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ON THE DEFICIENCIES OF MEROMORPHIC MAPPINGS OF C^n INTO $P^N C$

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1. Introduction

Let $f(z)$ be a non-degenerate meromorphic mapping of the n-dimensional complex Euclidean space C^n into the N-dimensional complex projective space $P^N C$. A generalization of results of Edrei-Fuchs [2] for meromorphic mappings of C into $P^N C$ was given by Toda [5], and an estimate of $K(\lambda)$ for meromorphic mappings of C^n into $P^N C$ was done by Noguchi [4]. In this note we generalize several results of Edrei-Fuchs [2] in the case of meromorphic mappings of C^n into $P^N C$.

Let $(z_1, \cdots, z_n)$ be the natural coordinate system in C^n. We put

$$\|z\|^2 = \sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha\ , \quad B(r) = \{z \in C^n : \|z\| < r\}\ , \quad \partial B(r) = \{z \in C^n : \|z\| = r\}$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)\ , \quad \psi = dd^c \log \|z\|^2\ , \quad \psi_k = \underbrace{\psi \wedge \cdots \wedge \psi}_{k}\ ,$$

and

$$\sigma = d^c \log \|z\|^2 \wedge \psi_{n-1}\ .$$

We note that $\int_{\partial B(r)} \sigma = 1$ for any $r > 0$. (See Carlson-Griffiths [1], p. 562).

For a divisor D in C^n ($\not\ni 0$), we write

$$n(t, D) = \int_{D \cap B(t)} \psi_{n-1} \quad \text{and} \quad N(r, D) = \int_0^r \frac{n(t, D)}{t} dt\ .$$

Let F be a line bundle over $P^N C$ and let $\{U_j\}_{j=1}^m$ be an open covering of $P^N C$ such that the restrictions $F|_{U_j}$ are trivial. Then F is determined by the 1-cocycles $\{\theta_{jk}\}$ which are non-zero holomorphic functions on $U_j \cap U_k$ and satisfying $\theta_{jk}(w) = \theta_{j\ell}(w) \cdot \theta_{\ell k}(w)$ for $w \in U_j \cap U_k \cap U_\ell$.

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Let $\phi = \{\phi_j\} \in H^0(P^N C, \mathcal{O}(F))$ be a holomorphic section of F and $a = \{a_j(w)\}$ an Hermitian metric in F, that is, every $a_j(w)$ is a positive C^∞-function and $a_j(w) = |\theta_{jk}(w)|^2 a_k(w)$ on $U_j \cap U_k$. Since $\frac{|\phi_j(w)|^2}{a_j(w)} = \frac{|\phi_k(w)|^2}{a_k(w)}$ on $U_j \cap U_k$, we put $|\phi|^2(w) = \frac{|\phi_j(w)|^2}{a_j(w)}$ and call it the norm of ϕ. We put $\omega = \omega_F = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log a_j(w)$ which represents a Chern class $c(F)$ of F.

The quantity

$$T(r, f) = \int_0^r \frac{dt}{t} \int_{B(t)} f^*\omega \wedge \psi_{n-1}$$

is called the characteristic function of f, where $f^*\omega$ denotes the pull back of the form ω by f. Sometimes we write $T(r)$ instead of $T(r, f)$ for simplicity. We note that $T(r, f)$ is independent of a choice of the form ω_F of F up to an 0(1)-term. (See Griffiths-King [3], p. 182)

For a hyperplane H in $P^N C$, we choose always a global holomorphic section $\phi \in H^0(P^N C, \mathcal{O}(F))$ such that the divisor (ϕ) of ϕ is equal to H and $|\phi|^2 \leqslant 1$.

We put

$$m(r, H) = \int_{\partial B(r)} u_\phi(z)\sigma \qquad (\geqslant 0) ,$$

where $u_\phi(z) = \log \frac{1}{|\phi|^2 (f(z))}$. Then by Nevanlinna's first main theorem, we have

$$T(r, f) = N(r, f^*H) + m(r, H) - m(0, H) ,$$

provided that $f(0) \notin H$.

