G. Ikegami
Nagoya Math. J.
Vol. 67 (1977), 15-34

# EXISTENCE OF REGULAR COVERINGS ASSOCIATED WITH LEAVES OF CODIMENSION ONE FOLIATIONS

GIKŌ IKEGAMI

## § 1. Statement of results

In this paper we are concerned with transversely orientable codimension one foliations. Let $\mathscr{F}$ be a $C^r$-foliation as above in a smooth manifold $M$, $r \geqq 1$, and let $F_0$ be a closed leaf of $\mathscr{F}$. A neighborhood $U$ of $F_0$ is called a *bicollar* of $F_0$ in this paper if there is a normal line bundle $\nu: U \to F_0$ with respect to a fixed Riemannian metric on $M$ such that each fibre of $\nu$ is transverse to $\mathscr{F}$. For a bicollar $U$ of $F_0$, $U_+ = F_0 \cup$ (a component of $U - F_0$) is called a *collar* of $F_0$. A leaf $F \in \mathscr{F}$ is said to be *asymptotic* to $F_0$ in $U_+$ if $F \cap V \neq \phi$ for any neighborhood $V$ of $F_0$ in $U_+$. Let $F_V$ be a leaf asymptotic to $F_0$ of the restricted foliation $\mathscr{F} | V$, where $V$ is a neighborhood of $F_0$ in $U_+$. A *plaque* of $F$ is a leaf of $F | N$ diffeomorphic to an open $(n-1)$-ball, where $N$ is a sufficiently small open $n$-ball in the $n$-manifold $M$. A $C^r$-covering $\tilde{\nu}: \tilde{F} \to F_0$ is said to be *associated* with $F_V$ if there is an injection $i: F_V \to \tilde{F}$ such that $\tilde{\nu} i = \nu | F_V$ and that $i$ maps any plaque of $F_V$ $C^r$-diffeomorphically into $\tilde{F}$. The *one sided holonomy group* $\Phi_+(F_0)$ of $F_0$ is the holonomy group of $F_0$ defined by the restricted foliation $\mathscr{F} | U_+$.

The main purpose of this paper is to prove Theorem 2, which is an existence theorem of associated regular coverings. Theorem 1 is used in the proofs of Theorem 2 and Theorem 5. Theorem 3 and Theorem 4 are the properties of associated regular coverings. As an application we show Theorem 5, which is an unstability theorem of foliations.

THEOREM 2. *Let $\mathscr{F}$ be a transversely orientable $C^r$-foliation of codimension one, $r \geqq 1$, $F_0$ be an orientable closed leaf of $\mathscr{F}$, and let $U_+$ be a collar of $F_0$. Suppose that the one sided holonomy group $\Phi_+(F_0)$ is abelian. Then there is a neighborhood $V_0$ of $F_0$ in $U_+$ such that any*

Received February 10, 1975.
Revised November 5, 1976.

*neighborhood $V$ of $F_0$ in $V_0$ satisfies the followings.*

For each asymptotic leaf $F$ to $F_0$ in $U_+$ let $F_V$ be an asymptotic leaf of $\mathscr{F}|V$ to $F_0$ contained in $F$. Then, an unique (in the sense of the equivalence of coverings) $C^r$-regular covering $\tilde{\nu}: \tilde{F} \to F_0$ is associated with $F_V$ and $\nu_*(\pi_1(F_V)) = \tilde{\nu}_*(\pi_1(\tilde{F}))$ in $\pi_1(F_0)$. Furthermore, the equivalence class of $\tilde{\nu}$ does not depend on $V$, and so an unique normal subgroup $G(F) = \nu_*(\pi_1(F_V))$ of $\pi_1(F_0)$ is associated with $F$.

$\tilde{\nu}$ and $G(F)$ are considered as invariants on the behavior of $F$ in a neighborhood of $F_0$ in $U_+$. There is an example of $\mathscr{F}$, $F_0$, and an asymptotic leaf $F$ to $F_0$ such that, for any one sided neighborhood $V$ of $F_0$, no regular covering is associated with $F_V$.

THEOREM 1. *Suppose that $\mathscr{F}$, $F_0$, and $U_+$ satisfy the same conditions as Theorem 2. Then, there are connected orientable codimension one submanifolds $N_1, \cdots, N_\ell$ of $F_0$ satisfying the followings.*

(i) *$F_0 - N_1 \cup \cdots \cup N_\ell$ is connected.*

(ii) *Let $F_*$ be the manifold obtained by cutting open $F_0$ along $N_1, \cdots, N_\ell$, and let $g: F_* \to F_0$ be the map pasting $F_*$ on $F_0$ naturally. (There are definitions of $F_*$ and $g$ in* §3.) *Thus $\partial F_* = \bigcup_{i=1}^{\ell} N_i' \cup N_i''$, $g^{-1}(N_i) = N_i' \cup N_i''$, and $g(N_i') = N_i = g(N_i'')$. Then, there are injective diffeomorphisms $f_i: [0, \varepsilon] \to [0, \varepsilon]$, $i = 1, \cdots, \ell$ with the following properties.*

(a) *$f_i(0) = 0$ and $f_i f_j(t) = f_j f_i(t)$ for any $i, j = 1, \cdots, \ell$ and $t$ such that $f_i f_j(t)$ and $f_j f_i(t)$ are defined.* (b) *Denote by $X_f$ the quotient manifold obtained from $F_* \times [0, \varepsilon]$ by identifying $(x, t) \in N_i' \times [0, \varepsilon]$ and $(x, f_i(t)) \in N_i'' \times [0, \varepsilon]$ for all $i = 1, \cdots, \ell$ and $t \in [0, \varepsilon]$. By the commutativity of $f_i$ and $f_j$, $X_f$ is well defined. The product foliation of $F_* \times [0, \varepsilon]$ induces a foliation $\mathscr{F}_f$ on $X_f$. Then, there is a neighborhood $V$ of $F_0$ in $U_+$ such that there is a leaf preserving $C^r$-diffeomorphism from $V$ onto $X_f$.* (c) *The germs of $f_1, \cdots, f_\ell$ at $0$ generate $\Phi_+(F_0)$. Moreover, if* $\dim F_0 > 2$, *they are chosen so that the germs of $f_1, \cdots, f_\ell$ are a basis of $\Phi_+(F_0)$.*

The following results are consequence of Theorem 1 and Theorem 2.

THEOREM 3. *Let $\mathscr{F}$ be a transversely orientable $C^1$-foliation of codimension one, and let $F_0$ be an orientable closed leaf of $\mathscr{F}$. Suppose that $\pi_1(F_0) = \boldsymbol{Z}^m \times G$ for a finite group $G$ and that* $\{\log h'_{\alpha_1}, \cdots, \log h'_{\alpha_m}\}$

*is rationally independent for a basis* $\alpha_1, \cdots, \alpha_m$ *of* $Z^m$*, where* $h'_{\alpha_i}$ *is the derivative of the holonomy of* $\alpha_i$.

*Then there are collars* $U_+$ *and* $U_-$ *in the both sides of* $F_0$ *such that any leaf meeting* $U_\sigma$ *is asymptotic to* $F_0$ *in* $U_\sigma$ *and that, for any neighborhood* $V$ *of* $F_0$ *in* $U_\sigma$ *and for any* $F \in \mathscr{F}$ *meeting* $U_\sigma$*, an unique regular covering* $\tilde{F}$ *with* $\pi_1(\tilde{F}) \cong G$ *is associated with* $F_V$. *Here* $\sigma$ *denotes* $+$ *or* $-$.

THEOREM 4. *Let* $\mathscr{F}$ *be a transversely orientable codimension one foliation of class* $C^r$*, for* $r \geqq 2$*, and let* $F_0$ *be an orientable closed leaf of* $\mathscr{F}$. *Suppose that the holonomy group* $\Phi(F_0)$ *of* $F_0$ *is abelian and that there is* $\tilde{f} \in \Phi(F_0)$ *such that the derivative* $\tilde{f}'$ *of* $\tilde{f}$ *at* 0 *satisfies* $\tilde{f}' \neq 1$.

*Then, there is a bicollar* $U = U_+ \cup U_-$ *of* $F_0$ *satisfying the followings. Let* $\sigma$ *denote* $+$ *or* $-$. (i) *Any leaf meeting* $U_\sigma$ *is asymptotic to* $F_0$ *in* $U_\sigma$. (ii) *For any neighborhood* $V$ *of* $F_0$ *in* $U_\sigma$ *and for any leaf* $F$ *meeting* $U_\sigma$*, an unique regular covering* $\tilde{\nu}$ *of* $F_0$ *is associated with* $F_V$ *and the normal subgroup* $G(F)$ *of* $\pi_1(F_0)$ *is well defined. Moreover,* (iii) $\tilde{\nu}$ *and* $G(F)$ *do not depend on* $U_+, U_-$*, and* $F$.

This theorem shows that, under the above assumptions, all leaves near $F_0$ in a collar are in the same situation and $\mathscr{F} | U_+, \mathscr{F} | U_-$ have the same structure.

Let $F$ be a closed submanifold of $M$, and let $\mathscr{F}, \mathscr{F}'$ be foliations on a neighborhood of $F$ in $M$ having $F$ as a leaf. We say that $\mathscr{F}$ and $\mathscr{F}'$ are *locally equivalent at* $F$, if there are neighborhoods $U$ and $U'$ of $F$ such that there is a homomorphism from $U$ onto $U'$ mapping any leaf of $\mathscr{F} | U$ onto a leaf of $\mathscr{F}' | U'$.

