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HARTMAN'S THEOREM FOR HYPERBOLIC SETS

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§ 1. Introduction, notation and definitions

Hartman proved that a diffeomorphism is topologically conjugate to a linear map on a neighbourhood of a hyperbolic fixed point ([3]). In this paper we study the topological conjugacy problem of a diffeomorphism on a neighbourhood of a hyperbolic set, and prove that for any hyperbolic set there is an arbitrarily slight extension to which a subshift of finite type is semi-conjugate. In the sequel, M denotes a compact C^∞ manifold with some Riemannian metric $|\cdot|$.

THEOREM 2. *Let $f: M \to M$ be a diffeomorphism with $\Lambda \subset M$ a hyperbolic set. Then there is a neighbourhood U of the zero-section of $T_\Lambda M$ and a bundle map $h: U \to \Lambda \times M$ such that $(f \times f) \circ h = h \circ Tf$. (We regard U and $\Lambda \times M$ as a microbundle).*

THEOREM 3. *Let f, Λ be as above, and W a neighbourhood of Λ. Then there are a hyperbolic set Λ' with $\Lambda \subset \Lambda' \subset W$ and a subshift of finite type which is semi-conjugate to Λ'.*

DEFINITION. Let E be a vector bundle with norms $\|\cdot\|$ on each fibre. A vector bundle map $T: E \to E$ is hyperbolic if E splits into

$$E = E^s \oplus E^u$$

where E^s and E^u are T invariant subbundles, and there are $0 < \lambda < 1$, $c > 0$ such that for $n \geqq 0$,

$$\|T^n v\| \leqq c\lambda^n \|v\| \qquad \text{if } v \in E^s$$
$$\|T^{-n} v\| \leqq c\lambda^n \|v\| \qquad \text{if } v \in E^u .$$

We may assume $c = 1$ ([4]).
Skewness of T is $\min\{\|T | E^s\|, \|T^{-1} | E^u\|\}$.

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Let $f: M \to M$ be a diffeomorphism. $M \supset \Lambda$ is a hyperbolic set if Λ is a closed f-invariant subset, and $Tf|\Lambda$ is hyperbolic. When its splitting is $Tf|\Lambda = E^s \oplus E^u$, define

$$\begin{aligned} B^s(r) &= \{v \in E^s \mid |v| \le r\} \\ B^u(r) &= \{v \in E^u \mid |v| \le r\} \\ B^s_x(r) &= B^s(r) \cap T_xM \\ B^u_x(r) &= B^u(r) \cap T_xM \,. \end{aligned}$$

Let $p: E \to \Lambda$ be a vector bundle with norms. $\Gamma = \Gamma(E)$ denotes the Banach space consisting of all bounded cross sections on Λ (not necessarily continuous) with sup norms. Let $\mathfrak{M}(\Gamma) = \{\text{maps}: \Gamma \to \Gamma\}$. For any $y \in E$, $\sigma_y \in \Gamma$ is given by

$$\sigma_y(x) = \begin{cases} y & \text{if } x = py \\ 0 & \text{otherwise}\,. \end{cases}$$

We define

$\tilde{\mathfrak{M}}_f(\Gamma) = \{H \in \mathfrak{M}(\Gamma) \mid H$ satisfies the following condition (I), (II)$\}$.

Condition (I), $H(\sigma_z)(x) = 0$ for $x \neq fp(z)$

(II), $H(\sigma_z)(fp(z)) = H(\sigma)(fp(z))$

for any σ with $\sigma(p(z)) = z$.

For any $H \in \tilde{\mathfrak{M}}_f(\Gamma)$, let a map

$$\Phi(H): E \to E$$

be given by $\Phi(H)(z) = H(\sigma_z)(fp(z))$. Then we define

$\mathfrak{M}_f(\Gamma) = \{H \in \tilde{\mathfrak{M}}_f(\Gamma) \mid \Phi(H)$ satisfies the following condition (III)$\}$.

Condition (III). $\Phi(H)$ is continuous.

Define

$\mathfrak{M}^b(\Gamma) = \{H \in \mathfrak{M}(\Gamma) \mid H$ is bounded$\}$, $\tilde{\mathfrak{M}}^b{}_f(\Gamma) = \tilde{\mathfrak{M}}_f(\Gamma) \cap \mathfrak{M}^b(\Gamma)$ and $\mathfrak{M}^b{}_f(\Gamma) = \mathfrak{M}_f(\Gamma) \cap \mathfrak{M}^b(\Gamma)$.