For a hyperplane H in $P^N C$, the quantity

$$\delta(H, f) = 1 - \limsup_{r \to \infty} \frac{N(r, f^*H)}{T(r, f)}$$

is called the deficiency of H. We define the order λ and the lower order μ of f as follows:

$$\lambda = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \mu = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r} .$$

Let $f: C^n \to P^N C$ be a meromorphic mapping and $w = (w_0; \cdots; w_N)$ a homogeneous coordinate system in $P^N C$. Then f can be represented as $f = (f_0; \cdots; f_N)$, where f_j are entire functions and $\operatorname{codim}\{z \in C^n : f_0(z) = \cdots = f_N(z) = 0\} \geqslant 2$. If $f = (g_0; \cdots; g_N)$ is another representation of f, then there is an entire function $\alpha(z)$ such that $g_j = e^\alpha \cdot f_j$ $(j = 0, \cdots, N)$. We now take the standard line bundle as F and, taking the metric $a(w) = \sum_{j=0}^N |w_j|^2/|w_i|^2$ $(w_i \neq 0)$ in F, we see $\omega = dd^c \log a(w)$ and obtain

$$(1) \qquad T(r, f) = \int_{\partial B(r)} \log \Big(\sum_{j=0}^N |f_j|^2\Big)^{1/2} \sigma - \log \Big(\sum_{j=0}^N |f_j(0)|^2\Big)^{1/2},$$

provided that $\sum_{j=0}^N |f_j(0)|^2 \neq 0$.

Let $\gamma_\rho(z, z_0)$ be an automorphism of $B(\rho)$ such that $\gamma_\rho(z_0, z_0) = 0$ for $z_0 \in B(\rho)$. We now write

$$\psi_\rho(z, z_0) = \psi \circ \gamma_\rho(z, z_0) \quad \text{and} \quad \sigma_\rho(z, z_0) = \sigma \circ \gamma_\rho(z, z_0) .$$

If $z_0 = (r, 0, \cdots, 0)$, $\zeta = (\zeta_1, \cdots, \zeta_n)$ and if

$$\gamma_\rho(\zeta, z_0) = \frac{\rho}{\rho - \frac{r}{\rho}\zeta_1}\left(\zeta_1 - r, \left(1 - \left(\frac{r}{\rho}\right)^2\right)^{1/2}\zeta_2, \cdots, \left(1 - \left(\frac{r}{\rho}\right)^2\right)^{1/2}\zeta_n\right),$$

then, by elementary calculation, we see

$$\psi_\rho(\zeta, z_0) = \frac{\rho^2 - r^2}{\|\gamma_\rho(\zeta, z_0)\|^2} dd^c \log \|z\|^2$$

and

$$d^c \log \|\gamma_\rho(\zeta, z_0)\|^2 = \frac{\rho^2 - r^2}{\left|\rho - \left(\frac{r}{\rho}\right)\zeta_1\right|^2} d^c \log \|z\|^2$$

on $\partial B(\rho)$, since $d\|z\|^2 = \sum_{\alpha=1}^n (\bar z_\alpha dz_\alpha + z_\alpha d\bar z_\alpha) = 0$ on $\partial B(\rho)$. Hence we have

$$\frac{\left(1 - \left(\frac{r}{\rho}\right)^2\right)^n}{\left(1 + \frac{r}{\rho}\right)^{2n}} \sigma(\zeta) \leqslant \sigma_\rho(\zeta, z_0) \leqslant \frac{\left(1 - \left(\frac{r}{\rho}\right)^2\right)^n}{\left(1 - \frac{r}{\rho}\right)^{2n}} \sigma(\zeta)$$

for $\zeta \in \partial B(\rho)$.

