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AN APPLICATION OF RITT'S LOW POWER THEOREM

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Abstract

Consider an algebraic differential equation $F = 0$ of the first order. A rigorous definition will be given to the classical concept of "particular solutions" of $F = 0$. By Ritt's low power theorem we shall prove that a singular solution of $F = 0$ belongs to the general solution of F if and only if it is a particular solution of $F = 0$.

§ 0. Introduction

Let $k\{y\}$ be the differential polynomial algebra in a single indeterminate y over an algebraically closed differential field k of characteristic zero, and F be an algebraically irreducible element of $k\{y\}$ of the first order. The totality Π of those elements A of $k\{y\}$ such that the remainder of A with respect to F is zero is an essential prime divisor of the perfect ideal $\{F\}$ in $k\{y\}$ generated by F. Let $\Pi, \Sigma_1, \cdots, \Sigma_s$ be the essential prime divisors of $\{F\}$. Then, each of the Σ_i contains the separant S of F (Cf. [5, pp. 30–32]). Take and fix a universal extension Ω of k, the existence of which was proved by Kolchin [3, p. 771]. The manifold of Π in Ω is called the general solution of F. A zero of F in Ω is called a singular solution of $F = 0$ if it is a zero of S. The manifold of Σ_i in Ω consists of a single point for each i (Cf. [5, p. 63]). A singular solution of $F = 0$ is an element of k, because it is either a zero of the discriminant of F with respect to y' or a zero of the initial of F.

Take a generic point w of the general solution of F. Then, w is transcendental over k. Hence, $k(w, w')$ is a one-dimensional algebraic function field over k, which will be denoted by K. We shall give a rigorous definition to the classical concept of "particular solutions" of $F = 0$ as follows (Cf. [1, p. 257]):

DEFINITION. A singular solution η of $F = 0$ will be called a partic-

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ular solution of $F = 0$ if there exists a prime divisor P of K such that

$$(1) \qquad \nu_P(w' - \eta') \geqq \nu_P(w - \eta) > 0 \,,$$

where ν_P is the normalized valuation belonging to P.

This definition is independent of the choice of a generic point w of the general solution of F.

By Ritt's low power theorem we shall prove the following:

THEOREM. *A singular solution η of $F = 0$ belongs to the general solution of F if and only if η is a particular solution of $F = 0$.*

§ 1. Proof of Theorem

Suppose that η is a singular solution of $F = 0$. Then, η is an element of k. Let G denote the polynomial in u, v obtained from F by the replacement of $y = u + \eta$, $y' = v + \eta'$. Suppose that

$$G = a_0(u)v^n + a_1(u)v^{n-1} + \cdots + a_n(u) \,,$$

where the a_i are elements of $k[u]$. Unless $a_i = 0$, we define s_i as the least exponent of u in a_i. If $a_i = 0$, we do not define s_i. For $i = n$, s_n can be defined, and $s_n > 0$. The following lemma is a corollary of Ritt's low power theorem (Cf. [5, p. 65]):

LEMMA. *The singular solution η belongs to the general solution of F if and only if we have the inequality*

$$(2) \qquad s_n \geqq s_i + n - i$$

for some i different from n $(0 \leqq i < n)$.

Let us make Puiseux diagram in G. Then, we have rational numbers $\mu_1, \cdots, \mu_m$ and subscripts $i_0, i_1, \cdots, i_m$ of the a such that they satisfy the following four conditions:

(i) $0 \leqq i_0 < i_1 < \cdots < i_m = n$;

(ii) $0 < \mu_1 < \cdots < \mu_m$;

(iii) for each j $(1 \leqq j \leqq m)$,

$$(3) \qquad s_p + \mu_j(n - p) = s_q + \mu_j(n - q) \,, \quad p = i_{j-1} \,, \quad q = i_j \,;$$

(iv) $s_i + \mu_j(n - i) \geqq \tau_j$

for all i, j $(0 \leqq i \leqq n, 1 \leqq j \leqq m)$, where τ_j is the number given by the equality (3).

Let P be a prime divisor of K such that

$$(4) \qquad \nu_P(w-\eta) > 0\ , \qquad \nu_P(w'-\eta') > 0\ .$$

Then, we have

$$(5) \qquad \nu_P(w'-\eta') = \mu_h \nu_P(w-\eta)$$

for some h. Conversely, for each h $(1 \leqq h \leqq m)$, there exists some prime divisor P of K which satisfies (4) and (5) (Cf. [2, Chap. 2], [4, Chap. 13]).

Because of (ii), there exists a prime divisor P of K satisfying (1) if and only if $\mu_m \geqq 1$. The inequality (2) holds for some i different from n $(0 \leqq i < n)$ if and only if $\mu_m \geqq 1$. Hence, we have our Theorem by Definition and Lemma.

§2. An example

Let k_0 be an algebraically closed field of characteristic zero, and $k_0(x)$ be the one-dimensional rational function field over k_0. We set $x' = 1$, and $a' = 0$ for all elements a of k_0. Suppose that k is the algebraic closure of $k_0(x)$, and that

$$F = x^2(y')^2 + (2x+y)yy' + y^2\ .$$

Then, the singular solutions of $F = 0$ are 0 and $-4x$. The former is a particular solution of $F = 0$, and the latter is not.

Let t denote $x + w/w'$. Then, t is a constant. We have $w = t^2(x-t)^{-1}$ and $w' = -t^2(x-t)^{-2}$. Hence, $k(w, w') = k(t)$ with $t' = 0$.

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