

## A GENERALIZATION OF THE LEVINSON-MASSERA'S EQUALITIES

KENICHI SHIRAIWA

In his study of non-linear differential equations of the second order, N. Levinson [3] defined the dissipative systems ( $D$ -systems) which arise in many important cases in practice. To a dissipative system a transformation  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  called the Poincaré transformation is associated. Levinson used the Poincaré transformation in the qualitative study of dissipative systems, and he [3] and Massera [5] obtained certain equalities between the number of subharmonic solutions of a dissipative systems under suitable conditions. We call these the Levinson-Massera's equalities.

In this paper we define a class of non-linear differential equations of order  $n$  named  $D'$ -systems, which is a subclass of  $D$ -systems for  $n = 2$  and still contains many important systems in practice. For a  $D'$ -systems of order  $n$ , we associate a transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ , which we call the Poincaré transformation associated to the  $D'$ -system. By a qualitative study of the Poincaré transformation  $T$ , we obtain some equalities between the number of subharmonic solutions of a  $D'$ -system under suitable conditions. These equalities coincide with the Levinson-Massera's equalities for  $n = 2$ .

In the final section of this paper, we give an extension of the above results to a certain class of the time dependent vector fields on a compact differentiable manifold of arbitrary dimension.

### § 1. The Poincaré transformation

Consider the following differential equation (1).

$$(1) \quad \frac{dx}{dt} = f(t, x), \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^n$$

We assume the following condition (A) throughout the paper.

(A) (a)  $f(t, x)$  is an  $\mathbf{R}^n$ -valued function of class  $C^1$ .

(b)  $f(t, x)$  is periodic of period 1 with respect to the variable  $t$ .

That is

$$f(t + 1, x) = f(t, x) \quad \text{for } t \in \mathbf{R}, \quad x \in \mathbf{R}^n.$$

(c) There exists a solution  $x = \varphi(t; t_0, x_0)$  of the equation (1) defined on  $-\infty < t < +\infty$  with any initial condition  $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ .

By the assumption (A) the above solution  $x = \varphi(t; t_0, x_0)$  is unique with respect to the initial value  $(t_0, x_0)$ .

Now, a transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is defined by

$$(2) \quad T(x) = \varphi(1; 0, x), \quad x \in \mathbf{R}^n.$$

We call this  $T$  the Poincaré transformation associated to the equation (1).

**PROPOSITION 1.**  *$T$  is a diffeomorphism of class  $C^1$  and is isotopic to the identity. Especially,  $T$  is homotopic to the identity and orientation preserving.*

*Proof.* It is easy to see that  $T$  is a bijection with  $T^{-1}(x) = \varphi(0; 1, x)$ ,  $x \in \mathbf{R}^n$ . Since  $f(t, x)$  is of class  $C^1$ ,  $T$  and  $T^{-1}$  are of class  $C^1$  by the smooth dependence of the solutions of the differential equation with respect to the initial conditions. Therefore,  $T$  is a diffeomorphism of class  $C^1$ .

Let  $T_t: \mathbf{R}^n \rightarrow \mathbf{R}^n$  ( $0 \leq t \leq 1$ ) be a map defined by  $T_t(x) = \varphi(t; 0, x)$ ,  $x \in \mathbf{R}^n$ . Then  $T_t$  ( $0 \leq t \leq 1$ ) is also a diffeomorphism of class  $C^1$ , and  $T_0 = 1$ , the identity map of  $\mathbf{R}^n$ , and  $T_1 = T$ . Now, define a map  $\tilde{T}: \mathbf{R}^n \times [0, 1] \rightarrow \mathbf{R}^n$  by  $\tilde{T}(x, t) = T_t(x) = \varphi(t; 0, x)$ ,  $(x, t) \in \mathbf{R}^n \times [0, 1]$ . Then  $\tilde{T}$  is of class  $C^1$  by the smooth dependence of the solutions with respect to the initial conditions. Therefore,  $T$  is isotopic to the identity.

Since any diffeomorphism which is isotopic to the identity is homotopic to the identity and orientation preserving, the latter half of the proposition is proved.

**PROPOSITION 2.**  $\varphi(t + 1; 0, x) = \varphi(t; 0, T(x))$ ,  $t \in \mathbf{R}$ ,  $x \in \mathbf{R}^n$ .

*Proof.* By our assumption (A) (b) it is easy to see that  $\psi(t) = \varphi(t + 1; 0, x)$ ,  $t \in \mathbf{R}$  is a solution of (1) for any  $x \in \mathbf{R}^n$ . Since  $\psi(0) = \varphi(1; 0, x) = T(x)$ ,  $\psi(t) = \varphi(t; 0, T(x))$  by the uniqueness of the solution of (1) with respect to the initial condition.

The following corollaries are easy consequences of Proposition 2.

COROLLARY 1. *For any integer  $k$ , the following equality holds.*

$$\varphi(t+k; 0, x) = \varphi(t; 0, T^k(x)), \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^n,$$

where  $T^k: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is the  $k$ -fold iterate of  $T$ .

COROLLARY 2. *Let  $k$  be a positive integer. Then the following three conditions are equivalent.*

- (a)  $x(t)$  is a periodic solution of (1) of period  $k$ .
- (b)  $x(0)$  is a fixed point of  $T^k$ .
- (c)  $x(0)$  is a periodic point of  $T$  of period  $k$ .

*Remark 1.* If  $x(t)$  is a periodic solution of rational period, then  $x(0)$  is a periodic point of  $T$ . If  $x(t)$  is a periodic solution of irrational period, then  $x(0)$  is a recurrent point of  $T$ . Therefore, it is a non-wandering point of  $T$ .

From now on, we represent a point  $x$  of  $\mathbf{R}^n$  as a column vector

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ , and we identify an  $n \times n$  matrix  $A$  with the linear map  $\mathbf{R}^n \rightarrow \mathbf{R}^n$  which assigns  $Ax \in \mathbf{R}^n$  to each  $x \in \mathbf{R}^n$ .

Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $p \in \mathbf{R}^n$  be a fixed point of  $T$ . Then the derivative  $DT(p): \mathbf{R}^n \rightarrow \mathbf{R}^n$  of  $T$  at  $p$  is a linear map corresponding to the Jacobian matrix of  $T$  at  $p$ , and it is a non-singular  $n \times n$  matrix.