2. We now prove the following theorem which yields a relation between the lower order and the deficiencies:

THEOREM 1. *Let* $f: C^n \to P^N C$ *be a meromorphic mapping of lower order* μ *such that* $\lim_{r\to\infty} (T(r, f)/\log r) = \infty$ *and let* H_j $(j = 0, \cdots, N)$ *be* $N+1$ *hyperplanes in* $P^N C$ *in general position. If* $\gamma = \max_{0\leq j\leq N} (1 - \delta(H_j, f)) < 1$, *then*

$$\mu \geqslant \frac{\log\left(\dfrac{1}{\gamma(2-\gamma)}\right)}{\log \tau} \qquad \textit{for } \gamma \neq 0$$

and

$$\mu \geqslant 1 \qquad \textit{for } \gamma = 0 ,$$

where $\tau = \max\left(\tau_0, \dfrac{5n}{\gamma(1-\gamma)}\right)$ *and* $\tau_0 \in \boldsymbol{R}$ *is the maximum real number of* τ_0 *such that* $((\tau_0 + 1)^n - (\tau_0 - 1)^n)\cdot(\tau_0 - 1)^{-n} = \dfrac{5}{2} n\cdot\tau_0^{-1}$.

The following is a direct result of Theorem 1.

COROLLARY 1. *Under the same assumption as in Theorem* 1, *if there are* $N+1$ *hyperplanes* $H_j \subset P^N C$ *in general position such that* $\delta(H_j, f) > 0$ $(j = 0, \cdots, N)$, *then the lower order* μ *of* f *is positive or infinity.*

To prove Theorem 1, we prepare a lemma.

LEMMA 1. *Let* $f: C^n \to P^N C$ *be a meromorphic mapping and* $H_j \subset P^N C$ $(j = 0, \cdots, N)$ $N+1$ *hyperplanes in general position. If* $\tau > \tau_0$, *then*

$$(2) \qquad T(r, f) \leqslant \frac{5n}{\tau} T(\tau r, f) + \max_{0\leq j\leq N} N(\tau r, H_j) + O(\log r) , \qquad (r \to \infty) .$$

Proof. Since $N+1$ hyperplanes H_j $(j = 0, 1, \cdots, N)$ in general position, we may take a homogeneous coordinate system $w = (w_0; \cdots; w_N)$ in $P^N C$ such that $H_j = \{w \in P^N C: w_j = 0\}$ $(j = 0, 1, \cdots, N)$, so we fix such homogeneous coordinate w and represent f as $f = (f_0; \cdots; f_N)$.

Let $\gamma_\rho(z, z_0)$ be an automorphism of $B(\rho)$ such that $\gamma_\rho(z_0, z_0) = 0$ for $z_0 \in B(\rho)$. For any j $(= 0, 1, \cdots, N)$ and $\rho > 0$, we have

$$\left|\int_{\partial B(\rho)} \log |f_j(z)|\, \sigma(z)\right| = \left|\int_{\partial B(\rho)} \left(\log^+ |f_j(z)| - \log^+ \frac{1}{|f_j(z)|}\right)\sigma(z)\right|$$
$$< T_1(\rho, f_j) + O(1) < \infty ,$$

where $T_1(\rho, f_j)$ denotes the characteristic function of $f_j: \boldsymbol{C}^n \to \boldsymbol{P}^1\boldsymbol{C}$. Hence we see that $\log|f_j(z)|$ is integrable on $\partial B(\rho)$ for $\rho > 0$ and $j = 0, \cdots, N$.

Putting $x_\alpha = (z_\alpha - \bar{z}_\alpha)/2$ and $y_\alpha = (z_\alpha + \bar{z}_\alpha)/2\sqrt{-1}$, we can regard $B(R)$ as the open ball in the $2n$-dimensional real Euclidean space with radius R and the center at the origin. Consider a Dirichlet problem

$$\begin{cases} \sum_{\alpha=1}^{n} \left(\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial y_\alpha^2}\right)\Omega_j = 0 & \text{in } B(R) , \\ \qquad \Omega_{j|\partial B(R)} = \log|f_j(z)| . \end{cases}$$

Then we see that there is a harmonic function $\Omega_j(z)$ in $B(R)$ satisfying

$$\Omega_j(\zeta) = \lim_{\substack{z\to\zeta \\ z\in B(R)}} \Omega_j(z) = \log|f_j(\zeta)|$$

for $\zeta \in \partial B(R)\backslash \mathrm{supp}\,(f_j)$, where (f_j) denotes the divisor of f_j, $(j = 0, \cdots, N)$.