The norm of $\mathfrak{M}^b(\Gamma)$ is defined by the sup norm. For a Lipschitz map f, Lip(f) denotes its Lipschitz number.

§ 2. Hartman's theorem for hyperbolic sets

LEMMA 1. (1) *$\tilde{\mathfrak{M}}^b{}_f(\Gamma)$ is a closed linear subspace of $\mathfrak{M}^b(\Gamma)$.*

(2) *$\mathfrak{M}^b{}_f(\Gamma)$ is a closed linear subspace of $\tilde{\mathfrak{M}}^b{}_f(\Gamma)$.*

Proof. Proof of (1) is easy. $\mathfrak{M}^b{}_f(\Gamma)$ is non-empty because a continuous, bounded, fiber and zero-section preserving map from E into E over f induces an element of $\mathfrak{M}^b{}_f(\Gamma)$. Let $\tilde{\mathfrak{M}}^b{}_f(E)=\{h:E\to E\,|\,h$ is a bounded map over f (not necessarily continuous)$\}$. $\mathfrak{M}^b{}_f(E)=\{h\in\tilde{\mathfrak{M}}^b{}_f(E)\,|\,h$ is continuous$\}$. Then $\mathfrak{M}^b{}_f(E)$ is a closed linear subspace. And the map

$$\Phi:\tilde{\mathfrak{M}}^b{}_f(\Gamma)\to\tilde{\mathfrak{M}}^b{}_f(E)$$

is a continuous linear map. Thus $\mathfrak{M}^b{}_f(\Gamma)=\Phi^{-1}(\mathfrak{M}^b{}_f(E))$ is a closed linear subspace.

LEMMA 2. (1) *If* $H\in\tilde{\mathfrak{M}}_f(\Gamma), G\in\tilde{\mathfrak{M}}_g(\Gamma)$, *then* $G\circ H\in\tilde{\mathfrak{M}}_{g\circ f}(\Gamma)$ *and* $\Phi(G\circ H)=\Phi(G)\circ\Phi(H)$.
(2) *If* $H\in\mathfrak{M}_f(\Gamma), G\in\mathfrak{M}_g(\Gamma)$, *then* $G\circ H\in\mathfrak{M}_{g\circ f}(\Gamma)$.
(3) *For a homeomorphism* $H\in\tilde{\mathfrak{M}}_f(\Gamma), H^{-1}\in\tilde{\mathfrak{M}}_{f^{-1}}(\Gamma)$ *and* $\Phi(H)\in\mathfrak{M}_f(E)$ *is an invertible map with* $\Phi(H)^{-1}=\Phi(H^{-1})$.

Proof. (1), (2) are obvious.

For any $x\in\Lambda$ and $\sigma\in\Gamma, H(\sigma)(x)=0$ if and only if $\sigma(f^{-1}(x))=\sigma_{\sigma f^{-1}(x)}(f^{-1}(x))=0$ because $H(\sigma)(x)=H\sigma_{\sigma f^{-1}(x)}(x)$ and H is injective. Thus $H^{-1}\sigma_z(x)=0$ for $x\neq f^{-1}p(z)$.

For any $z_0\in E, \sigma\in\Gamma$ with $\sigma(pz_0)=z_0$, define $z'=(H^{-1}\sigma_{z_0})(f^{-1}p(z_0))$, $z''=(H^{-1}\sigma)(f^{-1}p(z_0))$. Then $\sigma(pz_0)=(H\circ H^{-1}\sigma)(pz_0)=(H\sigma_{z''})(pz_0)$, and $\sigma_{z_0}(pz_0)=(H\circ H^{-1}\sigma_{z_0})(pz_0)=(H\sigma_{z'})(pz_0)$. On the other hand $(H\sigma_{z'})(x)=(H\sigma_{z''})(x)=0$ for $x\neq pz_0$. Then $H\sigma_{z'}=H\sigma_{z''}$. Because H is injective we have $\sigma_{z'}=\sigma_{z''}$, that is $z'=z''$. Thus $H^{-1}\in\tilde{\mathfrak{M}}_{f^{-1}}(\Gamma)$. $\Phi(H^{-1})=\Phi(H)^{-1}$ follows from (1).