LEMMA 1. *If  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is a Poincaré transformation associated to a differential equation, then  $\det DT^k(p) > 0$  for any periodic point of  $T$  of period  $k$ .*

*Proof.* By Proposition 1,  $T^k$  is orientation preserving for any positive integer  $k$ . Therefore,  $\det DT^k(p) > 0$  for any  $p \in \mathbf{R}^n$  such that  $T^k(p) = p$ .

An isomorphism (or a non-singular square matrix)  $L: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is called hyperbolic if all the absolute values of its eigenvalues are different from 1.

Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $p \in \mathbf{R}^n$  be a fixed point of  $T$ . We call  $p$  a hyperbolic fixed point of  $T$ , if  $DT(p)$  is hyperbolic.

Now, we shall give a criterion for a fixed point of a Poincaré transformation associated to a differential equation to be hyperbolic.

Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be the Poincaré transformation associated to (1), and let  $p \in \mathbf{R}^n$  be a fixed point of  $T$ . Then  $x = \varphi(t; 0, p)$  is a periodic solution of (1) of period 1 by Proposition 2. Now, consider the variation equation (3) of the equation (1) with respect to the periodic solution  $x = \varphi(t; 0, p)$  of period 1.

$$(3) \quad \frac{dx}{dt} = D_x f(t, \varphi(t; 0, p))x,$$

where  $D_x f(t, x)$  is the Jacobian matrix of  $f(t, x)$  with respect to  $x$ . The variation equation (3) is a linear differential equation with continuous periodic coefficients of period 1.

**PROPOSITION 3.** *Let  $W(t)$  be a fundamental matrix of the linear system (3). Then  $DT(p) = W(1)W(0)^{-1}$ .*

*Proof.* Put  $R(t) = D_x \varphi(t; 0, p)$ ,  $t \in \mathbf{R}$ , where  $D_x \varphi(t; 0, x)$  is the Jacobian matrix of  $\varphi(t; 0, x)$  with respect to  $x$ . Then  $R(t)$  is a matrix solution of (3) with the initial condition  $R(0) = E_n$ , the unit matrix.

Put  $S(t) = W(t)W(0)^{-1}$ ,  $t \in \mathbf{R}$ . Then  $S(t)$  is also a matrix solution of (3) with the initial condition  $S(0) = E_n$ . Therefore,  $R(t) = S(t)$  for all  $t$  by the uniqueness of the solution of (3) with respect to the initial condition. Especially  $R(1) = S(1)$ , that is  $DT(p) = W(1)W(0)^{-1}$ .

**COROLLARY.**  *$p$  is a hyperbolic fixed point of  $T$  if and only if all the real parts of the characteristic exponents of (3) are different from 0.*

*Proof.* This follows from the definition of a hyperbolic fixed point and the Floquet theory using Proposition 3.

Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $p \in \mathbf{R}^n$  be a hyperbolic fixed point of  $T$ . Now, let  $E^u$  be the intersection of  $\mathbf{R}^n$  and the direct sum of the generalized eigenspaces of  $DT(p)$  corresponding to the eigenvalues  $\lambda$  such that  $|\lambda| > 1$ . Similarly, let  $E^s$  be the intersection of  $\mathbf{R}^n$  and the direct sum of the generalized eigenspaces of  $DT(p)$  corresponding to the eigenvalues  $\lambda$  such that  $|\lambda| < 1$ .

**PROPOSITION 4.** *Under the above hypothesis and notations, the following properties hold.*

$$(a) \quad \mathbf{R}^n = E^u \oplus E^s$$

(b)  $E^u$  and  $E^s$  are  $DT(p)$ -invariant subspaces.

(c) Let  $\lambda_1, \dots, \lambda_n$  be the characteristic roots of  $DT(p)$ .

Then  $\dim E^u = \#\{\lambda_i; |\lambda_i| > 1\}$  and  $\dim E^s = \#\{\lambda_i; |\lambda_i| < 1\}$ ,  
 where  $\#M$  denotes the cardinality of the set  $M$ .

(d) Let  $\|\cdot\|$  be a norm of  $\mathbf{R}^n$ , and let  $L = DT(p)$ .

Then there exist constant  $c$  ( $c > 0$ ) and  $\lambda$  ( $0 < \lambda < 1$ ) such that

$$\begin{aligned} \|L^{-k}(x)\| &\leq c\lambda^k \|x\|, & x \in E^u \\ \|L^k(x)\| &\leq c\lambda^k \|x\|, & x \in E^s, \end{aligned}$$

where  $k$  is any positive integer.

This is a standard results in Smale's theory on differentiable dynamical systems (cf. [8]).

Let  $p$  be a hyperbolic fixed point of a diffeomorphism  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ .

Set

$$\begin{aligned} W^u(p) &= \left\{ x \in \mathbf{R}^n; \lim_{k \rightarrow \infty} T^{-k}(x) = p \right\} \quad \text{and} \\ W^s(p) &= \left\{ x \in \mathbf{R}^n; \lim_{k \rightarrow \infty} T^k(x) = p \right\}. \end{aligned}$$

We call  $W^u(p)$  (resp.  $W^s(p)$ ) the unstable (resp. stable) manifold of  $p$ . The following theorem is a standard results in Smale's theory (Cf. [8]).

**THEOREM.** *There exist suitable 1-1 immersions  $\phi^u: E^u \rightarrow \mathbf{R}^n$  and  $\phi^s: E^s \rightarrow \mathbf{R}^n$  such that*

- (a)  $\phi^u(E^u) = W^u(p)$ ,  $\phi^s(E^s) = W^s(p)$ ,
- (b)  $\phi^u(0) = \phi^s(0) = p$ , and
- (c)  $\dim W^u(p) = \dim E^u$ ,  $\dim W^s(p) = \dim E^s$ .

**COROLLARY.** (d)  $\dim W^u(p)$  is equal to the number of characteristic roots  $\lambda$  of  $DT(p)$  such that  $|\lambda| > 1$ .