For $\|z\| = r$ and any $\rho: r < \rho < R$, we have

$$\Omega_j(z) - \Omega_0(z) = \int_{\partial B(\rho)} (\Omega_j(\zeta) - \Omega_0(\zeta))\sigma_\rho(\zeta, z) ,$$

so

$$\log|f_j(z)| \leqq \Omega_j(z) \leqq \int_{\partial B(\rho)} (\Omega_j(\zeta) - \Omega_0(\zeta))\sigma_\rho(\zeta, z) + \Omega_0(z) .$$

By a homogeneity of a sphere $B(\rho)$, the upper bound and the lower bound of $\sigma \circ \gamma_\rho(\zeta, z)$ on $\partial B(\rho)$ can be replaced by those of $\sigma \circ \gamma_\rho^0(\zeta, z)$, where

$$\gamma_\rho^0(\zeta, z) = \frac{\rho}{\rho - \left(\frac{r}{\rho}\right)\zeta_1}(\zeta_1 - r, \sqrt{1-(r/\rho)^2}\zeta_2, \cdots, \sqrt{1-(r/\rho)^2}\zeta_n) .$$

Hence we have

$$\sigma_\rho(\zeta, z) = (1 + Q)\sigma(\zeta) ,$$

where

$$|Q| \leqq \frac{(\tau_\rho + 1)^n - (\tau_\rho - 1)^n}{(\tau_\rho - 1)^n} = \frac{2n\tau_\rho^{n-1} + O(\tau_\rho^{n-3})}{(\tau_\rho - 1)^n} , \qquad \tau_\rho = \frac{\rho}{r} > 1 .$$

Therefore, we obtain

$$\log|f_j(z)| \leqq \int_{\partial B(\rho)} (\Omega_j(\zeta) - \Omega_0(\zeta))\sigma(\zeta)$$

$$(3) \qquad + \frac{(\tau_\rho + 1)^n - (\tau_\rho - 1)^n}{(\tau_\rho - 1)^n} \int_{\partial B(\rho)} |\Omega_j(\zeta) - \Omega_0(\zeta)|\, \sigma(\zeta) + \Omega_0(z)$$

$$(j = 0, \cdots N) .$$

Let χ_ρ be the characteristic function of $B(\rho)$. Then the first term in the right hand side of (3) is equal to

$$\int_{\partial B(\rho)} (\Omega_j(\zeta) - \Omega_0(\zeta))\sigma(\zeta) = \int_{B(\rho)} d\{(\Omega_j(\zeta) - \Omega_0(\zeta))\sigma(\zeta)\}$$
$$= \int_{B(R)} \chi_\rho d\{(\Omega_j(\zeta) - \Omega_0(\zeta))\sigma(\zeta)\} ,$$

which is converges to

$$\int_{B(R)} d\{(\Omega_j(\zeta) - \Omega_0(\zeta))\sigma(\zeta)\} = \int_{\partial B(R)} (\Omega_j(\zeta) - \Omega_0(\zeta))\sigma(\zeta) \qquad (\rho \to R) .$$

This is easily verified by Lebesgue's convergence theorem.

