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ANALYTIC TORSION OF SPACE FORMS OF CERTAIN COMPACT SYMMETRIC SPACES

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Introduction

Let M be a compact, oriented Riemannian manifold of dimension d, and let Γ be the fundamental group of M. For a finite dimensional representation ρ of Γ on a vector space F, Ray and Singer [10] have defined the *analytic torsion* $T(M,\rho)$ as follows: We denote by E the vector bundle over M with typical fibre F defined by the representation ρ. Let $A^p(E)$ be the space of E-valued p forms on M. Let Δ^p be the Laplacian (cf. § 1) on $A^p(E)$, and let $H^p(E)$ be the space of harmonic forms in $A^p(E)$. Then

$$\zeta_p(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\{\mathrm{tr}\ e^{-t\Delta^p} - \dim H^p(E)\}dt$$

is (cf. [10]) an analytic function of s for large $\mathrm{Re}\,(s)$ and it extends (cf. [10]) to a meromorphic function in the s-plane which is analytic at $s=0$. The analytic torsion $T(M,\rho)$ is defined (cf. [10]) as the positive root of

$$\log T(M,\rho) = \frac{1}{2}\sum_{p=0}^{d}(-1)^p p\zeta_p'(0)\ .$$

They have showed (cf. [10]) that if $H^p(E)=(0)$ $(0\le p\le d)$, then the analytic torsion $T(M,\rho)$ does not depend on the Riemannian metrics on M. Ray [9] has calculated the analytic torsion $T(M,\rho)$ for lens spaces, and also obtained that $T(M,\rho)$ coincides the Reidemeister torsion (cf. [10]) for lens spaces.

The purpose of this paper is to compute the analytic torsion $T(M,\rho)$ for space forms of certain compact symmetric spaces.

Let G be a compact simply connected Lie group, and let $\tilde{M}=G/K$ be a simply connected compact globally symmetric space (cf. [5]). Let

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Γ be a discrete subgroup of G acting fixed point freely on $\tilde{M}$. Then the fundamental group of the orbit space $M = \Gamma\backslash\tilde{M}$ (called a *space form* of $\tilde{M}$ [16]) of Γ in $\tilde{M}$ is isomorphic to Γ. Let ρ_Γ be the representation restricted to Γ of a finite dimensional unitary representation ρ of G. Then our main result (cf. Corollary 3.1 in § 3) can be stated that

$$\textit{if } \operatorname{rank} G - \operatorname{rank} K \neq 1 , \quad \textit{then } T(M, \rho_\Gamma) = 1 ,$$

which is proved in § 3 using the explicit formula (cf. Theorem 2.2 in § 2) of the fundamental solution of the heat equation. To obtain this formula we devote in § 1 and a part of § 2 to review the harmonic theory in [7] for $A^p(E)$ in case of a compact symmetric space $\tilde{M}$.

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§ 1. Preliminary

1.1. Analytic torsion

Let M be a compact orientable Riemannian manifold of dimension d, and Γ the fundamental group of M. We denote by $\tilde{M}$ the universal covering manifold of M, and by ϖ the projection of $\tilde{M}$ onto M. The fundamental group Γ of M operates on $\tilde{M}$, and we denote by τ_γ the operation on $\tilde{M}$ of an element $\gamma \in \Gamma$. Let ρ be a representation of Γ in a vector space F. Γ operates on $\tilde{M} \times F$ by

$$\gamma(x, u) = (\tau_\gamma x, \rho(\gamma)u) , \qquad x \in \tilde{M} , \ u \in F , \quad \gamma \in \Gamma .$$

The quotient manifold $E = = \Gamma\backslash(\tilde{M} \times F)$ has a vector bundle structure over M with typical fibre F. Let $A^p(E)$ be the space of all E-valued p-forms on M. Since the vector bundle E is locally constant i.e. it is given by a system of locally constant transition functions, a coboundary operator d of degree 1 on the graded module $A(E) = \sum_{p=0}^{d} A^p(E)$ can be defined in a natural way. Let E^* be the dual vector bundle of E. Then for $\theta \in A^p(E)$ and $\omega \in A^q(E^*)$, a differentiable real valued $(p + q)$ form ${}^t\theta \wedge \omega$ on M is defined as usual (cf. Part I § 2, [7]). We assume that an inner product is given on each fibre of E which depends differentiably on the base manifold M (cf. [7]). The Riemannian metric of M and the inner product of the fibre bundle E give (cf. [7]) the linear isomorphism

$$\sharp : A^p(E) \longrightarrow A^p(E^*) .$$

The Riemannian metric of M defines the operator $*$ on real valued forms on M as usual, and we extend (cf. [7]) this operator $*$ linearly to $A^p(E)$. For $\theta, \omega \in A^p(E)$, we can define

$$(\theta, \omega) = \int_M {}^t\theta \wedge *\sharp\omega .$$

We define the operator ∂ of degree 1 on the graded module $A(E) = \sum_{p=0}^{d} A^p(E)$ so that $\sharp(\partial\theta) = d(\sharp\theta)$ holds for all $\theta \in A(E)$. Put

$$\delta\theta = (-1)^{dp+d+1} * \partial * \theta$$

for all $\theta \in A^p(E)$. Then δ is an operator of degree -1 on $A(E)$ and

$$(\delta\theta, \omega) = (\theta, d\omega)$$

holds for all $\theta, \omega \in A^p(E)$. We define the Laplacian Δ^p on $A^p(E)$ by putting

$$\Delta^p = d\delta + \delta d .$$

Let $L_2^p(E)$ be the completion of $A^p(E)$ with respect to the inner product (,) and let

$$A_\lambda^p(E) = \{\theta \in A^p(E) : \Delta^p\theta = \lambda\theta\}$$

for $\lambda \in \boldsymbol{R}$. Put $H^p(E) = A_0^p(E)$. Then it is known (cf. [1]) that each $A_\lambda^p(E)$ is finite dimensional $(\lambda \in \boldsymbol{R}), A_\lambda^p(E) = 0$ except for a discrete set of non-negative λ's and this countable sequence of subspaces $A_\lambda^p(E)$ gives an orthogonal direct sum decomposition of $L_2^p(E)$:

$$L_2^p(E) = \sum_\lambda A_\lambda^p(E) .$$

Moreover the series

$$Z^p(t) = \sum_\lambda e^{-\lambda t} \dim (A_\lambda^p(E)) \tag{1.1}$$

converges (cf. [10]) for every $t > 0$ and

$$\begin{aligned}\zeta_p(s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(Z^p(t) - \dim H^p(E))dt \\ &= \sum_{\lambda>0} \lambda^{-s} \dim A_\lambda^p(E)\end{aligned}$$

is (cf. [10]) an analytic function of s for large Re(s) and it can be extended (cf. [10]) to a meromorphic function of s-plane, which is analytic at $s = 0$.

DEFINITION. The *analytic torsion* $T(M, \rho)$ of the Riemannian manifold M is defined (cf. [10]) as the positive real root of

$$\log T(M, \rho) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p p \zeta_p'(0) . \tag{1.2}$$

1.2. The space form of Riemannian symetric space

Let G be a compact simply connected (necessarily semisimple) Lie group of dimension n. Let θ be a C^∞ involutive automorphism of G. Let K be the subgroup of G consisting of all fixed points of θ. Then K is connected and the coset space $\tilde{M} = G/K$ is a simple connected, compact, globally symmetric space (cf. [5] Theorem 7.2 Ch. VII). Let Γ be a discrete subgroup of G acting fixed point freely on $\tilde{M}$. Then $\tilde{M}$ is the universal covering manifold of the quotient manifold $M = \Gamma\backslash\tilde{M}$ which is called a *space form* of a symmetric space $\tilde{M}$ (cf. [16]). The fundamental group of M is isomorphic to Γ. Let ρ be a finite dimensional unitary representation of G on a complex vector space F. Let $E = E_\rho$ be the vector bundle over M with typical fibre F associated to the representation restricted to Γ of ρ. The projections of $\tilde{M}$ onto M, of G onto $\Gamma\backslash G$ are denoted respectively by ϖ and ϖ_0 and the projections of $\Gamma\backslash G$ onto M, of G onto $\tilde{M}$ are denoted respectively by π and π_0. Then $\Gamma\backslash G$ has a principal fibre bundle of a group K with a projection π. Let ρ_K be the restriction of ρ to K. Then the vector bundle E is (cf. [7] Prop. 3.1) associated to the principal fibre bundle $\Gamma\backslash G$ by the representation ρ_K of the group K. Let $(\ , \ )_F$ be the inner product in the space F invariant under $\rho(g)$, $g \in G$. Since $(\ , \ )_F$ is invariant under $\rho(K)$, it may define canonically a metric in the fibres of E.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to K. Let $\mathfrak{p} = \{X \in \mathfrak{g}; \theta X = -X\}$. In this paper we use the same letter for a differential mapping and its differential. Let B be the Killing form of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (the direct sum) and $B(X, Y) = 0$ $(X \in \mathfrak{k},\ Y \in \mathfrak{p})$. We may identify $\mathfrak{p}$ with the tangent space $T_0\tilde{M}$ at the origin $0 = \{K\} \in \tilde{M}$ in a natural way. Then the Killing form B which is negative definite and invariant under the $\mathrm{Ad}(K)$ action on $\mathfrak{p}$ allows us to define a Riemannian metric $\tilde{g}$ on $\tilde{M}$ such that $\tilde{g}_0 = -B$ on $T_0\tilde{M} \times T_0\tilde{M}$. Γ preserves this metric $\tilde{g}$ on $\tilde{M}$ and, so, there is a Riemannian metric g on M so that $\varpi^* g = \tilde{g}$.

Let $\{X_1, \cdots, X_d, X_{d+1}, \cdots, X_n\}$ be a basis of $\mathfrak{g}$ such that i) $B(X_i, X_j)$

$= -\delta_{ij}$ ii) $\{X_1, \cdots, X_d\}$ spans $\mathfrak{p}$ and iii) $\{X_{d+1}, \cdots, X_n\}$ spans $\mathfrak{k}$. Since the element X of $\mathfrak{g}$ can be considered as a left invariant vector field on G, the vector field X is projectable to a vector field $\varpi_0(X)$ on $\Gamma\backslash G$. Since this mapping $X \mapsto \varpi_0(X)$ is an injective homomorphism of $\mathfrak{g}$ into the Lie algebra of all vector fields on $\Gamma\backslash G$, we shall identify X with $\varpi_0(X)$.