LEMMA 3. *If* $H\in\mathfrak{M}_f(\Gamma)$ *is a homeomorphism and* H^{-1} *is a Lipschitz map, then* $\Phi(H)$ *is a homeomorphism.*

Proof. By Lemma 2, $\Phi(H)$ is an injection and $\Phi(H)^{-1}=\Phi(H^{-1})$. For any $r>0$, define $B(r)=\{z\in E\,|\,\|z\|\leq r\}$. It is sufficient to prove that for any $r>0$

$$\Phi(H)\,|\,\Phi(H)^{-1}(B(r)):\Phi(H)^{-1}(B(r))\to B(r)$$

is a homeomorphism. We have

$$\|\Phi(H)^{-1}(B(r))\|=\|H^{-1}(B'(r))\|\leq r\,\mathrm{Lip}\,(H^{-1})$$

where $B'(r)=\{\sigma_z\in\Gamma\,|\,\|\sigma_z\|\leq r\}$. $\Phi(H)^{-1}(B(r))$ is compact because $\Phi(H)^{-1}\cdot(B(r))$ is a closed subset of $B(r\,\mathrm{Lip}\,(H^{-1}))$. Then $\Phi(H)\,|\,\Phi(H)^{-1}(B(r))$ is a homeomorphism.

LEMMA 4. *Let $T \in \mathfrak{M}_f(\Gamma)$ be a hyperbolic linear homeomorphism. Then there is $\varepsilon > 0$ such that for any $\psi, \phi \in \mathfrak{M}^b{}_f(\Gamma)$ with* $\mathrm{Lip}(\psi)$, $\mathrm{Lip}(\phi) < \varepsilon$, *there is a unique map $H_{\psi\phi} \in \mathfrak{M}^b{}_{\mathrm{id}}(\Gamma)$ satisfying*

$$(T + \psi) \circ (\mathrm{id} + H_{\psi\phi}) = (\mathrm{id} + H_{\psi\phi}) \circ (T + \phi) .$$

Proof. The proof is essentially due to Pugh ([5], [6]). Let $0 < \varepsilon < \min\{1 - \lambda, \|T^{-1}\|^{-1}\}$ where λ is a skewness of T, and

$$\mu : \mathfrak{M}^b(\Gamma) \to \mathfrak{M}^b(\Gamma)$$

be defined by

$$\mu(H) = (\mathscr{L}^*_\phi - \mathrm{id})^{-1}(\phi - \psi \circ (\mathrm{id} + H)) \circ (T + \phi)^{-1}$$

where $\mathscr{L}^*_\phi(H) = T \circ H \circ (T + \phi)^{-1}$ with $(\mathscr{L}^*_\phi - \mathrm{id})$ being invertible. Then $\mathrm{Lip}(\mu) < 1$ and there is a unique fixed point $H_{\psi\phi}$ (c.f. [5], [6]). Because $T + \phi$ is a homeomorphism and $(T + \phi)^{-1}$ is a Lipschitz map, $(T + \phi)^{-1} \in \mathfrak{M}_{f^{-1}}(\Gamma)$ follows from Lemma 2. Then $(\phi - \psi \circ (\mathrm{id} + H)) \circ (T + \phi)^{-1} \in \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$ if $H \in \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$.

Similarly $\mathscr{L}^*_\phi(\mathfrak{M}^b_{\mathrm{id}}(\Gamma)) \subset \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$.
Thus a linear map

$$\mathscr{L}^*_\phi \,|\, \mathfrak{M}^b_{\mathrm{id}}(\Gamma) : \mathfrak{M}^b_{\mathrm{id}}(\Gamma) \to \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$$

is well defined, and $\mathscr{L}^*_\phi \,|\, \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$ is hyperbolic with an associated splitting

$$\mathfrak{M}^b_{\mathrm{id}}(\Gamma) = \mathfrak{M}^b_{\mathrm{id}}(\Gamma\,;\Gamma^u) \oplus \mathfrak{M}^b_{\mathrm{id}}(\Gamma\,;\Gamma^s)$$

where

$$\Gamma^a = \{\sigma \in \Gamma \,|\, \sigma(x) \in E^a \text{ for } x \in \Lambda\}$$
$$\mathfrak{M}^b_{\mathrm{id}}(\Gamma\,;\Gamma^a) = \{H \in \mathfrak{M}^b_{\mathrm{id}}(\Gamma) \,|\, H(\sigma) \in \Gamma^a \text{ for } x \in \Gamma\}$$

for $a = u, s$.

