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PICARD PRINCIPLE FOR FINITE DENSITIES ON SOME END

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Consider a parabolic *end* Ω of a Riemann surface in the sence of Heins [2] (cf. Nakai [3]). A *density* $P = P(z)dxdy$ $(z = x + iy)$ is a 2-form on $\bar{\Omega} = \Omega \cup \partial\Omega$ with nonnegative locally Hölder continuous coefficients $P(z)$. A density P is said to be *finite* if the integral

$$(1) \qquad \int_{\Omega} P(z)dxdy < \infty .$$

The *elliptic dimension* of a density P at the ideal boundary point δ, $\dim P$ in notation, is defined (Nakai [5], [6]) to be the 'dimension' of the half module of nonnegative solutions of the equation

$$(2) \qquad L_p u \equiv \Delta u - Pu = 0 \qquad (\text{i.e. } d^*du - uP = 0)$$

on an end Ω with the vanishing boundary values on $\partial\Omega$. The elliptic dimension of the particular density $P \equiv 0$ at δ is called the *harmonic dimension* of δ. After Bouligand we say that the *Picard principle* is valid for a density P at δ if $\dim P = 1$. For the punctured disk $V: 0 < |z| < 1$, Nakai [6] showed that the Picard principle is valid for any finite density P on $0 < |z| \leqq 1$ at the ideal boundary $z = 0$, and he conjectured that the above theorem is valid for every general end of harmonic dimension one. The purpose of this paper is to give a partial answer in the affirmative.

Heins [2] showed that the harmonic dimension of the ideal boundary δ of an end is one if Ω satisfies the condition $[H]$: There exists a sequence $\{A_n\}$ of disjoint annuli with analytic Jordan boundaries on Ω satisfying the condition that for each n, A_{n+1} separates A_n from the ideal boundary, and A_1 separates the relative boundary $\partial\Omega$ from the ideal boundary, and

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$$\sum_{n=1}^{\infty} \operatorname{mod} A_n = \infty . \tag{3}$$

We shall prove the following

THEOREM. *The Picard principle is valid at δ for any finite density P on an end Ω with the condition* [H].

The proof of the theorem will be given in no. 5 after three lemmas in no. 2–4. Although the essence of the proofs of these lemmas is found in Nakai [6], we include here their proofs for the sake of completeness. However the lemma in no. 4 requires an entirely different considerations for ends with infinite genus.

1. We always assume that an end Ω has a single ideal boundary component δ and that $\partial\Omega$ consists of a finite number of disjoint closed simple analytic curves on R. Let u be a bounded solution of (2) on Ω with continuous boundary values on $\partial\Omega$. We first note that

$$\sup_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u| . \tag{4}$$

In fact, since u^2 is subharmonic on Ω and Ω is a parabolic end, by the maximum principle for bounded subharmonic functions, we have the identity (4). The *P-unit* $e = e_P$ is the bounded solution of (2) on Ω with boundary values 1 on $\partial\Omega$. By (4) such a e_P is unique. Next consider the *associated operator* $\hat{L}_P$ with L_P which is introduced by Nakai ([5], [6]);

$$(\hat{L}_P u) dxdy = d^*du + 2d(\log e_P) \wedge {}^*du$$

for $u \in C^2(\Omega)$ where e_P is the P-unit on $\bar{\Omega}$. We say that the *Riemann theorem* is valid for $\hat{L}_P$ at δ if $\lim_{z\to\delta} u(z)$ exists for every bounded solution u of

$$\hat{L}_P u = 0 \tag{5}$$

on Ω. Nakai ([5], [6]), showed the following duality theorem (cf. also Heins [2], Hayashi [1], Nakai [4]): The Picard principle is valid for the operator L_P at δ if and only if the Riemann theorem is valid for the associated operator $\hat{L}_P$ at δ.

2. Concerning the valuation of the Dirichlet integral of $\log e_P$ we shall first prove (Nakai [6]):

LEMMA. *The P-unit e_P of a density P on an end $\bar{\Omega}$ satisfies the following inequality*

$$D_\Omega(\log e_P) \equiv \int_\Omega d \log e_P \wedge {}^*d \log e_P \leqq \int_\Omega (1 - e_P)P \ . \tag{6}$$