(e)  $\dim W^s(p)$  is equal to the number of characteristic roots  $\lambda$  of  $DT(p)$  such that  $|\lambda| < 1$ .

This is an easy consequence of the above theorem and Proposition 4.

Let  $p$  be a periodic point of  $T$  of minimal period  $n_0$ . Then  $p$  is a fixed point of  $T^{n_0}$ . If  $DT^{n_0}(p)$  is hyperbolic, we call  $p$  a hyperbolic periodic point of  $T$ . Using  $T^{n_0}$  instead of  $T$ , the similar theory can be developed as above for a periodic point of  $T$ .

## § 2. The fixed point index

Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a continuous map, and let  $p \in \mathbf{R}^n$  be an isolated fixed point of  $T$ . Define a map  $1 - T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  by  $(1 - T)(x) = x - T(x)$ ,  $x \in \mathbf{R}^n$ . Then  $(1 - T)(p) = 0$ , and there exists a neighborhood  $V$  of  $p$  such that  $(1 - T)(V - \{p\}) \subset \mathbf{R}^n - \{0\}$  by our assumption. Therefore,  $1 - T$  induces a homomorphism

$$(1 - T)_*: H_n(V, V - \{p\}) \longrightarrow H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}),$$

where  $H_n(A, B)$  denotes the  $n$ -dimensional homology group of a pair  $(A, B)$  with coefficients in the group of integers  $\mathbf{Z}$ .

The groups  $H_n(V, V - \{p\})$  and  $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$  are isomorphic to  $\mathbf{Z}$ , and if we fix an orientation of  $\mathbf{R}^n$ , then there correspond unique generators  $O_V$  and  $O_{\mathbf{R}^n}$  of  $H_n(V, V - \{p\})$  and  $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$  respectively. Using these generators  $O_V$  and  $O_{\mathbf{R}^n}$ , we get

$$(1 - T)_*(O_V) = mO_{\mathbf{R}^n}$$

for a suitable integer  $m$ . It is easily shown that  $m$  does not depend on the choice of a neighborhood  $V$  and an orientation of  $\mathbf{R}^n$ . The integer determined above is called the fixed point index of  $T$  at  $p$  and is denoted by  $\text{index}_T(p)$ .

**PROPOSITION 5.** *Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $p \in \mathbf{R}^n$  be a hyperbolic fixed point of  $T$ . Then  $p$  is an isolated fixed point, and the fixed point index is given as follows.*

$$\text{index}_T(p) = \begin{cases} 1 & \text{if } \det(1 - DT(p)) > 0, \\ -1 & \text{if } \det(1 - DT(p)) < 0, \end{cases}$$

where  $\det(1 - DT(p))$  is the determinant of the matrix  $1 - DT(p)$  and  $1$  is the unit matrix.

*Proof.* By a theorem of Hartman ([2], p. 245, Lemma 8.1),  $T$  restricted on a suitable neighborhood of  $p$  is topologically equivalent to  $DT(p)$  restricted on some neighborhood of the origin  $0$  of  $\mathbf{R}^n$ . Therefore, it is sufficient to prove Proposition 5 in case that  $T = DT(p)$  and  $p = 0$ .

Now assume that  $T$  is a hyperbolic linear isomorphism and  $p = 0$ . Then  $T = DT(p)$  and all the eigenvalues of  $T$  are different from 1. Therefore,  $\det(1 - T) \neq 0$ . Thus,  $0 \in \mathbf{R}^n$  is the only fixed point of  $T$ , and  $p = 0$  is an isolated fixed point of  $T$ .

It is well known that a linear isomorphism  $L: \mathbf{R}^n \rightarrow \mathbf{R}^n$  induces an isomorphism  $L_*: H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}) \rightarrow H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$ , and

$$L_*(O_{\mathbf{R}^n}) = \begin{cases} O_{\mathbf{R}^n} & \text{if } \det L > 0, \\ -O_{\mathbf{R}^n} & \text{if } \det L < 0, \end{cases}$$

Putting  $L = 1 - T = 1 - DT(p)$ , we obtain Proposition 5.

**PROPOSITION 6.** *Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $p \in \mathbf{R}^n$  be a hyperbolic fixed point of  $T$ . Let  $r$  be the number of the real characteristic roots  $\lambda_i$  of  $DT(p)$  such that  $\lambda_i > 1$ . Then  $\text{index}_T(p) = (-1)^r$ .*

*Proof.* Let  $\lambda_1, \dots, \lambda_n$  be the characteristic roots of  $DT(p)$ . Then  $\det(1 - DT(p)) = (1 - \lambda_1) \cdots (1 - \lambda_n)$ . If  $\lambda_i$  is a complex number, there exists some  $j$  such that  $\lambda_j = \bar{\lambda}_i$ . Therefore, it does not affect on the sign of  $\det(1 - DT(p))$ . Now, it is clear that the sign of  $\det(1 - DT(p))$  is equal to that of  $(-1)^r$ . Now Proposition 6 follows from Proposition 5.

**PROPOSITION 7.** *Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $p \in \mathbf{R}^n$  be a hyperbolic fixed point of  $T$ . If we put  $u = \dim E^u$  and  $L_u = DT(p)|_{E^u}: E^u \rightarrow E^u$ , then the following properties hold for any positive integer  $k$ .*

- (a) *If  $\det L_u > 0$ , then  $\text{index}_{T^k}(p) = (-1)^u$ .*
- (b) *If  $\det L_u < 0$ , then  $\text{index}_{T^{2k-1}}(p) = (-1)^{u+1}$  and  $\text{index}_{T^{2k}}(p) = (-1)^u$ .*

*Proof.* Let  $\lambda_1, \dots, \lambda_n$  be the characteristic roots of  $DT(p)$ . Then  $|\lambda_i| \neq 1$  for any  $i$  by our hypothesis. Since  $DT^k(p) = (DT(p))^k$ , the characteristic roots of  $DT^k(p)$  are  $\lambda_1^k, \dots, \lambda_n^k$ . Therefore, the absolute values of any eigenvalues of  $DT^k(p)$  are different from 1. Therefore,  $p$  is a hyperbolic fixed point of  $T^k$ .