Therefore $(\mathscr{L}^*_\phi - \mathrm{id}) \,|\, \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$ is invertible, and $(\mathscr{L}^*_\phi - \mathrm{id})^{-1}(H) \in \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$ for $H \in \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$. Thus we have $\mu(\mathfrak{M}^b_{\mathrm{id}}(\Gamma)) \subset \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$. Because $\mathfrak{M}^b_{\mathrm{id}}(\Gamma)$ is a closed linear subspace of $\mathfrak{M}^b(\Gamma)$, a unique fixed point $H_{\psi\phi}$ of μ is in $\mathfrak{M}^b_{\mathrm{id}}(\Gamma)$.

LEMMA 5. *Let T be as above. Then there is $\varepsilon > 0$ such that for any $\psi \in \mathfrak{M}^b{}_f(\Gamma)$ with* $\mathrm{Lip}(\psi) < \varepsilon$ *there is a unique map $H \in \mathfrak{M}^b_{\mathrm{id}}(\Gamma)$ satisfying*

$$(T + \psi) \circ (\mathrm{id} + H) = (\mathrm{id} + H) \circ T .$$

Moreover $\mathrm{id} + H$ *is a homeomorphism.*

LEMMA 6. *Let* $p: E \to \Lambda$ *be a vector bundle. Let* Λ *be compact,* $V \subset E$ *be a neighbourhood of the zero-section. Assume* $\phi: V \to E$ *is a fiber preserving map and*

(1) $\phi | (V \cap p^{-1}(x))$ *is differentiable for* $x \in \Lambda$

(2) $T_z\phi$ *is continuous with respect to* $z \in V$

(3) $\phi(0_x) = 0_{f(x)}$

(4) $T_{0_x}\phi = 0$,

where $T_z\phi$ *is the differential of* $\phi | (V \cap p^{-1}(x))$ *at* $z \in V \cap p^{-1}(x)$ *and* 0_x *is the zero vector at* $x \in \Lambda$.

Then for any $\varepsilon > 0$, *there is a neighbourhood* W *of the zero section, and a fiber preserving map*

$$\tilde{\phi}: E \to E$$

such that

(5) $\tilde{\phi} | W = \phi | W$

(6) $\mathrm{Lip}(\tilde{\phi}) < \varepsilon$

(7) $\tilde{\phi}$ *is bounded with the sup norm.*

Lemma 5 follows from Lemma 4, and Lemma 6 is a vector bundle version of ([5] p. 79).

THEOREM 1. *Let* $p: E \to \Lambda$ *be a vector bundle with* Λ *compact,* $f: \Lambda \to \Lambda$ *be a homeomorphism. Let* $T: E \to E$ *be a hyperbolic vector bundle map over* f.

Then there is $\varepsilon > 0$ *satisfying the followings; for any fiber and zero-section preserving map* $\phi: E \to E$ *over* f *such that* ϕ *is bounded with sup norm and* $\mathrm{Lip}(\phi) < \varepsilon$, *there is a unique fiber preserving map* $h_\phi: E \to E$ *over* id *such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{T} & E \\ {\scriptstyle \mathrm{id}+h_\phi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}+h_\phi} \\ E & \xrightarrow{T+\phi} & E \end{array}$$

is commutative. Moreover $\mathrm{id} + h_\phi$ *is a homeomorphism.*

Proof. Let λ be a skewness of $T, \varepsilon > 0$ be such that $\varepsilon < \min\{1 - \lambda, \|T^{-1}\|^{-1}\}$. A map

$$T_*: \Gamma(E) \to \Gamma(E)$$

given by

$$T_*(\sigma) = T \circ \sigma \circ f^{-1}$$

is a hyperbolic isomorphism with an associated splitting

$$\Gamma(E) = \Gamma(E^s) \oplus \Gamma(E^u) .$$

Then

$$\|T_* | \Gamma(E^s)\| = \|T | E^s\| < \lambda$$
$$\|T_*^{-1} | \Gamma(E^u)\| = \|T^{-1} | E^u\| < \lambda .$$