Let $\mathscr{F}^1_F$ be the set of germs at $F$ of codimension $k$ $C^1$-foliations $\mathscr{F}$ defined on neighborhoods $U_{\mathscr{F}}$ of $F$ in $M$ such that $\mathscr{F}$ has $F$ as a leaf, and let $\mathscr{F}^1_F$ have a suitable topology defined by the germ of the section into the Grassmannian which defines the foliation. H. Levine and M. Shub show an unstability theorem [2] as follows: If $\pi_1(F)$ has the form $Z^m \times G$ for $m > 1$ and an arbitrary group $G$, there are no stable elements in $\mathscr{F}^1_F$ with respect to local equivalence at $F$.

Here, we show an unstability theorem for foliations defined on a fixed neighborhood $U$ of $F$ in $M$. Let $\mathrm{Fol}^r_F(U)$ be the space of $C^r$-foliations $\mathscr{F}$ of codimension one defined on a neighborhood $U$ of $F$ in $M$ such that $\mathscr{F}$ has $F$ as a leaf. Let $\mathrm{Fol}^r_F(U)$ have the $C^r$-topology defined in [1] using the charts $\{\varphi: I^{n-1}\times I\to M^n\}$.

THEOREM 5. *Let $F$ be an orientable closed submanifold of $M$ of codimension one such that $\pi_1(F)=\boldsymbol{Z}^m\times G$ for $m>1$ and a finite group $G$. Let $\mathscr{F}$ be a transversely orientable codimension one foliation of class $C^r$ on a neighborhood of $F$ in $M$ with $F$ as a leaf. Then,*
(i) *if $r=2$, there is a neighborhood $U$ of $F$ such that for any neighborhood $N$ of $\mathscr{F}|U$ in* $\mathrm{Fol}_F^1(U)$ *there is $\mathscr{F}'$ in $N$ which is not locally equivalent at $F$ to $\mathscr{F}$. Moreover,*
(ii) *if $r>2$, assume that there is $\alpha$ in $\pi_1(F)$ such that $|h'_\alpha|\neq 1$, where $h'_\alpha$ is the derivative of the holonomy of $\alpha$. Then, the same result as* (i) *holds for* $\mathrm{Fol}_F^{r-1}(U)$.

In the preparation for this research the papers, [4] of Nishimori and [3] of Nakatsuka, were very helpful to the author.

## § 2. Preparation for Theorem 1

This section will be in the version of class $C^\infty$. Let $M$ be an oriented $n$-manifold, $n\geqq 3$, and let $N$ be an oriented closed smooth submanifold of $M$ with codimension one. Let $F: B^{n-1}\times I\to M$ be an orientation preserving embedding such that $F(B^{n-1}\times I)\cap N=F(B^{n-1}\times\partial I)$, where $B^{n-1}$ denotes an $(n-1)$-ball in $\boldsymbol{R}^{n-1}$ with origin 0, $I=[0,1]$, and $\partial$ denotes the boundary. We obtain an $(n-1)$-submanifold

$$N_*=\{N-\operatorname{int}F(B^{n-1}\times\partial I)\}\cup F(\partial B^{n-1}\times I)\,.$$

By smoothing the corners, $N_*$ can be regarded as a smooth manifold. Define a simple arc $f: I\to M$ by $f(t)=F(0,t)$, $t\in I$. We shall say that $N_*$ is obtained from $N$ *by attaching a 1-handle along* a simple arc $f$. If the intersection number of $N$ and $f$ is zero, $N_*$ is orientable. In this case we assume that $N_*$ has the orientation compatible with that of $N$. Then, $[N_*]=[N]$ in $H_{n-1}(M;\boldsymbol{Z})$, where [ ] denotes the homology class.

LEMMA 1. *Let $M$ be an oriented manifold of dimension $n\geqq 3$, and let $N'$ be a connected oriented closed $(n-1)$ submanifold of $M$. Then, for a simple closed path $c$ in $M$ which intersects $N'$ at finite points, there is a connected oriented closed $(n-1)$ submanifold $N$ of $M$ satisfying the following conditions.*

(i) $[N]=[N']$ *in* $H_1(M;\boldsymbol{Z})$.

(ii) *$N$ intersects $c$ at only $|[c]\cdot[N]|$ points.*

(iii) *For a small neighborhood $U$ of $c$ in $M$, $N$ is included in $N' \cup U$.*

*Proof.* We may assume $[c]\cdot[N'] \geqq 0$ and that $N'$ intersects with $c$ transversely at more than $[c]\cdot[N']$ points, $x_1 = c(t_1), \cdots, x_r = c(t_r)$, $0 < t_1 < \cdots < t_r < 1$. We construct by induction on $r$ the desired manifold $N$. There is $i$ such that $1 \leqq i \leqq r-1$ and that the intersection number of $N'$ and $c|[t_i, t_{i+1}]$ is zero. By attaching a 1-handle to $N'$ along the simple subarc $c|[t_i, t_{i+1}]$, we obtain $N'_*$ which intersects at $(r-2)$ points and with $[N'_*] = [N']$. Then $N'_*$ has the inductive property.

LEMMA 2. *Let $N \subset M$ be a pair of oriented connected manifolds of codimension one. If there is $\gamma$ in $H_1(M; \boldsymbol{Z})$ such that the intersection number $\gamma\cdot[N]$ is 1, then $M - N$ is connected.*

*Proof.* First, we show that there is a closed path $u: I \to M$, $u(0) = u(1)$, such that $u$ intersects with $N$ at a single point. Let $c$ be any closed path with $[c] = \gamma$. We may assume that $c$ meets $N$ transversely, and hence $c$ meets $N$ at finitely many points, $x_1 = c(t_1), \cdots, x_r = c(t_r)$, $0 < t_1 < \cdots < t_r < 1$. We shall construct by induction on $r$ a closed path $u$ as above. We may assume $r \geqq 3$. There is $i$ with $1 \leqq i \leqq r-1$ such that the intersection number of $N$ and $c|[t_i, t_{i+1}]$ is zero. Since $N$ is connected there is a path $d$ from $x_i$ to $x_{i+1}$ in $N$. Let $\varepsilon$ be a sufficiently small positive real number. Then, we can take a path $d'$ from $c(t_i - \varepsilon)$ to $c(t_{i+1} + \varepsilon)$ along $d$ so that $d'$ does not intersect with $N$. $c([0, t_i - \varepsilon]) \cup (\text{image } d') \cup c([t_{i+1} + \varepsilon, 1])$ is an image of a path $c': I \to M$ which meets $N$ at $(r-2)$ points. Moreover, we have $[c']\cdot[N] = \gamma\cdot[N] = 1$, where $[c']$ denotes the homology class of $c'$. Then $c'$ has the inductive property, and therefore $u$ is constructed.

For any two points $p_0$ and $p_1$ in $M - N$ there is a path $c$ from $p_0$ to $p_1$. We may assume as above that $c$ intersects $N$ transversely, and hence $c$ meets $N$ at finite points, $y_1 = c(s_1), \cdots, y_r = c(s_r)$, $0 < s_1 \cdots < s_r < 1$. We shall construct by induction on $r$ a path $v$ from $p_0$ to $p_1$ such that $v$ does not intersect $N$. Let $u$ be a closed path such that $u$ intersects $N$ at a single point $y_0 = u(t_0)$ for $t_0 \in (0, 1)$. There is a path $d$ from $y_1$ to $y_0$ in $N$. Let $\varepsilon > 0$ be sufficiently small. Then, there is a path $d_-$ in $M - N$ from $c(s_1 - \varepsilon)$ to $u(t_0 - \delta)$ along $d$, where $\delta$ is a positive or negative real number with a sufficiently small absolute value. Similarly, there is $d_+$ from $c(s_1 + \varepsilon)$ to $u(t_0 + \delta)$. $c([0, s_1 - \varepsilon]) \cup (\text{image } d_-)$

$\cup\, u(I-(t_0-\delta, t_0+\delta)) \cup (\text{image}\, d_+) \cup c([s_1+\varepsilon, 1])$ is an image of a path $c'$ from $p_0$ to $p_1$ which intersects $N$ at $(r-1)$ points. Then $c'$ has the inductive property. This proves Lemma 2.

Let $H: H_1(M; Z) \to Z_{(1)} + \cdots + Z_{(m)}$ be an epimorphism onto a free abelian group of rank $m$, $Z_{(i)} \cong Z$ $(i=1, \cdots, m)$. Let $p_i: Z_{(1)} + \cdots + Z_{(m)} \to Z_{(i)}$ be the projection onto the $i$-th factor. By Künneth's theorem the map $\kappa: H^1(M; Z) \to \operatorname{Hom}(H_1(M; Z), Z)$ induced from slant operation is an isomorphism since $H_0(M; Z)$ is free abelian. Assume $\partial M = \phi$. Let $\delta: H^1(M; Z) \to H_{n-1}(M; Z)$ be the Poincaré duality isomorphism, and let $\theta_i = \delta\kappa^{-1}(p_i H)$. For $\gamma \in H_1(M; Z)$,

$$\begin{aligned} \gamma\cdot\theta_i &= \gamma \cap \delta^{-1}(\theta_i) \\ &= \gamma \cap (\kappa^{-1}(p_i H)) \\ &= p_i H(\gamma)\ , \end{aligned}$$

where $\cap$ denotes cup product. Thus we have

$$\gamma\cdot\theta_i = p_i H(\gamma) \qquad \text{for } \gamma \in H_1(M; Z)\ . \tag{1}$$

Now, we set the following result of Nakatsuka.