Similarly, the second term in the right hand side of (3) converges to

$$\frac{(\tau_R + 1)^n - (\tau_R - 1)^n}{(\tau_R - 1)^n} \int_{\partial B(R)} |\Omega_j(\zeta) - \Omega_0(\zeta)|\, \sigma(\zeta) \qquad (\rho \to R) .$$

Hence, for any $j\ (= 0, 1, \cdots, N)$ we obtain from (3)

$$\log |f_j(z)| \leqq \int_{\partial B(R)} \log \left| \frac{f_j(\zeta)}{f_0(\zeta)} \right| \sigma(\zeta) + \frac{5n}{2\tau} \int_{\partial B(R)} \log \left| \frac{f_j(\zeta)}{f_0(\zeta)} \right| \sigma(\zeta) + \Omega_0(z) ,$$

so

$$(4) \qquad \max_{0 \leqq j \leqq N} \log |f_j(z)| \leqq \max_{0 \leqq j \leqq N} (N(R, (f_j)) - N(R, (f_0)))$$
$$+ \max_{0 \leqq j \leqq N} \frac{5n}{2\tau} \int_{\partial B(R)} \left| \log \left| \frac{f_j(\zeta)}{f_0(\zeta)} \right| \right| \sigma(\zeta) + \Omega_0(z) .$$

On the other hand, by (1) we have

$$T(r, f) = \int_{\partial B(r)} \log \left(\sum_{j=0}^{N} |f_j|^2 \right)^{1/2} \sigma - \log \left(\sum_{j=0}^{N} |f_j(0)|^2 \right)^{1/2}$$

provided that $\sum_{j=0}^{N} |f_j(0)|^2 \neq 0$. Hence, by integrating (4) on $\partial B(r)$, we have

$$T(r, f) \leqq \int_{\partial B(r)} \max_{0 \leqq j \leqq N} \log |f_j(z)|\, \sigma(z) + O(1)$$

$$\leqq \max_{0 \leqq j \leqq N} (N(R, (f_j)) - N(R, (f_0))) + \frac{5n}{\tau} T(R, f) + \int_{\partial B(r)} \Omega_0(z)\sigma(z) + O(\log r) , \qquad (r \to \infty) .$$

Since $\Omega_0(z)$ is harmonic in $B(R)$, we see

$$\begin{aligned} \int_{\partial B(r)} \Omega_0(z)\sigma(z) &= \lim_{r' \to R} \int_{\partial B(r')} \Omega_0(z)\sigma(z) \\ &= \int_{\partial B(R)} \Omega_0(z)\sigma(z) = \int_{\partial B(R)} \log |f_0(z)| \sigma(z) \\ &= N(R, (f_0)) , \end{aligned}$$

whence

$$T(r, f) \leqq \max_{0 \leqq j \leqq N} (N(R, (f_j))) + \frac{5n}{\tau} T(R, f) + O(\log r) .$$

Thus we obtain

$$T(r, f) \leqq \max_{0 \leqq j \leqq N} (N(R, (H_j))) + \frac{5n}{\tau} T(R, f) + O(\log r) , \qquad (r \to \infty) ,$$

since $N(R, (f_j)) = N(R, H_j)$ $(j = 0, 1, \cdots, N)$.
Therefore we have Lemma 1.

Now we shall prove Theorem 1. By Lemma 1, we have

$$T(r, f) \leqq \max_{0 \leqq j \leqq N} (N(R, H_j)) + \frac{5n}{\tau} T(R, f) + O(\log r) \tag{5}$$

for $\tau > \tau_0, R = \tau r$. We now choose c and c' such that $\gamma < c' < c < 1$. Since $1 - \delta(H_j, f) = \limsup_{r \to \infty} N(r, H_j)/T(r, f) \leqq \gamma$ $(j = 0, 1, \cdots, N)$, we have

$$N(r, H_j) < c' T(r, f) \qquad (j = 0, 1, \cdots, N) \tag{6}$$

for all sufficiently large values of r. We take

$$\tau = \max \left(\tau_0, \frac{5n}{c(1 - c)} \right) , \tag{7}$$

where τ_0 is determined such as in the statement of Theorem 1. Then we have from (5), (6) and (7)

$$T(r, f) \leqq c(2 - c) T(\tau r, f) .$$

Hence, by a similar method to Edrei-Fuchs [2], we have

$$\mu = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} \geqq \log\left\{\frac{1}{c(2-c)}\right\} \Big/ \log\tau .$$

By letting $c \to \gamma$, we obtain the conclusion of Theorem 1.

