m. We want to prove, by induction on r, that for $r \leq p$

(7.14) $\quad f(a_1 \otimes \otimes^{p-r} e \otimes a_2 \otimes \cdots \otimes a_r) = 0 \qquad \text{for all } a_1, \cdots, a_r \in A .$

This, of course, means that $Q^p_\mu = 0$, for $p = 2, \cdots, m-1$—which is the first assertion. Assume that (7.14) is valid for $r \leq q$, and evaluate (7.3) on $'b_{k,\ell} \otimes \otimes^{p-q+m-2} e \otimes b'_{i,j} \otimes a_2 \otimes \cdots \otimes a_q$. Using (7.11) and (7.13) we obtain the relation

$$\begin{aligned} & f(\tau'_1 b_{k,\ell} \otimes \otimes^{p-q-1} e \otimes b'_{i,j} \otimes a_2 \otimes \cdots \otimes a_q) \\ (7.15) \qquad & \quad + (p-q-1) f('b_{k,\ell} \otimes \otimes^{p-q-1} e \otimes b'_{i,j} \otimes a_2 \otimes \cdots \otimes a_q) \\ & \quad + f('b_{k,\ell} \otimes \otimes^{p-q-1} e \otimes \tau_m b'_{i,j} \otimes a_2 \otimes \cdots \otimes a_q) = 0 . \end{aligned}$$

As before, we define a F-homomorphism $L: A \otimes_F A \to A$ by putting

$$L('b_{k,\ell} \otimes b'_{ij}) = f('b_{k,\ell} \otimes \otimes^{p-q-1} e \otimes b_{i,j} \otimes a_2 \otimes \cdots \otimes a_q) .$$

An easy computation shows that (7.14) now becomes

$$(p-q-1)L + L \circ (\tau_1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \tau_m) = 0 .$$

Since $q \leq p-1$ holds, assumption (iii) implies $L = 0$. Thus the induction argument is finished, and the first assertion is proved.

(2) As for the second claim, let λ be an element of ${}_FU(A)^{[m]}$. By definition we have $\lambda^2 = \cdots = \lambda^m = 0$. We want to prove, by induction on p, that $\lambda^p = 0$, for all $p \geq m$. If $\lambda^q = 0$, for $2 \leq q < p$, then it follows from (5.11)—by computing the $(p+m-1)^{st}$ component—that $f = \lambda^p$

satisfies (7.3). As in (1) we obtain $f(\otimes^p e) = 0$, as $p \neq m$ holds. This, however, makes it possible to repeat the previous argument, which leads to $f = 0$.

(3) Again, (6.8) and (7.9), (1) and (2), render the isomorphism ${}_F U(A) \cong Q_\mu^m$. And, as before, we have the canonical F-homomorphism

$$Q_\mu^m \ni f \to f(\otimes^m e) \in A \ .$$

And, again, we will have to prove that $f(\otimes^m e) = 0$ implies $f = 0$. The proof proceeds literally as in (1), since the assumption $f(\otimes^m e) = 0$ furnishes the starting point for the induction argument.

(4) An easy computation shows that $f \in Q_\mu^m$ implies

$$mf(\otimes^m e) = \sum_{i=1}^{m} \tau_i(f(\otimes^m e)) \ .$$

In other words, $f(\otimes^m e)$ is an eigenvector, with eigenvalue m, of $\sum_{i=1}^m \tau_i$. Since $\mu_A(\otimes^m e)$ is such an eigenvector, condition (iv) implies that $f(\otimes^m e)$ is a scalar multiple of $\mu_A(\otimes^m e)$. Hence Q_μ^m is one-dimensional, and (1), (2), (3) imply (4).

(7.16) COROLLARY. *Suppose that, under the general assumptions of* (7.10), (ii) *and* (iii) *are satisfied and* (i) *is valid for all* $p > p_0$, *for some* p_0. *Then* ${}_F U(A)^{[p_0]} = {}_F U(A)^{[p_0+1]} = \cdots = 0$, *and hence* ${}_F U(A)$ *can be obtained by finitely many successive extensions of certain finite dimensional* F*-vector spaces; moreover,* ${}_F U(A)$ *is a unipotent algebraic group. In particular, if* $F = \boldsymbol{R}$ *or* $\boldsymbol{C}$, *then—under the stated conditions—*${}_F U(A)$ *is a unipotent, simply connected real resp. complex Lie group which, in the complex case, is a Stein manifold.*

Proof. The proof of (7.10) shows that the assumptions imply $Q_\mu^{p_0+1} = Q_\mu^{p_0+2} = \cdots = 0$. Hence ${}_F U(A)^{[p_0]} = {}_F U(A)^{[p_0+1]} = \cdots$. Since the intersection of these groups is trivial, they all have to be trivial. Algebraicity of ${}_F U(A)$ follows from (5.12), and the remaining assertions are now either obvious or a matter of definition.