LEMMA 3 ([3]). *Let $M$ be a compact connected orientable manifold of dimension $n \geqq 3$ and $\theta \in H_{n-1}(M; Z)$. Then, there is a connected orientable $(n-1)$-submanifold $N$ in $M$ such that $\theta = [N]$ if and only if there is a homology class $\gamma \in H_1(M; Z)$ such that the intersection number $\gamma\cdot\theta = 1$.*

PROPOSITION 1. *Let $M$ be a connected orientable closed manifold of dimension $n \geqq 3$, and let $H: H_1(M; Z) \to Z_{(1)} + \cdots + Z_{(m)}$ be an epimorphism. Then, there are connected closed codimension one submanifolds $N_1, \cdots, N_m$ of $M$ satisfying the followings.*

(i) *$N_1, \cdots, N_m$ are in general position in $M$.*

(ii) *$\gamma\cdot[N_i] = p_i H(\gamma)$ for any $\gamma \in H_1(M; Z)$, $i = 1, \cdots, m$.*

(iii) *$N_i - N_1 \cup \cdots \cup N_{i-1}$ is connected for $i = 2, \cdots, m$.*

(iv) *$M - N_1 \cup \cdots \cup N_m$ is connected.*

(v) *$Hj_*(H_1(M - N_1 \cup \cdots \cup N_i; Z)) = Z_{(i+1)} + \cdots + Z_{(m)}$ for $i = 1, \cdots, m-1$, and $=0$ for $i = m$. Here, $j$ is the inclusion $M - N_1 \cup \cdots \cup N_i \to M$.*

*Proof.* Since $p_i H: H_1(M; Z) \to Z_{(i)}$ is an epimorphism, there is $\gamma_i \in H_1(M; Z)$ such that $H(\gamma_i)$ is the generator of $Z_{(i)}$, $i = 1, \cdots, m$. Then, by Lemma 3, $\gamma_i \cdot \theta_i = p_i H(\gamma_i) = 1$ implies that there are connected orientable closed $(n-1)$-submanifolds $N'_1, \cdots, N'_m$ in $M$ such that $[N'_i] = \theta_i$, $i = 1, \cdots, m$. $N'_1, \cdots, N'_m$ may be assumed to be in general position.

We vary $N'_i$ to $N_i$, $i = 1, \cdots, m$, by induction on $i$ so that $N_1, \cdots, N_i$ satisfy the following condition $C$(i). Denote $M_i = M - N_1 \cup \cdots \cup N_i$.

$C$(i) ( i ) $N_1, \cdots, N_i$ are in general position in $M$.

(ii) $[N_k] = \theta_k$, $k = 1, \cdots, i$.

(iii) $N_k - N_1 \cup \cdots \cup N_{k-1}$ is connected for $k = 2, \cdots, i$ if $i \geqq 2$.

(iv) $M_i$ is connected.

( v ) $H \circ j_*(H_1(M_k; \boldsymbol{Z}) = \boldsymbol{Z}_{(k+1)} + \cdots + \boldsymbol{Z}_{(m)}$ for $k = 1, \cdots, i$.

First, we construct $N_1$ as follows. Since $n \geqq 3$, there are simple closed paths $c_2, \cdots, c_m$ such that $[c_2] = \gamma_2, \cdots, [c_m] = \gamma_m$ and that they are mutually disjoint. By Lemma 1, there is a manifold $N_1$ such that $[N_1] = [N'_1]$ and that $N_1$ does not intersects $c_2, \cdots, c_m$. By Lemma 2, the existence of $\gamma_1$ implies that $M - N_1$ is connected. Since $c_2, \cdots, c_m$ are contained in $M_1$ and $0 = \gamma \cdot [N_1] = p_1 H(\gamma)$ for $\gamma \in H_1(M_1; \boldsymbol{Z})$, it is not difficult to see that $Hj_*(H_1(M_1; \boldsymbol{Z})) = \boldsymbol{Z}_{(2)} + \cdots + \boldsymbol{Z}_{(m)}$. Then, $N_1$ satisfies the condition $C(1)$.

Next, suppose that $N_1, \cdots, N_i$ are constructed so that the condition $C$(i) is satisfied. Now, we construct $N_{i+1}$ so that $N_1, \cdots, N_i, N_{i+1}$ satisfy $C(i+1)$. By (v) of $C$(i) there is a simple closed path $c_{i+1}$ in $M_i$ realizing $\gamma_{i+1} \in H_1(M; \boldsymbol{Z})$, and hence the intersection number $c_{i+1} \cdot (N'_{i+1} - N_1 \cup \cdots \cup N_i) = c_{i+1} \cdot N'_{i+1} = [c_{i+1}] \cdot [N'_{i+1}] = \gamma_{i+1} \cdot \theta_{i+1}$ is 1. We can take $c_{i+1}$ so that it intersects $N'_{i+1}$ transversely. Then, by the method of the proof of Lemma 3 in [3], there is a closed manifold $N''_{i+1}$ such that (i) $N''_{i+1} \cap M_i$, so $N''_{i+1}$, is connected and (ii) $[N''_{i+1}] = [N'_{i+1}]$ in $H_{n-1}(M; \boldsymbol{Z})$ and $[N'_{i+1} \cap M_i] = [N'_{i+1} \cap M_i]$ in $H_{n-1}(M_i; \boldsymbol{Z})$. Here, $N''_{i+1}$ is obtained by attaching slender 1-handles to $N'_{i+1}$ along simple arcs in $M_i$. Next, we vary $N''_{i+1}$ to construct $N_{i+1}$ so that $N_1, \cdots, N_i$ and $N_{i+1}$ satisfy the condition $C(i+1)$. By (v) of $C$(i), there are simple closed paths $c_{i+2}, \cdots, c_m$ in $M_i$ realizing $\gamma_{i+2}, \cdots, \gamma_m$, respectively. We may assume that they intersect $N''_{i+1}$ transversely and that they are mutually disjoint. Similarly as the construction of $N_1$, we obtain $N_{i+1}$ from $N''_{i+1}$ by attaching slender 1-handles along simple arcs contained in $c_{i+2}, \cdots, c_m$ so that $N_{i+1}$ does not intersects $c_{i+2}, \cdots, c_m$, that $[N_{i+1}] = [N''_{i+1}]$, and that $H \circ j_*(H_1(M - N_1 \cup \cdots \cup N_i \cup N_{i+1}; \boldsymbol{Z})) = \boldsymbol{Z}_{(i+2)} + \cdots + \boldsymbol{Z}_{(m)}$. Since $c_{i+1}$ is a path in $M_i$ and $c_{i+1} \cdot (N_{i+1} \cap M_i) = c_{i+1} \cdot N_{i+1} = 1$, Lemma 2 implies that $M_{i+1} = M_i$

$-N_{i+1}$ is connected. From the above, we can see that $N_1, \cdots, N_{i+1}$ satisfy the condition $C(i+1)$. This proves Proposition 1.

## § 3. Proof of Theorem 1

Let $\mathscr{F}$ be a codimension one foliation of class $C^r$ of an orientable $(n+1)$-manifold $M$, and suppose that an orientable $n$-manifold $F_0$ is a closed leaf of $\mathscr{F}$. Let $\nu: U \to F_0$ is an $\boldsymbol{R}$-bundle of a bicollar $U$ of $F_0$, and let $\nu_+: U_+ \to F_0$ is an $\boldsymbol{R}_+$-bundle of a collar $U_+$ of $F_0$, $\boldsymbol{R} = (-\infty, \infty)$ and $R_+ = [0, \infty)$. $F_0$ is identified with the zero section of $\nu$ or $\nu_+$, and the fibres of $\nu$ and $\nu_+$ are identified with $\boldsymbol{R}$ and $\boldsymbol{R}_+$ respectively.

A curve $u: [0,1] \to U$ is called a *leaf curve* from $u(0)$ to $u(1)$ if the image of $u$ is contained in a leaf. Let $y \in \nu^{-1}u(0)$ and let $u_y: [0,1] \to U$ be a leaf curve such that $u_y(0) = y$ and $\nu u_y(t) = u(t)$ for any $t \in [0,1]$. We call $u_y$ the *y-lift* of $u$. There exists at most one $y$-lift of $u$. If there is the $y$-lift of $u$ for any $y$ in $[y_1, y_2] \subset \boldsymbol{R} = \nu^{-1}u(0)$ the *holonomy map* $h_u$ from $[y_1, y_2]$ into $\boldsymbol{R} = \nu^{-1}u(b)$ is defined by $h_u(y) = u_y(b)$.

