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# THE DECAY OF THE LOCAL ENERGY FOR WAVE EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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## § 0. Introduction

The exponential decay of the local energy for wave equations in exterior domains of the odd dimensional space has been proved in [1] $\sim$ [6] etc. under the Dirichlet boundary condition and in [5], [7] under the Neumann condition and the other conditions. In this paper, we shall consider this problem for the following equation:

$$\text{(I)} \qquad \frac{\partial^2}{\partial t^2}u = \frac{1}{\rho(x)}\nabla\cdot\rho(x)\nabla u \,, \qquad \text{in } R^n \times (0, \infty)$$

with the initial data

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x) \,,$$

where $n \geq 3$ is the space dimension, $f(x)$ and $g(x)$ are of compact support, and $\rho(x)$ is the discontinuous function defined as follows:

$$\rho(x)\begin{cases} = \rho > 1 \,, & \text{in } \mathcal{O} \\ = 1 & \text{in } \mathcal{E} = R^n - \bar{\mathcal{O}} \,. \end{cases}$$

It is convenient to regard the problem (I) as follows: Let $v = u|_{\mathcal{E}\times(0,\infty)}$ and $w = u|_{\mathcal{O}\times(0,\infty)}$. Then, $v$ and $w$ satisfy the equations $\square\, v = 0$ and $\square\, w = 0$ in $\mathcal{E} \times (0, \infty)$ and $\mathcal{O} \times (0, \infty)$, respectively, and the relation between $v$ and $w$

$$\text{(0.1)} \qquad v|_{\partial\mathcal{E}} = w|_{\partial\mathcal{O}} \,,$$

$$\text{(0.2)} \qquad \left.\frac{\partial v}{\partial n}\right|_{\partial\mathcal{E}} = \rho\left.\frac{\partial w}{\partial n}\right|_{\partial\mathcal{O}}$$

holds on $\partial\mathcal{O} = \partial\mathcal{E}$, where $n = (n_1, \cdots, n_n)$ denotes the unit normal on $\partial\mathcal{E}$

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which points into $\mathscr{E}$. (From now on, we use $n=(n_1,\cdots,n_n)$ in this sense in order to fix the notation.)

By a $C^2$-solution $u$, we mean that $u$ belongs to $C^2(\bar{\mathscr{O}}\times[0,\infty))\cap C^2(\bar{\mathscr{E}}\times[0,\infty))$ and satisfies (0.1) and (0.2) on $\partial\mathscr{E}$ and that $u$ is real valued. In fact, such a solution exists: We set $A=-\frac{1}{\rho(x)}\nabla\cdot\rho(x)\nabla$. Then, the operator $A$ is a positive self-adjoint operator in $L^2(\rho(x)dx)$ with weight $\rho(x)$ whose domain is given by

$$\mathscr{D}(A)=\{u\in H^1(R^n)\,|\,w=u|_{\mathscr{O}}\in H^2(\mathscr{O}),\ v=u|_{\mathscr{E}}\in H^2(\mathscr{E}),\ w \text{ and } v \text{ satisfy } (0.1) \text{ and } (0.2) \text{ in } H^{3/2}(\partial\mathscr{E}) \text{ and } H^{1/2}(\partial\mathscr{E}), \text{ respectively}\}\,,$$

$H^1(\mathscr{O})$ and $H^2(\mathscr{E}),\cdots$ being the usual Sobolev spaces. Hence, this implies that for given $f\in H^1(R^n)$ and $g\in L^2(R^n)$ of problem (I), there exist a unique weak solution $u(x,t)$ such that $u(x,t)\in C^1((0,T);L^2(R^n))\cap C((0,T);H^1(R^n))$ for any $T>0$. Moreover, if $\partial\mathscr{E}$ is smooth enough, the following regularity theorem holds for $A$:

$$\mathscr{D}(A^N)\subset\{u\in H^1(R^n)\,|\,w\in H^{2N}(\mathscr{O}),\,v\in H^{2N}(\mathscr{E})\}\,.$$

Hence, if we choose the initial data $f$ and $g$ as $f\in\mathscr{D}(A^N)$ and $g\in\mathscr{D}(A^N)$, $N$ being large enough, we can find a desired solution by the imbedding theorem of Sobolev. We note that a weak solution is obtained as a limit of such a solution in the energy norm.

As is easily seen, the total energy

$$\int_{R^n}\rho(x)(|u_t(t)|^2+|\nabla u(t)|^2)dx$$

is conserved in $t$. We denote this quantity by $G_0(u)$, so that

$$\frac{1}{\rho}G_0(u)\le\int_{R^n}(|u_t(t)|^2+|\nabla u(t)|^2)dx\le G_0(u)\,,$$

since $\rho>1$. We define $E(u;h,T)$ as follows:

$$E(u;h,T)=\int_{|x|\le h}(|u_t(T)|^2+|\nabla u(T)|^2)dx\,.$$

Before stating the main theorem, we make the following assumption on $\mathscr{O}$:

Assumption (A). (i) $\mathscr{O}$ is a convex open bounded domain with smooth boundary which contains the origin. For brevity, (0.3) $$\mathcal{O} \subset \{x \,|\, |x| < \tfrac{1}{2}\}\,.$$

(ii) There exists a $C^4$-function $\chi(x)$ such that

(a.1) $\chi(x) = \text{const} > 0$, on $\partial\mathcal{E}$;

(a.2) $\chi_n = \dfrac{\partial\chi}{\partial n} = (\chi_j \cdot n_j) > \beta > 0$, $\chi_j = \dfrac{\partial\chi}{\partial x_j}$, on $\partial\mathcal{E}$;

(a.3) $(\chi_{ij})$, $\chi_{ij} = \partial^2\chi/\partial x_i \partial x_j$, is a positive definite matrix at each point of $R^n$;

(a.4) $\chi = (1 - r^{-\delta})x_j/r$, $r = |x|$, $0 < \delta < 1$, for $r \geq r_0$ large enough.

If $\mathcal{O}$ is strictly convex, we can find such a function (see [5] p. 246).

MAIN THEOREM. *Let $n \geq 3$. Assume that Assumption* (A) *is satisfied. Let $u$ be the $C^2$-solution of problem* (I) *with the initial data $f$ and $g$ of compact support: support of $f$ and $g \subset |x| \leq \gamma$. Then, if $n$ is odd*

$$E(u; h, T) \leq k_1 e^{-\theta T} G_0(u)\,,$$

*and if $n$ is even,*

$$E(u; h, T) \leq k_2 T^{-1} G_0(u)\,,$$

*where $k_1, k_2$ and $\theta$ are constants depending only on $h$ and $\gamma$.*

The above main theorem is proved by a modification or generalization of methods used in Morawetz [4] and Strauss [6]. In § 1, we show that $E(u; h, t)$ is integrable in $t$ and in § 2, we prove that $E(u; h, t)$ decays at the rate of $t^{-1}$. In § 3, we prove the exponential dacay.

Finally we note the following facts throughout this paper: (a) $k$, $k_1, k_2, \cdots$ are used to denote positive constants, which are not necessarily the same. (b) Integration with no domain attached is taken over the whole space. (c) we use the summation convention. (d) we write simply $\chi_n, v_n, \cdots$ instead of $\dfrac{\partial\chi}{\partial n}, \dfrac{\partial v}{\partial n}, \cdots$.

## 1. Integrability of the local energy

We state some preliminary lemmas.