Let $\{\omega^1, \cdots, \omega^n\}$ be the dual basis of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ with respect to $\{X_1, \cdots, X_n\}$. Then they can be considered as left invariant forms on G and so are Γ invariant; then there is a form on $\Gamma\backslash G$ which induces ω^i through ϖ_0. We shall denote also this form by ω^i. Let h be a Riemannian metric on $\Gamma\backslash G$ such that $\varpi_0^* h = g$. The volume element dv associated to this metric h is given by $dv = \omega^1 \wedge \cdots \wedge \omega^n$. Since K is connected, we can define a G invariant orientation on $\tilde{M}$ so that $\{X_1, \cdots, X_d\}$ is positively oriented. Since Γ preserves this orientation, we can define an orientation of M such that the projection ϖ is orientation preserving. Let dm be the volume element on M defined by g. Moreover we denote by $dk^{\#}$ the invariant volume element $\omega^{d+1} \wedge \cdots \wedge \omega^n$ on K, where $\omega^{d+1}, \cdots, \omega^n$ are considered as left invariant 1 forms on K. Then for every continuous function f on $\Gamma\backslash G$, we have (cf. [7] Lemma 5.2)

$$\int_{\Gamma\backslash G} f(y)dv = \int_M dm\Big(\int_K f(R_k y)dk^{\#}\Big) \tag{1.3}$$

where R_k is the action of $k \in K$ on $\Gamma\backslash G$ and $\int_K f(R_k y)dk^{\#}$ is regarded as a function on M. In particular, if f' is a continuous function on M, then we have (cf. [7] Lemma 5.3)

$$\int_M f' dm = \frac{1}{\mathrm{vol}\,(K)} \int_{\Gamma/G} (f' \circ \pi)dv\ . \tag{1.4}$$

1.3. The inner product of $A^p(E)$

Let $A^p(\Gamma, \tilde{M}, \rho)$ be the space of all F valued p forms on $\tilde{M}$ such that

$$\tau_\gamma^* \eta = \rho(\gamma)\eta\ , \qquad \gamma \in \Gamma\ .$$

We denote also by d the exterior differentiation on $A^p(\Gamma, \tilde{M}, \rho)$ which defines a coboundary operator of degree 1 on the graded module $A(\Gamma, \tilde{M}, \rho) = \sum_{p=0}^d A^p(\Gamma, \tilde{M}, \rho)$. For $\eta \in A^p(\Gamma, \tilde{M}, \rho)$, define θ in $A^p(E)$ by

$$\theta_{\varpi(x)}(\varpi(L_1), \cdots, \varpi(L^p)) = \varpi_x(\eta_x(L_1, \cdots, L^p))$$

for $x \in \tilde{M}$ and $L_1, \cdots, L^p \in T_x(\tilde{M})$ where ϖ_x is the linear isomorphism of F onto the fibre $E_{\varpi(x)}$ of E over $\varpi(x)$ defined by $\varpi_x(u) = \varpi(x, u)$, $u \in F$. Here ϖ is the natural projection of $\tilde{M} \times F$ onto E. Then the mapping $\eta \to \theta$ defines (cf. [7] p. 369) an isomorphism of the complex $A(\Gamma, \tilde{M}, \rho)$ onto the complex $A(E)$.

Let $A^p(\Gamma \backslash G, K, \rho)$ be the space of all F valued p forms on $\Gamma \backslash G$ such that (i) $\theta(X)\eta^0 = -\rho(X)\eta^0$, $X \in \mathfrak{k}$ (ii) $i(X)\eta^0 = 0$, $X \in \mathfrak{k}$ where $\theta(X)$ is the Lie derivation by X and $i(X)$ is the interior product by X.

For $\eta \in A^p(\Gamma, \tilde{M}, \rho)$, define $\tilde{\eta}$ by

$$\tilde{\eta}_g = \rho(g^{-1})(\pi_0^*\eta)_g\,, \qquad g \in G\,.$$

Then there exists uniquely an element $\eta^0 \in A^p(\Gamma \backslash G, K, \rho)$ such that $\tilde{\eta} = \varpi_0^*\eta^0$. The mapping $\eta \mapsto \eta^0$ defines (cf. [7] p. 376) a linear isomorphism of $A^p(\Gamma, \tilde{M}, \rho)$ onto $A^p(\Gamma \backslash G, K, \rho)$. Define a coboundary operator d^0 on the graded module $A(\Gamma \backslash G, K, \rho) = \sum_{p=0}^{d} A^p(\Gamma \backslash G, K, \rho)$ such a way that $d^0\eta^0 = (d\eta)^0$ for $\eta \in A^p(\Gamma, \tilde{M}, \rho)$.

For an F valued p form η^0 on $\Gamma \backslash G$, we define a system of F valued functions $\{\tilde{\eta}_{i_1 \cdots i_p}; 1 \leq i_1 < \cdots < i^p \leq d\}$ on $\Gamma \backslash G$ by

$$\tilde{\eta}_{i_1 \cdots i_p} = \eta^0(X_{i_1}, \cdots, X_{i_p})\,.$$

For $\eta^0 \in A^p(\Gamma \backslash G, K, \rho), \tilde{\eta}_{i_1 \cdots i_p} = 0$ if there exists some $i_\nu > d$.

There corresponds to each form $\theta \in A^p(E)$ a form $\eta \in A^p(\Gamma, \tilde{M}, \rho)$ and to each form $\eta \in A^p(\Gamma, \tilde{M}, \rho)$ corresponds a form $\eta^0 \in A^p(\Gamma \backslash G, K, \rho)$. Moreover the form η^0 is determined by the system $\{\tilde{\eta}_{i_1 \cdots i_p}\}$. Then the inner product (,) in $A^p(E)$ is given as follows: For $\theta, \omega \in A^p(E)$, then

$$(1.5) \qquad (\theta, \omega) = \frac{1}{\mathrm{vol}\,(K)p\,!} \sum_{i_1, \cdots, i_p = 1}^{d} \int_{\Gamma \backslash G} (\tilde{\eta}_{i_1 \cdots, p}, \tilde{\zeta}_{i_1 \cdots i_p})_F dv$$

where $\{\tilde{\eta}_{i_1 \cdots i_p}\}$ (resp. $\{\tilde{\zeta}_{i_1 \cdots i_p}\}$) is the system of F valued functions on $\Gamma \backslash G$ corresponding to θ (resp. ω) (cf. [7] Prop. 5.1),

Let the inner product (,) in $A^p(\Gamma, \tilde{M}, \rho)$ by $(\eta, \zeta) = (\theta, \omega)$ where η (resp. ζ) $\in A^p(\Gamma, \tilde{M}, \rho)$ corresponds to θ (resp. ω) $\in A^p(E)$. Let $L_2^p(\Gamma, \tilde{M}, \rho)$ be the completion of $A^p(\Gamma, \tilde{M}, \rho)$ with respect to this inner product.

1.4. The Laplacian on $A^p(\Gamma, \tilde{M}, \rho)$

We shall use the following convection for the ranges of indices: $1 \leqq \lambda, \mu, \cdots \leq n$; $1 \leq i, j, \cdots \leq d$ and $d + 1 \leq a, b, \cdots \leq n$. Let $[X_\lambda, X_\mu] = \sum c_{\lambda\mu}^{\nu} X_\nu$. Then in case of G compact, we have the following relation:

$$\begin{cases} c^k_{ij} = c^b_{ka} = c^k_{ab} = 0 \\ c^a_{ij} = -c^i_{aj} = c^i_{ja} = -c^j_{ia} \,. \end{cases}$$

LEMMA 1.1. *For* $\eta \in A^p(\Gamma, \tilde{M}, \rho)$, *we have*

$$(d\eta)^{\sim}_{i_1\cdots i_{p+1}} = \sum_{u=1}^{p+1} (-1)^{u-1}(X_{i_u} + \rho(X_{i_u}))\tilde{\eta}_{i_1\cdots\hat{i}_u\cdots i_{p+1}} \,.$$

For a proof, see [7] Prop. 4.1.

LEMMA 1.2. *There exists an operator* δ *of degree* -1 *on the complex* $A(\Gamma, \tilde{M}, \rho)$ *such that*

$$(\delta\eta, \zeta) = (\eta, d\zeta) \,, \qquad \text{for } \eta, \zeta \in A(\Gamma, \tilde{M}, \rho) \,.$$

Moreover for $A^p(\Gamma, \tilde{M}, \rho)$, *we have*

$$\begin{aligned} &(\delta\eta)^{\sim}_{i_1\cdots i_{p-1}} = -\sum_{k=1}^{d} (X_k + \rho(X_k))\tilde{\eta}_{k i_1\cdots i_{p-1}} && (p \geq 1) \,, \\ &\delta\eta = 0 && (p = 0) \,. \end{aligned}$$

Proof. Since the case $p = 0$ is trivial, we may assume $p \geqq 1$. Let $\zeta \in A^{p-1}(\Gamma, \tilde{M}, \rho)$. By (1.5) and Lemma 1.2,

$$\begin{aligned} (\eta, d\zeta) &= \frac{1}{\mathrm{vol}\,(K)p\,!} \\ &\quad \times \sum_{i_1,\cdots,i_p=1}^{d} \int_{\Gamma\backslash G} \left(\tilde{\eta}_{i_1\cdots i_p}, \sum_{u=1}^{p} (-1)^{u-1}(X_{i_u} + \rho(X_{i_u}))\tilde{\zeta}_{i_1\cdots\hat{i}_u\cdots i_p}\right)_F dv \\ &= \frac{1}{\mathrm{vol}\,(K)p\,!} \\ &\quad \times \sum_{i_1,\cdots,i_p=1}^{d} \sum_{u=1}^{p} \int_{\Gamma\backslash G} (\eta_{i_u i_1\cdots i_p}, (X_{i_u} + \rho(X_{i_u}))\tilde{\zeta}_{i_1\cdots\hat{i}_u\cdots i_p})_F dv \\ &= \frac{1}{\mathrm{vol}\,(K)(p-1)\,!} \\ &\quad \times \sum_{j_1,\cdots,j_{p-1}=1}^{d} \sum_{k=1}^{d} \int_{\Gamma\backslash G} (\tilde{\eta}_{k j_1\cdots j_{p-1}}, (X_k + \rho(X_k))\tilde{\zeta}_{j_1\cdots j_{p-1}})_F dv \\ &= \frac{1}{\mathrm{vol}\,(K)(p-1)\,!} \\ &\quad \times \sum_{j_1,\cdots,j_{p-1}=1}^{d} \int_{\Gamma\backslash G} \left(-\sum_{k=1}^{d} (X_k + \rho(X_k))\tilde{\eta}_{k j_1\cdots j_{p-1}}, \tilde{\zeta}_{j_1\cdots j_{p-1}}\right)_F dv \end{aligned}$$

since the last equality follows from that $(\rho(X)u, v)_F = -(u, \rho(X)v)_F$ $X \in \mathfrak{g}$, $u, v \in F$ and that $\int_{\Gamma\backslash G} (Xf_1, f_2)_F dv = -\int_{\Gamma\backslash G} (f_1, Xf_2)_F dv$ for $X \in \mathfrak{g}$, F valued C^∞ functions f_1, f_2 on $\Gamma\backslash G$ (cf. [7] Lem. 5.1).