Let $x_* \in F_0$ and $u$ be a closed leaf curve with base point $x_*$. The germ of $h_u$ at 0 is called the *holonomy* of $u$. The holonomy of $u$ is determined by the homotopy class $[u]$ of $u$ in $\pi_1(F_0, x_*)$ and is independent of the choice of $\nu$ up to conjugations by origin preserving diffeomorphism of $\boldsymbol{R}$. Let $G^r$ be the group of the germs at 0 of all orientation-preserving local $C^r$-diffeomorphisms of $\boldsymbol{R}$ which leave the origin fixed. A homomorphism $h: \pi_1(F_0, x_*) \to G^r$ is defined by corresponding the holonomy of $u$ to $[u] \in \pi_1(F_0, x_*)$. The image of the homomorphism $h$ is called the *holonomy group* of $F_0$ and denoted by $\Phi(F_0)$. The *one-sided holonomy group* $\Phi_+(F_0)$ of $F_0$ is defined similarly by replacing $\nu$ and $\boldsymbol{R}$ by $\nu_+$ and $\boldsymbol{R}_+$.

A proof of the following Lemma 4 is found in the proof of Lemma 2 in [4].

LEMMA 4. *If $\Phi_+(F_0)$ is the trivial group there is a neighborhood $U_0$ of $F_0$ in $U_+$ such that the restricted foliation $\mathscr{F} | U_0$ is trivial; i.e. for each leaf $F$ of $\mathscr{F} | U_0$, $\nu: F \to F_0$ is a diffeomorphism.*

In this paper, we assume that $\Phi_+(F_0)$ is abelian, then $\Phi_+(F_0)$ is free abelian since $G^r$ has no torsion element. Let $\iota: \Phi_+(F_0) \to \boldsymbol{Z}_{(1)} + \cdots + \boldsymbol{Z}_{(m)}$ be an isomorphism and let $\eta: \pi_1(F_0, x_*) \to H_1(F_0 : \boldsymbol{Z})$ be the Hurewicz homomorphism. Then, there is an epimorphism $H: H_1(F_0; \boldsymbol{Z}) \to \boldsymbol{Z}_{(1)} + \cdots$

$+ Z_{(m)}$ such that $H\eta = \iota h$. Let $p_i$ be the projection from $Z_{(1)} + \cdots + Z_{(m)}$ onto the $i$-th factor. Thus we have the following diagram.

$$\begin{tikzcd} \pi_1(F_0, x_*) \arrow[r, "h"] \arrow[d, "\eta"] & \Phi(F_0) \arrow[r, "\iota"] & Z_{(1)} + \cdots + Z_{(m)} \arrow[r, "p_i"] & Z_{(i)} \\ H_1(F_0; Z) \arrow[urr, "H"'] & & & \end{tikzcd}$$

Let $N_1, \cdots, N_m$ be codimension one smooth submanifolds in $F_0$ such that they are in the general position and that $F_0 - N_1 \cup \cdots \cup N_m$ is connected. Denote by $F_1$ the compact manifold with boundary obtained by attaching two copies $N_1'$ and $N_1''$ of $N_1$ to $F_0 - N_1$, so that $\partial F_1 = N_1' \cup N_1''$. Then, a local diffeomorphism $g_1 : F_1 \to F_0$ is defined by $g_1(x) = x$ for $x \notin \partial F_1$ and $g_1(y') = g_1(y'') = y$ for $y \in N_1$, where $y' \in N_1'$ and $y'' \in N_1''$ are the copies of $y \in N_1$. $g_1^{-1}(N_i) \subset F_1$ is denoted also by $N_i$, $i = 2, \cdots, m$. Inductively we define $F_2, \cdots, F_m$ and $g_i : F_i \to F_{i-1}$, $i = 2, \cdots, m$, similarly as above. The boundaries of $F_2, \cdots, F_m$ have possibly corners. Let $g : F_m \to F_0$ be the composition $g_j \cdots g_1$. $F_m$ is said to be the manifold which is *obtained by cutting open* $F_0$ *along* $N_1, \cdots, N_m$. $g$ is said to be the map *pasting* $F_m$ *on* $F_0$.

*Proof of Theorem* 1. If $n = 1$, this theorem is well known in the theory of dynamical system. If $n > 2$, let $N_1, \cdots, N_m$ be the manifolds obtained by Proposition 1 for the epimorphism $H : H_1(F_0; Z) \to Z_{(1)} + \cdots + Z_{(m)} \cong \Phi_+(F_0)$ defined above. If $n = 2$, let $p$ be the genus of $F_0$. Then we can take simple closed curves $N_1, \cdots, N_{2p}$ in $F_0$ such that $N_i \cap N_j$ is at most one point for any different $i, j$ and that $F_0 - N_1 \cup \cdots \cup N_{2p}$ is an open 2-ball. We define $N_1, \cdots, N_\ell$ in the theorem as above.

Since $F_0 - N_1 \cup \cdots \cup N_\ell$ is a 2-ball for $n = 2$, $\Phi_+(F_0 - N_1 \cup \cdots \cup N_\ell) = 0$ in $\mathscr{F} | \nu_+^{-1}(F_0 - N_1 \cup \cdots \cup N_\ell)$. When $n > 2$, let $c$ be a simple closed path in $F_0 - N_1 \cup \cdots \cup N_m$. Let $\gamma$ be the homotopy class of $c$ in $\pi_1(F_0, x_*)$, $x_* \in F_0 - N_1 \cup \cdots \cup N_m$, and $[N_i]$ be the homology class of $N_i$ in $H_1(F_0; Z)$. Then, by Proposition 1,

$$\begin{aligned} p_i \iota h(\gamma) &= p_i H_\eta(\gamma) , \\ &= p_i H([c]) , \qquad [c] \in H_1(F_0; Z) \\ &= [c] \cdot [N_i] = 0 \end{aligned}$$

since $c \cap N_i = \phi$, for $i = 1, \cdots, m$. This implies $\Phi_+(F_0 - N_1 \cup \cdots \cup N_\ell) = 0$, if $n > 2$. By using Lemma 4, wee see that there is a injective

$C^r$-diffeomorphism $\xi:(F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta] \to U_+$ such that (i) $\xi$ maps each $(F_0 - N_1 \cup \cdots \cup N_\ell) \times \{t\}$ into a leaf of $\mathscr{F}|U_+$ and that (ii) $\nu_+\xi(x, t) = x$ for $x \in F_0 - N_1 \cup \cdots \cup N_\ell$ and $t \in [0, \delta]$. Put $\xi((F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta]) = \tilde{F}_* \subset \nu_+^{-1}(F_0 - N_1 \cup \cdots \cup N_\ell)$. By identifying $\xi(x, t)$ with $(x, t)$, $(x, t) \in (F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta]$ is a coordinates of $\tilde{F}_*$. Putting $V' = \mathrm{cl}\, F_*$, $V'$ is a closed neighborhood of $F_0$ in $U_+$. We are dealing with the holonomy maps and the holonomies for closed paths in $F_0$ with the fixed base point $x_* \in \mathrm{int}\, \tilde{F}_*$. From now on in this section a holonomy maps are considered as local diffeomorphisms of $[0, \delta]$ by identifying $[0, \delta]$ with $x_* \times [0, \delta]$, where $x_* \times [0, \delta]$ is the expression of the above coordinates.

The number of the connected components of $N_i - N_1 \cup \cdots \cup N_{i-1} \cup N_{i+1} \cup \cdots \cup N_\ell$ is only one if $n = 2$. For $n > 3$, let $N_{ij}$ be one of these components. For any $x$ in $N_{ij}$ there is a closed path $v_x$ in $F_0$ with base point $x_*$ such that $v_x$ intersects $N_1 \cup \cdots \cup N_\ell$ at only one point $x$, since $F_0 - N_1 \cup \cdots \cup N_\ell$ is connected. There is $\varepsilon_x$ with $0 < \varepsilon_x \leqq \delta$ such that there is a leaf curve of $\mathscr{F}|V'$ which is the lift of $v_x$ starting from $(x_*, \varepsilon_x) \in v_+^{-1}(x_*)$. So, the holonomy map $f_x$ of $v_x$ is defined on $[0, \varepsilon_x]$. Let $\tilde{v}_x$ be a lift of $v_x$ and let $\tilde{v}_x(0) = s'$, $v_x(1) = s''$ in $\{x_*\} \times [0, s_x] \subset v_+^{-1}(x_*)$. Let $\tilde{v}_x(t_0) \in v_+^{-1}(x)$. For any $t', t''$ with $0 \leqq t' < t_0 < t'' \leqq 1$, we have $\tilde{v}_x(t') = (v_x(t'), s')$ and $\tilde{v}_x(t'') = (v_x(t''), s'')$ in the coordinates $\tilde{F}_* = (F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta]$, since $\mathscr{F}|\tilde{F}_*$ is trivial. Hence, we have $f_x(s') = s''$. Let $N_{ij}$ have the orientation which is compatible with the inclusion $N_{ij} \subset N_i$ and the given orientation of $N_i$. For another point $y$ in $N_{ij}$ let $v_y$ be a closed curve as above such that $[v_y]\cdot[N_{ij}] = [v_x]\cdot[N_{ij}]$. From the triviality of $\mathscr{F}|\tilde{F}_*$ it is easy to see that the source of the holonomy map $f_y$ of $v_y$ is same as $f_x$ and that $f_y = f_x$ on it, i.e. $f_y(s) = f_x(s)$ for any $s \in [0, \varepsilon_x]$. Therefore, there are $\varepsilon_{ij}$ with $0 < \varepsilon_{ij} < \delta$ and an injective diffeomorphism $f_{ij}: [0, \varepsilon_{ij}] \to [0, \delta]$ satisfying the following property; for any $x$ in $N_{ij}$ and any closed path $v_x$ in $F_0$ with base point $x_*$ such that $v_x$ intersects $N_1 \cup \cdots \cup N_\ell$ at only one point $x$ and that $[v_x]\cdot[N_i] = 1$, the holonomy map of $v_x$ is defined on $[0, \varepsilon_{ij}]$ and is equal to $f_{ij}$. For two components $N_{ij}$ and $N_{ik}$ of $N_i - N_1 \cup \cdots \cup N_{i-1} \cup N_{i+1} \cup \cdots \cup N_\ell$ the holonomy maps $f_{ij}$ and $f_{ik}$ are coincide on a small neighborhood of 0, since $[v_x]\cdot[N_i] = [v_y]\cdot[N_i] = 1$ so the holonomies of $v_x$ and $v_y$ are coincide. Hence, there are $\varepsilon_i$ with $0 < \varepsilon_i < \delta$ and an injective diffeomorphism $f_i: [0, \varepsilon_i] \to [0, \delta]$ satisfying the same property as above. Therefore, there are $0 < \varepsilon < \delta$ and injective diffeomorphisms $f_1, \cdots, f_\ell$ for $N_1, \cdots, N_\ell$ satisfying the following property; for any $x$ in $N_i$ and any closed path $v_x$ in $F_0$ with base point $x_*$ such that $v_x$ intersects $N_1 \cup \cdots \cup N_\ell$ at only one point $x$ and that $[v_x]\cdot[N_i] = 1$, the holonomy map of $v_x$ is defined on $[0, \varepsilon]$ and is equal to $f_i$. Since $\Phi(F_0)$ is abelian, we may assume that $f_1, \cdots, f_\ell$ are mutually commutative by choosing $\varepsilon$ sufficiently small.