For a map $\phi: E \to E$ with $\mathrm{Lip}\,(\phi) < \varepsilon$, we define a map

$$\phi_*: \Gamma(E) \to \Gamma(E)$$

by $\phi_*(\sigma) = \phi \circ \sigma \circ f^{-1}$.
Then

$$\mathrm{Lip}\,(\phi_*) = \mathrm{Lip}\,(\phi)$$

and

$$\phi_* \in \mathfrak{M}^b{}_f(\Gamma(E)) .$$

By Lemma 5, there is a unique map $H \in \mathfrak{M}^b_{\mathrm{id}}(\Gamma(E))$ with the commutative diagram;

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{T_*} & \Gamma(E) \\ \downarrow{\scriptstyle \mathrm{id}+H} & & \downarrow{\scriptstyle \mathrm{id}+H} \\ \Gamma(E) & \xrightarrow{T_*+\phi_*} & \Gamma(E) . \end{array}$$

The map $h_\phi: E \to E$ defined by

$$h_\phi(z) = H(\sigma_z)(p(z))$$

is the required map. The uniqueness follows from the unique

THEOREM 2. (*Hartman's theorem for hyperbolic sets*)
Let $f: M \to M$ *be a diffeomorphism,* $\Lambda \subset M$ *be a hyperbo*
Then there are a neighbourhood U *of the zero section in*
a map

$$h : U \to \Lambda \times M$$

which maps U homeomorphically onto a neighbourhood of the diagonal in $\Lambda \times M$ with

1) $pr_2 \circ h = p$
2) $h(0_x) = (x, x)$
3) *the diagram*

$$\begin{array}{ccc} U \cap Tf^{-1}(U) & \xrightarrow{Tf} & U \\ h\downarrow & & h\downarrow \\ \Lambda \times M & \xrightarrow{f \times f} & \Lambda \times M \end{array} \quad \textit{commutes.}$$

Proof. Let $\varepsilon > 0$ satisfy $\varepsilon < \min\{1 - \lambda, \|T^{-1}\|^{-1}\}$ where λ is the skewness of $Tf \mid \Lambda$. Assume that $W \subset T_\Lambda M$ is a neighbourhood of the zero section of $T_\Lambda M$.

By taking W sufficiently small, a map $F : W \to T_\Lambda M$ can be given by

$$(f \times f) \circ (p, \exp) = (p, \exp) \circ F .$$

Define $\phi = F - T_0 F \mid W$, where

$$T_0 F : T_\Lambda M \to T_\Lambda M$$

is the differential of F on the zero sections. By Lemma 6 there are a neighbourhood of the zero section $U \subset W$, and a fiber preserving map over f

$$\tilde{\phi} : T_\Lambda M \to T_\Lambda M$$

such that

(1) $\tilde{\phi} \mid U = \phi \mid U$
(2) $\mathrm{Lip}\,(\tilde{\phi}) < \varepsilon$
(3) $\tilde{\phi}$ is bounded with the sup norms.

Define $\tilde{F} = T_0 F + \tilde{\phi}$. Then

$$\tilde{F} = F \text{ on } U$$
$$\mathrm{Lip}\,(\tilde{F} - T_0 F) < \varepsilon$$
$$\tilde{F} - T_0 F \text{ is bounded on } T_\Lambda M \text{ with the sup norms .}$$

By Theorem 1, there is a fiber preserving map over id_Λ

$$\tilde{h} : T_\Lambda M \to T_\Lambda M$$

with

$$\tilde{h} \circ T_0F = \tilde{F} \circ \tilde{h}$$
$$\tilde{h} \text{ is a homeomorphism}$$
$$pr \circ \tilde{h} = p\ .$$

Because the derivative of the exponential map at the zero-section is the identity, $T_0F = Tf$. Thus $h = (p, \exp) \circ \tilde{h} | V$ is the required map.

§ 3. Semi-conjugacies of subshifts to hyperbolic sets

Let $\mathscr{A} = \{A_1, \cdots, A_n\}$ be a finite set and $T = (t_{ij})$ be a $n \times n$ $0-1$ matrix. $\mathscr{A}^Z$ denotes the space of maps from integers Z into $\mathscr{A}$ with compact-open topology ($\mathscr{A}$ and Z have the discrete topologies). The shift transformation $\rho : \mathscr{A}^Z \to \mathscr{A}^Z$ is defined by

$$\rho((x_i)_{i \in Z}) = (x_i')_{i \in Z} \quad \text{where} \quad x_i' = x_{i+1}$$

for $(x_i)_{i \in Z} \in \mathscr{A}^Z$.