Put

$$\tilde{\theta}_{j_1\cdots j_{p-1}} = -\sum_{k=1}^{d} (X_k + \rho(X_k))\tilde{\eta}_{kj_1\cdots j_{p-1}}$$

and define an F valued $(p-1)$ form θ^0 on $\Gamma\backslash G$ by

$$\theta^0 = \frac{1}{(p-1)!} \sum_{j_1,\cdots,j_{p-1}=1}^{d} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{p-1}} .$$

Then $\theta^0(X_{j_1}, \cdots, X_{j_{p-1}}) = \tilde{\theta}_{j_1\cdots j_{p-1}}$ and $\theta^0 \in A^{p-1}(\Gamma\backslash G, K, \rho)$. Let $\theta \in A^{p-1}(\Gamma, \tilde{M}, \rho)$ which corresponds to θ^0, and define the operator δ by $\delta\eta = \theta$. Then we have $(\delta\eta)^{\sim}_{j_1\cdots j_{p-1}} = \tilde{\theta}_{j_1\cdots j_{p-1}}$ and $(\delta\eta, \zeta) = (\eta, d\zeta)$. Q.E.D.

We define the Laplacian operator Δ^p by $\Delta^p = d\delta + \delta d$ on $A^p(\Gamma, \tilde{M}, \rho)$. Then the isomorphism $A^p(E) \ni \theta \mapsto \eta \in A^p(\Gamma, \tilde{M}, \rho)$ transforms the operators δ, Δ^p in $A^p(E)$ to the operators δ, Δ^p in $A^p(\Gamma, \tilde{M}, \rho)$. For $\lambda \in \boldsymbol{R}$, let $A^p_\lambda(\Gamma, \tilde{M}, \rho) = \{\eta \in A^p(\Gamma, \tilde{M}, \rho) : \Delta^p\eta = \lambda\eta\}$. Then this isomorphism induces the isomorphism of $A^p_\lambda(E)$ onto $A^p_\lambda(\Gamma, \tilde{M}, \rho)$.

PROPOSITION 1.1. *For $\eta \in A^p(\Gamma, \tilde{M}, \rho)$, we have*

$$(\Delta^p\eta_{i_1\cdots i_p})^{\sim} = -\sum_{\nu=1}^{n} (X_\nu + \rho(X_\nu))^2\tilde{\eta}_{i_1\cdots i_p} .$$

Proof. Let $p \geqq 1$. For $\eta \in A^p(\Gamma, \tilde{M}, \rho)$, we have

$$\begin{aligned} (\Delta^p\eta)^{\sim}_{i_1\cdots i_p} = &-\sum_{k=1}^{d} (X_k + \rho(X_k))^2\tilde{\eta}_{i_1\cdots i_p} \\ &+ \sum_{k=1}^{d}\sum_{u=1}^{p} (-1)^{u-1}\{[X_k, X_{i_u}] + \rho([X_k, X_{i_u}])\}\tilde{\eta}_{ki_1\cdots \hat{i}_u\cdots i_p} \end{aligned} \tag{1.6}$$

from Lemma 1.2 and Lemma 1.2. Since η^0 satisfies $\theta(X)\eta^0 = -\rho(X)\eta^0$, $X \in \mathfrak{k}$ and $c^k_{ai_u} = -c^a_{ki_u}$, we have

$$(X_a + \rho(X_a))\tilde{\eta}_{i_1\cdots i_p} = -\sum_{u=1}^{p}\sum_{k=1}^{d} c^a_{ki_u}\tilde{\eta}_{i_1\cdots (k)_u\cdots i_p} \tag{1.7}$$

where $(k)_u$ denotes that the index i_u is replaced by the index k. Then by (1.7), the second term of (1.6) coincides with

$$\begin{aligned} &\sum_{a=d+1}^{n} (X_a + \rho(X_a))\left(\sum_{k=1}^{d}\sum_{u=1}^{p} c^a_{ki_u}\tilde{\eta}_{i_1\cdots(k)_u\cdots i_p}\right) \\ &= -\sum_{a=d+1}^{n} (X_a + \rho(X_a))^2\tilde{\eta}_{i_1\cdots i_p} . \end{aligned}$$

For $p = 0$, if $\eta \in A^0(\Gamma, \tilde{M}, \rho)\eta^0$ satisfies

$$(X_a + \rho(X_a))\eta^0 = 0 .$$

Then $(\Delta^p\eta)^0 = -\sum_{\nu=1}^n (X_\nu + \rho(X_\nu))^2\eta^0$. Q.E.D.

§ 2. Fundamental solution of the heat equation

2.1. Space $C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)^0$

To calculate the series $Z^p(t)$ (1.1), we have to estimate the fundamental solution (cf. [6]) of the heat equation

$$\frac{\partial u_t}{\partial t} = -\Delta^p u_t \qquad (t > 0)\ ,\ u_t \in A^p(E)\ .$$

But we shall transform this equation to the equation on the space $C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)^0$ which is isometrically isomorphic to $A^p(E)$, and construct (cf. Theorem 2.1) the fundamental solution of this transformed equation on $C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)^0$ which will be used to calculate the series $Z^p(t)$.

Let $\mathfrak{p}^*$ be the dual space of $\mathfrak{p}$. The adjoint action of K on $\mathfrak{p}$ induces the action of K on the exterior tensor product $\bigwedge^p \mathfrak{p}^*$ of $\mathfrak{p}^*$ such that for $1 \leq i_1 < \cdots < i^p \leq d$,

$$\mathrm{Ad}_p^*(k)(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) = \mathrm{Ad}^*(k)_{\mathfrak{p}}\omega^{i_1} \wedge \cdots \wedge \mathrm{Ad}^*(k)_{\mathfrak{p}}\omega^{i_p}$$

where $\mathrm{Ad}^*(k)_{\mathfrak{p}}\omega = {}^t\mathrm{Ad}(k^{-1})_{\mathfrak{p}}\omega$, $\omega \in \mathfrak{p}^*$, $k \in K$. Here ${}^t\mathrm{Ad}(k)^p$ is the transposed action of the adjoint action $\mathrm{Ad}(k)_{\mathfrak{p}}$ of K on $\mathfrak{p}$. The product group $\Gamma \times K$ acts on $F \otimes \bigwedge^p \mathfrak{p}^*$ by

$$(\gamma, k)(u \otimes \eta) = (\rho(\gamma) \otimes \mathrm{Ad}_p^*(k))(u \otimes \eta) = \rho(\gamma)u \otimes \mathrm{Ad}_p^*(k)\eta$$

for $(\gamma, k) \in \Gamma \times K$, $u \in F$ and $\eta \in \bigwedge^p \mathfrak{p}^*$.

DEFINITION 2.1. Let $C(G, F \otimes \bigwedge^p \mathfrak{p}^*)$ denote the set of all $F \otimes \bigwedge^p \mathfrak{p}^*$ valued continuous functions on G and let $C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)$ be the set of all $F \otimes \bigwedge^p \mathfrak{p}^*$ valued C^∞ function on G. Define

$$C(G, F \otimes \textstyle\bigwedge^p \mathfrak{p}^*)^0 = \{\varphi \in C(G, F \otimes \bigwedge^p \mathfrak{p}^*)\,;\, \varphi(\gamma g k) = (\gamma, k^{-1})\varphi(g) \text{ for all } \gamma \in \Gamma,\ k \in K\}\ .$$

$$C^\infty(G, F \otimes \textstyle\bigwedge^p \mathfrak{p}^*)^0 = \{\varphi \in C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)\,;\, \varphi(\gamma g k) = (\gamma, k^{-1})\varphi(g) \text{ for all } \gamma \in \Gamma,\ k \in K\}\ .$$

Now we define an injective mapping

$$\varepsilon : A^p(\Gamma, \tilde{M}, \rho) \longrightarrow C^\infty(G, F \otimes \textstyle\bigwedge^p \mathfrak{p}^*)$$

by

$$\varepsilon(\eta)(g) = \sum_{1\leq i_1<\cdots<i_p\leq d} \eta_{i_1\cdots i_p}(g) \otimes \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \qquad (g \in G) .$$

Here $\eta_{i_1\cdots i_p}(g) = \eta(\tau_g X_{i_1}, \cdots, \tau_g X_{i_p})$ and the tangent vector $\tau_g X_i$ of $\tilde{M}$ at $\pi_0(g)$ is the image of $X_i \in T_0\tilde{M} = \mathfrak{p}$ under the differential of the translation τ_g at 0.