Since $f_i$ and $f_i^{-1}$ are monotonously increasing, $f_i(\varepsilon) > \varepsilon$ implies $\varepsilon > f_i^{-1}(\varepsilon)$. So, replacing $f_i$ by $f_i^{-1}$ (i.e. replacing the orientation of $N_i$) if necessary, we can suppose that $\varepsilon \geqq f_i(\varepsilon)$ for all $i$. Notice that $N_i = \bigcup_j \mathrm{cl}\, N_{ij}$ and $g^{-1}(N_{ij}) = N'_{ij} \cup N''_{ij}$. Here, $g: F_* \to F_0$ is the diffeomorphism pasting $F_*$ on $F_0$, $F_*$ is the manifold obtained by cutting open $F_0$ along $N_1, \cdots, N_\ell$, and $N'_{ij}, N''_{ij}$ are diffeomorphic manifolds such that $g(N'_{ij}) = N_{ij} = g(N''_{ij})$. Then, $g^{-1}(N_i) = N'_i \cup N''_i$, where $N'_i$ and $N''_i$ are diffeomorphic manifolds such that $N'_i = \bigcup_j \mathrm{cl}\, N'_{ij}$, $N''_i = \bigcup_j \mathrm{cl}\, N''_{ij}$ and $g(N'_i) = N_i = g(N''_i)$. Since $N_1, \cdots, N_\ell$ are in general position and $f_1, \cdots, f_\ell$ are mutually commutative, it is not difficult to show that a quotient manifold $X_f$ is well defined from $F_* \times [0, \varepsilon]$ by identifying $(x, s) \in N'_i \times [0, \varepsilon]$ and $(x, f_i(s)) \in N''_i \times [0, \varepsilon]$. Let $\mathscr{F}_f$ be the foliation on $X_f$ induced from the trivial foliation of $F_* \times [0, \varepsilon]$. Since $\mathrm{int}\, F_*$ is diffeomorphic to $F_0 - N_1 \cup \cdots \cup N_\ell$, we can see from the above facts that there is a $C^r$-diffeomorphism from a neighborhood $V$ of $F_0$ in $U_+$ onto $X_f$ mapping each leaf of $\mathscr{F} | V$ onto a leaf of $\mathscr{F}_f$.

By the constructions of $f_1, \cdots, f_\ell$, these maps satisfies the property (ii)–(c) in the theorem. This completes the proof of Theorem 1.

## § 4. Proof of Theorem 2

LEMMA 5. *Let $f_1, \cdots, f_\ell$ be injective homeomorphisms from $[0, \varepsilon]$ into $[0, \varepsilon]$ such that $f_i(0) = 0$ for $i = 1, \cdots, \ell$. Suppose*

$$f_i f_j(t) = f_j f_i(t) , \qquad i, j = 1, \cdots, \ell .$$

*Put*

$$(1) \qquad h_1(t) = f_{i_\alpha}^{\sigma_\alpha} \cdots f_{i_1}^{\sigma_1}(t) , \qquad \sigma_a = \pm 1 ,$$

$$(2) \qquad h_2(t) = f_{j_\beta}^{\tau_\beta} \cdots f_{j_1}^{\tau_1}(t) , \qquad \tau_b = \pm 1 .$$

*Then $h_1(t) = h_2(t)$ for any $t$ such that $h_1(t)$ and $h_2(t)$ are defined if*

(3) $\sum_{i_a=i}\sigma_{i_a} = \sum_{j_b=i}\tau_{j_b}$ , $i = 1, \cdots, \ell$ , $a = 1, \cdots, \alpha$ , $b = 1, \cdots, \beta$ .

*Here, $f_i^{-1}$ is considered to be defined on* $[0, f_i(\varepsilon)]$.

*Proof.* By the assumption we have

$$f_i^\sigma f_j^\tau(t) = f_j^\tau f_i^\sigma(t) , \qquad \sigma, \tau = \pm 1 , i, j = 1, \cdots, \ell$$

for any $t$ such that both sides of the expression are defined. We define a linear order $<$ in the set $\{f_1, \cdots, f_\ell, f_1^{-1}, \cdots, f_\ell^{-1}\}$ as follows; for $f_i$, $f_j$ and $f_i^{-1}, f_j^{-1}$, we define $f_i < f_j$ and $f_i^{-1} < f_j^{-1}$ respectively if $i < j$, and we define $f_i < f_j^{-1}$ for any $f_i$ and $f_j^{-1}$. It is not difficult to see that $<$ is a linear order.

Next, we show that if $f_i^{\sigma_i} f_j^{\sigma_j}(t)$ is defined and $f_i^{\sigma_i} < f_j^{\sigma_j}$, $f_j^{\sigma_j} f_i^{\sigma_i}(t)$ is also defined and $f_i^{\sigma_i} f_j^{\sigma_j}(t) = f_j^{\sigma_j} f_i^{\sigma_i}(t)$. This property is trivial for $f_i$ and $f_j$. For $f_i^{-1}$ and $f_j^{-1}$ it is shown as follows. Suppose $f_i^{-1} < f_j^{-1}$. If $f_i^{-1} f_j^{-1}(t)$ is defined, $f_j^{-1}(t) \leqq f_i(\varepsilon)$, so $t \leqq f_j f_i(\varepsilon)$. Since $f_j f_i(\varepsilon) = f_i f_j(\varepsilon)$, $t \leqq f_i f_j(\varepsilon)$. Hence, $f_i^{-1}(t) \leqq f_j(\varepsilon)$, and so $f_j^{-1} f_i^{-1}(t)$ is defined. Then $f_i^{-1} f_j^{-1}(t) = f_j^{-1} f_i^{-1}(t)$. Finally, for $f_i$ and $f_j^{-1}$ it is shown as follows. Suppose $f_i < f_j^{-1}$. If $f_i f_j^{-1}(t)$ is defined, $t \leqq f_j(\varepsilon)$, so $t_i(t) \leqq f_i f_j(\varepsilon)$. Since $f_i f_j(\varepsilon) = f_j f_i(\varepsilon) \leqq f_j(\varepsilon)$, $f_i(t) \leqq f_j(\varepsilon)$. Then, $f_j^{-1} f_i(t)$ is defined, and so $f_i f_j^{-1}(t) = f_j^{-1} f_i(t)$.

If $f_j(t)$ or $f_j^{-1}(t)$ is defined, $f_j(f_i^{-1} f_i)(t)$ or $(f_i^{-1} f_i) f_j^{-1}(t)$ is defined and $f_j(t) = f_j(f_i^{-1} f_i)(t)$ or $f_j^{-1}(t) = (f_i^{-1} f_i) f_j^{-1}(t)$, respectively. Next, we interplate $f_i^{-1} f_i$ in the right hand of the expressions of (1) and (2) if necessary so that the same number of $f_i$ and $f_i^{-1}$ are contained in these expressions for each $i = 1, \cdots, \ell$. Finally, we change the order in the rows of the terms in these expressions to the order induced from $<$. Then, the obtained expressions are identical. This proves $h_1(t) = h_2(t)$.