Let Σ be the ρ invariant set of $\mathscr{A}^Z$ given by

$$\Sigma = \left\{ (a_i)_{i \in Z} \in \mathscr{A}^Z \,\middle|\, \begin{matrix} t_{n_i n_{i+1}} = 1 \quad \text{where} \\ a_i = A_{n_i} \end{matrix} \right\} .$$

Σ is called a subshift of finite type on the symbol $\mathscr{A}$ determined by the intersection matrix T.

Bowen ([1]) proved that when $\Lambda \subset M$ is a basic set of an Axiom A diffeomorphism there are a subshift of finite type Σ and a semiconjugacy $\Pi : \Sigma \to \Lambda$, i.e. Π is surjective and $f\Pi = \Pi\rho$ ([1]).

In this section we consider the case when Λ is a hyperbolic set. Our result is the following.

THEOREM 3. *Suppose $f : M \to M$ is a diffeomorphism, $\Lambda \subset M$ is a compact hyperbolic set and W is a neighbourhood of Λ.*

Then there are a finite set $\mathscr{B} = \{B_i\}_{i=1,\dots,N}$, and a matrix $T = (t_{ij})$ satisfying the followings;

1) *for any $1 \leqq i \leqq N$, B_i is a closed m-disk and $\Lambda \subset \bigcup_i \operatorname{int} B_i$*
2) *$T = (t_{ij})$ is a $N \times N$ $0-1$ matrix*
3) *the diagram*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Pi} & M \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma & \xrightarrow{\Pi} & M \end{array}$$

is commutative, where Σ is the subshift of finite type on the symbol $\mathscr{B}$ determined by the intersection matrix T, and ρ is the shift transformation.

4) *Π is a continuous map given by $\Pi((a_i)_{i\in Z}) = \bigcap_{i\in Z} f^{-i}(a_i)$.*

5) *$\Lambda' = \Pi(\Sigma)$ is a closed hyperbolic set with $\Lambda \subset \Lambda' \subset W$.*

Proof. Step 1. We may assume that W is a neighbourhood of Λ so small that any invariant set contained in W is hyperbolic ([4]). Let ε be a positive number such that an expansive constant of a hyperbolic set in W is greater than ε. Let U be a neighbourhood of the zero-section of $T_\Lambda M$ on which the map $h: U \to \Lambda \times M$ with $(f \times f)\circ h = h \circ Tf$ is defined by Theorem 2. $\bar{h}$ is given by

$$\bar{h} = pr_2 \circ h : U \xrightarrow{h} \Lambda \times M \xrightarrow{pr_2} M .$$

Choose $r_1 > r > 0$ such that

$$\begin{aligned} &Tf(B^u(r)) \subset \operatorname{int} B^u(r_1) \\ &Tf^{-1}(B^s(r)) \subset \operatorname{int} B^s(r_1) \\ &U \supset B^s(r_1) \oplus B^u(r_1) \\ &W \supset \bar{h}(B^s_x(r) \times B^u_x(r)) \\ &\operatorname{diam} \bar{h}(B^s_x(r) \times B^u_x(r)) < \varepsilon \qquad \text{for } x \in \Lambda . \end{aligned}$$

Step 2. For any $x \in \Lambda$, let $V_x \subset \Lambda$ be a neighbourhood of x in Λ such that

1) for any $y \in f^{-1}V_x$
$f\bar{h}(B^s_y(r) \times B^u_y(r)) \subset \operatorname{int} \bar{h}(B^s_x(r) \times B^u_x(r_1))$, and f maps $\bar{h}(B^s_y(r) \times \partial B^u_y(r)$ into $\operatorname{int} \bar{h}(B^s_x(r) \times (B^u_x(r_1) - B^u_x(r)))$ with degree ± 1. (Here "f maps with degree ± 1" means that the homomorphism f_* between the homology groups is of degree ± 1. This does not depend on isomorphisms: $H_{u-1}(\bar{h}(B^s_y(r) \times \partial B^u_y(r))) \approx H_{u-1}(\operatorname{int} \bar{h}(B^s_x(r) \times (B^u_x(r_1) - B^u_x(r))) \approx Z$. Here u denotes also the fiber dimension of the unstable bundle E^u.)