Then the mapping ε defines an isomorphism of $A^p(\Gamma, \tilde{M}, \rho)$ into $C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0$. Let Δ_0^p be an operator of $C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0$ defined by

$$\Delta_0^p \varepsilon(\eta) = \varepsilon(\Delta^p \eta) \tag{2.1}$$

for $\eta \in A^p(\Gamma, \tilde{M}, \rho)$. For $\lambda \in \boldsymbol{R}$, let

$$C_\lambda^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 = \{\varphi \in C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 ; \Delta_0^p \varphi = \lambda\varphi\} .$$

Then for every $\lambda \in \boldsymbol{R}$, the mapping ε induces an isomorphism of $A_\lambda^p(\Gamma, \tilde{M}, \rho)$ onto $C_\lambda^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0$.

Moreover we define the metric (,) in $C(G, F \otimes \wedge^p \mathfrak{p}^*)$ by

$$(\varphi, \varphi') = C \sum_{1\leq i_1<\cdots<i_p\leq d} C \int_G (\varphi_{i_1\cdots i_p}(g), \varphi'_{i_1\cdots i_p}(g))_F dg$$

where dg is the Haar measure on G with total volume 1, the constant $C = \mathrm{vol}\,(G)/\mathrm{vol}\,(K)$ and

$$\varphi(g) = \sum_{1\leq i_1<\cdots<i_p\leq d} \varphi_{i_1\cdots i_p}(g) \otimes \omega_{i_1 \wedge} \cdots {}_\wedge \omega_{i_p} ,$$
$$\varphi'(g) = \sum_{1\leq i_1<\cdots<i_p\leq d} \varphi'_{i_1\cdots i_p}(g) \otimes \omega_{i_1 \wedge} \cdots {}_\wedge \omega_{i_p} .$$

Let $L_2(G, F \otimes \wedge^p \mathfrak{p}^*)$ be the completion of $C(G, F \otimes \wedge^p \mathfrak{p}^*)$ with respect to this inner product and let $L_2(G, F \otimes \wedge^p \mathfrak{p}^*)^0$ be the completion of $C(G, F \otimes \wedge^p \mathfrak{p}^*)^0$ be the completion of $C(G, F \otimes \wedge^p \mathfrak{p}^*)^0$ in $L_2(G, F \otimes \wedge^p \mathfrak{p}^*)$.

Notice that for $\eta \in A^p(\Gamma, \tilde{M}, \rho)$,

$$\eta_{i_1\cdots i_p}(g) = \rho(g)\tilde{\eta}_{i_1\cdots i_p}(\tilde{\omega}_0(g)) , \qquad g \in G . \tag{2.2}$$

For

$$\begin{aligned}
\tilde{\eta}_{i_1\cdots i_p}(\varpi_0(g)) &= \eta^0_{\varpi_0(g)}(X_{i_1}, \cdots, X_{i_p}) \\
&= (\varpi_0^* \eta^0)_g(X_{i_1}, \cdots, X_{i_p}) \\
&= \rho(g^{-1})(\pi_0^* \eta)_g(X_{i_1}, \cdots, X_{i_p}) \\
&= \rho(g^{-1})\eta_{\pi_0(g)}(\tau_g X_{i_1}, \cdots, \tau_g X_{i_p}) \\
&= \rho(g^{-1})\eta_{i_1\cdots i_p}(g)
\end{aligned}$$

where for each $X \in \mathfrak{p}$, the image of the tangent vector X_g of G at g under the projection π_0 coincides with the image of the tangent vector X_0 of M at 0 under the translation τ_g.

Then from (1.5), (2.2), the definition of the inner product in $A^p(\Gamma, \tilde{M}, \rho)$ and the invariantness of $(\ , \ )_F$ under the action ρ of G, the mapping ε induces the isometry of $L_2^p(\Gamma, \tilde{M}, \rho)$ onto $L_2(G, F \otimes \bigwedge^p \mathfrak{p}^*)^0$. Hence we have the decomposition

$$L_2(G, F \otimes \bigwedge{}^p \mathfrak{p}^*)^0 = \sum_\lambda C_\lambda^\infty(G, F \otimes \bigwedge{}^p \mathfrak{p}^*)^0 .$$

Therefore we have

$$Z^p(t) = \sum_\lambda e^{-\lambda t} \dim C_\lambda^\infty(G, F \otimes \bigwedge{}^p \mathfrak{p}^*)^0 . \tag{2.3}$$

2.2. The Laplacian in $C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)^0$

Now let r be the right regular representation of G on $L_2(G, F \otimes \bigwedge^p \mathfrak{p}^*)$; i.e.

$$(r_g\varphi)(x) = \varphi(xg) \qquad (x \in G)$$

for any $g \in G$, $\varphi \in L_2(G, F \otimes \bigwedge^p \mathfrak{p}^*)$. For any $X \in \mathfrak{g}$, we define $r(X)$ by

$$r(X)\varphi = X\varphi \qquad \varphi \in C^\infty(G, F \otimes \bigwedge{}^p \mathfrak{p}^*)$$

where $X\varphi(g) = [(d/dt)\varphi(g \exp tX)]_{t=0}$, $g \in G$. Then $X \mapsto r(X)$ $(X \in \mathfrak{g})$ is a representation of $\mathfrak{g}$ on $C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)$. Let $U(\mathfrak{g}^C)$ be the universal enveloping algebra of $\mathfrak{g}^C$. Then this representation extends uniquely to a representation of $U(\mathfrak{g}^C)$ which is denoted again by r. Let $\Omega = \sum_{\nu=1}^n X_\nu^2 \in U(\mathfrak{g}^C)$. Then the operator $r(\Omega)$ on $C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)$ commutes with the right and left translations of G on $C^\infty(G, \otimes \bigwedge^p \mathfrak{p}^*)$. Hence we have

$$r(\Omega)C^\infty(G, F \otimes \bigwedge{}^p \mathfrak{p}^*)^0 \subset C^\infty(G, F \otimes \bigwedge{}^p \mathfrak{p}^*)^0 .$$

Moreover we have

PROPOSITION 2.1. *For $\eta \in A^p(\Gamma, \tilde{M}, \rho)$, we have*

$$(\Delta^p\eta)_{i_1\cdots i_p} = -\sum_{\nu=1}^n X_\nu^2 \eta_{i_1\cdots i_p}$$

that is,

$$\Delta_0^p \varepsilon(\eta) = -r(\Omega)\varepsilon(\eta) .$$

Proof. By (2.2), we have for $X \in \mathfrak{g}$, $\eta \in A^p(\Gamma, \tilde{M}, \rho)$,

$$(X + \rho(X))(\tilde{\eta}_{i_1\cdots i_p} \circ \varpi_0)(g) = (X + \rho(X))(\rho^{-1} \circ \eta_{i_1\cdots i_p})(g)$$
$$= (X\tilde{\eta}_{i_1\cdots i_p}) \circ \varpi_0(g) .$$

Proposition 2.1 follows from Proposition 1.1. Q.E.D.

Let $H_0^p(G, F \otimes \wedge^p \mathfrak{p}^*) = C_0^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 = \{\varphi \in C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 ; \Delta_0^p \varphi = 0\}$. From Proposition 2.1, for $\varphi = \sum_{1 \leq i_1 < \cdots < i_p \leq d} \varphi_{i_1\cdots i_p} \otimes \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \in C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0$, we have

$$\Delta_0^p \varphi = r(\Omega)\varphi = \sum_{1 \leq i < \cdots < i_p \leq d} \Omega \varphi_{i_1\cdots i_p} \otimes \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} .$$

Then

$$\Delta_0^p \varphi = 0 \Longleftrightarrow \Omega \varphi_{i_1\cdots i_p} = 0 \qquad (1 \leq i_1 < \cdots < i_p \leq d)$$
$$\Longleftrightarrow \text{every } \varphi_{i_1\cdots i_p} \text{ is a constant mapping of } G \text{ into } F .$$

Hence $H_0^p(G, F \otimes \wedge^p \mathfrak{p}^*) \cong \{\eta \in F \otimes \wedge^p \mathfrak{p}^* : (\gamma, k)\eta = \eta \text{ for all } (\gamma, k) \in \Gamma \times K\}$.

Therefore we have the following theorem.

THEOREM 2.1. *Under the assumption in* §1, *for* $0 \leq p \leq d$, *we have*

$$\dim H^p(E) = [\rho_\Gamma : \boldsymbol{1}_\Gamma][\mathrm{Ad}_p^* : \boldsymbol{1}_K] .$$

Here ρ_Γ *is the representation of* ρ *restricted to* Γ, $[\rho_\Gamma : \boldsymbol{1}_\Gamma]$ (*resp.* $[\mathrm{Ad}_p^* : \boldsymbol{1}_K]$) *is the multiplicity with which the trivial representation* $\boldsymbol{1}_\Gamma$ (*resp.* $\boldsymbol{1}_K$) *of* Γ (*resp.* K) *occurs in* ρ_Γ (*resp.* Ad_p^*).

COROLLARY 2.1. *We preserve the notation and the assumption in* §1. *Then*

$$\sum_{p=0}^{d} (-1)^p p \dim H^p(E) = [\rho_\Gamma : \boldsymbol{1}_\Gamma] \int_K \chi(k) dk \tag{2.4}$$

where $\chi(k) = \sum_{p=0}^{d} (-1)^p p \chi_p^*(k)$, $\chi_p^*(k)$ *is the trace of* $\mathrm{Ad}_p^*(k)$ *on* $\wedge^p \mathfrak{p}^*$ *and* dk *is the Haar measure on* K *with total volume* 1.