LEMMA 6. *Let $\mathscr{F}$ be a transversely orientable $C^r$-foliation of codimension one, $r \geqq 1$, and let $F_0$ be a compact leaf of $\mathscr{F}$. Let $\nu$ be a normal $\boldsymbol{R}_+$-bundle map from a collar $U_+$ onto $F_0$ such that $\nu$ is transverse to $\mathscr{F}$, and let $F \in \mathscr{F}$ be asymptotic to $F_0$ in $U_+$. Then, the following properties are satisfied.*

(i) *For a neighborhood $V$ of $F_0$ in $U_+$, let $F_V$ be an asymptotic leaf of $\mathscr{F} | V$ to $F_0$ such that $F_V \cap F \neq \phi$. Then, an unique regular covering $\tilde{\nu}: \tilde{F} \to F_0$ is associated with $F_V$ and $\nu_*(\pi_1(F_V)) = \tilde{\nu}_*(\pi_1(\tilde{F}))$ in $\pi_1(F_0)$ if and only if the following condition $(*)$ is satisfied.*

$(*)$ *For a point $x_*$ in $F_0$ and any closed path $u$ in $F_0$ with the base* *points* $x_*$ *let* $y$ *and* $z$ *be any two points in* $\nu^{-1}(x_*) \cap F_V$ *such that* $h_u(y)$ *and* $h_u(z)$ *are defined, where* $h_u$ *is the holonomy map of* $u$. *Then,* $h_u(y) = y$ *if and only if* $h_u(z) = z$.

(ii) *Suppose* $F$ *and* $V$ *satisfies* $(*)$. *Then, for any neighborhood* $V'$ *of* $F_0$ *in* $V$, *the same regular covering as* $\tilde{\nu}$ *is associated with* $F_{V'}$.

*Proof.* Let $\tilde{\nu}: \tilde{F} \to F_0$ be a regular covering and let $u$ be a closed curve in $F_0$ with base point $x_*$. For $y$ and $z$ in $\tilde{\nu}^{-1}(x_*)$ let $u_y$ and $u_z$ be the lifts of $u$ starting from $y$ and $z$ respectively. Then, $u_y$ is a closed curve if and only if $u_z$ is so. Therefore, if there is an associated regular covering with $F_V$, condition $(*)$ is satisfied.

Next, we prove the converse. Define a subgroup $G(F_V)$ of $\pi_1(F_0, x_*)$ by

$$G(F_V) = \{\alpha \in \pi_1(F_0, x_*) \mid \text{there is a closed curve } \bar{u} \text{ in } F_V \text{ such that } [\nu\bar{u}] = \alpha\}\,,$$

where [ ] denotes the homotopy class. We must show that $G(F_V)$ is a subgroup of $\pi_1(F_0, x_*)$. For $\alpha$ and $\beta$ in $G(F_V)$ there are closed curves $\bar{u}$ and $\bar{v}$ in $F_V$ such that $[\nu\bar{u}] = \alpha$ and $[\nu\bar{v}] = \beta^{-1}$. Let $y, z \in \nu^{-1}(x_*)$ be the base point of $\bar{u}, \bar{v}$. Assume $x_* < y < z$ in the line $\nu^{-1}(x_*)$. Put $\nu\bar{u} = u$ and $\nu\bar{v} = v$. By the existence of $\bar{v}, h_v(y)$ is defined. Condition $(*)$ implies $h_v(y) = y$. So, there is the lift $\tilde{v}$ of $v$ starting from $y$. $\tilde{v}$ is a closed curve in $F_V$. Then, $\bar{u}\tilde{v}$ is a closed curve in $F_V$ such that $[\nu(\bar{u}\tilde{v})] = \alpha\beta^{-1}$. Therefore, $\alpha\beta^{-1} \in G(F_V)$.

To the conjugacy class of a subgroup of $\pi_1(F_0, x_*)$ an unique covering of $F_0$ exists. Let $\tilde{\nu}: \tilde{F} \to F_0$ be the covering corresponding to the conjugacy class including $G(F_V)$. Then, for $\tilde{y} \in \tilde{\nu}^{-1}(x_*)$, $\tilde{\nu}_*(\pi_1(F, y))$ is a subgroup of $\pi_1(F_0, x_*)$ which is conjugate to $G(F_V)$.

Next, we define the map $i: F_V \to \tilde{F}$. Fix two points $y_* \in F_V$ and $\tilde{y}_* \in \tilde{F}$ so that $\nu(y_*) = \tilde{\nu}(\tilde{y}_*) = x_*$ and that $\tilde{\nu}_*(\pi_1(\tilde{F}, \tilde{y}_*)) = G(F_V)$. For any point $y$ in $F_V$ there is a curve $u: [0,1] \to F_V$ such that $u(0) = y_*$ and $u(1) = y$. Let $\tilde{u}$ be the lift of $\nu u$ starting from $\tilde{y}_*$ for the covering $\tilde{\nu}$. We define $i(y) \in \tilde{F}$ by $i(y) = \tilde{u}(1)$. $i(y)$ is well defined, i.e. for another curve $v$ in $F_V$ from $y_*$ to $y$, $\tilde{v}(1) = \tilde{u}(1)$. In fact, since $[\nu(uv^{-1})] \in G(F_V)$ and $G(F_V) = \tilde{\nu}_*(\pi_1(\tilde{F}, \tilde{y}_*))$, the lift of $\nu(uv^{-1})$ starting from $\tilde{y}_* \in \tilde{F}$ is a closed curve. Hence, $\tilde{u}^{-1}\tilde{v}$ is a closed curve with the base point $\tilde{u}(1)$. This implies $\tilde{v}(1) = \tilde{u}(1)$. By the definition of $i$, $\tilde{\nu} \circ i = \nu$ is obvious.

If $\nu(y) \neq \nu(y')$, clearly $i(y) \neq i(y')$. Next, we show that $i(y) \neq i(y')$

when $\nu(y) = \nu(y')$ and $y \neq y'$. Let $u'$ and $v'$ be the curves in $F_V$ from $y_*$ to $y$ and $y'$ respectively. Put $\nu u' = u$ and $\nu v' = v$. We can assume that $y < y'$ in $\nu^{-1}(y)$. Since $h_{v^{-1}}(y') = y_*$, $h_{v^{-1}}(y)$ is defined and $h_{v^{-1}}(y) < y_*$ in $\nu^{-1}(x_*)$. Since $h_{uv^{-1}}(y_*) = h_{v^{-1}}(y) < y_*$, $[uv^{-1}] \notin G(F_V)$. So that, the lift of $uv^{-1}$ starting from $\tilde{y}_*$ in $\tilde{F}$ is never a closed curve. Hence, $i(y) = \tilde{u}(1) \neq \tilde{v}(1) = i(y')$. Therefore, $i$ is an injection.

It is obvious that $i$ maps any plaque of $F_V$ $C^r$-diffeomorphically into $\tilde{F}$.

To show $\tilde{\nu}$ is a regular covering we are sufficient to show that $G(F_V)$ is a normal subgroup of $\pi_1(F_0, x_*)$. Let $u$ and $v$ be closed curves in $F_0$ with the base point $x_*$. Assume $[u] \in G(F_V)$. Since $F_V$ is asymptotic to $F_0$ there is $y$ in $\nu^{-1}(x_*) \cap F_\nu$ such that $h_{vuv^{-1}}(y)$ is defined. Since $[u] \in G(F_V)$, $h_u h_v(y) = h_v(y)$. So that, $h_{vuv^{-1}}(y) = h_{v^{-1}} h_u h_v(y) = y$. Hence, $[vuv^{-1}] \in G(F_V)$. This implies that $G(F_V)$ is a normal subgroup. Therefore, (i) is proved.

To prove (ii) it is sufficient, if $G(F_V) = G(F_{V'})$ is shown. But, this is obvious since $F_V$ is asymptotic to $F_0$.

*Proof of Theorem* 2. By Theorem 1 we obtain $N_1, \cdots, N_\ell \subset F_0, V$, and the functions $f_1, \cdots, f_\ell$. Let $x_* \in F_0 - N_1 \cup \cdots \cup N_\ell$. For an asymptotic leaf $F$ of $\mathscr{F}|V$ to $F_0$, let $F_V$ be an asymptotic leaf of $\mathscr{F}|V$ to $F_0$ such that $F_V \subset F$.

First, we show that, if $u, v$ are closed paths in $F_0$ with base point $x_*$ in a same homology class of $H_1(F_0; Z)$, $h_u(y) = h_v(y)$ for any $y \in \nu^{-1}(x_*) \cap V$ such that $h_u(y), h_v(y)$ are defined. Let $\tilde{u}, \tilde{v}$ be the leaf curves of $\mathscr{F}|V$ which are lifts of $u, v$ starting from $y$. We may assume that $\tilde{u}, \tilde{v}$ intersect $\nu^{-1}(N_1 \cup \cdots \cup N_\ell)$ transversely. So, since $F_0 - N_1 \cup \cdots \cup N_\ell$ is connected, $\tilde{u}$ and $\tilde{v}$ are homotopic to $\tilde{u}_1 \cdots \tilde{u}_\alpha$ and $\tilde{v}_1 \cdots \tilde{v}_\beta$ by homotopies such that the homotopies preserve the end points of the paths and that each homotopy level is a leaf curve of $\mathscr{F}|V$, where $\tilde{u}_1 \cdots \tilde{u}_\alpha$ and $\tilde{v}_1 \cdots \tilde{v}_\beta$ are the paths which are the compositions of the paths $\tilde{u}_a, \tilde{v}_b$ with end points in $\nu^{-1}(x_*)$ such that putting $\nu\tilde{u}_a = u_a$ and $\nu\tilde{v}_b = v_b$, $u_a$ and $v_b$ are closed paths in $F_0$ each of which intersect $N_1 \cup \cdots \cup N_\ell$ at one point. Here, the composition of paths is defined by

$$uv(x) = \begin{cases} u(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2} \\ v(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1 \,. \end{cases}$$