LEMMA 1.1. *Let $\chi(x)$ be a $C^4$-function. Then, the identity*

(1.1) $$(u_{tt} - u_{jj})(\chi_i u_i + \tfrac{1}{2}\chi_{ii} u) = X_t(u) + \nabla \cdot Y(u) + Z(u)$$

*holds, where*

$$X(u) = u_t(\chi_i u_i + \tfrac{1}{2}\chi_{ii}u)\,,$$
$$Y_j(u) = -u_j(\chi_i u_i + \tfrac{1}{2}\chi_{ii}u) + \tfrac{1}{2}\chi_j(|\nabla u|^2 - u_t^2) + \tfrac{1}{4}\chi_{iij}u^2\,.$$
$$Z(u) = \chi_{ij}u_i u_j - \tfrac{1}{4}\chi_{iijj}u^2\,.$$

LEMMA 1.2. *Assume that $\chi(x)$ is a $C^4$-function satisfying* (a.4). *Then, we have*

$$\chi_{ij}u_i u_j \geq \delta r^{-1-\delta}|\nabla u|^2\,, \tag{1.2}$$

$$\chi_{iijj} \leq -\delta(1+\delta)r^{-3-\delta}\,, \tag{1.3}$$

*for $r = |x| \geq r_0$ large enough.*

By a direct calculation, we obtain Lemmas 1.1 and 1.2. (see Lemmas 1 and 2 of Strauss [6])

LEMMA 1.3. *Let $u$ be a $C^2$-solution of problem* (I). *Suppose that Assumption* (A) *is satisfied. Then, for any $\varepsilon > 0$ small enough,*

$$\int_0^T \int e^{-2\varepsilon t}(|\nabla u|^2 + (1+r)^{-2}u^2)(1+r)^{-1-\delta}dxdt \leq k_1 G_0(u) + k_2 \int_0^T \int_{|x|\leq r_0} e^{-2\varepsilon t}u^2 dxdt\,,$$

*where $k_1$ and $k_2$ are constants independent of $\varepsilon$ and $T$, and $G_0(u)$ is the total energy.*

*Proof.* We set $v = u|_{\mathscr{E}\times(0,T)}$ and $w = u|_{\mathscr{O}\times(0,T)}$. We multiply the identity (1.1) with $\chi(x)$ satisfying (a.1) ~ (a.4) by $e^{-2\varepsilon t}$ and integrate over $\mathscr{E}\times(0,T)$ and $\mathscr{O}\times(0,T)$, separately. We have

$$\begin{aligned}\int_0^T\int_{\mathscr{E}} e^{-2\varepsilon t}Z(v)dxdt &= -\int_0^T\int_{\mathscr{E}} e^{-2\varepsilon t}X_t(v)dxdt \\ &\quad + \int_0^T\int_{\partial\mathscr{E}} e^{-2\varepsilon t}(Y_j(v)\cdot n_j)d\sigma dt \\ &= I_1 + I_2\,,\end{aligned} \tag{1.4}$$

$$\begin{aligned}\int_0^T\int_{\mathscr{O}} e^{-2\varepsilon t}Z(w)dxdt &= -\int_0^T\int_{\mathscr{O}} e^{-2\varepsilon t}X_t(w)dxdt \\ &\quad - \int_0^T\int_{\partial\mathscr{E}} e^{-2\varepsilon t}(Y_j(w)\cdot n_j)d\sigma dt \\ &= II_1 - II_2\,.\end{aligned} \tag{1.5}$$

Integration by parts yields

$$I_1 = -\int_{\mathscr{E}} e^{-2\varepsilon T} X(v, T)dx + \int_{\mathscr{E}} X(v, 0)dx - 2\varepsilon \int_0^T \int_{\mathscr{E}} e^{-2\varepsilon t} X(v) dxdt$$
$$= I_{11} + I_{12} + I_{13} .$$

Recalling the expression of $X(v)$ in Lemma 1.1, we have

$$X(v) \leq k(v_t^2 + |\nabla v|^2 + r^{-2}v^2)$$

for some $k > 0$, since $\chi_{ii} = O(r^{-1})$ as $r \to \infty$. Integrating $X(v)$ over $\mathscr{E}$, we have

$$\int_{\mathscr{E}} X(v)dx \leq k \int (u_t^2 + |\nabla u|^2 + r^{-2}u^2)dx .$$

Note that if $n \geq 3$, $\int r^{-2}u^2 dx \leq k \int |\nabla u|^2 dx$. Then, it follows that

(1.6) $$I_{11}, I_{12} \leq kG_0(u) .$$

Moreover, we have

(1.7) $$I_{13} \leq k\varepsilon \int_0^T e^{-2\varepsilon t} dt G_0(u) \leq k_1 G_0(u) .$$

Combining (1.6) and (1.7), we obtain

(1.8) $$I_1 \leq kG_0(u) .$$

Similarly we have

(1.9) $$II_1 \leq kG_0(u) .$$

Next, we consider the terms $I_2$ and $II_2$. Making use of the fact that $\chi_i v_i = \chi_n v_n$ on $\partial\mathscr{E}$ by (a.1) and writing $|\nabla v|^2 = v_n^2 + |\nabla_{\tan} v|^2$ on $\partial\mathscr{E}$, we have

$$(Y_j(v)\cdot n_j) = -\tfrac{1}{2}\chi_n v_n^2 - \tfrac{1}{2}\chi_{ii} v_n v + \tfrac{1}{2}\chi_n(|\nabla_{\tan} v|^2 - v_t^2) + \tfrac{1}{4}\chi_{iin} v^2 .$$

We obtain a similar expression also for $(Y_j(w)\cdot n_j)$. In view of relations (0.1) and (0.2), we see that

$$(Y_j(v)\cdot n_j) - (Y_j(w)\cdot n_j) = \tfrac{1}{2}(1-\rho^2)\chi_n w_n^2 + \tfrac{1}{2}(1-\rho)\chi_{ii} w_n w .$$

Since $(1-\rho^2)\chi_n < 0$ by $\rho > 1$ and (a.2), it follows that on $\partial\mathscr{E}$

$$(Y_j(v)\cdot n_j) - (Y_j(w)\cdot n_j) \leq kw^2 ,$$

for $k > 0$. Furthermore we have for any $\eta > 0$ small enough,

$$\int_{\partial\mathscr{E}} w^2 d\sigma \le \eta \int_{\mathcal{O}} |\nabla w|^2 dx + k(\eta) \int_{\mathcal{O}} w^2 dx \, .$$

Hence, we obtain

$$I_2 - II_2 \le \eta \int_0^T \int_{\mathcal{O}} e^{-2\varepsilon t} |\nabla w|^2 \, dx + k(\eta) \int_0^T \int_{\mathcal{O}} e^{-2\varepsilon t} w^2 dx \tag{1.10}$$

for any $\eta > 0$ small enough.

Now, by (1.2) and (1.3),

$$Z(v) \ge \delta r^{-1-\delta} |\nabla v|^2 + \tfrac{1}{4}\delta(1+\delta) r^{-3-\delta} v^2 \, , \qquad \text{for } |r| \ge r_0 \, . \tag{1.11}$$

And by (a.3),

$$Z(v) \ge k_1 |\nabla v|^2 - k_2 v^2 \, , \qquad \text{in } |x| \le r_0 \cap \mathscr{E} \, , \tag{1.12}$$

$$Z(w) \ge k_3 |\nabla w|^2 - k_4 w^2 \, , \qquad \text{in } \mathcal{O} \, . \tag{1.13}$$

Taking $\eta$ in (1.10) small enough and combining (1.8) $\sim$ (1.12) with (1.13), we finally obtain

$$\int_0^T \int e^{-2\varepsilon t} (|\nabla u|^2 + (1+r)^{-2} u^2)(1+r)^{-1-\delta} dx dt$$
$$\le k_1 G_0(u) + k_2 \int_0^T \int_{|x| \le r_0} e^{-2\varepsilon t} u^2 dx dt \, .$$

LEMMA 1.4. *Under the same assumption as in Lemma* 1.3, *the following estimate holds*:

$$\int_0^T e^{-2\varepsilon t} (1+r)^{-1-\delta} u_t^2 dx dt$$
$$\le k_1 G_0(u) + k_2 \int_0^T \int_{|x| \le r_0} e^{-2\varepsilon t} u^2 dx dt \, .$$

*Proof*. Let $p(x) = (1+r^2)^{-(1+\delta)/2}$. Then, $|\Delta p| \le k(1+r)^{-3-\delta}$. As in the proof of Lemma 1.3, we set $v = u|_{\mathscr{E} \times (0,T)}$ and $w = u|_{\mathcal{O} \times (0,T)}$. We multiply the equation $\Box \, v = 0$ by $e^{-2\varepsilon t} p(x) v$ and integrate over $\mathscr{E} \times (0, T)$. Then, we have

$$0 = \int_{\mathscr{E}} e^{-2\varepsilon t} p(x) v_t v dx \Big|_0^T + \int_0^T \int_{\mathscr{E}} e^{-2\varepsilon t} (|\nabla v|^2 - v_t^2) p(x) dx dt$$
$$+ 2\varepsilon \int_0^T \int_{\mathscr{E}} e^{-2\varepsilon t} p(x) v_t v dx dt + \int_0^T \int_{\partial\mathscr{E}} e^{-2\varepsilon t} p(x) v_n v d\sigma dt$$
$$- \frac{1}{2} \int_0^T \int_{\partial\mathscr{E}} e^{-2\varepsilon t} p_n v^2 d\sigma dt - \frac{1}{2} \int_0^T e^{-\varepsilon t} \Delta p v^2 dx dt \, .$$