2.3. The fundamental solution of the heat equation on $C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0$

Now let T be a maximal torus of G and let $\mathfrak{t}$ be the subalgebra of $\mathfrak{g}$ corresponding to T. Let $\Gamma_0 = \{H \in \mathfrak{t} : \exp H = 1\}$ be the kernel of the homomorphism $\exp : \mathfrak{t} \to T$. Let I be the set of all G-integral forms on $\mathfrak{t}$:

$$I = \{\lambda \in \mathfrak{t} : \lambda(H) \in 2\pi \boldsymbol{Z} \qquad \text{for all } H \in \Gamma_0\} .$$

Let (,) be an $\mathrm{Ad}(G)$ invariant positive definite inner product on $\mathfrak{g}$

defined by $(X, Y) = -B(X, Y)$, $X, Y \in \mathfrak{g}$. Let Φ be the set of all non-zero roots of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to the complexification $\mathfrak{t}^C$ of $\mathfrak{t}$. We choose an arbitrary lexicographic order in $\mathfrak{t}$. Let Φ^+ be the positive root of Φ with respect to this order. Let D be the set of all dominant G-integral forms on $\mathfrak{t}$:

$$D = \{\lambda \in I : (\lambda, \alpha) \geqq 0 \qquad \text{for all } \alpha \in \Phi^+\} .$$

Since an irreducible representation of G is uniquely determined, up to equivalence, by its highest weight, there exists a bijection of D onto the set of equivalence classes of irreducible representations of G. For $\lambda \in D$, let χ_λ (resp. d_λ) be the trace (resp. degree) of the irreducible representation with the highest weight λ.

Define (cf. [14]) an absolutely convergent series $Z_t(g)$ by

$$Z_t(g) = \sum_{\lambda \in D} d_\lambda \, e^{-(\lambda+2\delta,\lambda)t} \chi_\lambda(g) , \qquad t > 0 \tag{2.5}$$

where $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

PROPOSITION 2.2. *For $\varphi \in C(G, F \otimes \bigwedge^p \mathfrak{p}^*)$, the unique solution of the equation*

$$\begin{cases} \dfrac{\partial \varphi_t}{\partial t} = r(\Omega)\varphi_t , & \varphi_t \in C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*) \\ \lim\limits_{t \downarrow 0} \varphi_t = \varphi & \textit{(pointwise convergence)} \end{cases} \tag{2.6}$$

is given by

$$\varphi_t(g) = \int_G Z_t(x^{-1}g)\varphi(x)dx \tag{2.7}$$

where $Z_t(g)$ is the function (2.5) *and dx is the Haar measure on G with total volume* 1. *Moreover we denote by K_t the mapping* (2.7) $\varphi \mapsto \varphi_t$. *Then we have*

$$K_t C(G, F \otimes \bigwedge^p \mathfrak{p}^*)^0 \subset C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)^0 . \tag{2.8}$$

Proof. Since $\Omega\chi_\lambda = -(\lambda + 2\delta, \lambda)\chi_\lambda$, $\lambda \in D$ (cf. [13]), we have $(\partial/\partial t)Z_t = \Omega Z_t$. Then for $\varphi \in C(G, F \otimes \bigwedge^p \mathfrak{p}^*)$, we have $\varphi_t \in C^\infty(G, F \otimes \bigwedge^p \mathfrak{p}^*)$ and

$$\begin{aligned} r(\Omega)\varphi_t(g) &= \int_G (\Omega Z_t)(x^{-1}g)\varphi(x)dx \\ &= \int_G \frac{\partial}{\partial t} Z_t(x^{-1}g)\varphi(x)dx = \frac{\partial}{\partial t}\varphi_t(g) . \end{aligned}$$

By Peter-Weyl's theorem, for every complex continuous function f on G, we have

$$\lim_{t\downarrow 0}\int_G Z_t(x^{-1}g)f(x)dx = f(g) .$$

Then for every $F\otimes\bigwedge^p\mathfrak{p}^*$ valued function φ, we have also

$$\lim_{t\downarrow 0}\int_G Z_t(x^{-1}g)\varphi(x)dx = \varphi(x) .$$

The last statement follows from that for $\varphi\in C(G,F\otimes\bigwedge^p\mathfrak{p}^*)$, and $g_1, g_2, g\in G$,

$$\varphi_t(g_1gg_2) = \int_G Z_t(x^{-1}g)\varphi(g_1xg_2)dx .$$

Q.E.D.

Define the operator P on $C(G,F\otimes\bigwedge^p\mathfrak{p}^*)$ by

$$P\varphi(g) = \sum_{\gamma\in\Gamma}\int_K \rho(\gamma)\otimes \mathrm{Ad}_p^*\,(k)(\varphi(\gamma^{-1}gk))dk$$

for $\varphi\in C(G,F\otimes\bigwedge^p\mathfrak{p}^*)$. Then the operator P satisfies the following conditions:

(i) P maps $C(G,F\otimes\bigwedge^p\mathfrak{p}^*)$ onto $C(G,F\otimes\bigwedge^p\mathfrak{p}^*)^0$.

(ii) $P^2 = P$.

Moreover for $\varphi\in C(G,F\otimes\bigwedge^p\mathfrak{p}^*)$, by means of Propositions 2.1 and 2.2, $K_tP\varphi$ $(t>0)$ has the following properties:

(i) $K_tP\varphi\in C^\infty(G,F\otimes\bigwedge^p\mathfrak{p}^*)^0$,

(ii) $\dfrac{\partial}{\partial t}(K_tP\varphi) = r(\Omega)(K_tP\varphi) = -\Delta_0^p(K_tP\varphi)$ and

(iii) $\lim_{t\downarrow 0} K_tP\varphi = P\varphi$.

On the other hand, for $\varphi\in C(G,F\otimes\bigwedge^p\mathfrak{p}^*)$,

$$\begin{aligned} K_tP\varphi(x) &= \int_G Z_t(y^{-1}x)P\varphi(y)dy \\ &= \sum_{\gamma\in\Gamma}\int_{G\times K} Z_t(y^{-1}x)\rho(\gamma)\otimes\mathrm{Ad}_p^*\,(k)\varphi(\gamma^{-1}yk)dkdy \\ &= \int_G\Big(\sum_{\gamma\in\Gamma}\int_K Z_t(ky^{-1}\gamma^{-1}x)\rho(\gamma)\otimes\mathrm{Ad}_p^*\,(k)dk\Big)\varphi(y)dy .\end{aligned} \tag{2.9}$$

Put

$$Z_t^p(x, y) = \sum_{\gamma \in \Gamma} \int_K Z_t(ky^{-1}\gamma^{-1}x)\rho(\gamma) \otimes \mathrm{Ad}_p^*(k)dk . \tag{2.10}$$

Therefore we obtain the following theorem.

THEOREM 2.2. *For $t > 0$, let $Z_t^p : G \times G \to \mathrm{End}(F \otimes \wedge^p \mathfrak{p}^*)$ be the smooth map defined by* (2.10). *Then Z_t^p is the fundamental solution of the heat equation $\partial\varphi_t/\partial t = -\Delta_0^p \varphi_t$ $(t > 0)$, $\varphi_t \in C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0$, that is, for $\varphi \in C(G, F \otimes \wedge^p \mathfrak{p}^*)$, put*

$$\varphi_t(x) = \int_G Z_t^p(x, y)\varphi(y)dy , \qquad x \in G .$$

Then φ_t satisfies the following properties:

(i) $\varphi_t \in C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0$,

(ii) $\dfrac{\partial \varphi_t}{\partial t} = -\Delta\varphi_t$ *and*

(iii) $\lim_{t \downarrow 0} \varphi_t(x) = \varphi(x)$ *for every* $x \in G$.

COROLLARY 2.2. *Let $Z^p(t)$ be the series* (1.1). *Then we have*

$$Z^p(t) = \sum_{\gamma \in \Gamma} \chi_\rho(\gamma) \int_{G \times K} Z_t(\gamma^{-1}gkg^{-1})\chi_p^*(k)dkdg \tag{2.11}$$

where $\chi_\rho(\gamma)$ is the trace of $\rho(\gamma)$.

Proof. By (2.3) and Theorem 2.2, we have

$$\begin{aligned} Z^p(t) &= \sum_\lambda \mathrm{e}^{-\lambda t} \dim C_\lambda^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 \\ &= \text{trace of the operator } \mathrm{e}^{-t\Delta_0^p} : C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 \\ &\qquad\qquad \longrightarrow C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 \\ &= \text{trace of the operator } \mathrm{e}^{-t\Delta_0^p} \circ P : C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*) \\ &\qquad\qquad \longrightarrow C^\infty(G, F \otimes \wedge^p \mathfrak{p}^*)^0 \\ &= \text{trace of } K_t \circ P \\ &= \int_G \mathrm{tr}\, Z_t^p(g, g)dg \end{aligned}$$

where $\mathrm{tr}\, Z_t^p(g, g)$ is the trace of the endomorphism $Z_t^p(g, g)$ of $F \otimes \wedge^p \mathfrak{p}^*$. The last equality follows from (2.10). Q.E.D.

Remark. In case of $\Gamma = \{1\}$, we have due to Corollary 2.2,

$$Z^p(t) = \int_K Z_t(k)\chi_p^*(k)dk . \tag{2.12}$$

If $p = 0$, this formula has been obtained in [2].

The following Corollary is obtained immediately from Corollary 2.2.

COROLLARY 2.3. *We preserve the above notations. Then we have*

$$\sum_{p=0}^{d} (-1)^p p Z^p(t) = \sum_{\gamma \in \Gamma} \chi_\rho(\gamma) \int_{G \times K} Z_t(\gamma^{-1} g k g^{-1}) \chi(k) dk dg \,. \tag{2.13}$$

where $\chi(k) = \sum_{p=0}^{d} (-1)^p p \chi_p^*(k)$, $\chi_p^*(k)$ *is the trace of* $\mathrm{Ad}_p^*(k)$ *on* $\bigwedge^p \mathfrak{p}^*$.

§ 3. Computation of Analytic Torsion

3.1. To calculate analytic torsion, we have to compute $\chi(k) = \sum_{p=0}^{d} (-1)^p p \chi_p^*(k)$, $k \in K$. For this purpose, we prepare a lemma as follows.