Define $N_{i_a}$ and $N_{j_b}$ by $u_a \cap (N_1 \cup \cdots \cup N_\ell) = u_a \cap N_{i_a}$ and $v_b \cap (N_1$

$\cup \cdots \cup N_\ell) = v_b \cap N_{j_b}$. Let the intersection numbers be $[u_a]\cdot[N_{i_a}] = \sigma_a$ and $[v_b]\cdot[n_{j_b}] = \tau_b$, where $\sigma_a, \tau_b = \pm 1$. Here, $N_1, \cdots, N_\ell$ are imposed the orientations such that if a closed path $u$ intersects $N_1 \cup \cdots \cup N_\ell$ at only one point in $N_i$ with the intersection number $[u]\cdot[N_i] = \sigma$, as in the proof of Theorem 1, then the holonomy map $h_u$ of $u$ is coincide with $f_i^\sigma$. Thus, we have

$$h_u(y) = h_{u_1 \cdots h_\alpha}(y) = h_{u_\alpha} \cdots h_{u_1}(y)$$
$$= f_{i_\alpha}^{\sigma_\alpha} \cdots f_{i_1}^{\sigma_1}(y) .$$

Similarly,

$$h_v(y) = f_{j_\beta}^{\tau_\beta} \cdots f_{j_1}^{\tau_1}(y) .$$

Since $u$ and $v$ are in the same homology class, Lemma 5 implies $h_u(y) = h_v(y)$.

If we can show that $V$ and $F_V$ satisfy the condition $(*)$ in Lemma 6, the proof of Theorem 2 is completed by Lemma 6. $(*)$ is shown as follows. Let $y, z$ be two points in $\nu^{-1}(x_*) \cap F_V$ such that $h_u(y)$ and $h_u(z)$ are defined, where $u$ is a closed path in $F_0$ with end points $x_*$. We can assume $y \geqq h_u(y)$; if $y < h_u(y)$, consider the curve $u^{-1}$ with the inverse direction of $u$. Here, $<$ is considered in the coordinates $\nu^{-1}(x_*) \cap V = x_* \times [0, \varepsilon]$. Let $y > z$. Since $h_u$ is a homomorphism, $h_u(y) > h_u(z)$. There is a path $\tilde{w}$ in $F_V$ from $y$ to $z$. Put $w = \nu\tilde{w}$. Since $h_u(y) \leqq y$ and $h_w h_u(y)$ is defined. $h_u(y) \leqq y$ implies $z = h_w(y) \geqq h_w h_u(y)$. Notice that $y = h_u(y)$ if and only if $z = h_w h_u(y)$. We have $h_{w^{-1}uw}(z) = h_w h_u h_{w^{-1}}(z) = h_w h_u(y)$. Since $w^{-1}uw$ and $u$ are in the same homology class, $h_{w^{-1}uv}(z) = h_u(z)$ by the fact that we proved above. Thus, $h_u(z) = h_w h_u(y)$. Since $z = h_w(y)$, we have $y = h_u(y)$ if and only if $z = h_u(z)$. This proves Theorem 2.

## § 5. Proof of Theorem 3

Let $\nu_+ : U'_+ \to F_0$ be a collar. Since $\{\log h'_{\alpha_1}, \cdots, \log h'_{\alpha_m}\}$ is rationally independent, there is a closed curve $u$ in $F_0$ such that $h'_u \neq 1$. We can assume that $0 < h'_u < 1$. Let $x$ be the base point of $u$. There is an interval $[t, z)$ in $\nu_+^{-1}(x)$ and a positive number $r < 1$ such that for any $y$ in $[x, z)$ $h_u(y)$ is defined and that $h'_u(y) < r$. Hence, $\lim_{i \to \infty} (h_u)^i(y) = x$ for any $y$ in $[x, z)$. Therefore, by taking a sufficiently small collar $U_+$, any leaf meeting $U_+$ is asymptotic to $F_0$. We can $U_-$ similarly.

By the assumption of $\pi_1(F_0)$, the one sided holonomy group $\Phi_\sigma(F_0)$ is abelian for $\sigma = +$ or $-$. Let $V$ be any neighborhood of $F_0$ in $U_\sigma$. Then, for any leaf $F$ meeting $U_\sigma$ a regular covering $\tilde{\nu}: \tilde{F} \to F_0$ is associated with $F_V$, by Theorem 2.

Since holonomy has no torsion element, $G(F) = \nu_* \pi_1(F_V) = \tilde{\nu}_* \pi_1(\tilde{F}) \supset G$. $\nu_*$ and $\tilde{\nu}_*$ are injections. Suppose that there is a leaf $F$ such that, for the associated covering $\tilde{\nu}: \tilde{F} \to F_0$ with $F_V$, $G(F) \neq G$. Then, there is a closed curve $\tilde{u}$ in $\tilde{F}$ with base point in $\tilde{\nu}^{-1}(x)$ such that the homotopy class $\alpha = [\tilde{\nu}\tilde{u}]$ is not contained in $G$. By the definition of $\tilde{F}$, there is a closed curve $u$ in $F_V$ starting from a point $y$ in $\nu_\sigma^{-1}(x)$ such that $[\nu_\sigma u] = \alpha$. Then, for any $y'$ in the interval $[x, y]$ in $\nu_\sigma^{-1}(x)$, the holonomy map $h_\alpha(y')$ is defined. As above, there is a sequence of points $y_0 = y, y_1, y_2, \cdots$ in $[x, y] \cap F_V$ such that $\lim_{i \to \infty} y_i = x$. By condition (*) of Lemma 6, $h_\alpha(y_i) = y_i$ for each $y_i$. Since $\pi_1(F_0, x) = \boldsymbol{Z}_{(1)} + \cdots + \boldsymbol{Z}_{(m)} + G$, and $\nu_* \pi_1(F_V) \supset G$, we can put

$$\alpha = a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_m \alpha_m$$

for the integers $a_1, \cdots, a_m$ with $(a_1, \cdots, a_m) \neq (0, \cdots, 0)$. Let $u_1, \cdots, u_m$ be the closed curves with base point $x$ realizing the homotopy classes $\alpha_1, \cdots, \alpha_m$ respectively. Then, the multiple $v = u_1^{a_1} \cdots u_m^{a_m}$ realizes $\alpha$, so that, $[v] = [\nu_\sigma u]$. Let $v_t$ be a homotopy from $u$ to $v$, $t \in [0, 1]$. Since $h_{v_t}(y')$ is defined for arbitrary $y' \in [x, y]$ which is sufficiently close to $x$, we have $h_u(y') = h_v(y')$. Hence, for such $y'$

$$h_\alpha(y') = (h_{\alpha_m})^{a_m} \cdots (h_{\alpha_1})^{a_1}(y') .$$

Since $\lim_{i \to \infty} y_i = x$ and $h_\alpha(y_i) = y_i$, we have $h'_\alpha = 1$. Hence,

$$(h'_{\alpha_m})^{a_m} \cdots (h'_{\alpha_1})^{a_1} = 1 .$$

Therefore,

$$a_1 \log h'_{\alpha_1} + \cdots + a_m \log h'_{\alpha_m} = 0$$

with $(a_1, \cdots, a_m) \neq (0, \cdots, 0)$. But, this contradicts to the assumption of the theorem. This proves Theorem 3.

## § 6. Proof of Theorem 4

The proof of (i) and (ii) of Theorem 4 is contained in the proof of Theorem 3.

Next we prove (iii). Since $f$ is a local diffeomorphism of class $C^2$ with $f(0)=0$ and $f'(0)>1$, by a theorem of Sterenberg [5], there is a $C^1$-diffeomorphism $g$ from a neighborhood of 0 of $\boldsymbol{R}$ into $\boldsymbol{R}$ such that $f'(0)\cdot t = gfg^{-1}(t)$ for any $t$ in the image of $g$. Hence, by a $C^1$-alternation of the coordinate of $\nu^{-1}(x)\cap U$, we may assume that $f(t)=dt$, where $d=f'(0)<1$. Hereafter we use the new coordinate of $\nu^{-1}(x)\cap U$ translated by $g$. Let $f_1,\cdots,f_\ell$ be local diffeomorphisms of $\boldsymbol{R}$ generating $\Phi(F_0)$. Since $\Phi(F_0)$ is abelian, we may assume $f_i f = f f_i$ for $i=1,\cdots,m$ by choosing $U$ sufficiently small. Hence, $f_i'(f(t))\cdot f'(t) = f'(f_i(t))\cdot f_i'(t)$, and so $f_i'(f(t)) = f_i'(t)$, for $f'(t)=d$. Then, $f_i'(t)=f_i'(0)$, since $\lim_{n\to\infty} f^n(t)=0$ and $f_i$ is of class $C^1$. Therefore, $f_i(t)=d_i\cdot t$, where $d_i = f_i'(0)$. To show (iii), it is sufficient if $G(F)=G(F')$ is shown for any asymptotic leaves $F$ and $F'$ to $F_0$. Let $\alpha$ be a closed curve realizing an element of $G(F)$ and let $h_\alpha$ be the holonomy map defined by $\alpha\in\pi_1(F_0,x)$. Then, $h_\alpha$ can be written as $h_\alpha = f_1^{k_1}\cdots f_\ell^{k_\ell}$. By the definition of $G(F)$, there is a closed curve $\beta$ in $F\cap U$ with the end point $t$ in $\nu^{-1}(x)$ such that $\alpha=\nu\circ\beta$. Hence, $t=h_\alpha(t)=f_1^{k_1}\cdots f_\ell^{k_\ell}(t)=d_1^{k_1}\cdots d_\ell^{k_\ell}\circ t$. Thus, $h_\alpha = id$. since $d_1^{k_1}\cdots d_\ell^{k_\ell}=1$. Therefore, a lift of $\alpha$ to $F'$ is a closed curve, and so the holonomy class of $\alpha$ is contained in $G(F')$. This implies $G(F) = G(F')$. This completes the proof of Theorem 3.