A similar identity for $w$ is obtained by multiplying $\square\, w = 0$ by $\rho e^{-2\varepsilon t} p(x) w$ and integrating over $\mathcal{O} \times (0, T)$. By the definition of $p(x)$, we can prove in the same way as in the proof of Lemma 1.3 that

$$\int_{\mathcal{E}} e^{-2\varepsilon t} p(x) v_t v dx \Big|_0^T \leq k G_0(u) \,, \tag{1.14}$$

$$2\varepsilon \int_0^T \int_{\mathcal{E}} e^{-2\varepsilon t} p(x) v_t v dx dt \leq k G_0(u) \,. \tag{1.15}$$

The same estimates as (1.14) and (1.15) are obtained for $w(x)$ with domain of integration $\mathcal{O}$. Thus, by taking account of relations (0.1) and (0.2), and by adding up the two identities obtained for $v$ and $w$, the boundary integral is estimated by

$$k \int_0^T \int_{\mathcal{O}} e^{-2\varepsilon t} (|\nabla w|^2 + w^2) dx dt \,,$$

so that we have

$$\begin{aligned} &\int_0^T \int e^{-2\varepsilon t} (1 + r)^{-1-\delta} u_t^2 dx dt \\ &\qquad \leq k_1 G_0(u) + k_2 \int_0^T \int e^{-2\varepsilon t} (|\nabla u|^2 + (1 + r)^{-2} u^2)(1 + r)^{-1-\delta} dx dt \,. \end{aligned}$$

Combining this estimate with Lemma 1.3, we obtain the conclusion.

LEMMA 1.5. *Suppose that the same assumption as in Lemma 1.3 is satisfied. Let $R$ be a positive fixed number. Then, for any $\eta > 0$ small enough, there exists a constant $k = k(\eta)$ independent of $\varepsilon$ such that*

$$\int_0^\infty \int_{|x| \leq R} e^{-2\varepsilon t} u^2 dx dt \leq k G_0(u) + \eta \int_0^\infty \int e^{-2\varepsilon t} (1 + r)^{-1-\delta} u_t^2 dx dt \,,$$

*where we note that the constant $k$ may depend on the support of the initial data $f$ and $g$.*

This lemma will be proved in Appendix.

Combining Lemmas 1.3 and 1.4 with Lemma 1.5, and letting $T \to \infty$ and $\varepsilon \to 0$, we immediately obtain the following result.

THEOREM 1. *Let $n \geq 3$ and let $u$ be a $C^2$-solution of problem* (I) *with initial data of compact support. Suppose that Assumption* (A) *is satisfied. Then,*

$$\int_0^\infty \int (u_t^2 + |\nabla u|^2)(1+r)^{-1-\delta} dxdt \leq kG_0(u)$$

*for $k>0$ depending only on $\delta$ and the support of the initial data. Therefore, we have that*

$$E(u\,;h,t) = \int_{|x|\leq h} (u_t^2 + |\nabla u|^2)dx$$

*is integrable in $t$.*

## § 2. Uniform decay of the local energy

In this section, we shall prove the uniform decay of the local energy. We introduce the following function: Let $\ell(x)$ be a $C^4$-function such that

$$\ell(x) = \text{const} > 0\,, \qquad \text{on } \partial\mathscr{E} \tag{2.1}$$

$$\ell_n = (\ell_j \cdot n_j) > \beta > 0\,, \qquad \text{on } \partial\mathscr{E} \tag{2.2}$$

$$\ell(x) = r^2 \qquad \text{for } |x| \geq r_1 \ (r_1 \text{ large enough})\,. \tag{2.3}$$

We begin with the following identity (cf. Morawetz [4] and Zachmanoglou [9]): Let $A(x,t)$ be a $C^\infty$-function of $x$ and $t$.

$$(u_{tt} - u_{jj})(Au_t + t\ell_j u_j + (n-1)tu) = F_t(u) + \nabla\cdot G(u) + H(u)\,, \tag{2.4}$$

where

$$\begin{aligned}
F(u) &= \tfrac{1}{2}A(u_t^2 + |\nabla u|^2) + t\ell_j u_j u_t + (n-1)tu_t u - \tfrac{1}{2}(n-1)u^2 \\
G_j(u) &= -u_j(Au_t + t\ell_j u_j + (n-1)tu) + \tfrac{1}{2}t\ell_j(|\nabla u|^2 - u_t^2) \\
H(u) &= \tfrac{1}{2}u_t^2(t\ell_{jj} - A_t - 2(n-1)t) + u_t u_j(A_j - \ell_j) \\
&\quad + \tfrac{1}{2}(2t\ell_{jk}u_j u_k + 2(n-1)t|\nabla u|^2 - t\ell_{jj}|\nabla u|^2 - A_t|\nabla u|^2)\,.
\end{aligned}$$

LEMMA 2.1. *Let $u$ be a $C^2$-solution of problem* (I) *with initial data of compact support and let $w = u|_{\mathscr{E}\times(0,T)}$. Assume that $\ell(x)$ satisfies* (2.1) ~ (2.3). *Then,*

$$\begin{aligned}
\frac{1}{2}T^2 \int_{\mathscr{E}} (w_t(T)^2 + |\nabla w(T)|^2)dx \\
\leq kTG_0(u) + \int_0^T \int_{\partial\mathscr{E}} \alpha(t) d\sigma dt
\end{aligned}$$

*for $k>0$ independent of $T$, where*

$$\alpha(t) = \tfrac{1}{2}t\ell_n w_n^2 + \tfrac{1}{2}t\ell_n(w_t^2 - |\nabla_{\tan} w|^2) + (n-1)tw_n w + \rho(r^2+t^2)w_n w_t \ ,$$

*and* $|\nabla w|^2 = w_n^2 + |\nabla_{\tan} w|^2$ *on* $\partial\mathscr{E}$.

*Proof.* We integrate (2.4) with $A = \rho(r^2 + t^2)$ over $\mathcal{O} \times (0, T)$ to obtain

$$(2.5) \qquad 0 = \int_{\mathcal{O}} F(w)dx\Big|_0^T + \int_0^T\int_{\partial\mathscr{E}} (G_j(w)\cdot n_j)d\sigma dt + \int_0^T\int_{\mathcal{O}} H(w)dxdt \ .$$

We note the following estimates:

$$(2.6) \qquad H(w) \le kt(w_t^2 + |\nabla w|^2)$$

$$(2.7) \qquad \begin{aligned}\int_{\mathcal{O}} |w_t w|\, dx &\le k\Big(\int_{\mathcal{O}} w_t^2 dx + \int_{\mathcal{O}} w^2 dx\Big)\\ &\le k_1\Big(\int u_t^2 dx + \int r^{-2}u^2 dx\Big) \le k_2 G_0(u)\end{aligned}$$

$$(2.8) \qquad \int_{\mathcal{O}} w^2 dx \le kG_0(u) \ .$$

Making use of these estimates, we see from (2.5) and the expression of $F(w)$ that

$$(2.9) \qquad \begin{aligned}&\frac{1}{2}T^2\int_{\mathcal{O}} (w_t(T)^2 + |\nabla w(T)|^2)dx\\ &\qquad \le kTG_0(u) - \int_0^T\int_{\partial\mathscr{E}} (G_j(w)\cdot n_j)d\delta dt \ .\end{aligned}$$

On the other hand, by (2.1), we have

$$(2.10) \qquad \begin{aligned}-(G_j(w)\cdot n_j) &= \frac{1}{2}t\ell_n w_n^2 + \frac{1}{2}t\ell_n(w_t^2 - |\nabla_{\tan} w|^2)\\ &\quad + \rho(r^2+t^2)w_n w_t + (n-1)tw_n w \ .\end{aligned}$$

Combining (2.10) with (2.9), we obtain the desired estimate.