2) for any $y \in fV_x$

$f^{-1}\bar{h}(B^s_y(r)\times B^u_y(r))\subset \operatorname{int}\bar{h}(B^s_x(r_1)\times B^u_x(r))$, and f^{-1} maps $\bar{h}(\partial B^s_y(r)\times B^u_y(r))$ into $\operatorname{int}\bar{h}((B^s_x(r_1)-B^s_x(r))\times B^u_x(r))$ with degree ± 1.

The existence of neighbourhoods V_x satisfying 1), 2) follows from the continuity of h and the fact that the homomorphisms

$$f_*: H_{u-1}(f\bar{h}(B^s_{f^{-1}(x)}(r)\times \partial B^u_{f^{-1}(x)}(r))) \\ \to H_{u-1}(\operatorname{int}\bar{h}(B^s_x(r)\times (B^u_x(r_1)-B^u_x(r)))$$

and

$$f_*^{-1}: H_{s-1}(f^{-1}\bar{h}(\partial B^s_{f(x)}(r)\times B^u_{f(x)}(r)))) \\ \to H_{s-1}(\operatorname{int}\bar{h}((B^s_x(r_1)-B^s_x(r))\times B^u_x(r)))$$

are isomorphic.

Let $\{U_x\}_{x\in\Lambda}$ be a refinement of $\{V_x\}_{x\in\Lambda}$ such that $f(U_y)\cap U_x\neq\phi$ (resp. $f^{-1}(U_y)\cap U_x\neq\phi$) implies $f(U_y)\subset V_x$ (resp. $f^{-1}(U_y)\subset V_{,x}$).

Choose $X_1,\cdots,X_N\in\Lambda$ such that $\{U_{x_i}\}_{i=1,\dots,N}$ is a covering of Λ.

Step 3. An intersection matrix $T=(t_{ij})$ is given as follows. $t_{ij}=1$ if

1) $f\bar{h}(B^s_{x_i}(r)\times B^u_{x_i}(r)\subset \operatorname{int}\bar{h}(B^s_{x_j}(r)\times B^u_{x_j}(r_1))$ and f maps $\bar{h}(B^s_{x_i}(r)\times \partial B^u_{x_i}(r))$ into $\operatorname{int}\bar{h}(B^s_{x_j}(r)\times(B^u_{x_j}(r_1)-B^u_{x_j}(r)))$ with degree ±1.
2) $f^{-1}\bar{h}(B^s_{x_j}(r)\times B^u_{x_j}(r))\subset\operatorname{int}\bar{h}(B^s_{x_i}(r_1)\times B^u_{x_i}(r))$ and f^{-1} maps $\bar{h}(\partial B^s_{x_j}(r)\times B^u_{x_j}(r))$ into $\operatorname{int}\bar{h}((B^s_{x_i}(r_1)-B^u_{x_i}(r))\times B^u_{x_i}(r))$ with degree ±1.

$t_{ij}=0$ otherwise.

Step 4. Suppose that $i_0,\cdots,i_m$ satisfy $t_{i_n i_{n+1}}=1$ for $n=0,\cdots,m-1$. Define maps

$$H^{(i_n)}:\bar{h}(B^s_{x_{i_n}}(r_1)\times B^u_{x_{i_n}}(r))\to\bar{h}(B^s_{x_{i_n}}(r)\times B^u_{x_{i_n}}(r))$$

by

$$H^{(i_n)}(\bar{h}(z_1,z_2))=\begin{cases}\bar{h}((z_1,z_2)) & \text{if } |z_1|\le r\\ \bar{h}\left(\left(\dfrac{rz_1}{|z_1|},z_2\right)\right) & \text{if } |z_1|>r\,,\end{cases}$$

and a map

$$H:\bar{h}(B^s_{x_{i_m}}(r)\times B^u_{x_{i_m}}(r))\to\bar{h}(B^s_{x_{i_0}}(r)\times B^u_{x_{i_0}}(r))$$

by

$$H=H^{(i_0)}\circ f^{-1}\circ\cdots\circ H^{(i_{m-2})}\circ f^{-1}\circ H^{(i_{m-1})}\circ f^{-1}\,.$$

Then we have

$$H(\bar{h}(B^s_{x_{i_m}}(r) \times B^u_{x_{i_m}}(r))) \cap \bar{h}(\operatorname{int} B^s_{x_{i_0}}(r) \times B^u_{x_{i_0}}(r))$$
$$= \bigcap_{n=0}^{m} f^{-n}(\operatorname{int} B^s_{x_{i_n}}(r) \times B^u_{x_{i_n}}(r)) \,.$$