3. We shall next show that, if $K(f) = \limsup_{r\to\infty} \sum_{j=0}^{N} N(r, H_j)/T(r,f)$ is sufficiently small, then the order λ is close to the lower order μ and that, if, in addition, μ is finite, then λ and μ are both close to a positive integer. First we shall prove

LEMMA 2. *Let $f: C^n \to P^N C$ be a meromorphic mapping. Then*

$$\begin{aligned} 2T(r,f) - 2N(r) &< (q+1)r^q \int_\rho^R N(\alpha t)t^{-q-1}\phi\left(\frac{t}{r}\right)dt \\ &\quad + 8.5(N+1)\left(\frac{r}{\rho}\right)^q T(\alpha\rho) + 8.5(N+1)\left(\frac{r}{R}\right)^{q+1} T(\alpha R) + O(1) \\ &\qquad (r\to\infty) , \end{aligned} \tag{8}$$

where

$$\phi(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(t^2 - 2t\cos\theta + 1)^{1/2}} , \qquad N(r) = \sum_{j=0}^{N} N(r, H_j) ,$$

$$\alpha = e^{1/q+1} , \quad \tau = (35(N+1))^{1/\beta} , \quad \rho = \frac{r}{\alpha\tau} , \quad R = \frac{\tau r}{\alpha}$$

and q denotes the largest integer not exceeding λ.

Proof. Let $f = (f_0; \cdots; f_N)$, where f_j $(j = 0, 1, \cdots, N)$ are entire functions and ℓ be a complex line in C^n through the origin. Using the inequality (10.2) in Edrei-Fuchs [2, p. 317], we have for $u \in \ell$ with $\|u\| = r$

$$\begin{aligned} &2T_\ell(r, f_j) - 2N_\ell(r, 0, f_j) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f_j(ue^{i\theta})|\, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f_j(ue^{i\theta})|} d\theta \\ &\leqq (q+1)r^q \int_\rho^R N_\ell(\alpha t, 0, f_j)t^{-q-1}\phi\left(\frac{t}{r}\right)dt + 8.5\left(\frac{r}{\rho}\right)^q T_\ell(\alpha\rho, f_j) \\ &\quad + 8.5\left(\frac{r}{R}\right)^{q+1} T_\ell(\alpha R, f_j) , \end{aligned} \tag{9}$$

where $N_\ell(r)$ and $T_\ell(r)$ denote the counting function and the characteristic function of a meromorphic function of one complex variable obtained by restricting of f to $\ell \subset C^n$.

Let $\nu(\ell)$ be the standard volume form on $P^{n-1}C$ defined by ψ and consider ℓ as a point of $P^{n-1}C$ in natural manner. Then we have from (9)

$$\begin{aligned}
&2T(r,f_j) - 2N(r,0,f_j)\\
&\quad = \int_{\partial B(r)} \log^+ |f_j|\,\sigma + \int_{\partial B(r)} \log^+ \frac{1}{|f_j|}\sigma\\
&\quad = \int_{P^{n-1}C} \nu(\ell)\left\{\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f_j(ue^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f_j(ue^{i\theta})|}d\theta\right\}\\
&\quad < (q+1)r^q \int_\rho^R N(\alpha t, (f_j))t^{-q-1}\phi\left(\frac{t}{r}\right)dt\\
&\qquad + 8.5\left(\frac{r}{\rho}\right)^q T(\alpha\rho, f_j) + 8.5\left(\frac{r}{R}\right)^{q+1} T(\alpha R, f_j)\\
&\qquad\qquad (j = 0, \cdots, N)\ ,
\end{aligned}$$

by noting $n(t,(f_j)) = \int_{\ell \in P^{n-1}C} n_\ell(t,0,f_j)\nu(\ell)$ and by using Fubini's theorem, where $u \in \ell$ with $\|u\| = r$. Hence, by summing up those with respect to j, we have