(7.17) *Remark.* Condition (iv) of (7.10) is only needed to prove assertion (4). In the absence of (iv) one still has that ${}_F U(A)$ is isomorphic to a finite dimensional F-vector space, namely Q_μ^m.

(7.18) THEOREM. *Let* F *be a field of characteristic zero. Then the as-*

sertions of (7.10) *are Zariski-generically valid for F-algebras of arity m. That is*

(1) ${}_FU(A) = {}_FU(A)^{[1]} = \cdots = {}_FU(A)^{[m-1]}$

(2) ${}_FU(A)^{[m]} = {}_FU(A)^{[m+1]} = \cdots = 0$

(3) ${}_FU(A)$ *is a F-vector space of dimension* 1.

Proof. Due to (6.8), these assertions are equivalent with

$$Q^p_\mu = 0 \qquad \text{for } p \neq m \ , \quad \text{and} \quad \dim_F Q^m_\mu = 1 \ .$$

Let $e^1, \cdots, e^n$ be the unit vectors in F^n. For $f \in T^p(A, A)$, put

$$f(e^{k_1} \otimes \cdots \cdot \otimes e^{k_p}) = \sum_{i=1}^{n} f_i^{k_1, \cdots, k_p} e^i \ .$$

Then it follows from (6.4) that $f \in Q^p_\mu$ is equivalent with the $f_k^{k_1, \cdots, k_p}$ satisfying a certain system L_p of homogeneous linear equations whose coefficients are $\boldsymbol{Z}$-linear combinations of the structure coefficients of the F-algebra A. One checks easily that the number of these linear equations equals n^{m+p}. Therefore, $Q^p_\mu = 0$, for $p \neq m$, is equivalent with rank $L_p = n^{1+p}$; this in turn is equivalent with the structure coefficients lying in a certain Zariski-open set Z_p. (7.2) shows that none of the sets Z_p, $p \neq m$, is empty as for any m there is a F-algebra of arity m on F^n which has a unit element (e.g. $\prod_1^n E_m$). Finally, $\dim_F Q^m_\mu = 1$ is equivalent with rank $L_m = n^{1+p} - 1$; this is equivalent with the structure coefficients lying in a certain Zariski-open set Z_m which, by the previous argument, is not empty. Hence our claim is established.

Finally, we shall state—again without giving detailed proofs—the changes that occur as one switches from ${}_R\mathscr{A}lg_m$ to ${}_R\mathscr{A}lg_{mc}$. (7.1) remains valid for ${}_RU(A)^{[p]}_c$ rather than ${}_RU(A)^{[p]}$. (7.2) also stays in force. Here one has to observe that for $f \in T^p(A, A)$ and f_c its canonical image in $T^p(A, A)_c$,

$$\delta^*_{\mu, Ac} f_c = \delta^*_{f, Ac} \mu_c$$

is equivalent with

$$\delta^*_{\mu, A} f - \delta^*_{f, A} \mu \in C^*(A, A) \ . \tag{7.19}$$

One verifies easily that (7.4) still holds. This starts, as in the proof of (7.2), an induction argument on q which shows that f vanishes on all elements of the form $c_p(a_2 \otimes \cdots \otimes a_q)$ (see proof of (4.18)). Then the reasoning contained in the proof of (3.21) leads to the desired result. Similarly, (7.5) remains valid with an analogous adjustment in proof. It is easy to see that (7.7) stays true and that ω maps $S \otimes_R C^p(A, A)$ onto $C^p(S \otimes_R A, S \otimes_R A)$, thus inducing an isomorphism

$$\omega_c : S \otimes_R T^p(A, A)_c \to T^p(S \otimes_R A, S \otimes_R A)_c \ .$$

The validity of (7.8) for $Q^p_{\mu c}$ and $Q^p_{S \otimes_R \mu c}$ follows exactly as in the proof of (7.8). Now, in place of (7.10) we get