By the definition of $t_{i_n i_{n+1}} = 1$, the map

$$H_* : H_{s-1}(\bar{h}(\partial B^s_{x_{i_m}}(r) \times B^u_{x_{i_m}}(r))) \to H_{s-1}(\bar{h}(\partial B^s_{x_{i_0}}(r) \times B^u_{x_{i_0}}(r))$$

is isomorphic. This implies

$$H(\bar{h}(B^s_{x_{i_m}}(r) \times B^u_{x_{i_m}}(r))) \cap \bar{h}(\operatorname{int} B^s_{x_{i_0}}(r) \times B^u_{x_{i_0}}(r)) \neq \phi \,.$$

Hence

$$\bigcap_{n=0}^{m} f^{-n}(\bar{h}(B^s_{x_{i_n}}(r) \times B^u_{x_{i_n}}(r))) \neq \phi \,.$$

For a finite sequence $i_{-\ell} \cdots i_m$ satisfying $t_{i_n i_{n+1}} = 1\ (-\ell \leq n \leq m-1)$

$$\bigcap_{n=-\ell}^{m} f^{-n}\bar{h}(B^s_{x_{i_n}}(r) \times B^u_{x_{i_n}}(r)))$$
$$= f^{-\ell}\Big(\bigcap_{n=0}^{m+\ell} f^{-n}(\bar{h}(B^s_{x_{i_{n-\ell}}}(r) \times B^u_{x_{i_{n-\ell}}}(r)))\Big)$$
$$\neq \phi$$

because

$$\bigcap_{n=0}^{m+\ell} f^{-n}(\bar{h}(B^s_{x_{i_{n-\ell}}}(r) \times B^n_{x_{i_{n-\ell}}}(r))) \neq \phi \,.$$

This implies

$$\bigcap_{n\in Z} f^{-n}(\bar{h}(B^s_{x_{i_n}}(r) \times B^u_{x_{i_n}}(r))) \neq \phi$$

if $t_{i_n i_{n+1}} = 1$ $(n \in Z)$.

Put $J = \{\{j_n\}_{n\in Z} \mid j_n \in \{1, \cdots, N\},\ t_{j_n j_{n+1}} = 1\}$, $\Lambda' = \bigcup_{\{j_n\}\in J} \bigcap_n f^{-n}\bar{h}(B^s_{x_{j_n}}(r) \times B^u_{x_{j_n}}(r))$. Then Λ' is a hyperbolic set contained in W, and $\bigcap_n f^{-n}\bar{h}(B^s_{x_{i_n}}(r) \times B^u_{x_{i_n}}(r)) \subset \Lambda'$. An expansive constant of $f|\Lambda'$ is greater than $\operatorname{diam} \bar{h}(B^s_{x_i}(r) \times B^u_{x_i}(r))$ for any $1 \leqq i \leqq N$. Thus $\cap f^{-n}\bar{h}(B^s_{x_{j_n}}(r) \times B^u_{x_{j_n}}(r))$ = one point if $\{j_n\} \in J$. $\Pi(\Sigma) = \Lambda'$. For any $x \in \Lambda$ and $n \in Z$, there is $U_{x_{i_n}}$ with $f^n(x) \in U_{x_{i_n}} \subset \bar{h}(B^s_{x_{i_n}}(x) \times B^u_{x_{i_n}}(x))$. Thus $x \in \cap f^{-n}\bar{h}(B^s_{x_{i_n}}(r) \times B^u_{x_{i_n}}(r))$. Therefore $\Lambda \subset \Lambda'$.

Put $B_i = \bar{h}(B^s_{x_i}(r) \times B^u_{x_i}(r))$, $\mathscr{B} = \{B_i\}_{i=1,\ldots,N}$.

Then $\mathscr{B}$ and $T = (t_{ij})$ define the required subshift. This completes the proof.

COROLLARY. *Let* $f: M \to M$ *be an Anosov diffeomorphism. Then there are a subshift of finite type* Σ *and a semi-conjugacy* $\Pi: \Sigma \to M$.

In the case when f satisfies $\Omega(f) = M$, the above was proved by Sinai ([7]). But we don't know that the above semiconjugacy can be chosen such that there is an integer N with the cardinal number of $\pi(x) \leqq N$ for any $x \in M$([2]).

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