$$\begin{aligned}
2\sum_{j=0}^N T(r,f_j) - 2\sum_{j=0}^N N(r,H_j) &\leqq (q+1)r^q \int_\rho^R \sum_{j=0}^N N(\alpha t, H_j)t^{-q-1}\phi\left(\frac{t}{r}\right)dt\\
&\quad + 8.5\left(\frac{r}{\rho}\right)^q \sum_{j=0}^N T(\alpha\rho, f_j) + 8.5\left(\frac{r}{R}\right)^{q+1} \sum_{j=0}^N T(\alpha R, f_j)\ .
\end{aligned}$$

This implies

$$\begin{aligned}
2T(r,f) - 2N(r) - O(1) &\leqq (q+1)r^q \int_\rho^R \sum_{j=0}^N N(\alpha t, H_j)t^{-q-1}\phi\left(\frac{t}{r}\right)dt\\
&\quad + 8.5(N+1)\left(\frac{r}{\rho}\right)^q T(\alpha\rho, f) + 8.5(N+1)\left(\frac{r}{R}\right)^{q+1} T(\alpha R, f)\ .
\end{aligned}$$

This proves the lemma.

LEMMA 3. *Under the same assumption as in Lemma* 1, *suppose further that there are a non-negative integer* q *and a positive number* $\beta\ (0 < \beta < \frac{1}{2})$ *such that*

(10) $$K(f) = \limsup_{r\to\infty} \sum_{j=0}^N N(r,H_j)/T(r,f) < \beta/5e(q+1)\ .$$

I. *If*

(11) $$\lambda > q + 1 - \beta ,$$

they every interval

(12) $$x \leqq r \leqq (35(N+1))^{1/\beta} x \qquad (x > x_0)$$

contains a point s *such that*

(13) $$T(u)u^{-q-1+\beta} \leqq T(s)s^{-q-1+\beta} \qquad (x_0 \leqq u \leqq s) ,$$

where x_0 *is a suitable positive number satisfying* $N(x) < \tau T(x)$ *for all* $x \geqq x_0$.

II. *If*

(14) $$\mu < q + \beta ,$$

then every interval (12) *contains a point* t *such that*

$$T(t)t^{-q-\beta} \geqq T(v)v^{-q-\beta} . \qquad (v \geqq t) .$$

From this lemma, we easily have

COROLLARY 2. *If* (10) *and* (11) *hold, then* $\mu \geqq q + 1 - \beta$. *If* (10) *and* (14) *hold, then* $\lambda \leqq q + \beta$.

Here we shall give a proof of Lemma 3. Let $\tau = (35(N+1))^{1/\beta}$ and $q + \beta \leqq c \leqq q + 1 - \beta$. Then we see

(15) $$T(r, f)/r^c < \sup_{r/\tau \leqq u \leqq \tau r} T(u, f)/u^c$$

for all sufficiently large values of r. In fact, if we take $\kappa = \beta/5e(q+1)$, then (10) implies

(16) $$N(u) < \kappa T(u)$$

for all large u. Suppose that (15) is violate, that is, suppose

(17) $$T(u) \leqq \left(\frac{u}{r}\right)^c T(r) \qquad \left(\frac{r}{\tau} \leqq u \leqq \tau r\right) .$$

Then Lemma 2, (16), (17) and a similar method to that of Edrei-Fuchs [2] imply the following contradiction:

$$2 \leqq 2\kappa + \frac{2.2e}{\beta}(q+1)\kappa + 17(N+1)e/35(N+1) < 2 .$$

Thus we have the desired assertion.