Now assume the following (4).

$$(4) \quad \begin{cases} \lambda_i \text{ is real and } \lambda_i > 1 \text{ for } 1 \leq i \leq r, \\ \lambda_i \text{ is real and } \lambda_i < -1 \text{ for } r + 1 \leq i \leq r + s, \\ \lambda_i \text{ is complex and } |\lambda_i| > 1 \text{ for } r + s + 1 \leq i \leq r + s + t \text{ and} \\ |\lambda_i| < 1 \text{ for } r + s + t + 1 \leq i < n \end{cases}$$

Then  $t$  is even for complex characteristic roots appear in pair, and  $u = r + s + t$  by Proposition 4(c). Since  $\det L_u = \lambda_1 \cdots \lambda_{r+s+t}$  and the product of complex characteristic roots  $\lambda_{r+s+1}, \dots, \lambda_{r+s+t}$  is positive, the

sign of  $\det L_u$  is equal to that of  $(-1)^s$ .

If  $\det L_u > 0$ , then  $s$  is even. Therefore,  $(-1)^u = (-1)^{r+s+t} = (-1)^r$ . Now, let  $r(k)$  be the number of the real characteristic roots  $\lambda_i^k$  of  $DT^k(p)$  such that  $\lambda_i^k > 1$ . Then by (4) and the fact that the complex characteristic roots appear in pair with their conjugates, we conclude that  $r(k) \equiv r \pmod{2}$ . Therefore,  $\text{index}_{T^k}(p) = (-1)^{r(k)} = (-1)^r = (-1)^u$  by Proposition 6 and the above stated facts. This proves (a).

If  $\det L_u < 0$ , then  $s$  is odd. Therefore,  $(-1)^u = (-1)^{r+s+t} = (-1)^{r+1}$ . And if  $k$  is even, then  $r(k) \equiv r + s \equiv r + 1 \pmod{2}$  as above. Thus,  $\text{index}_{T^k}(p) = (-1)^{r(k)} = (-1)^u$  if  $k$  is even. If  $k$  is odd, then  $r(k) \equiv r \pmod{2}$ . Therefore,  $\text{index}_{T^k}(p) = (-1)^{r(k)} = (-1)^r = (-1)^{u+1}$ . This completes the proof.

EXAMPLE 1. Let  $n = 2$ , and let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be a Poincaré transformation associated to a differential equation. Let  $p \in \mathbf{R}^2$  be a hyperbolic fixed point of  $T$ , and let  $\lambda_1, \lambda_2$  ( $|\lambda_1| \leq |\lambda_2|$ ) be the characteristic roots of  $DT(p)$ . Then the following four cases occur since  $\det DT(p) > 0$  by Lemma 1.

(i) The point  $p$  is a completely unstable fixed point of  $T$  if  $1 \leq |\lambda_1| \leq |\lambda_2|$ . In this case,  $\text{index}_{T^k}(p) = 1$  for any positive integer  $k$ .

(ii) The point  $p$  is a completely stable fixed point of  $T$  if  $|\lambda_1| \leq |\lambda_2| < 1$ . In this case,  $\text{index}_{T^k}(p) = 1$  for any positive integer  $k$ .

(iii) The point  $p$  is a directly unstable fixed point of  $T$  if  $0 < \lambda_1 < 1 < \lambda_2$ . In this case,  $\text{index}_{T^k}(p) = -1$  for any positive integer  $k$ .

(iv) The point  $p$  is an inversely unstable fixed point if  $\lambda_2 < -1 < \lambda_1 < 0$ . In this case,  $\text{index}_{T^{2k-1}}(p) = 1$  and  $\text{index}_{T^{2k}}(p) = -1$  for any positive integer  $k$ .

Similar statements hold for hyperbolic periodic points of  $T$ .

DEFINITION 1. Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $p$  be a hyperbolic fixed point of  $T$ . Let  $\mathbf{R}^n = E^u \oplus E^s$  be the direct sum decomposition of  $\mathbf{R}^n$  with respect to  $L = DT(p)$  as in Proposition 4, and let  $L_u = DT(p)|_{E^u}: E^u \rightarrow E^u$ .

(i) If  $\dim E^u$  is even and  $\det L_u > 0$ , then we call  $p$  a fixed point of type *PD*.

(ii) If  $\dim E^u$  is odd and  $\det L_u > 0$ , then we call  $p$  a fixed point of type *ND*.

(iii) If  $\dim E^u$  is even and  $\det L_u < 0$ , then we call  $p$  a fixed point of type *PI*.



(iv) If  $\dim E^u$  is odd and  $\det L_u < 0$ , then we call  $p$  a fixed point of type *NI*.

For a hyperbolic periodic point  $p$ , we define its type similarly.

Using the above terminology Proposition 7 is restated as follows.

PROPOSITION 7'. *Under the same assumption of Proposition 7, the following properties hold for any positive integer  $k$ .*

- (i) *If  $p$  is of type PD, then  $\text{index}_{T^k}(p) = 1$ .*
- (ii) *If  $p$  is of type ND, then  $\text{index}_{T^k}(p) = -1$ .*
- (iii) *If  $p$  is of type PI, then  $\text{index}_{T^{2k-1}}(p) = -1$  and  $\text{index}_{T^{2k}}(p) = 1$ .*
- (iv) *If  $p$  is of type NI, then  $\text{index}_{T^{2k-1}}(p) = 1$  and  $\text{index}_{T^{2k}}(p) = -1$ .*

EXAMPLE 2. If  $n = 2$ , case (iii) of Definition 1 does not occur for any Poincaré transformations associated to differential equations by Lemma 1.

If  $p$  is a completely unstable or stable fixed (or periodic) point of  $T$ , then it is of type *PD*. If  $p$  is a directly unstable fixed (or periodic) point of  $T$ , then it is of type *ND*. If  $p$  is an inversely unstable fixed (or periodic) point of  $T$ , then it is of type *NI*.

Remark 2. The notions of the completely unstable, completely stable, directly unstable and inversely unstable fixed point classify the local topological types of the hyperbolic fixed point for  $n = 2$ . But our types defined above do not classify the local topological types of the hyperbolic fixed points.