Let V be a d dimensional real vector space and let A be an endomorphism of V. For $1 \leq p \leq d$, $\bigwedge^p A$ is a linear operator of $\bigwedge^p V$ into itself,

$$(\textstyle\bigwedge^p A)(v_1 \wedge \cdots \wedge v_p) = Av_1 \wedge \cdots \wedge Av_p \,, \qquad v_i \in V \,.$$

We define $\bigwedge^0 A$ to be the identity endomorphism of the field of scalars. Let $\mathrm{tr}(\bigwedge^p A)$ be the trace of the endomorphism $\bigwedge^p A$. Then it is known that

$$\det(xI - A) = \sum_{p=0}^{d} (-1)^p \,\mathrm{tr}(\textstyle\bigwedge^p A) x^{d-p}$$

where I is the identity endomorphism of V and x is an indeterminate. So we have

$$\left[\frac{d}{dx}\left\{x^d \det\left(\frac{1}{x} I - A\right)\right\}\right]_{x=1} = \sum_{p=0}^{d} (-1)^p p \,\mathrm{tr}(\textstyle\bigwedge^p A) \,. \tag{3.1}$$

Hence we obtain

LEMMA 3.1. *We preserve the notation in* § 1. *For* $k \in K$, *we have*

$$\chi(k) = \sum_{p=1}^{d} (-1)^p p \chi_p^*(k) = \left[\frac{d}{dx}\left\{x^d \det\left(\frac{1}{x} I_{\mathfrak{p}} - \mathrm{Ad}(k^{-1})_{\mathfrak{p}}\right)\right\}\right]_{x=1}$$

where $I_{\mathfrak{p}}$ *is the identity operator on* $\mathfrak{p}$, $\mathrm{Ad}(k)_{\mathfrak{p}}$ *is the adjoint action of* K *on* $\mathfrak{p}$ *and* $d = \dim G/K = \dim \mathfrak{g}$.

Proof. By the definition and (3.1), Lemma 3.1 is obtained immediately.

Let $\mathfrak{t}_{\mathfrak{k}}$ be a Cartan subalgebra of $\mathfrak{k}$. Let $\mathfrak{t}$ be the centralizer of $\mathfrak{t}_{\mathfrak{k}}$ in $\mathfrak{g}$. Then $\mathfrak{t}$ is (cf. [3] Lemma 32) a θ-stable Cartan subalgebra of $\mathfrak{g}$ and

$$(3.2) \qquad \mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} + \mathfrak{t}_{\mathfrak{p}}\,, \qquad \mathfrak{t}_{\mathfrak{p}} = \mathfrak{t} \cap \mathfrak{p}\,.$$

So, $\dim \mathfrak{t}_{\mathfrak{p}} = \operatorname{rank} G - \operatorname{rank} K$. Let T_K be the analytic subgroup of K corresponding to $\mathfrak{t}_{\mathfrak{k}}$. Then T_K is a maximal torus of K since K is connected. We choose once for all a lexicographic order in $\mathfrak{t}_{\mathfrak{k}}$. Let $\Phi_{\mathfrak{k}}$ be the root system of $(\mathfrak{k}^C, \mathfrak{t}_{\mathfrak{k}})$, i.e. the set of non-zero elements β of the dual space $\mathfrak{t}_{\mathfrak{k}}^*$ of $\mathfrak{t}_{\mathfrak{k}}$ such that $\{E \in \mathfrak{k}^C : [H, E] = \sqrt{-1}\beta(H)E \text{ for any } H \in \mathfrak{t}_{\mathfrak{k}}\}$ is not zero. Let $\Phi_{\mathfrak{k}}^+$ be the set of all positive roots of $\Phi_{\mathfrak{k}}$ with respect to this order. For every continuous function f on K such that $f(k_1 k k_1^{-1}) = f(k)$ for every $k_1, k \in K$, it follows (cf. [5] Ch X) that (Weyl's integral formula for K)

$$\int_K f(k)dk = \frac{1}{w_K}\int_{T_K} D_K(h)f(h)dh$$

where w_K is the order of the Weyl group of the compact group K, dh is the Haar measure on T_K with total volume 1 and

$$D_K(h) = \left|\prod_{\beta\in\Phi_{\mathfrak{k}}^+}\left(\exp\left(\frac{\sqrt{-1}}{2}\beta(H)\right) - \exp\left(-\frac{\sqrt{-1}}{2}\beta(H)\right)\right)^2\right|$$

for $h = \exp H \in T_K$.

By means of this formula, Corollaries 2.1 and 2.3, we have

$$(3.3) \quad \sum_{p=0}^{d}(-1)^p p Z^p(t) = \frac{1}{w_K}\sum_{\gamma\in\Gamma}\chi_\rho(\gamma)\int_{G\times T_K} D_K(h)Z_t(\gamma^{-1}yhy^{-1})\chi(h)dhdy$$

$$(3.4) \qquad \sum_{p=0}^{d}(-1)^p p \dim H^p(E) = \frac{[\rho_\Gamma : I_\Gamma]}{w_K}\int_{T_K} D_K(h)\chi(h)dh\,.$$

So, using Lemma 3.1, to calculate $\chi(h)$ for $h \in T_K$, we have to investigate the action of $\operatorname{ad} H$ on $\mathfrak{p}$ for $H \in \mathfrak{t}_{\mathfrak{k}}$.

3.2. For $\lambda \in \mathfrak{t}^*$, let $\lambda_{\mathfrak{k}}$ (resp. $\lambda_{\mathfrak{p}}$) be the restriction of λ to $\mathfrak{t}_{\mathfrak{k}}$ (resp. $\mathfrak{t}_{\mathfrak{k}}$). We choose once for all a lexicographic order on $\mathfrak{t}_{\mathfrak{p}}^*$. We define an order on $\mathfrak{t}^*$ in such a way that

$$\lambda \in \mathfrak{t}^*, \lambda > 0 \Longleftrightarrow \begin{array}{ll}\text{(i)} & \lambda_{\mathfrak{p}} > 0 \quad \text{or} \\ \text{(ii)} & \lambda_{\mathfrak{p}} = 0 \quad \text{and} \quad \lambda_{\mathfrak{k}} > 0\,.\end{array}$$

Let Φ be the root system of $(\mathfrak{g}^C, \mathfrak{t})$, i.e. the set of non-zero elements α of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ such that $\mathfrak{g}_\alpha = \{E \in \mathfrak{g}^C : [H, E] = \sqrt{-1}\alpha(H)E$ for any $H \in \mathfrak{t}\}$ is not zero. Let Φ^+ be the set of positive roots of Φ with respect to this order. For $\alpha \in \Phi$, define $\alpha^\theta \in \Phi$ by $\alpha^\theta(H) = \alpha(\theta H)$, $H \in \mathfrak{t}$. Let $\mathfrak{g}_\alpha$ be a root subspace of $\mathfrak{g}_C$ for $\alpha \in \Phi$. Then we have that

$$\alpha \in \Phi \Longleftrightarrow \alpha^\theta \in \Phi \quad \text{and} \quad \theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha^\theta} . \tag{3.5}$$

The root α vanishes identically on $\mathfrak{t}_\mathfrak{p}$ (resp. $\mathfrak{t}_\mathfrak{k}$) if and only if $\alpha = \alpha^\theta$ (resp. $\alpha = -\alpha^\theta$). Let $\Phi_I = \{\alpha \in \Phi : \alpha^\theta = \alpha\}$ and let $\Phi_C = \{\alpha \in \Phi : \alpha^\theta \neq \alpha$ and $\alpha \neq -\alpha^\theta\}$. Then $\Phi = \Phi_I \cup \Phi_C$ (a disjoint union) since there is no $\alpha \in \Phi$ which vanishes identically on $\mathfrak{t}_\mathfrak{k}$ (cf. Lemma 33 [3]). Let $\Phi_{I,\mathfrak{k}} = \{\alpha \in \Phi_I : \mathfrak{g}_\alpha \subset \mathfrak{k}^C\}$ and let $\Phi_{I,\mathfrak{p}} = \{\alpha \in \Phi_I : \mathfrak{g}_\alpha \subset \mathfrak{p}^C\}$. We denote the intersection of Φ_I (resp. $\Phi_{I,\mathfrak{k}}, \Phi_{I,\mathfrak{p}}, \Phi_C$) with Φ^+, by Φ_I^+ (resp. $\Phi_{I,\mathfrak{k}}^+, \Phi_{I,\mathfrak{p}}^+, \Phi_C^+$). Let τ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. For every $\alpha \in \Phi$, we choose a root vector E_α such that $\tau E_\alpha = -E_{-\alpha}$. By (3.5), we can take a non-zero complex number $c_\alpha (\alpha \in \Phi_C)$ such that $\theta E_\alpha = c_\alpha E_{\alpha\theta}$. Then each c_α $(\alpha \in \Phi_C)$ satisfies

$$c_\alpha c_{\alpha\theta} = 1 , \qquad c_{-\alpha} = \overline{c_{\alpha\theta}} . \tag{3.6}$$

For $\alpha \in \Phi_C^+$, we have

$$\begin{aligned} E_{-\alpha} &= \tfrac{1}{2}(\theta E_{-\alpha} + \theta(\theta E_{-\alpha})) - \tfrac{1}{2}(\theta E_{-\alpha} - \theta(\theta E_{-\alpha})) \\ &= \tfrac{1}{2}(c_{-\alpha} E_{-\alpha\theta} + c_{-\alpha\theta} E_{-\alpha\theta}) - \tfrac{1}{2}(c_{-\alpha} E_{-\alpha\theta} - c_{-\alpha\theta} E_{-\alpha\theta}) \\ &= \frac{c_{-\alpha}}{2}(E_{-\alpha\theta}\theta + \theta E_{-\alpha\theta}\theta) - \frac{c_{-\alpha}}{2}(E_{-\alpha\theta} - \theta E_{-\alpha\theta}) . \end{aligned}$$

By the choice of the order of $\mathfrak{t}^*$,

$$\alpha \in \Phi_C^+ \Rightarrow -\alpha^\theta \in \Phi_C^+ . \tag{3.7}$$

Hence we have

$$\mathfrak{g}^C = \mathfrak{t}^C + \sum_{\alpha \in \Phi_I} CE_\alpha + \sum_{\alpha \in \Phi_C^+} C(E_\alpha + \theta E_\alpha) + \sum_{\alpha \in \Phi_C^+} C(E_\alpha - \theta E_\alpha) ,$$

that is

$$\begin{cases} \mathfrak{k}^C = \mathfrak{t}_\mathfrak{k}^C + \sum_{\alpha \in \Phi_{I,\mathfrak{k}}} CE_\alpha + \sum_{\alpha \in \Phi_C^+} C(E_\alpha + \theta E_\alpha) , \\ \mathfrak{p}^C = \mathfrak{t}_\mathfrak{p}^C + \sum_{\alpha \in \Phi_{I,\mathfrak{p}}} CE_\alpha + \sum_{\alpha \in \Phi_C^+} C(E_\alpha - \theta E_\alpha) . \end{cases} \tag{3.8}$$