THEOREM 2. *Let* $f: \boldsymbol{C}^n \to P^N\boldsymbol{C}$ *be a meromorphic mapping of order* λ *and of lower order* μ. *Let* p *be the integer such that* $p - \frac{1}{2} \leqq \mu < p + \frac{1}{2}$. *If* $\beta: 0 < \beta \leqq \frac{1}{2}$ *and*

$$K(f) = \limsup_{r\to\infty} \sum_{j=0}^{N} N(r, H_j)/T(r, f) < \beta/\max(20n + 1, 2\tau_0)(p + 1) , \tag{18}$$

then $p \geqq 1$, $|\lambda - p| < e\beta/2\max(20n + 1, 2\tau_0)$ *and*

$$p - \beta \leqq \mu \leqq \lambda \leqq p + \{e\beta/2\max(20n + 1, 2\tau_0)\} .$$

To prove Theorem 2, we need the following lemma.

LEMMA 4 (*Noguchi* [4]). *Let* $f: \boldsymbol{C}^n \to P^N\boldsymbol{C}$ *be a meromorphic mapping of finite order* λ *which is not a positive integer. Then, for any* $N + 1$ *hyperplanes* $H_j \subset P^N\boldsymbol{C}$ $(j = 0, 1, \cdots, N)$ *in general position,*

$$K(f) \geqq 2\Gamma^4(\tfrac{3}{4})\,|\sin \pi\lambda|/\{\pi^2\lambda + \Gamma^4(\tfrac{3}{4})\,|\sin \pi\lambda|\} . \tag{19}$$

Now we can give a proof of Theorem 2. If $K(f) = 0$, then $\gamma = 0$ and $\mu \geqq 1$. If $\gamma \neq 0$, then by Theorem 1 we have

$$\mu \geqq \log \frac{1}{\gamma(2-\gamma)} \Big/ \log \tau > \log(1/2\gamma)/\log \max\left(\tau_0, \frac{5n}{\gamma(1-\gamma)}\right) .$$

Since

$$\gamma = \max_{0\leqq j \leqq N}(1 - \delta(H_j, f)) \leqq K(f) < 1/\max(2\tau_0, 20n + 1)(p + 1) ,$$

we see

$$\log 2\tau_0 < \log(1/2\gamma) \quad \text{and} \quad \log(5n/\gamma(1-\gamma)) < 2\log(1/2\gamma) .$$

Hence we have $\mu \geqq \frac{1}{2}$, so $p \geqq 1$.

We now show that

$$\lambda \leqq p + 1 - \beta . \tag{20}$$

Suppose that (20) is violate. Then, from (18), we see $K(f) < \beta/5e(p + 1)$. Hence we can apply Corollary 2 with $q = p$ and obtain $\mu \geqq p + 1 - \beta$. This contradicts our hypothesis. Hence (20) is valid. By (18) and Lemma 4, we see

$$\beta/\max(20n + 1, 2\tau_0)(p + 1) > K(f) > |\sin \pi\lambda|/e(p + 1) ,$$

whence

$$|\sin \pi\lambda| < e\beta/\max(20n+1, 2\tau_0) .$$

If k is the integer defined by $|k-\lambda| \leqq \min(\lambda-[\lambda], [\lambda]+1-\lambda)$, then

$$2|k-\lambda| \leqq |\sin \pi(k-\lambda)| = |\sin \pi\lambda| < e\beta/\max(20n+1, 2\tau_0) .$$

Since $p-\frac{1}{2} \leqq \mu \leqq \lambda < p+1-\beta$, this leaves the only possibility $k=p$, so $|\lambda-p| < e\beta/2\max(20n+1, 2\tau_0)$.

On the other hand, if we apply Collorary 2 with $q+1=p \geqq 1$, then we see $\mu \geqq p-\beta$. This completes the proof of Theorem 2.

COROLLARY 3. *Let $f: C^n \to P^N C$ be a meromorphic mapping of order λ and of lower order μ and suppose $\lim_{r\to\infty} T(r,f)/\log r = \infty$. If there are $N+1$ hyperplanes $H_j \subset P^N C$ $(j=0,1,\cdots,N)$ in general position such that $\delta(H_j,f)=1$ $(j=0,1,\cdots,N)$, then λ is identical with μ and is a positive integer or infinity.*

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