The local topological type of a hyperbolic fixed point is classified by the dimension of  $E^u$  and the signs of  $\det L_u$  and  $\det L_s$ .

If  $T$  is a Poincaré transformation associated to a differential equation, then  $\det DT(p) > 0$  for any fixed point  $p$  of  $T$  by Lemma 1. Since  $\det DT(p) = \det L_u \cdot \det L_s$  for a hyperbolic fixed point  $p$ , the local topological type of a hyperbolic fixed point is classified by the dimension of  $E^u$  and the sign of  $L_u$ .

### §3. Levinson-Massera's equalities

The following theorem is well known.

THEOREM. (*Poincaré-Hopf-Lefschetz*) *Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a continuous map such that all the fixed points of  $T$  are isolated. Suppose that there exists a subset  $K$  of  $\mathbf{R}^n$  such that*

- (i)  $K$  is homeomorphic to a closed  $n$ -disk,
  - (ii)  $T(K) \subset K$  and
  - (iii) all the fixed points of  $T$  are contained in  $K$ .
- Then the following equality holds.

$$\sum_{T(p)=p} \text{index}_T(p) = 1$$

For a proof, see A. Dold [1] for example.

As an easy application, we have the following proposition.

**PROPOSITION 8.** *Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$ , and let  $k$  be a positive integer. Suppose that all the periodic points of  $T$  of period  $k$  are hyperbolic. Further, assume that there exists a subset  $K$  of  $\mathbf{R}^n$  such that*

- (i)  $K$  is homeomorphic to a closed  $n$ -disk,
- (ii)  $T^k(K) \subset K$  and
- (iii) all the periodic points of  $T$  of period  $k$  are contained in  $K$ .

Then the following equality holds.

$$\sum_{T^k(p)=p} \text{index}_{T^k}(p) = 1$$

Now we shall state the main theorem of this paper.

**THEOREM 1.** *Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a diffeomorphism of class  $C^1$  such that every periodic point of  $T$  is hyperbolic. Further, assume that there exists a subset  $K$  of  $\mathbf{R}^n$  such that*

- (i)  $K$  is homeomorphic to a closed  $n$ -disk,
- (ii)  $T(K) \subset K$  and
- (iii) every periodic point belongs to  $K$ .

For each positive integer  $q$ , let  $PD(q)$  (resp.  $ND(q)$ ,  $PI(q)$ ,  $NI(q)$ ) denote the number of the periodic points of  $T$  of minimal period  $q$  of type  $PD$  (resp.  $ND$ ,  $PI$ ,  $NI$ ), and let  $N(q)$  be the number of periodic points of  $T$  of minimal period  $q$ . Then the following equalities hold.

$$(5) \quad N(q) = PD(q) + ND(q) + PI(q) + NI(q) \quad \text{for any } q.$$

$$(6) \quad PD(1) + NI(1) = ND(1) + PI(1) + 1.$$

$$(7) \quad N(1) = 2(ND(1) + PI(1)) + 1.$$

$$(8) \quad PD(q) + NI(q) = ND(q) + PI(q) \quad \text{if } q \text{ is odd and } q > 1.$$

$$(9) \quad N(q) = 2(PD(q) + NI(q)) = 2(ND(q) + PI(q)) \quad \text{if } q \text{ is odd and } q > 1.$$

$$(10) \quad PD(q) + NI(q) + 2PI(q/2) = ND(q) + PI(q) + 2NI(q/2) \quad \text{if } q \text{ is even.}$$

$$(11) \quad N(q) = 2(ND(q) + PI(q) + NI(q/2) - PI(q/2)) \quad \text{if } q \text{ is even.}$$

The following corollary is immediate from Theorem 1.

**COROLLARY.** *Under the same assumption of Theorem 1, the following properties hold.*

(i)  $N(1)$  is odd.

(ii) If  $q$  is odd and  $q > 1$ , then  $N(q)$  is divisible by  $2q$ .

(iii) If  $q$  is even and  $PI(q/2) = NI(q/2)$ , then  $N(q)$  is divisible by  $2q$ . Especially, if  $q$  is even and  $PI(q/2) = NI(q/2) = 0$ , then  $N(q)$  is divisible by  $2q$ .

*Proof of Theorem 1.* By the assumption of Theorem 1, all the hypothesis of Proposition 8 are satisfied for any positive integer  $k$ , and the equality (5) holds by the definition.

Putting  $k = 1$  in Proposition 8, we obtain the following equality by Proposition (7)′.

$$(12) \quad PD(1) - ND(1) - PI(1) + NI(1) = 1.$$

This proves (6), and (7) is an easy consequence of (5) and (6).

In order to prove (8) and (10), we write down the equality of Proposition 8 in terms of  $PD(q)$ ,  $ND(q)$ ,  $PI(q)$  and  $NI(q)$ . For this purpose, the following lemma is useful.

**LEMMA 2.** *Let  $p$  be a periodic point of  $T$  of minimal period  $r$ , and let  $k$  be a positive integer. Then the following properties hold.*

(a) If  $p$  is of type  $PD$ , then  $p$  is a periodic point of  $T$  of period  $kr$  of type  $PD$  for any  $k$ .

(b) If  $p$  is of type  $ND$ , then  $p$  is a periodic point of  $T$  of period  $kr$  of type  $ND$  for any  $k$ .

(c) If  $p$  is of type  $PI$ , then  $p$  is a periodic point of  $T$  of period  $kr$  of type  $PI$  (resp.  $PD$ ) for any odd  $k$  (resp. even  $k$ ).

(d) If  $p$  is of type  $NI$ , then  $p$  is a periodic point of  $T$  of period  $kr$  of type  $NI$  (resp.  $ND$ ) for any odd  $k$  (resp. even  $k$ ).

*Proof.* Since  $DT^{kr}(p) = (DT^r(p))^k$  for each  $k$ , Lemma 1 is easily derived from the definition.

Now we come back to the proof of Theorem 1.

Let  $q$  be any odd integer greater than 1. Let  $q = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ , where  $p_1, \dots, p_m$  are odd primes and  $\alpha_1, \dots, \alpha_m$  are positive integers. Now, we shall prove (8) by induction on  $s(q) = \alpha_1 + \cdots + \alpha_m$ .