Proof. Take a sequence $\{\Omega_n\}$ of ends such that $\bar{\Omega}_{n+1} \subset \Omega_n$ $(n = 1, 2, \cdots)$, $\bigcap_{n=1}^\infty \Omega_n = \emptyset$. Let e_n be a continuous function on $\bar{\Omega}$ such that $L_P e_n = 0$ on $\Omega - \bar{\Omega}_n$ and $e_n = 1$ on $\bar{\Omega}_n \cup \partial\Omega$. Since e_n is decreasing as $n \to \infty$, by the Harnack principle, e_n converges to the P-unit e_P on $\bar{\Omega}$ uniformly on each compact subset of $\bar{\Omega}$, and the same is true for de_n and $^*de_n$. Observe that

$$\begin{aligned} d(e_n^{-1}{}^*de_n) &= e_n^{-1}d^*de_n + de_n^{-1} \wedge {}^*de_n \\ &= P + d \log e_n \wedge {}^*d \log e_n \end{aligned}$$

on $\Omega - \bar{\Omega}_n$. Since $e_n^{-1} = 1$ on $\bar{\Omega}_n \cup \partial\Omega$, we deduce the identity

$$\int_\Omega d \log e_n \wedge {}^*d \log e_n = \int_\Omega (1 - e_n)P \tag{7}$$

from the Stokes formula. Observe that $(1 - e_n)P$ is increasing as $n \to \infty$. On taking the inferior limit as $n \to \infty$ on the both sides of (7) and applying the Fatou lemma and the Lebesgue theorem, we conclude that

$$D_\Omega(\log e_P) \leqq \liminf_{n\to\infty} \int_\Omega d \log e_n \wedge {}^*d \log e_n = \int_\Omega (1 - e_P)P$$

Q.E.D.

3. Let u be a bounded solution of (5). The Dirichlet integral of u is finite if the density P is finite, i.e. we state the following (Nakai [6]):

LEMMA. *If a density P is finite on Ω, then any bounded solution u of $\hat{L}_P u = 0$ on $\bar{\Omega}_0$ has a finite Dirichlet integral on any end Ω_0 with $\bar{\Omega}_0 \subset \Omega$.*

Proof. Let $\{\Omega_n\}_1^\infty$ be a sequence as in no. 2 with $\bar{\Omega}_1 \subset \Omega_0$ and u_n be a continuous function on $\bar{\Omega}_0$ such that $\hat{L}_P u_n = 0$ on $\Omega_0 - \bar{\Omega}_n$, $u_n = u$ on $\partial\Omega_0$ and $u_n = 0$ on $\bar{\Omega}_n$. Then we have the identity

$$\begin{aligned} d(u_n{}^*du_n) &= du_n \wedge {}^*du_n + u_n d^*du_n \\ &= du_n \wedge {}^*du_n - 2u_n d \log e \wedge {}^*du_n \end{aligned}$$

on $\Omega_0 - \bar{\Omega}_n$, where e is the P-unit of P on Ω. The Stokes formula yields

$$D_{\Omega_0}(u_n) = \int_{\partial\Omega_0} u_n{}^*du_n + 2\int_{\Omega_0} u_n d\log e \wedge {}^*du_n$$

where

$$D_{\Omega_0}(u_n) = \int_{\Omega_0} du_n \wedge {}^*du_n \;.$$

The function u_n converges to u uniformly on every compact subset of $\bar{\Omega}_0$ and ${}^*du_n$ converges to ${}^*du$ uniformly on $\partial\Omega_0$. In fact, $v_n = eu_n$ is a bounded solution of (2) on $\Omega_0 - \bar{\Omega}_n$ and $|v_n| \leqq \sup_{\bar{\Omega}_0}|u|$. Then v_n converges to a bounded solution v of (2) uniformly on every compact subset of $\bar{\Omega}_0$. Since v and eu are both bounded solutions of (2) with the same boundary values on $\partial\Omega_0$, we have that $v = eu$, i.e. $u_n \to u$ as $n \to \infty$ uniformly on every compact subset of $\bar{\Omega}_0$. Similarly we have the last assertion. Since u_n is bounded and $u_n = u$ on $\partial\Omega_0$, by the Schwarz inequality, we deduce the inequality

$$D_{\Omega_0}(u_n) \leqq \left|\int_{\partial\Omega_0} u^*du_n\right| + kD_{\Omega_0}(\log e)^{1/2}D_{\Omega_0}(u_n)^{1/2} \tag{8}$$

for some constant $k > 0$. Observe that the first term of the right hand side of (8) is bounded. On the other hand, since P is a finite density, by Lemma in no. 2, $D_{\Omega_0}(\log e)$ is finite. Therefore $D_{\Omega_0}(u_n)$ is bounded. The Fatou lemma yields

$$D_{\Omega_0}(u) \leqq \liminf_{n\to\infty} D_{\Omega_0}(u_n) < \infty \;. \qquad \text{Q.E.D.}$$