Since $\alpha \neq \alpha^\theta$ $(\alpha \in \Phi_C)$, we can define non-zero vectors X_α, Y_α $(\alpha \in \Phi_C)$ by $X_\alpha = E_\alpha + \theta E_\alpha$, $Y_\alpha = E_\alpha - \theta E_\alpha$ for $\alpha \in \Phi_C$. By means of $\theta\tau = \tau\theta$ and τE_α

$= -E_{-\alpha}$, we have $\tau X_\alpha = -X_{-\alpha}$ and $\tau Y_\alpha = -Y_{-\alpha}$. Then we have

$$\begin{cases} W_\alpha = X_\alpha - X_{-\alpha}\,, & Z_\alpha = \sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{k} \\ \tilde{W}_\alpha = Y_\alpha - Y_{-\alpha}\,, & \tilde{Z}_\alpha = \sqrt{-1}(Y_\alpha + Y_{-\alpha}) \in \mathfrak{p} \end{cases} \tag{3.9}$$

for $\alpha \in \Phi_C^+$. Since $\alpha^\theta \neq \alpha$, $-\alpha(\alpha \in \Phi_C^+)$, all $W_\alpha, Z_\alpha, \tilde{W}_\alpha$ and $\tilde{Z}_\alpha$ are non-zero for $\alpha \in \Phi_C^+$. Moreover we have, for $\alpha \in \Phi_C^+$,

$$\begin{cases} W_{-\alpha^\theta} = -\dfrac{1}{2}\Big(\dfrac{1}{c_\alpha} + \dfrac{1}{c_{-\alpha}}\Big)W_\alpha + \dfrac{\sqrt{-1}}{2}\Big(\dfrac{1}{c_\alpha} - \dfrac{1}{c_{-\alpha}}\Big)Z_\alpha\,, \\ Z_{-\alpha^\theta} = \dfrac{\sqrt{-1}}{2}\Big(\dfrac{1}{c_\alpha} - \dfrac{1}{c_{-\alpha}}\Big)W_\alpha + \dfrac{1}{2}\Big(\dfrac{1}{c_\alpha} + \dfrac{1}{c_{-\alpha}}\Big)Z_\alpha\,, \\ \tilde{W}_{-\alpha^\theta} = \dfrac{1}{2}\Big(\dfrac{1}{c_\alpha} + \dfrac{1}{c_{-\alpha}}\Big)\tilde{W}_\alpha - \dfrac{\sqrt{-1}}{2}\Big(\dfrac{1}{c_\alpha} - \dfrac{1}{c_{-\alpha}}\Big)\tilde{Z}_\alpha \quad \text{and} \\ \tilde{Z}_{-\alpha^\theta} = \dfrac{\sqrt{-1}}{2}\Big(\dfrac{1}{c_\alpha} + \dfrac{1}{c_{-\alpha}}\Big)\tilde{W}_\alpha - \dfrac{1}{2}\Big(\dfrac{1}{c_\alpha} + \dfrac{1}{c_{-\alpha}}\Big)\tilde{Z}_\alpha\,, \end{cases} \tag{3.10}$$

where all coefficients $\pm\frac{1}{2}(1/c_\alpha + 1/c_{-\alpha})$, $\pm\sqrt{-1}/2(1/c_\alpha - 1/c_{-\alpha})$ are real numbers due to (3.6).

Now we choose any root α_1 of Φ_C^+. If $\Phi_C^+\backslash\{\alpha_1, -\alpha_1^\theta\}$ is non-empty, we choose any root α_2 belonging to $\Phi_C^+\backslash\{\alpha_1, -\alpha_1^\theta\}$. Then $-\alpha_2^\theta$ belongs to $\Phi^+\backslash\{\alpha_1, -\alpha_1^\theta, \alpha_2\}$. Inductively we may choose a subset $\{\alpha_1, \cdots, \alpha_r\}$ of Φ_C^+ such that $\{\alpha_1, \cdots, \alpha_r, -\alpha_1^\theta, \cdots, -\alpha_r^\theta\} = \Phi_C^+$. Then by (3.9), (3.10) and the choice of $\{\alpha_1, \cdots, \alpha_r\}$, $\sum_{i=1}^r (\boldsymbol{R}W_{\alpha_i} + \boldsymbol{R}Z_{\alpha_i})$ (resp. $\sum_{i=1}^r (\boldsymbol{R}\tilde{W}_{\alpha_i} + \boldsymbol{R}\tilde{Z}_{\alpha_i})$) is a real form of $\sum_{\alpha\in\Phi_C^+} \boldsymbol{C}(E_\alpha + \theta E_\alpha)$ (resp. $\sum_{\alpha\in\Phi_C^+} \boldsymbol{C}(E_\alpha - \theta E_\alpha)$).

On the other hand, for $\alpha \in \Phi_I^+$, we put $U_\alpha = E_\alpha - E_{-\alpha}$, $V_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Then $\sum_{\alpha\in\Phi_{I,\mathfrak{k}}^+} (\boldsymbol{R}U_\alpha + \boldsymbol{R}V_\alpha)$ (resp. $\sum_{\alpha\in\Phi_{I,\mathfrak{p}}^+} (\boldsymbol{R}U_\alpha + \boldsymbol{R}V_\alpha)$) is a real form of $\sum_{\alpha\in\Phi_{I,\mathfrak{k}}^+} \boldsymbol{C}E_\alpha$ (resp. $\sum_{\alpha\in\Phi_{I,\mathfrak{p}}^+} \boldsymbol{C}E_\alpha$).

Therefore together with (3.8) we obtain the following lemma:

LEMMA 3.2. *We preserve the above notation. Then we have the following direct sum decomposition*:

$$\mathfrak{k} = \mathfrak{t}_\mathfrak{k} + \sum_{\alpha\in\Phi_{I,\mathfrak{k}}^+} (\boldsymbol{R}U_\alpha + \boldsymbol{R}V_\alpha) + \sum_{i=1}^r (\boldsymbol{R}W_{\alpha_i} + \boldsymbol{R}Z_{\alpha_i})\,,$$

$$\mathfrak{p} = \mathfrak{t}_\mathfrak{p} + \sum_{\alpha\in\Phi_{I,\mathfrak{p}}^+} (\boldsymbol{R}U_\alpha + \boldsymbol{R}V_\alpha) + \sum_{i=1}^r (\boldsymbol{R}\tilde{W}_{\alpha_i} + \boldsymbol{R}\tilde{Z}_{\alpha_i})\,.$$

LEMMA 3.3. *For each $H \in \mathfrak{t}_\mathfrak{k}$, we have*

$$\det(xI_\mathfrak{p} - \mathrm{Ad}(h)_\mathfrak{p}) = (x-1)^{\ell_\mathfrak{p}} \prod_{\alpha\in\Phi_{I,\mathfrak{p}}^+ \cup \{\alpha_1,\cdots,\alpha_r\}} \{(x - \cos\alpha(H))^2 + \sin^2\alpha(H)\}$$

where $\ell_{\mathfrak{p}} = \dim \mathfrak{t}_{\mathfrak{p}} = \operatorname{rank} G - \operatorname{rank} K$.

Proof. For $\alpha \in \Phi_I$, we have by the definition of U_α, V_α,

$$[H, U_\alpha] = \alpha(H)V_\alpha\ , \quad [H, V_\alpha] = -\alpha(H)U_\alpha \qquad (H \in \mathfrak{t}_{\mathfrak{k}})\ .$$

On the other hand we have for $\alpha \in \Phi_C$,

$$[H, X_\alpha] + [H, Y_\alpha] = \sqrt{-1}\alpha(H)X_\alpha + \sqrt{-1}\alpha(H)Y_\alpha$$

by $E_\alpha = (X_\alpha + Y_\alpha)/2$. For $H \in \mathfrak{t}_{\mathfrak{k}}$, we compare the $\mathfrak{k}^C$ (resp. $\mathfrak{p}^C$) component of this equality to obtain $[H, X_\alpha] = \sqrt{-1}\alpha(H)X_\alpha$ (resp. $[H, Y_\alpha] = \sqrt{-1}\alpha(H)Y_\alpha$). Then we have

$$\begin{aligned} &[H, W_\alpha] = \alpha(H)Z_\alpha\ , && [H, Z_\alpha] = -\alpha(H)W_\alpha\ , \\ &[H, \tilde{W}_\alpha] = \alpha(H)\tilde{Z}_\alpha \quad \text{and} && [H, \tilde{Z}_\alpha] = -\alpha(H)\tilde{W}_\alpha \end{aligned}$$

by the definition of $W_\alpha, Z_\alpha, \tilde{W}_\alpha$ and $\tilde{Z}_\alpha$. Hence from Lemma 3.2, we have Lemma 3.3. Q.E.D.

PROPOSITION 3.1. *We preserve the above notation. Then for* $h = \exp H, H \in \mathfrak{t}_{\mathfrak{k}}$, *we have*

(i) $\chi(h) = 0 \qquad (\ell_{\mathfrak{p}} > 1)$

(ii) $\chi(h) = -\prod_{\alpha \in \Phi_{I,\mathfrak{p}}^+ \cup \{\alpha_1, \cdots, \alpha_r\}} (2 - 2\cos\alpha(H)) \quad (\ell_{\mathfrak{p}} = 1)$ and

(iii) $\chi(h) = \prod_{\alpha \in \Phi_{I,\mathfrak{p}}^+} (2 - 2\cos\alpha(H)) \times \sharp(\Phi_{I,\mathfrak{p}}^+) \qquad (\ell_{\mathfrak{p}} = 0)$.