If  $s(q) = 1$ , then  $q$  is an odd prime, and the number of the fixed points of  $T^q$  of type  $PD$  (resp.  $ND, PI, NI$ ) is equal to  $PD(1) + PD(q)$  (resp.  $ND(1) + ND(q), PI(1) + PI(q), NI(1) + NI(q)$ ) by Lemma 2. Therefore, by Proposition 8 applied to case  $k = q$ , we obtain the following equality.

$$(PD(1) + PD(q)) - (ND(1) + ND(q)) - (PI(1) + PI(q)) \\ + (ND(1) + ND(q)) = 1.$$

Subtracting (12) from the above equality, we obtain (8).

Assume that (8) holds for odd integers  $r$  with  $s(r) < s$  ( $s > 1$ ). We shall prove (8) for odd  $q$  such that  $s(q) = s$ .

By Lemma 2, the number of the fixed points of  $T^q$  of type  $PD$  (resp.  $ND, PI, NI$ ) is equal to  $\sum_{r|q} PD(r)$  (resp.  $\sum_{r|q} ND(r), \sum_{r|q} PI(r), \sum_{r|q} NI(r)$ ). Therefore, we have the following equality by Proposition 8.

$$\sum_{r|q} PD(r) - \sum_{r|q} ND(r) - \sum_{r|q} PI(r) + \sum_{r|q} NI(r) = 1$$

By our inductive assumption, we have the following equality

$$PD(r) - ND(r) - PI(r) + NI(r) = 0 \quad \text{if } r|q \text{ and } 1 < r < q.$$

Therefore, from the above equalities and (6), we obtain (8). Thus, (8) is proved for any odd  $q$  with  $q > 1$ , and (9) is an easy consequence of (8) and (5).

Let  $q$  be an even integer, and let  $q = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ , where  $p_1, \dots, p_m$  are odd primes and  $\alpha_0, \alpha_1, \dots, \alpha_m$  are positive integers. Now, we shall prove (10) by induction on  $t(q) = \alpha_0 + \alpha_1 + \cdots + \alpha_m$ .

If  $t = 1$ , then  $q = 2$ , and the number of the fixed point of  $T^q$  of type  $PD$  (resp.  $ND, PI, NI$ ) is equal to  $PD(1) + PD(2) + PI(1)$  (resp.  $ND(1) + ND(2) + NI(1), PI(2), NI(2)$ ) by Lemma 2. Therefore, the following equality holds by Proposition 8.

$$(PD(1) + PD(2) + PI(1)) - (ND(1) + ND(2) + NI(1)) \\ - PI(2) + NI(2) = 1$$

Subtracting (12) from the above equality, we obtain (10) for  $q = 2$ .

Assume that (10) holds for even integers  $r$  such that  $t(r) < t$  ( $t > 1$ ). We shall prove (10) for even  $q$  such that  $t(q) = t$ .

Let  $A = \{2^{\beta_0} p_1^{\beta_1} \cdots p_m^{\beta_m}; 1 \leq \beta_0 \leq \alpha_0 - 1, 0 \leq \beta_i \leq \alpha_i, i = 1, \dots, m\}$   
 $B = \{p_1^{\beta_1} \cdots p_m^{\beta_m}; 0 \leq \beta_i \leq \alpha_i, i = 1, \dots, m\}$  and  $C = \{2^{\alpha_0} p_1^{\beta_1} \cdots p_m^{\beta_m}; 0 \leq \beta_i \leq \alpha_i, i = 1, \dots, m\}$ . Then, the set  $\{r; r|q\}$  is a disjoint union of A, B and C.

By Lemma 2 the number of the fixed points of  $T^q$  of type  $PD$  (resp.  $ND, PI, NI$ ) is equal to  $\sum_{r|q} PD(r) + \sum_{r \in A} PI(r) + \sum_{r \in B} PI(r)$  (resp.  $\sum_{r|q} ND(r) + \sum_{r \in A} NI(r) + \sum_{r \in B} NI(r)$ ,  $\sum_{r \in C} PI(r)$ ,  $\sum_{r \in C} NI(r)$ ). Therefore, the following equality holds by Proposition 8.

$$\left( \sum_{r|q} PD(r) + \sum_{r \in A} PI(r) + \sum_{r \in B} PI(r) \right) - \left( \sum_{r|q} ND(r) + \sum_{r \in A} NI(r) + \sum_{r \in B} NI(r) \right) - \sum_{r \in C} PI(r) + \sum_{r \in C} NI(r) = 1$$

This is rewritten as follows.

$$(13) \quad \begin{aligned} & \sum_{r \in A} (PD(r) - ND(r) + PI(r) - NI(r)) \\ & + \sum_{r \in B} (PD(r) - ND(r) + PI(r) - NI(r)) \\ & + \sum_{r \in C} (PD(r) - ND(r) - PI(r) + NI(r)) = 1 \end{aligned}$$

By (6), (8) and the inductive hypothesis, we have the following equalities.

$$PD(r) - ND(r) = PI(r) - NI(r) - 2(PI(r/2) - NI(r/2)) \quad \text{for } r \in A \cup C, r \neq q.$$

$$PD(r) - ND(r) = PI(r) - NI(r) \quad \text{for } r \in B, r \neq 1.$$

$$PD(1) - ND(1) = PI(1) - NI(1) + 1.$$

Putting these equalities into (13), we obtain the following equality.

$$\begin{aligned} & 2 \sum_{r \in A} (PI(r) - NI(r)) - 2 \sum_{r \in A} (PI(r/2) - NI(r/2)) \\ & + 2 \sum_{r \in B} (PI(r) - NI(r)) - 2 \sum_{\substack{r \in C \\ r \neq q}} (PI(r/2) - NI(r/2)) \\ & + PD(q) - ND(q) - PI(q) + NI(q) = 0. \end{aligned}$$

By simplifying the above equality, we obtain the following equality.

$$2(PI(q/2) - NI(q/2)) + PD(q) - ND(q) - PI(q) + NI(q) = 0$$

This proves (10), and (11) is an easy consequence of (5) and (10).