LEMMA 2.2. *Let $u$ be a $C^2$-solution of problem* (I) *with initial data of compact support and let $v = u|_{\mathscr{E}\times(0,T)}$. Assume that $\ell(x)$ satisfies* (2.1) $\sim$ (2.3). *Then, for fixed $h > 0$, there exists a constant $k = k(h)$ independent of $T$ such that*

$$\begin{aligned}&\frac{1}{8}T^2\int_{|x|\le h\cap\mathscr{E}} (v_t(T)^2 + |\nabla v(T)|^2)dx\\ &\qquad \le kTG_0(u) + \int_0^T\int_{\partial\mathscr{E}} \beta(t)d\sigma dt\end{aligned}$$

*for any $T > 0$ large enough, where*

$$\beta(t) = -\frac{1}{2} t\ell_n v_n^2 - \frac{1}{2} t\ell_n(v_t^2 - |\nabla_{\tan} v|^2) - (r^2 + t^2)v_n v_t - (n-1)tv_n v ,$$

*and the constant $k(h)$ may depend on the support of the initial data.*

*Proof.* First, we rewrite $F(v)$ and $G(v)$. To do so, we consider the following identity:

$$-\tfrac{1}{2}(n-1)(v^2)_t = -\tfrac{1}{4}(n-1)\nabla\cdot(r^{-2}((r^2+t^2)v^2)_t x) + \tfrac{1}{2}(n-1)((r^{-2}(r^2+t^2)((\nabla v\cdot x)v + \tfrac{1}{2}(n-2)v^2))_t) ,$$

$x = (x_1, \cdots, x_n)$ being a position vector. By use of this identity, we rewrite the last term of $F(v)$, $-\frac{1}{2}(n-1)v^2$, so that we have

$$(v_{tt} - v_{jj})(Av_t + t\ell_j v_j + (n-1)tv) = \tilde{F}_t(v) + \nabla\cdot\tilde{G}(v) + \tilde{H}(v) \tag{2.11}$$

with $A(x,t) = (r^2 + t^2)$, where

$$\begin{aligned}\tilde{F}(v) &= \tfrac{1}{2}(r^2+t^2)(v_t^2 + |\nabla v|^2) + t\ell_j v_j v_t + (n-1)tv_t v \\ &\quad + \tfrac{1}{2}(n-1)(r^{-2}(r^2+t^2)((\nabla v\cdot x)v + \tfrac{1}{2}(n-2)v^2)) \\ \tilde{G}_j(v) &= -v_j((r^2+t^2)v_t + t\ell_j v_j + (n-1)tv) + \tfrac{1}{2}t\ell_j(|\nabla v|^2 - v_t^2) \\ &\quad - \tfrac{1}{4}(n-1)r^{-2}((r^2+t^2)v^2)_t x_j . \\ \tilde{H}(v) &= H(v) .\end{aligned}$$

We integrate (2.11) over $\mathscr{E} \times (0, T)$ to obtain

$$0 = \int_{\mathscr{E}} \tilde{F}(v)dx\Big|_0^T - \int_0^T\int_{\partial\mathscr{E}} (\tilde{G}_j(v)\cdot n_j)d\sigma dt + \int_0^T\int_{\mathscr{E}} \tilde{H}(v)dxdt . \tag{2.12}$$

Now, by (2.3), we have in $|x| \geq r_1$,

$$\ell_{jk}v_j v_k = 2|\nabla v|^2$$
$$\ell_{jj} = 2n ,$$

so that

$$\tilde{H}(v) = 0 , \qquad \text{in } |x| \geq r_1 .$$

Hence, we have in $\mathscr{E}$

$$\tilde{H}(v) \leq kt(1+r)^{-1-\delta}(v_t^2 + |\nabla v|^2)$$

for $k > 0$ independent of $t$, so that by Theorem 1,

$$\int_0^T \int_{\mathscr{E}} \tilde{H}(v)dxdt \leq kTG_0(u) \tag{2.13}$$

with $k > 0$ independent of $T$. Clearly,

$$\int_{\mathscr{E}} |\tilde{F}(v)|\, dx \bigg|_0 \leq kG_0(u) \ , \tag{2.14}$$

for $k > 0$ depending only on the support of the initial data, where we have used that $\int r^{-2}u^2 dx \leq k \int |\nabla u|^2\, dx$ for $n \geq 3$. On the other hand, $\tilde{F}(v)|_T$ can be rewritten as follows:

$$\tilde{F}(v)|_T = K_1(v, T) + K_2(v, T) \ ,$$

where

$$\begin{aligned} K_1(v, T) = {} & \tfrac{1}{2}(r^2 + T^2)(|\nabla v|^2 - v_r^2) \\ & + \tfrac{1}{4}r^{-2m}((r + T)^2((r^m v)_r + (r^m v)_t)^2 \\ & + (r - T)^2((r^m v)_r - (r^m v)_t)^2) \\ & + (\tfrac{1}{4}(n-1)(n-2) - \tfrac{1}{8}(n-1)^2)r^{-2}(r^2 + T^2)v^2 \ , \\ & \qquad m = (n-1)/2 \ , \\ K_2(v, T) = {} & (\ell_j v_j - 2rv_r)Tv_t \ . \end{aligned}$$

Note that for $n \geq 3$, $\frac{1}{4}(n-1)(n-2) - \frac{1}{8}(n-1)^2 \geq 0$ and that $K_1(v, T) \geq 0$. By (2.3),

$$\ell_j v_j = 2rv_r \ , \qquad \text{in } |x| \geq r_1 \ ,$$

so that

$$K_2(v, T) = 0 \ , \qquad \text{in } |x| \geq r_1 \ .$$

Hence, we have

$$\int_{\mathscr{E}} |K_2(v, T)|\, dx \leq kTG_0(u) \tag{2.15}$$

for $k > 0$ independent of $T$. Moreover, when $|x| \leq h$, $h < \frac{1}{2}T$,

$$\begin{aligned} K_1(v, T) \geq {} & \tfrac{1}{2}T^2(|\nabla v|^2 - v_r^2) \\ & + \tfrac{1}{8}r^{-2m}T^2((r^m v)_r^2 + (r^m v)_t^2) \\ & + \tfrac{1}{8}(n-1)(n-3)r^{-2}(r^2 + T^2)v^2 \\ \geq {} & \tfrac{1}{8}T^2(|\nabla v|^2 + v_t^2 + \tfrac{1}{2}(n-1)\nabla \cdot (r^{-2}v^2 x) \\ & - \tfrac{1}{4}(n-1)(n-3)r^{-2}v^2) + \tfrac{1}{8}(n-1)(n-3)r^{-2}(r^2 + T^2)v^2 \end{aligned} \tag{2.16}$$

$$\geq \tfrac{1}{8}T^2(|\nabla v|^2 + v_t^2 + \tfrac{1}{2}(n-1)\nabla\cdot(r^{-2}v^2x)) .$$

With the above estimates (2.13) ~ (2.16), we have from (2.12)

$$\begin{aligned}(2.17)\qquad &\frac{1}{8}T^2\int_{|x|\leq h\cap\mathscr{E}}\Big(|\nabla v|^2 + v_t^2 + \frac{1}{2}(n-1)\nabla\cdot(r^{-2}v^2x)\Big)dx \\ &\leq kTG_0(u) + \int_0^T\int_{\partial\mathscr{E}}(\tilde{G}_j(v)\cdot n_j)d\sigma dt .\end{aligned}$$

Recalling the expression of $\tilde{G}_j(v)$ and writing $|\nabla v|^2 = v_n^2 + |\nabla v_{\tan} v|^2$ on $\partial\mathscr{E}$, we have by (2.1)

$$(\tilde{G}_j(v)\cdot n_j) = \beta(t) - \tfrac{1}{4}(n-1)r^{-2}((r^2+t^2)v^2)_t(x_j\cdot n_j) ,$$

where $\beta(t)$ is the function defined in this lemma. Hence,

$$\begin{aligned}\int_0^T\int_{\partial\mathscr{E}}(\tilde{G}_j(v)\cdot n_j)d\sigma dt &= \int_0^T\int_{\partial\mathscr{E}}\beta(t)d\sigma dt \\ &\quad - \frac{1}{4}(n-1)\int_{\partial\mathscr{E}} r^{-2}(r^2+t^2)v^2(x_j\cdot n_j)d\sigma\Big|_0^T .\end{aligned}$$