4. Consider an end Ω with the condition [H], i.e. there exists a sequence $\{A_n\}$ of disjoint annuli on Ω with the condition (3). Let $\lambda(\gamma)$ denote the oscillation of $u \in C^1(\Omega)$ on a set $\gamma \subset \Omega$, i.e.

$$\lambda(\gamma) = \max_{\gamma} u(z) - \min_{\gamma} u(z) \;.$$

We prove the following

LEMMA. *If a function $u \in C^1(\Omega)$ has a finite Dirichlet integral on Ω with the condition* [H], *then there exists a sequence $\{\Omega_n\}$ of ends such that $\lambda_n = \lambda(\Omega_n) \to 0$ as $n \to \infty$.*

Proof. Choose a strictly decreasing sequence $\{a_n\}$ $(n = 0, 1, 2, \cdots)$ of positive numbers a_n such that $a_0 = 1$ and that

$$\operatorname{mod} A_n = \log\,(a_{n-1}/a_n) \tag{9}$$

for $n = 1, 2, \cdots$. By the condition (3), we have that $a_n \to 0$ as $n \to \infty$. Take a sequence $\{C_n\}$ of concentric circles $|z| = a_n$ $(n = 1, 2, \cdots)$ on the complex plane. A_n is conformally equivalent to $a_n < |z| < a_{n-1}$ $(n = 1, 2, \cdots)$ by (9). Therefore the restriction of u to $\bigcup_{n=1}^{\infty} A_n$ is considered as a function on $0 < |z| < 1$ by giving the values of u on C_n as follows:

$$u(z_0) = \lim_{z \to z_0} u(z) \qquad (z_0 \in C_n \text{ and } a_n < |z| < a_{n-1}) .$$

Let $\lambda(r)$ be the oscillation of u on $|z| = r$ $(0 < r < 1)$. Then we have

$$\lambda(r) \leqq \int_0^{2\pi} \left| \frac{\partial}{\partial \theta} u(re^{i\theta}) \right| d\theta .$$

The Schwarz inequality yields

$$\lambda(r)^2 \leqq 2\pi \int_0^{2\pi} \left| \frac{\partial}{\partial \theta} u(re^{i\theta}) \right|^2 d\theta .$$

Therefore we have

$$\frac{\lambda(r)^2}{r} \leqq 2\pi \int_0^{2\pi} \left(\left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial u}{\partial \theta} \right|^2 \right) r d\theta .$$

We integrate the both sides of the above on $(0, 1)$ with respect to dr and obtain

$$\text{(10)} \qquad \frac{1}{2\pi} \int_0^1 \frac{\lambda(r)^2}{r} dr \leqq \int_{0<|z|<1} du \wedge {}^*du = \sum_{n=1}^{\infty} D_n$$

where D_n denotes the Dirichlet integral of u on A_n. By the assumption of Lemma the right hand side of (10) is finite and then the same is true for the left hand side of (10). This shows that $\liminf_{r \to 0} \lambda(r) = 0$, i.e. there exists a decreasing sequence r_n such that $\lambda(r_n) \to 0$ as $n \to \infty$. Since the image set on Ω of $|z| = r_n$ is a cycle of Ω separating $\partial\Omega$ from δ, there exist ends Ω_n such that $\partial\Omega_n$ are the images of $|z| = r_n$ $(n = 1, 2, \cdots)$.

Q.E.D.

5. *Proof of the theorem.* In view of the duality theorem in no. 1, we only have to show that any bounded solution u of $\hat{L}_P u = 0$ on Ω has the limit at δ. Since P is a finite density on Ω, by Lemmas 2, 3 and 4, there exists a sequence $\{\Omega_n\}$ of ends such that $\lambda_n = \lambda(\partial\Omega_n) \to 0$ as $n \to \infty$. Consider functions $m_n e$, $M_n e$ and eu on $\bar{\Omega}_n$ where $m_n = \min_{\partial\Omega_n} u(z)$, $M_n = \max_{\partial\Omega_n} u(z)$ and e is the P-unit of P on Ω. These functions are solutions

of (2) on Ω_n with continuous boundary values on $\partial\Omega_n$. Observe that

$$m_n e \leqq eu \leqq M_n e$$

on $\partial\Omega_n$. By (4), the same inequality is valid on Ω_n. Therefore $m_n \leqq u \leqq M_n$ on $\bar{\Omega}_n$, i.e.

$$0 \leqq \sup_{\bar{\Omega}_n} u(z) - \inf_{\bar{\Omega}_n} u(z) \leqq M_n - m_n = \lambda_n .$$

Since $\lambda_n \to 0$ as $n \to \infty$, u has the limit at δ. Q.E.D.

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