DEFINITION 2. Consider the equation (1) satisfying the condition (A). The equation (1) is called a  $D'$ -system if there exists a subset  $K$  of  $\mathbf{R}^n$  satisfying the following two conditions.

(i)  $K$  is homeomorphic to a closed  $n$ -disk.

(ii) For any solution  $x(t)$  of (1) there exists a suitable number  $t_0 \in \mathbf{R}$  such that  $x(t_0) \in K$ , and if  $x(t_1) \in K$  for some  $t_1 \in \mathbf{R}$ , then  $x(t) \in K$  for any  $t \geq t_1$ .

Levinson and Massera ([3], [5]) called the equation (1) satisfying the condition (A) a  $D$ -system if it satisfies the following condition.

(iii) There exist a positive number  $r$  and a positive integer  $N$  satisfying the following condition.

For any solution  $x(t)$  of (1) there exists a suitable number  $t_0 \in \mathbf{R}$  such that  $\|x(t_0)\| \leq r$  and  $\|x(t)\| \leq r$  for  $t \geq t_0 + N$ .

PROPOSITION 9. *If  $n = 2$ , a  $D'$ -system is a  $D$ -system.*

*Proof.* This is clear from the definitions.

EXAMPLE 3. (Duffing's Equation) In the equation

$$(14) \quad \frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = e(t),$$

We assume the following four conditions.

(i)  $f(x), g(x)$  and  $e(t)$  are of class  $C^1$ .

(ii)  $e(t)$  is periodic of period 1.

(iii) There exists a positive constant  $c$  such that  $f(x) \geq c$ .

(iv)  $g'(x) \geq 0$  and  $\lim_{x \rightarrow \infty} g(x) > E$ ,  $\lim_{x \rightarrow -\infty} g(x) < -E$ , where  $E = \max |e(t)|$ .

The equation (14) is equivalent to the following 2-dimensional system (14)'.  
(14)'

$$(14)' \quad \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -f(x)y - g(x) + e(t) \end{cases}$$

The equation (14)' (or (14)) is a  $D'$ -system (Cf. Loud [4], Shiraiwa [7]).

EXAMPLE 4. (Levinson-Langenhop-Opial) In the equation

$$(15) \quad \frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right) \frac{dx}{dt} + g(x) = e(t) ,$$

we assume the following five conditions.

- (i)  $f(x, v), g(x)$  and  $e(t)$  are of class  $C^1$ .
- (ii)  $e(t)$  is periodic of period 1.
- (iii) There exist positive numbers  $m$  and  $a$  such that

$$f(x, v) \geq m \quad \text{for } |x| \geq a, |v| \geq a .$$

- (iv) There exists a positive constant  $M$  such that

$$f(x, v) \geq -M .$$

- (v)  $\liminf_{x \rightarrow -\infty} g(x) > Ma + E$  and  $\limsup_{x \rightarrow -\infty} g(x) < -(Ma + E)$ , where  $E = \max |e(t)|$ .

The equation (15) is equivalent to the following (15)'.

$$(15)' \quad \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -f(x, y) - g(x) + e(t) \end{cases}$$

The equation (15)' (or (15)) is a  $D'$ -system (Cf. Opial [6]).

**THEOREM 2.** *Let the equation (1) be a  $D'$ -system, and let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the Poincaré transformation associated to the equation (1). Suppose that any periodic points of  $T$  is hyperbolic. Then the equalities (5)~(11) hold for the periodic points of  $T$ .*

*Proof.* By the definition of  $D'$ -system, the Poincaré transformation associated to a  $D'$ -system satisfies the assumptions of Theorem 1.

**COROLLARY 1.** *Under the same assumption of Theorem 2, Corollary of Theorem 1 holds for the Poincaré transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  associated to a  $D'$ -system (1).*

**COROLLARY 2.** (Levinson-Massera [3], [5]) *Suppose that the equation (1) is a  $D'$ -system and  $n = 2$ . Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the Poincaré transformation associated to the equation (1). Assume that all the periodic points of  $T$  are hyperbolic. We denote by  $C(q)$  (resp.  $D(q), I(q)$ ) the number of the completely unstable or stable (resp. directly unstable, inversely unstable) periodic points of  $T$  of minimal period  $q$  ( $q$ : a positive integer). Then, the following equalities hold.*

$$C(1) + I(1) = D(1) + 1.$$

$$C(q) + I(q) = D(q) \text{ if } q \text{ is odd and } q > 1.$$

$$C(q) + I(q) = D(q) + 2I(q/2) \text{ if } q \text{ is even.}$$

If we denote by  $N(q)$  the number of the periodic points of  $T$  of minimal period  $q$ , then the following equalities hold.

$$N(q) = C(q) + D(q) + I(q) \text{ for any } q.$$

$$N(1) = 2D(1) + 1.$$

$$N(q) = 2D(q) \text{ if } q \text{ is odd and } q > 1.$$

$$N(q) = 2D(q) + 2I(q/2) \text{ if } q \text{ is even.}$$

*Proof.* This follows easily from Theorem 2 and Example 2.

#### §4. An extension of Theorem 1 and 2

In this section we shall discuss an extension of Theorem 1 and 2 to the case where  $\mathbf{R}^n$  is replaced by a compact differentiable manifold of class  $C^1$ .

Let  $M$  be a compact differentiable  $n$ -dimensional manifold of class  $C^1$ , and let  $f: M \rightarrow M$  be a diffeomorphism of class  $C^1$ . For a fixed (or periodic) point of  $f$ , we can define the notion of hyperbolicity using a coordinate neighborhood. Also, the fixed point index can be defined similarly (Cf. [9], [1]).

The following theorem is well known (Cf. [1]).

**THEOREM.** (*Lefschetz*) Let  $f: M \rightarrow M$  be a continuous map such that all the fixed points of  $f$  are isolated. Let  $f_{*i}: H_i(M) \rightarrow H_i(M)$  be the induced homomorphism on the  $i$ -th homology group  $H_i(M)$  with coefficients in  $\mathbf{R}$ . Put  $L(f) = \sum_{i=0}^n (-1)^i \text{Trace } f_{*i}$  (the Lefschetz number).