Since

$$\int_{|x|\leq h\cap\mathscr{E}}\nabla\cdot(r^{-2}v^2x)dx = \int_{|x|=h} r^{-1}v^2 d\sigma - \int_{\partial\mathscr{E}} r^{-2}v^2(x_j\cdot n_j)d\sigma .$$

it follows from (2.17) that

$$\begin{aligned}(2.18)\qquad &\frac{1}{8}T^2\int_{|x|\leq h\cap\mathscr{E}}(|\nabla v|^2 + v_t^2)dx \\ &\leq kTG_0(u) + \int_0^T\int_{\partial\mathscr{E}}\beta(t)d\sigma dt + L(v) ,\end{aligned}$$

where

$$\begin{aligned}L(v) &= \frac{1}{16}(n-1)T^2\int_{\partial\mathscr{E}} r^{-2}v^2(x_j\cdot n_j)d\sigma\Big|_T \\ &\quad - \frac{1}{4}(n-1)\int_{\partial\mathscr{E}} r^{-2}(r^2+t^2)v^2(x_j\cdot n_j)d\sigma\Big|_0^T .\end{aligned}$$

Since $(x_j\cdot n_j)\geq 0$ on $\partial\mathscr{E}$ because of the convexity of $\mathscr{O}$,

$$L(v) \leq k\int_{\partial\mathscr{E}} v^2(x_j\cdot n_j)d\sigma\Big|_0 \leq kG_0(u) .$$

This, together with (2.18), completes the proof.

Combining Lemmas 2.1 and 2.2, we have the following theorem.

THEOREM 2. *Suppose that Assumption* (A) *is satisfied. Let $u$ be the $C^2$-solution of problem* (I) *with the initial data $f$ and $g$ such that the support of $f$ and $g$ is contained in $|x| \leq \gamma$. Then, there exists a constant $k = k(h, \gamma)$ independent of $T$ such that*

$$E(u; h, T) \leq kT^{-1}G_0(u) . \tag{2.19}$$

*Remark.* This result is valid for weak solutions, since a weak solution is obtained as a limit of $C^2$-solutions in the energy norm.

*Proof.* We add up the two inequalities obtained in Lemmas 2.1 and 2.2. Then, we have

$$\begin{aligned}\frac{1}{8}T^2 \int_{|x|\leq h} &(|\nabla u(T)|^2 + u_t(T)^2)dx \\ &\leq kTG_0(u) + \int_0^T \int_{\partial\mathscr{E}} (\alpha(t) + \beta(t))d\sigma dt ,\end{aligned}$$

$\alpha(t)$ and $\beta(t)$ being the functions defined in Lemmas 2.1 and 2.2, respectively. Recall the relations (0.1) and (0.2). Then, we have

$$\alpha(t) + \beta(t) = \tfrac{1}{2}(1 - \rho^2)t\ell_n w_n^2 + (n - 1)(1 - \rho)tw_n w .$$

Since $\rho > 1$ and $\ell_n > \beta > 0$ on $\partial\mathscr{E}$ by (2.2), it follows that

$$\alpha(t) + \beta(t) \leq ktw^2 , \qquad \text{on } \partial\mathscr{E} ,$$

for $k > 0$ independent of $t$. Moreover, we have by Theorem 1,

$$\int_0^T \int_{\partial\mathscr{E}} w^2 d\sigma dt \leq k \int_0^T \int_{\mathscr{E}} (|\nabla w|^2 + w^2)dxdt \leq kG_0(u) .$$

This completes the proof.

## § 3. Exponential decay of the local energy

In this section, we shall prove the exponential decay of the local energy when $n$ is odd, using Theorem 2 and following the procedure of Morawetz [4].

We recall the definition of $E(u; h, t)$:

$$E(u; h, t) = \int_{|x|\leq h} (u_t(t)^2 + |\nabla u(t)|^2)dx ,$$

and introduce the new notation:

(3.1) $$G(u; h, t) = \int_{|x| \leq h} \rho(x)(u_t(t)^2 + |\nabla u(t)|^2)dx .$$

Since $\rho > 1$, we have

(3.2) $$E(u; h, t) \leq G(u; h, t) \leq \rho E(u; h, t) .$$

In this section, by a solution we mean a weak solution. As was stated in Introduction, $G(u; \infty, t)$ $(= G_0(u))$ is conserved in $t$ for the solution $u$ of problem (I). For later use, we rewrite (2.19) as follows:

(3.3) $$E(u; h, T) \leq p(T, h, \gamma)E(u; \infty, 0)$$

with $p(T, h, \gamma) = \rho k(h, \gamma)T^{-1}$, $k(h, \gamma)$ being the constant in Theorem 2. By Remark after Theorem 2, (3.3) is valid for weak solutions.

LEMMA 3.1. *Let $u$ be the solution of problem* (I) *with the initial data $f$ and $g$ such that $f \in H^1(R^n)$ and $g \in L^2(R^n)$ and that the support of $f$ and $g$ is contained in $|x| < \gamma$.* ($\gamma > \frac{1}{2}$, $\mathcal{O} \subset |x| < \gamma$ *by* (0.3)). *Then, the solution $u$ may be written as*

$$u = R_0 + F_0 ,$$

*where $F_0$ is the free space solution with the same initial data as $u$. Furthermore,*

$$F_0 = 0 \qquad \text{for } r = |x| \leq t - \gamma .$$

*$R_0$ has compact support of at most $3\gamma$ at $t = 2\gamma$, and is a solution of problem* (I) *for $t > 2\gamma$. We have*

$$E(R_0; \infty, s) \leq 4G_0(u) , \qquad s \geq 0 .$$

*Proof.* It is clear that $F_0 = 0$ for $r \leq t - \gamma$ by Huyghen's principle. Hence, for $t \geq 2\gamma$, $F_0 = 0$ in $|x| \leq \gamma$, so that $F_0$ is a solution of problem (I) for $t > 2\gamma$. Since $u$ is a solution of problem (I), $R_0$ is also a solution for $t > 2\gamma$. We easily see that $R_0$ has compact support of at most $3\gamma$ at $t = 2\gamma$ by the dependence of domain. Moreover, we have for $s \geq 0$,

$$E(R_0; \infty, s) = E(u - F_0; \infty, s) \leq 2(E(u; \infty, s) + E(F_0; \infty, s)) .$$

Using (3.2) and the fact that $F_0$ is the free space solution with the same initial data as $u$, we conclude that

$$\begin{aligned} E(R_0; \infty, s) &\leq 2(G(u; \infty, s) + E(F_0; \infty, 0)) \\ &\leq 2(G_0(u) + G(F_0; \infty, 0)) = 4G_0(u) . \end{aligned}$$

LEMMA 3.2 (Morawetz [4], Lemma 2). *For* $T > 4\gamma, R_0 = R_1 + F_1$. *Here* $F_1$ *is the free space solution with the same initial data as* $R_0$ *at* $t = T$, *and*

$$F_1 = 0 \qquad \text{for } r < t - T - \gamma ,$$

*while* $R_1$ *is a solution of problem* (I) *for* $t > T + 2\gamma$ *and has compact support of at most* $3\gamma$ *at* $t = T + 2\gamma$. *Furthermore,*

$$E(R_1; \infty, T + 2\gamma) \leq kE(R_0; 5\gamma, T)$$

*with* $k = 2(\rho + 1)$.