Proof. From Lemma 3.1 and 3.2, we have, for $h = \exp H$ $(H \in \mathfrak{t}_{\mathfrak{k}})$,

$$\begin{aligned} \chi(h) &= \left[\frac{d}{dx}\left\{x_d \det\left(\frac{1}{x}I_{\mathfrak{p}} - \operatorname{Ad}(h^{-1})_{\mathfrak{p}}\right)\right\}\right]_{x=1} \\ &= \left[\frac{d}{dx}\left\{(1-x)^{\ell_{\mathfrak{p}}} \prod_{\alpha \in \Phi_{I,\mathfrak{p}}^+ \cup \{\alpha_1, \cdots, \alpha_r\}} (1 - 2x\cos\alpha(H) + x^2)\right)\right\}\right]_{j=1} \end{aligned}$$

by means of $d = \dim \mathfrak{p} = \ell_{\mathfrak{p}} + 2\sharp(\Phi_{I,\mathfrak{p}}^+) + 2r$ where $\sharp(\Phi_{I,\mathfrak{p}}^+) + 2r$ where $\sharp(\Phi_{I,\mathfrak{p}}^+)$ is the order of $\Phi_{I,\mathfrak{p}}^+$. In case of $\ell_{\mathfrak{p}} = 0$, then $\Phi = \Phi_I$. Hence Proposition 3.1 is obtained. Q.E.D.

On the other hand, the root system Φ_K of $\mathfrak{k}^C$ with respect to $\mathfrak{t}_{\mathfrak{k}}$ is given due to (3.8) by

$$\Phi_K = \{\alpha_{\mathfrak{k}} : \alpha \in \Phi_C \cup \Phi_{I,\mathfrak{k}}\}$$

where $\alpha_{\mathfrak{k}}$ is the restriction of α to $\mathfrak{t}_{\mathfrak{k}}$. For $\beta \in \Phi_K$, let E'_β be E_α if $\beta = \alpha_{\mathfrak{k}}$, $(\alpha \in \Phi_{I,\mathfrak{k}})$ or X_α if $\beta = \alpha_{\mathfrak{k}}$, $(\alpha \in \Phi_C)$. Then E_β is a root vector of

$\mathfrak{k}^C$ with respect to $\mathfrak{t}_\mathfrak{k}$ for β. Let U'_β be U_α if $\beta = \alpha_\mathfrak{t}$, $\alpha \in \Phi_{I,\mathfrak{k}}$ or W_α if $\beta = \alpha_\mathfrak{t}$, $\alpha \in \Phi_C$. Put $\mathfrak{m} = \sum_{\beta \in \Phi_K} CE_\beta \cap \mathfrak{k}$. Then we have

$$\begin{aligned}\sum_{\beta \in \Phi_K^+} (RU'_\beta + RV'_\beta) &= \mathfrak{m} \\ &= \Big(\sum_{\alpha \in \Phi_{I,\mathfrak{k}}} CE_\alpha + \sum_{\alpha \in \Phi_C} CX_\alpha\Big) \cap \mathfrak{k} \\ &= \sum_{\alpha \in \Phi_{I,\mathfrak{k}}^+} (RU_\alpha + RV_\alpha) + \sum_{i=1}^{r} (RW_{\alpha_i} + RZ_{\alpha_i}) .\end{aligned}$$

Hence for $h = \exp H \in T_K$,

$$\begin{aligned}\det (I_\mathfrak{m} - \operatorname{Ad}(h)_\mathfrak{m}) &= \prod_{\alpha \in \Phi_{I,\mathfrak{k}}^+ \cup \{\alpha_1, \cdots, \alpha_r\}} (2 - 2\cos \alpha(H)) \\ &= \prod_{\beta \in \Phi_\mathfrak{k}^+} (2 - 2\cos \beta(H)) .\end{aligned} \tag{3.11}$$

Then we have

PROPOSITION 3.2. *For* $h = \exp H \in T_K$,

$$\begin{aligned}D_K(h) &= \left|\prod_{\alpha \in \Phi_\mathfrak{k}^+} \left(\exp\left(\frac{\sqrt{-1}}{2}\beta(H)\right) - \exp\left(-\frac{\sqrt{-1}}{2}\beta(H)\right)\right)^2\right| \\ &= \left|\prod_{\alpha \in \Phi_{I,\mathfrak{k}}^+ \cup \{\alpha_1, \cdots, \alpha_r\}} \left(\exp\left(\frac{\sqrt{-1}}{2}\alpha(H)\right) - \exp\left(-\frac{\sqrt{-1}}{2}\alpha(H)\right)\right)^2\right| .\end{aligned}$$

Proof. For $h = \exp H \in T_K$, by means of (3.11),

$$\begin{aligned}D_K(h) &= \left|\prod_{\beta \in \Phi_\mathfrak{k}^+} \left(\exp\left(\frac{\sqrt{-1}}{2}\beta(H)\right) - \exp\left(-\frac{\sqrt{-1}}{2}\beta(H)\right)\right)^2\right| \\ &= \left|\prod_{\beta \in \Phi_\mathfrak{k}} (2 - 2\cos \beta(H))\right| \\ &= \left|\prod_{\alpha \in \Phi_{I,\mathfrak{p}}^+ \cup \{\alpha_1, \cdots, \alpha_r\}} (2 - 2\cos \alpha(H))\right| \\ &= \left|\prod_{\alpha \in \Phi_{I,\mathfrak{p}}^+ \cup \{\alpha_1, \cdots, \alpha_r\}} \left(\exp\left(\frac{\sqrt{-1}}{2}\alpha(H)\right) - \exp\left(-\frac{\sqrt{-1}}{2}\alpha(H)\right)\right)^2\right| .\end{aligned}$$

Q.E.D.

3.3. Main theorem

THEOREM 3.1. *We preserve the assumption in* § 1. *Then we have that*

Case (i) $\operatorname{rank} G - \operatorname{rank} K \neq 1$,

$$\sum_{p=1}^{d}(-1)^p p Z^p(t) = \sum_{p=0}^{d}(-1)^p p \dim H^p(E)$$
$$= \begin{cases} 0 & (\operatorname{rank} G - \operatorname{rank} K > 1) \\ 2^{-1}\dim M & (\operatorname{rank} G - \operatorname{rank} K = 0) . \end{cases}$$

Case (ii) $\operatorname{rank} G - \operatorname{rank} K = 1$,

$$(3.12)\qquad \sum_{p=0}^{d}(-1)^p p Z^p(t) = -\frac{1}{w_K}\sum_{\gamma\in\Gamma}\chi_\rho(\gamma)\int_{G\times T_K} Z_t(\gamma g h g^{-1}) D(h) dh dg ,$$

$$(3.13)\qquad \sum_{p=0}^{d}(-1)^p p \dim H^p(E) = \frac{-[\rho_\Gamma : I_\Gamma]}{w_K}\int_{T_K} D(h) dh$$

where $D(h) = \left|\prod_{\alpha\in\Phi^+}\left(\exp\left(\frac{\sqrt{-1}}{2}\alpha(H)\right) - \exp\left(-\frac{\sqrt{-1}}{2}\alpha(H)\right)\right)^2\right|$ for $h = \exp H \in T$.

Proof. If $\operatorname{rank} G - \operatorname{rank} K > 1$, then by means of (3.3), (3.4) and Proposition 3.1 (i), we obtain the results. If $\operatorname{rank} G - \operatorname{rank} K = 1$, by means of (3.3), (3.4), Proposition 3.1 (ii) and Proposition 3.2, we obtain (3.12) and (3.13). Let $\operatorname{rank} G - \operatorname{rank} K = 0$. Then $\mathfrak{k}$ has a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Let T be a Cartan subgroup of G corresponding to $\mathfrak{t}$. Then Γ consists only of the identity of G since every translation τ_g $(g \in G)$ has a fixed point and Γ is assumed to act on $\tilde{M}$ fixed point freely. In fact, $G = \bigcup_{g\in G} gKg^{-1}$ since G and K are connected and K has a maximal torus T of G. Then we have

$$(3.14)\qquad \sum_{p=0}^{d}(-1)^p p Z^p(t) = \int_T Z_t(h) D_K(h)\chi(h) dh$$

and

$$(3.15)\qquad \sum_{p=0}^{d}(-1)^p p \dim H^p(E) = \int_T D_K(h)\chi(h) dh .$$

From Proposition 3.1 (iii) and Proposition 3.2, we have $D_K(h)\chi(h) = D(h) \times \#(\Phi^+_{I,\mathfrak{p}}) = D(h)2^{-1}\dim(G/K)$. Therefore applying Weyl's integral formula for G to (3.14), (3.15), we have

$$(3.14) = \int_G Z_t(g) dg = 1 \quad \text{and}$$
$$(3.15) = \int_G dg = 1 . \qquad \text{Q.E.D.}$$

Due to Theorem 3.1., we have

COROLLARY 3.1. *Under the assumption in* §1, *we have*

$$T(M, \rho_\Gamma) = 1 \qquad \text{if } \operatorname{rank} G - \operatorname{rank} K \neq 1$$

where ρ_Γ *is the representation restricted to* Γ *of an arbitrary finite dimensional unitary representation* ρ *of* G.

Remark. Ray and Singer [10] showed in general that $T(M, \rho) = 1$ for every even dimensional Riemannnian manifold. The new fact obtained in this paper is that $T(M, \rho_\Gamma) = 1$ *in case of* $M = \Gamma\backslash\tilde{M}$ *where* $\tilde{M}$ *is an odd dimensional simply connected symmetric space* G/K *such that* G *is compact, semisimple and* $\operatorname{rank} G - \operatorname{rank} K > 1$. Such irreducible symmetric spaces $\tilde{M}$ are as follows: all odd dimensional compact simple Lie group except $SU(2)$; $SU(n)/SO(n)$, $n = 4m$ or $4m + 3$ $(m \geqq 1)$; $SU(2n)/Sp(n)$, $n = 2m$ $(m \geqq 1)$ (cf. [5] Ch. IX.). In the case $\tilde{M} = SO(2n)/SO(2n-1)$($(2n-1)$ dimensional sphere), $T(M, \rho)$ has been calculated in Ray [9]. The cases $\tilde{M} = SU(2)$; $SU(4)/SO(4)$; $SU(3)/SO(3)$; $SO(p+q)/SO(p) \times SO(q)$ $(p, q = \text{odd},\ p > 1,\ q > 1)$ are remained for a further study.

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