Then the following equality holds.

$$\sum_{f(p)=p} \text{index}_f(p) = L(f)$$

**COROLLARY.** In addition to the hypothesis of the above theorem, we assume that  $f$  is homotopic to the identity. Set  $\chi(M) = \sum_{i=0}^n (-1)^i \dim H_i(M)$  (Euler characteristic of  $M$ ), where  $\dim H_i(M)$  is the dimension of  $H_i(M)$  as a vector space over  $\mathbf{R}$ . Then,

$$\sum_{f(p)=p} \text{index}_f(p) = \chi(M).$$

*Proof.* If  $f$  is homotopic to the identity, then  $L(f) = \chi(M)$ .



**THEOREM 3.** *Let  $f: M \rightarrow M$  be a diffeomorphism of class  $C^1$  such that all the periodic points of  $f$  are hyperbolic. Further, we assume that  $L(f) = L(f^k)$  for any positive integer  $k$ . Then the following equalities hold.*

$$PD(1) + NI(1) = ND(1) + PI(1) + L(f).$$

$$PD(q) + NI(q) = ND(q) + PI(q) \text{ if } q \text{ is odd and } q > 1.$$

$$PD(q) + NI(q) + 2PI(q/2) = ND(q) + PI(q) + 2NI(q/2) \text{ if } q \text{ is even.}$$

*In the above equalities,  $PD(q)$  (resp.  $ND(q)$ ,  $PI(q)$ ,  $NI(q)$ ) is the number of the periodic points of  $f$  of minimal period  $q$  of type  $PD$  (resp.  $ND, PI, NI$ ). And if we denote by  $N(q)$  the number of periodic points of  $f$  of minimal period  $q$ , then the following equalities hold.*

$$N(q) = PD(q) + ND(q) + PI(q) + NI(q)$$

$$N(1) = 2(ND(1) + PI(1)) + L(f)$$

$$N(q) = 2(ND(q) + PI(q)) \text{ if } q \text{ is odd and } q > 1.$$

$$N(q) = 2(ND(q) + PI(q) + NI(q/2) - PI(q/2)) \text{ if } q \text{ is even.}$$

*Proof.* Theorem 3 is proved similarly to Theorem 1.

Let  $X_t$ ,  $t \in \mathbf{R}$  be a time dependent vector field of class  $C^1$  on  $M$ . Assume that  $X_t$  is periodic of period 1 with respect to the variable  $t$ . Then it is easy to see that there exists a unique solution  $x = \varphi(t; t_0, x)$  of  $X_t$  defined on  $-\infty < t < +\infty$  for any initial value  $(t_0, x_0) \in \mathbf{R} \times M$ .

Now define a transformation  $f: M \rightarrow M$  by  $f(x) = \varphi(1; 0, x)$ ,  $x \in M$ . We call this  $f$  the Poincaré transformation associated to the periodic time dependent system  $X_t$ .

As Proposition 1, we can prove that  $f$  is a diffeomorphism of class  $C^1$  and is isotopic to the identity. Therefore,  $f$  is homotopic to the identity, and  $f$  is orientation preserving if  $M$  is oriented.

**THEOREM 4.** *Let  $X_t$ ,  $t \in \mathbf{R}$  be a differentiable time dependent vector field of class  $C^1$  on a compact differentiable manifold  $M$  of dimension  $n$ . And assume that  $X_t$  is periodic of period 1 with respect to  $t$ . Now, let  $f: M \rightarrow M$  be the Poincaré transformation associated to  $X_t$ . Assume further that all the periodic points of  $f$  are hyperbolic. Then the following equalities hold.*

$$PD(1) + NI(1) = ND(1) + PI(1) + \chi(M).$$

$$PD(q) + NI(q) = ND(q) + PI(q) \text{ if } q \text{ is odd and } q > 1.$$

$$PD(q) + NI(q) + 2PI(q/2) = ND(q) + PI(q) + 2NI(q/2) \text{ if } q \text{ is even.}$$

$$N(q) = PD(q) + ND(q) + PI(q) + NI(q) \text{ for any } q.$$

$$N(1) = 2(ND(1) + PI(1)) + \chi(M).$$

$N(q) = 2(ND(q) + PI(q))$  if  $q$  is odd and  $q > 1$ .

$N(q) = 2(ND(q) + PI(q) + NI(q/2) - P(q/2))$  if  $q$  is even.

Here  $PD(q)$ ,  $ND(q)$ ,  $PI(q)$ ,  $NI(q)$ ,  $N(q)$  and  $\chi(M)$  are defined as above.

*Proof.* Theorem 4 is proved from Theorem 3 and the fact that  $L(f^k) = \chi(M)$  for any positive integer  $k$  in our case since  $f$  is homotopic to the identity.

#### REFERENCES

- [ 1 ] A. Dold: Fixed Point Index and Fixed Point Theorem for Euclidean Neighborhood Retracts, *Topology* **4** (1965), 1–8.
- [ 2 ] P. Hartman: Ordinary Differential Equations, John Wiley & Sons (1964).
- [ 3 ] N. Levinson: Transformation Theory of Non-linear Differential Equations of the Second Order, *Ann. of Math.* **45** (1944), 723–737. Corrections, *ibid.* **49** (1948), 738.
- [ 4 ] W. S. Loud: Boundedness and Convergence of Solutions of  $x'' + cx' + g(x) = e(t)$ , *Duke Math. J.* **24** (1957), 63–72.
- [ 5 ] J. L. Massera: The Number of Subharmonic Solutions of Non-Linear Differential Equations of the Second Order, *Ann. of Math.* **50** (1949), 118–126.
- [ 6 ] Z. Opial: Démonstration d'un Théorème de N. Levinson et C. Langenhop, *Ann. Polon. Math.* **7** (1960), 241–246.
- [ 7 ] K. Shiraiwa: Boundedness and Convergence of Solutions of Duffing's Equation, *Nagoya Math. J.*, **66** (1977), 151–166.
- [ 8 ] S. Smale: Stable Manifolds for Differential Equations and Diffeomorphisms, *Ann. Scuola Norm Sup. Pisa* (3) **17** (1963), 97–116.
- [ 9 ] S. Smale: Differentiable Dynamical Systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

*Department of Mathematics  
College of General Education  
Nagoya University*