*Proof.* We continue $F_1$ as $F_1 = R_0$ for $t < T$. Then, $\square F_1 = 0$ in the domain exterior to $|x| \leq \gamma \times (0, T)$. We apply Huyghen's principle to $F_1$ in this domain. Let $(x, t)$ be a point with $|x| < t - T - \gamma$. Then, the backward cone with vertex at $(x, t)$ does not intersect $|x| = \gamma \times (0, T)$, and intersect the plane $t = 2\gamma$ outside the sphere $|x| \leq 3\gamma$ where the support of $R_0$ is contained in virtue of Lemma 3.1. Thus we conclude that $F_1 = 0$ for $|x| < t - T - \gamma$. Consequently, when $t > T + 2\gamma$, $F_1$ is a solution of problem (I). By Lemma 3.1, $R_0$ is a solution of problem (I) for $t > 2\gamma$. Hence, $R_1$ is also a solution for $t > T + 2\gamma$, and the fact that $R_1$ has compact support of at most $3\gamma$ at $t = T + 2\gamma$ is easily obtained by the dependence of domain, since $\square R_1 = 0$ in $|x| > \gamma \times (T, \infty)$ and $R_1 = 0$ at $t = T$. Therefore, we have

$$\begin{aligned} E(R_1; \infty, T + 2\gamma) &= E(R_1; 3\gamma, T + 2\gamma) \\ &\leq 2(E(R_0; 3\gamma, T + 2\gamma) + E(F_1; 3\gamma, T + 2\gamma)) \\ &\leq 2(G(R_0; 3\gamma, T + 2\gamma) + E(F_1; 3\gamma, T + 2\gamma)) . \end{aligned}$$

On the other hand, making use of the fact that $R_0$ and $F_1$ are solutions of problem (I) and of the free space wave equation with the same initial data as $R_0$ at $t = T$, respectively, we can obtain by the standard method of energy estimate that

$$\begin{aligned} G(R_0; 3\gamma, T + 2\gamma) &\leq G(R_0; 5\gamma, T) , \\ E(F_1; 3\gamma, T + 2\gamma) &\leq E(R_0; 5\gamma, T) . \end{aligned}$$

Thus we conclude that

$$E(R_1; \infty, T + 2\gamma) \leq 2(\rho + 1)E(R_0; 5\gamma; T) .$$

This completes the proof.

THEOREM 3. *Suppose that Assumption* (A) *is satisfied. Let $u$ be the solution of problem* (I) *with the initial data $f$ and $g$ such that $f \in H^1(R^n)$ and $g \in L^2(R^n)$ and that the support of $f$ and $g$ is contained in $|x| < \gamma$. Let $\gamma_0 > \gamma$. Then, there exist constants $k = k(\gamma_0, \gamma)$ and $\theta = \theta(\gamma_0, \gamma)$ such that*

$$E(u; \gamma_0, T) \leq ke^{-\theta T} G_0(u) .$$

*Proof.* In Lemma 3.2, we decomposed $R_0$ into $R_0 = R_1 + F_1$. We apply the same procedure to $R_1$. We define $F_2$ as follows: $F_2 = R_1$ for $T < t \leq 2T$ and $F_2$ is continued for $t > 2T$ as the solution of the free space wave equation with the initial data $F_2(2T) = R_1(2T)$ and $F_{2t}(2T) = R_{1t}(2T)$. Exactly in the same way as in the proof of Lemma 3.2, we see that

$$F_2 = 0 , \qquad \text{for } |x| < t - 2T - \gamma .$$

We set $R_2 = R_1 - F_2$. Then, it follows from the above fact that $R_2$ is a solution of problem (I) for $t > 2T + 2\gamma$. Furthermore, $R_2$ has compact support of at most $3\gamma$ at $t = 2T + 2\gamma$, and

$$E(R_2; \infty, 2T + 2\gamma) \leq kE(R_1; 5\gamma, 2T)$$

with $k = 2(\rho + 1)$. We repeat this procedure. Then, for $t > nT$,

$$u = \sum_{j=0}^{n} F_j + R_n ,$$

where

$$F_j = 0 \qquad \text{for } |x| < t - jT - \gamma , \tag{3.4}$$

and

$$R_n \text{ is a solution of problem (I) for } t > nT + 2\gamma . \tag{3.5}$$

Let $\gamma_0 > \gamma$ and let $t > nT + \gamma + \gamma_0 > nT + 2\gamma$. Then, in view of (3.4), $u = R_n$ in $|x| < \gamma_0$, so that by (3.5) and (3.2),

$$\begin{aligned} E(u; \gamma_0, t) &= E(R_n; \gamma_0, t) \leq G(R_n; \gamma_0, t) \leq G(R_n; \infty, t) \\ &= G(R_n; \infty, nT + 2\gamma) \leq \rho E(R_n; \infty, nT + 2\gamma) . \end{aligned}$$

Moreover, by Lemma 3.2, it follows that

$$E(u; \gamma_0, t) \leq \rho E(R_n; \infty, nT + 2\gamma) \leq \rho k E(R_{n-1}; 5\gamma, nT)$$

for $k = 2(\rho + 1)$. Note that $R_{n-1}$ is a solution of problem (I) for $t > (n-1)T + 2\gamma$ and that $R_{n-1}$ has compact support of at most $3\gamma$ at $t = (n-1)T + 2\gamma$. Hence, we can apply (3.3) to $E(R_{n-1}; 5\gamma, nT)$ to obtain

$$E(R_{n-1}; 5\gamma, nT) \leq \rho k p(T, \gamma) E(R_{n-1}; \infty, (n-1)T + 2\gamma)$$

with $p(T, \gamma) = \rho k(5\gamma, 3\gamma)(T - 2\gamma)^{-1}$. Repeating this procedure and using Lemma 3.1, we conclude that

$$\begin{aligned} E(u; \gamma_0, t) &\leq \rho \exp\{n \log kp(T, \gamma)\} E(R_0; \infty, 2\gamma) \\ &\leq 4\rho \exp\{n \log kp(T, \gamma)\} G_0(u) . \end{aligned}$$

Here, we take $T$ so large that

$$\log kp(T, \gamma) = -\theta T$$

with $\theta > 0$. This is possible since $p(T, \gamma) \to 0$ as $T \to \infty$. Therefore,

$$E(u; \gamma_0, t) \leq 4\rho e^{-\theta n T} G_0(u) .$$

Thus, if for given $t > 0$ large enough, we choose the maximal integer $n$ such that $t \geq nT + \gamma + \gamma_0$, then $n \geq (t - \gamma - \gamma_0)T^{-1} - 1$. Hence, we obtain

$$E(u; \gamma_0, t) \leq k_1 e^{-\theta t} G_0(u)$$

with $k_1 = 4\rho \exp \theta(\gamma + \gamma_0 + T)$. This completes the proof.

Finally we note the following fact: The method presented here can be applied to a slightly more general problem of the following form:

$$\frac{\partial^2}{\partial t^2} u - \frac{1}{a(x)} \nabla \cdot \rho(x) \nabla u = 0 ,$$

where

$$\rho(x) = \begin{cases} \rho > 1 & \text{in } \mathcal{O} \\ 1 & \text{in } \mathcal{E} \end{cases} \quad \text{and} \quad a(x) = \begin{cases} a & \text{in } \mathcal{O} \\ 1 & \text{in } \mathcal{E} \end{cases} ,$$

and $\mathcal{O}$ satisfies Assumption (A). Then, if $a \leq \rho$, we can obtain the same result as Main Theorem.

## Appendix

We shall prove Lemma 1.5.

Let $s(t)$ be a $C^\infty$-function such that $s(t)=0$ for $0\le t\le t_0-1$ and and $s(t)=1$ for $t\ge t_0$, $t_0>1$. We put $\tilde{u}(x,t)=s(t)u(x,t)$. Then, $\tilde{u}(x,t)$ satisfies the following equation:

$$\tilde{u}_{tt}-\frac{1}{\rho(x)}\nabla\cdot\rho(x)\nabla\tilde{u}=p(x,t)$$

with the initial condition

$$\tilde{u}(x,0)=0 \quad\text{and}\quad \tilde{u}_t(x,0)=0\ ,$$

where $p(x,t)=2s_tu_t+s_{tt}u$. Using the conservation of energy for $u$ and the fact that the support of $\tilde{u}$ is bounded for $0\le t\le t_0$, we see that

$$\int_0^{t_0}\int u^2dxdt\le kt_0^3G_0(u)=k_1G_0(u)\ . \tag{4.1}$$

Hence, in order to prove Lemma 1.5, it is sufficient to show that

$$\int_0^\infty\int_{|x|\le R}e^{-2\varepsilon t}\tilde{u}^2dxdt\le k(\eta)G_0(u)+\eta\int_0^\infty\int e^{-2\varepsilon t}(1+r)^{-1-\delta}\tilde{u}_t^2dxdt\ , \tag{4.2}$$

where $G_0(u)$ is the total energy.