(7.20) THEOREM. *Let F be a field of characteristic zero and let A be a F-algebra of arity $m \geq 2$ with $\dim_F A < \infty$. Let e be a non-trivial idempotent of A such that for the F-endomorphism T of A, which is given by*

$$T(a) = \sum_{i=1}^{m} \mu(\otimes^{i-1} e \otimes a \otimes \otimes^{m-i} e) \ ,$$

the following conditions are satisfied:

(i) *for any $L \in \mathrm{Hom}_F (\otimes_F^q A, A)$ and any pair of integers p, q with $0 \leq q \leq p$ and $p \geq 2$, other than $p = m$ and $q = 0$*

$$T \circ L = (p - q)L + L \circ \sum_{i=1}^{q} \otimes^{i-1} \mathrm{id}_A \otimes T \otimes \otimes^{q-i} \mathrm{id}_A$$

implies $L = 0$

(ii)

$$\frac{\det (\mathrm{id}_A \cdot X - T)}{(X - m)}(m) \neq 0 \ .$$

Then

(1) $_F U(A)_c = {}_F U(A)_c^{[1]} = \cdots = {}_F U(A)_c^{[m-1]}$

(2) $_F U(A)_c^{[m]} = {}_F U(A)_c^{[m+1]} = \cdots = 0$

(3) *there is a canonical, injective F-homomorphism $Q^m_{\mu c} \to A$*

(4) $_F U(A)_c$ *is one-dimensional F-vector space.*

Proof. The proof proceeds along the lines of the proof of (7.10). It follows from (7.19) and hypothesis (i), for $q = 0$ and $p \neq m$, that for any $f \in T^p(A, A)$ with $f_c \in Q^p_{\mu c}$,

$$f(\otimes^p e) = 0 \ .$$

This is used as the anchor point for an induction, on q, that f vanishes on the image of $c_{p,q}\colon \otimes_F^q A \to \otimes_F^p A$ where $c_{p,q}$ is defined by (see proof of (4.18))

$$c_{p,q}(a_1 \otimes \cdots \otimes a_q) = c_p(a_1 \otimes \cdots \otimes a_q) .$$

The induction hypothesis, applied to (7.19), renders then

$$T \circ f \circ c_{p,q} = (p - q) f \circ c_{p,q} + f \circ c_{p,q} \circ \sum_{i=1}^{q} \otimes^{i-1} \mathrm{id}_A \otimes T \otimes \otimes^{p-i} \mathrm{id}_A .$$

Hence hypothesis (i) leads to the assertions (1)–(3) (see also proof of (7.10)). (ii), finally, implies (4)—just as in the proof of (7.10).

(7.16), (7.17), and (7.18) remain valid in the commutative situation.

BIBLIOGRAPHY

[1] Bourbaki, N.: Algebra I. Addison-Wesley Publ. 1974.

[2] Coleman, C.: Growth and Decay Estimates near Non-elementary Stationary Points. Can. J. Math. **XXII** (1970), 1156–1167.

[3] Courant, R. and D. Hilbert: Methods of Mathematical Physics, Vol. II. Interscience Publ. 1962.

[4] Koecher, M.: Über Abbildungen, die Lösungen von gewissen gewöhnlichen Differentialgleichungen erhalten. Ms.

[5] Kreisel, G. and Krivine: Modelltheorie. Springer Verlag 1972.

[6] Lefschetz, S.: Differential Equations: Geometric Theory, 2nd ed. Interscience Publ. 1963.

[7] Levin, J. J. and S. Shatz: On the Matrix Riccati Equation. Proc. AMS **10** (1959), 519–524.

[8] ——: Riccati Algebras. Duke Math. J. **30** (1963), 579–594.

[9] Markus, L.: Quadratic Differential Equations and Non-associative Algebras. Contributions to the Theory of Non-linear Oscillations, Vol. V, 185–213. Princeton University Press 1960.

[10] Pareigis, B.: Categories and Functors. Academic Press 1970.

[11] Röhrl, H.: Algebren und Riccati Gleichungen. Sitz. ber. d. Berliner Math. Ges. (1972/4).

[12] Röhrl, H. and M. B. Wischnewsky: Subalgebras that are Cyclic as Submodules. Manuscripta Math. **9** (1976), 195–209.

[13] Röhrl, H.: A theorem on non-associated algebras and its applications to differential equations. Manuscripta Math. **21** (1977), 181–187.

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