Now, we put $\tilde{v}=\tilde{u}|_{\mathscr{E}\times(0,\infty)}$ and $\tilde{w}=\tilde{u}|_{\mathscr{O}\times(0,\infty)}$ and define $\tilde{U}(x,\omega)$, $\tilde{V}(x,\omega)$ and $\tilde{W}(x,\omega)$ for $\omega=\mu+i\kappa$, $\kappa>0$, as follows:

$$\begin{cases}\tilde{U}(x,\omega)=\displaystyle\int_0^\infty e^{i\omega t}\tilde{u}(x,t)dt\\ \tilde{V}(x,\omega)=\displaystyle\int_0^\infty e^{i\omega t}\tilde{v}(x,t)dt\\ \tilde{W}(x,\omega)=\displaystyle\int_0^\infty e^{i\omega t}\tilde{w}(x,t)dt\ .\end{cases} \tag{4.3}$$

Then, $\tilde{V}(x,\omega)$ and $\tilde{W}(x,\omega)$ satisfy the following equation:

$$\begin{aligned}-\Delta\tilde{V}-\omega^2\tilde{V}&=P\ , &&\text{in } \mathscr{E}\ ,\\ -\Delta\tilde{W}-\omega^2\tilde{W}&=P\ , &&\text{in } \mathscr{O}\ ,\end{aligned} \tag{4.4}$$

$$\tilde{V}=\tilde{W}\qquad \text{on } \partial\mathscr{E} \tag{4.5}$$

$$\tilde{V}_n=\rho\tilde{W}_n\ ,\qquad \text{on } \partial\mathscr{E}\ , \tag{4.6}$$

where

$$P(x,\omega)=\int_0^\infty e^{i\omega t}p(x,t)dt\ .$$

Note that $p(x,t)=0$ for $t\geq t_0$ and that $p(x,t)$ is of compact support in $x$ for $0\leq t\leq t_0$, so that $P(x,\omega)$ is also of compact support uniformly in $\omega$. We can prove that if $\operatorname{Im}\omega>0$ and $P\in L^2(R^n)$, problem (4.4) $\sim$ (4.6) has a unique solution $U$ such that $U\in H^1(R^n)$, $V=U|_{\mathscr{E}}\in H^2(\mathscr{E})$ and $W=U|_{\mathscr{O}}\in H^2(\mathscr{O})$ and that (4.5) and (4.6) are satisfied in $H^{3/2}(\partial\mathscr{E})$ and $H^{1/2}(\partial\mathscr{E})$, respectively.

Before proving (4.2), we introduce the functional space $H^2(\mathscr{O},R)$: Let $B_R=\{x\,|\,|x|\leq R\}$.

$$H^2(\mathscr{O},R)=\{U\in H^1(B_R)\,|\,U|_{\mathscr{O}}\in H^2(\mathscr{O}),\,U|_{B_R\cap\mathscr{E}}\in H^2(B_R\cap\mathscr{E})\ .$$

The norm in $H^2(\mathscr{O},R)$ is given by

$$|||U|||_{2,R}^2=\|U\|_{1,R}^2+\|W\|_{2,\mathscr{O}}^2+\|V\|_{2,B_R\cap\mathscr{E}}^2\ ,$$

where $W=U|_{\mathscr{O}}$, $V=U|_{B_R\cap\mathscr{E}}$ and $\|\cdot\|_{1,R}$, $\|\cdot\|_{2,\mathscr{O}}$, and $\|\cdot\|_{2,B_R\cap\mathscr{E}}$ are the norms in the Sobolev spaces $H^1(B_R)$, $H^2(\mathscr{O})$ and $H^2(B_R\cap\mathscr{E})$, respectively.

With the above notation, we state the following lemma from which (4.2) follows.

LEMMA A.1. *Let* $\operatorname{Im}\omega>0$, $\omega=\mu+i\kappa$, *and let* $G(x)$ *be a function with compact support. Let* $U(x,\omega)$ *be the solution of problem* (4.4) $\sim$ (4.6) *with* $P=G$. *Then, we have the following statement*:
(i) $n$; *odd. Let* $|\mu|\leq\Lambda$ *and* $0<\kappa\leq 1$. *Then, there exists a constant* $k=k(\Lambda,R)$ *such that*

$$|||U|||_{2,R}\leq k\,\|G\|_0\ .$$

(ii) $n$; *even. Let* $\mu_1>0$. *Let* $\mu_1\leq|\mu|\leq\Lambda$ *and* $0<\lambda\leq 1$. *Then, there exists a constant* $k_1=k_1(\Lambda,\mu_1,R)$ *such that*

$$|||U|||_{2,R}\leq k_1\,\|G\|_0\ .$$

*Here* $\|\cdot\|_0$ *is the norm in* $L^2(R^n)$ *and the constants* $k$ *and* $k_1$ *may depend on the support of* $G$.

The proof of this lemma is rather long and is done in the same way as in the proof of Lemma 4.6, Wilcox [8], pp. 65, and so we omit it.

We shall proceed to the proof of Lemma 1.5.

*Proof of Lemma* 1.5. As is stated above, it is sufficient to prove (4.2). Using the Schwarz inequality and the fact that $p(x,t)=0$ for

$t \geq t_0$, we have

$$\int |P(x, \omega)|^2 \, dx \leq t_0 \int_0^{t_0} \int |p(x, t)|^2 \, dxdt \,.$$

Moreover, since $|p|^2 \leq k(u_t^2 + u^2)$, it follows from (4.1) that

$$\int |P(x, \omega)|^2 \, dx \leq kG_0(u) \tag{4.7}$$

for $k = k(t_0)$ independent of $\omega$. First suppose that $n$ is odd. Let $\tilde{U}(x, \omega)$ be the function defined by (4.3). Then, we have by Lemma A.1 and (4.7),

$$\begin{aligned} \int_0^\infty \int_{|x| \leq R} e^{-2\varepsilon t} |\tilde{u}(x, t)|^2 \, dxdt &= \int_{-\infty}^\infty \int_{|x| \leq R} |\tilde{U}(x, \mu + i\varepsilon)|^2 \, dxd\mu \\ &\leq \int_{-\Lambda}^{\Lambda} \int_{|x| \leq R} |\tilde{U}(x, \mu + i\varepsilon)|^2 \, dxd\mu \\ &\quad + \Lambda^{-2} \int_{-\infty}^\infty \int_{|x| \leq R} |\mu \tilde{U}(x, \mu + i\varepsilon)|^2 \, dxd\mu \\ &\leq k(\Lambda) G_0(u) \\ &\quad + k(R) \Lambda^{-2} \int_0^\infty \int e^{-2\varepsilon t} (1 + r)^{-1-\delta} \tilde{u}_t^2 dxdt \,. \end{aligned}$$

Hence, if we take $\Lambda$ sufficiently large, we obtain the desired result.

Next, we consider the even-dimensional case which is more complicated. Let $\delta < \delta' < \frac{3}{5}$ and let $\sigma = 1 + \delta'$. We choose $b$, $1 < b < 2$, so that $q(b) = (b^2 - b)(-\frac{1}{2}b^2 + b + \frac{1}{2})^{-1} = \sigma$. In fact, such a $b$ exists since $q(1) = 0$ and $q(2) = 4$. We set $C_0 = b^\sigma(-\frac{1}{2}b^2 + b + \frac{1}{2})$ for $b$ defined above and introduce the following function:

$$\varphi(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq 1 \\ -\frac{1}{2}r^2 + r + \frac{1}{2} & \text{for } 1 < r \leq b \\ C_0 r^{-\sigma} & \text{for } r > b \,, \; r = |x| \,. \end{cases} \tag{4.8}$$

By the definition of $b$ and $C_0$, we see that $\varphi(r)$ is a $C^1$-function and piecewise $C^2$-function and that $\Delta\varphi(r) \leq 0$.

Now, let $\tilde{U}(x, \omega)$, $\tilde{V}(x, \omega)$ and $\tilde{W}(x, \omega)$ be the functions defined by (4.3). We multiply the equation $-\Delta\tilde{V} - \omega^2\tilde{V} = P$ by $\varphi(r)\bar{\tilde{V}}$, integrate over $\mathscr{E}$ and take the real parts. Then, using the fact that $\varphi = 1$ and $\varphi_n = 0$ on $\partial\mathscr{E}$ by (0.3), we have

$$\operatorname{Re}\int_{\partial\mathscr{E}}\tilde{V}_n\bar{\tilde{V}}d\sigma+\int_{\mathscr{E}}\varphi(r)\,|\nabla\tilde{V}|^2\,dx-\frac{1}{2}\int_{\mathscr{E}}\Delta\varphi\,|\tilde{V}|^2\,dx$$
$$=\operatorname{Re}\omega^2\int_{\mathscr{E}}\varphi(r)\,|\tilde{V}|^2\,dx+\operatorname{Re}\int_{\mathscr{E}}P\varphi\bar{\tilde{V}}dx\,.$$

Similarly multiplying the equation $-\Delta\tilde{W}-\omega^2\tilde{W}=P$ by $\rho\bar{\tilde{W}}$, we obtain

$$-\operatorname{Re}\int_{\partial\mathscr{E}}\rho W_n\bar{\tilde{W}}d\sigma+\int_{\mathscr{O}}\rho\,|\nabla\tilde{W}|^2\,dx$$
$$=\operatorname{Re}\omega^2\int_{\mathscr{O}}\rho\,|\tilde{W}|^2\,dx+\operatorname{Re}\int_{\mathscr{O}}\rho P\bar{\tilde{W}}dx\,.$$

Taking account of relations (4.5) and (4.6), and adding up these two equalities, we have

$$\begin{aligned}k\int(1+r)^{-\sigma}\,|\nabla\tilde{U}|^2\,dx&\le\operatorname{Re}\omega^2\Big(\int_{\mathscr{E}}\varphi(r)\,|\tilde{V}|^2\,dx+\int_{\mathscr{O}}\rho\,|\tilde{W}|^2\,dx\Big)\\&\quad+\operatorname{Re}\int_{\mathscr{E}}P\varphi\bar{\tilde{V}}dx+\operatorname{Re}\int_{\mathscr{O}}\rho P\bar{\tilde{W}}dx\,,\end{aligned}\tag{4.9}$$

where we have used that $\Delta\varphi\le 0$ and that $\varphi(r)\ge k(1+r)^{-\sigma}$. We claim that if $n\ge 4$ and $1<\sigma<\frac{8}{5}$,

$$\int(1+r)^{-2-\sigma}\,|\tilde{U}|^2\,dx\le k\int(1+r)^{-\sigma}\,|\nabla\tilde{U}|^2\,dx\,.\tag{4.10}$$

This assertion will be proved later. The third and fourth terms on the right side of (4.9) are estimated as

$$\eta\int(1+r)^{-2-\sigma}\,|\tilde{U}|^2\,dx+k(\eta)\int|P|^2\,dx$$

for any $\eta>0$ small enough, where we have used the fact that $P$ is of compact support uniformly in $\omega$. Hence, in view of (4.9) and (4.10), it follows that there exist constants $k_1$ and $k_2$ such that

$$\int(1+r)^{-2-\sigma}\,|\tilde{U}|^2\,dx\le k_1\mu^2\int(1+r)^{-\sigma}\,|\tilde{U}|^2\,dx+k_2\int|P|^2\,dx\,,$$

since $\operatorname{Re}\omega^2=\mu^2-\kappa^2\le\mu^2$, $\omega=\mu+i\kappa$, and $\varphi\le k(1+r)^{-\sigma}$. We rewrite $k_1\mu^2\int(1+r)^{-\sigma}\,|\tilde{U}|^2\,dx$ as follows:

$$\begin{aligned}k_1\mu^2\int(1+r)^{-\sigma}\,|\tilde{U}|^2\,dx&=k_1\mu^2\int_{|x|\le M}(1+r)^{-2-\sigma}(1+r)^2\,|\tilde{U}|^2\,dx\\&\quad+k_1\mu^2\int_{|x|\ge M}(1+r)^{-1-\delta}(1+r)^{-\gamma'}\,|\tilde{U}|^2\,dx\,,\end{aligned}$$

where $\sigma = 1 + \delta'$ and $\eta' = \delta' - \delta > 0$. For $\eta > 0$ small enough, we first choose $M = M(\eta)$ so large that $k_1(1+r)^{-\eta'} \leq \eta$ for $|x| \geq M$, and next $\mu_0 = \mu_0(\eta)$ so small that $k_1\mu^2(1+r)^2 \leq \eta$ for $|x| \leq M$ and $|\mu| \leq \mu_0(\eta)$. Thus, we conclude that for any $\eta > 0$ small enough, there exist constants $k(\eta)$ and $\mu_0(\eta)$ such that

$$\int (1+r)^{-2-\sigma} |\tilde{U}|^2 \, dx \leq \eta\mu^2 \int (1+r)^{-1-\delta} |\tilde{U}|^2 \, dx + k(\eta) \int |P|^2 \, dx$$

for $|\mu| \leq \mu_0(\eta)$. Hence, for each fixed $R > 0$ and any $\eta > 0$ small enough, we have

$$\int_{|x| \leq R} |\tilde{U}|^2 \, dx \leq \eta\mu^2 \int (1+r)^{-1-\delta} |\tilde{U}|^2 \, dx + k(\eta, R) \int |P|^2 \, dx \tag{4.11}$$

for $|\mu| \leq \mu_0(\eta, R)$.

Now, we shall prove (4.2). As in the proof of the odd dimensional case, we have

$$\begin{aligned}\int_0^\infty \int_{|x| \leq R} e^{-2\varepsilon t} \tilde{u}^2 dx dt &= \int_{-\infty}^\infty \int_{|x| \leq R} |\tilde{U}(x, \mu + i\varepsilon|^2 \, dx d\mu \\ &= \int_{|\mu| \leq \mu_0(\eta)} dx d\mu + \int_{\mu_1 \leq |\mu| \leq \Lambda} dx d\mu + \int_{|\mu| \geq \Lambda} dx d\mu \\ &= I_1 + I_2 + I_3 .\end{aligned}$$

$I_1, I_2$ and $I_3$ are estimated as follows:

$$I_1 \leq \eta \int_{-\infty}^\infty \int (1+r)^{-1-\delta} |\mu \tilde{U}(x, \mu + i\varepsilon)|^2 \, dx d\mu + k_1(\eta) G_0(u)$$

by (4.11) and (4.7), if we take $\mu_0(\eta)$ sufficiently small for any $\eta > 0$.

$$I_2 \leq k_2(\eta, \Lambda) G_0(u)$$

by (ii) of Lemma A.1 and (4.7).

$$I_3 \leq k_3 \Lambda^{-2} \int_{-\infty}^\infty \int (1+r)^{-1-\delta} |\mu \tilde{U}(x, \mu + i\varepsilon)|^2 \, dx d\mu .$$

Here the constants $k_1, k_2$ and $k_3$ may depend on $R$. Thus, for any $\eta > 0$ small enough, we can choose $\Lambda$ so large that

$$\int_0^\infty \int_{|x| \leq R} e^{-2\varepsilon t} \tilde{u}^2 dx dt \leq k(\eta) G_0(u) + \eta \int_0^\infty \int e^{-2\varepsilon t} (1+r)^{-1-\delta} \tilde{u}_t^2 dx dt .$$

This proves (4.2). It remains to prove (4.10). We start with the following identity:

$$\int r^{2-n}|\nabla((1+r)^{-\sigma/2}r^{(n-2)/2}u)|^2\,dx = \int (1+r)^{-\sigma}|\nabla u|^2\,dx - C_1\int(1+r)^{-2-\sigma}|u|^2\,dx + 2C_2\int(1+r)^{-1-\sigma}r^{-1}|u|^2\,dx - C_3\int(1+r)^{-\sigma}r^{-2}|u|^2\,dx$$

for $u \in H^1(R^n)$, where

$$C_1 = \tfrac{1}{4}(\sigma^2 + 2\sigma)\,,\quad C_2 = \tfrac{1}{4}\sigma(n-1)\,,\quad \text{and}\quad C_3 = \tfrac{1}{4}(n-2)^2\,.$$

Furthermore, we have by the Schwarz inequality,

$$2C_2\int(1+r)^{-1-\sigma}r^{-1}|u|^2\,dx \le C_4\int(1+r)^{-2-\sigma}|u|^2\,dx + C_3\int(1+r)^{-\sigma}r^{-2}|u|^2\,dx$$

with $C_4 = \sigma^2(n-1)^2(2n-4)^{-2}$. Hence, if $\sigma < \frac{5}{8} \le 2(n-2)^2(2n-3)^{-1}$, then $C_1 - C_4 > 0$. This completes the proof.

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