*Remark* 1. For $\tilde{f}\in\Phi(F_0)$ let $\tilde{f}'\in\boldsymbol{R}$ be the derivative of $\tilde{f}$ at 0. Denoting $D\Phi(F_0)=\{\tilde{f}'\,|\,\tilde{f}\in\Phi(F_0)\}$, $D\Phi(F_0)$ is a multiplicative subgroup of $\boldsymbol{R}-\{0\}$. Let $D:\Phi(F_0)\to D\Phi(F_0)$ be the homomorphism defined by the derivation. Then, for any asymptotic leaf $F$ to $F_0$, we see that $G(F)\subset \ker Dh$, where $h$ is the homomorphism $\pi_1(F_0,x_*)\to\Phi(F_0)$ defined in §3.

*Remark* 2. If $\mathscr{F}$ is of class $C^2$, then, by the method used in the proof of Theorem 4, we see that the sequence

$$1 \longrightarrow G(F) \xrightarrow{\subset} \pi_1(F_0) \xrightarrow{h} \Phi(F_0) \longrightarrow 1$$

is exact for any asymptotic leaf $F$ to $F_0$.

## §7. Proof of Theorem 5

Assuming that $\pi_1(F)=\boldsymbol{Z}_{(1)}+\cdots+\boldsymbol{Z}_{(m)}+G$ for a finite group $G$, let $N_1,\cdots,N_m$ be the manifolds of $F$ obtained by Proposition 1 for the isomorphism $H:H_1(F:Z)\to\boldsymbol{Z}_{(1)}+\cdots+\boldsymbol{Z}_{(m)}$. Here, we may assume that $\dim F \geqq 2$, because if $\dim F = 2$, $F$ is a torus. By observing the proof of Theorem 1, the same conclusion of Theorem 1 is satisfied for these $N_1, \cdots, N_m$. Then, if $\mathscr{F}$ is a foliation of class $C^r$, there are injective $C^r$-diffeomorphisms $f_i^+ : [0, \varepsilon] \to [0, \varepsilon]$ for $i = 1, \cdots, m$ with the properties (a) and (b) of Theorem 1. By the proof of Theorem 1, $f_i$ can be identified with an one sided holonomy map $h_{\alpha_i}^+$ of a generator $\alpha_i$ of $Z_{(i)}$.

We divide the stage into Case 1 and Case 2. (i) of Theorem 5 is divided into the both cases and (ii) is contained in Case 1.

Case 1: The case that $\mathscr{F}$ is of class $C^r$, $r \geqq 2$, and that there is $i$ such that $(f_i^+)'(0) \neq 1$. Let $f_j$ be a (both sided) holonomy map of $\alpha_j$. Then $f_j'(0) = (f_j^+)'(0)$. By Sternberg's theorem, $f_1, \cdots, f_m$ are $C^{r-1}$-conjugate to linear functions by a same conjugation map $g$ in a small neighborhood of 0. (See the proof of Theorem 4.) Then, $g f_j g^{-1}(t) = f_j'(0) \cdot t$ if $|t|$ is sufficiently small. Let $U_-$ be a collar of $F$ such that $U_-$ is in the another side of $U_+$. Using Theorem 1 we get $f_i^- : [-\varepsilon', 0] \to [-\varepsilon', 0]$ for $i = 1, \cdots, m$. $f_i^-$ is the other sided holonomy map of a generator $\alpha_i'$ of $Z_{(i)}$. $|f_i(t)| \leqq |t|$ for sufficiently small $|t|$ if and only if $|f_i'(0)| \leqq 1$ since $\bar{f}_i = g f_i g^{-1}$ is linear and $\bar{f}_i(t) = f_i'(0) \cdot t$, $i = 1, \cdots, m$. Hence, by taking $\varepsilon'$ small, $\alpha_i' = \alpha_i$, i.e. $f_i^+$ and $f_i^-$ are the one sided holonomies of the same generator $\alpha_i$ of $Z_{(i)}$. Therefore, there are injective linear maps $\bar{f}_i : [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$, $i = 1, \cdots, m$ with the following properties: Let $N_i', N_i''$, and $F_*$ be the manifolds defined in Theorem 1. Denote by $X_{\bar{f}}$ the quotient manifold obtained from $F_* \times [-\varepsilon, \varepsilon]$ by identifying $(x, t) \in N_i' \times [-\varepsilon, \varepsilon]$ and $(x, \bar{f}_i(t)) \in N_i'' \times [-\varepsilon, \varepsilon]$ for all $i = 1, \cdots, m$ and $t \in [-\varepsilon, \varepsilon]$. The product foliation of $F_* \times [-\varepsilon, \varepsilon]$ induces a foliation $\mathscr{F}_{\bar{f}}$ on $X_{\bar{f}}$. Then, there is a neighborhood $V$ of $F$ such that there is a leaf preserving $C^{r-1}$-diffeomorphism $\varphi$ from $V$ onto $X_{\bar{f}}$ which maps $F$ onto $F_* \times 0/\sim$.

By Theorem 4, for any leaf $F'$ meeting $V$ an unique regular covering $\tilde{F}$ is associated with $F_V'$. Since $\bar{f}_i$ is linear, by Theorem 3, $\nu_* \pi_1(F_V') = \tilde{\nu}_* \pi_1(\tilde{F}) \cong \pi_1(\tilde{F}) \cong G$ if and only if $\log \bar{f}_1', \cdots, \log \bar{f}_m'$ are rationally independent. By an arbitrarily small perturbations of $\bar{f}_1, \cdots, \bar{f}_m$, we can take linear maps $\bar{g}_1, \cdots, \bar{g}_m : [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$ such that $\log \bar{g}_1', \cdots, \log \bar{g}_m'$ are rationally independent or dependent when $\log \bar{f}_1', \cdots, \log \bar{f}_m'$ are rationally dependent or independent, respectively.

Let $U$ be an open neighborhood of $F$ contained in $V$. Let $N$ be a neighborhood of $\mathscr{F}|U$ in $\mathrm{Fol}_F^{r-1}(U)$. $\varphi(U)$ is a neighborhood of $F_* \times 0/\sim$ in $X_{\bar{f}}$. Since $\bar{g}_1, \cdots, \bar{g}_m$ are close to $\bar{f}_1, \cdots, \bar{f}_m$ we may assume that $\varphi(U) \subset X_{\bar{g}} \subset X_{\bar{f}}$ and that $\mathscr{F}_{\bar{g}}|\varphi(U)$ is close to $\mathscr{F}_{\bar{f}}|\varphi(U)$. $\mathscr{F}_{\bar{g}}|\varphi(U)$ induce a foliation $\mathscr{F}'$ of $U$. By taking $\bar{g}_1, \cdots, \bar{g}_m$ sufficiently close to $\bar{f}_1, \cdots, \bar{f}_m$ we can $\mathscr{F}' \in N$.

Case 2. The case that $\mathscr{F}$ is of class $C^1$ and that $(f_i^+)'(0) = 1$ for all $i = 1, \cdots, m$. $f_i^+$ is the one sided holonomy map of $\alpha_i$ defined on $[0, \varepsilon]$. First, assume that there is no neighborhood $U$ of $F$ such that $F|U$ is a product foliation. For small $\delta > 0$ we define a $C^1$-diffeomorphism $g_i^+ : [0, \varepsilon + \delta] \to \boldsymbol{R}_+$ by

$$g_i^+(t) = \begin{cases} t & \text{for } 0 \leqq t < \delta \\ f_i^+(t - \delta) + \delta & \text{for } t > \delta\,. \end{cases}$$

Since $(f_i^+)'(0) = 1$, $g_i^+$ is of class $C^1$. It is easy to see that $g_1^+, \cdots, g_m^+$ are mutually commutative since $f_1^+, \cdots, f_m^+$ are so. $g_i^+|[0, \varepsilon]$ is a $C^1$-perturbation of $f_i^+$. Let $\mathscr{F}_f$ and $X_f$ be the ones defined in Theorem 1 from $f_i^+$ and $F_* \times [0, \varepsilon]$. Define $\mathscr{F}_g$ and $X_g$ similarly from $g_i^+$ and $F_* \times [0, \varepsilon + \delta]$. We can consider that $X_f \subset X_g$ and that $\mathscr{F}_g|X_f$ is $C^1$-close to $\mathscr{F}_f$ is $\delta$ is small enough. There is a neighborhood $V_+$ of $F$ in $U_+$ and a $C^1$-diffeomorphism $\varphi: V_+ \to X_f$ mapping $\mathscr{F}|V_+$ to $\mathscr{F}_f$. Let $\mathscr{F}'_+$ be the foliation induced by $\varphi^{-1}$ from $\mathscr{F}_g|X_f$. $\mathscr{F}'_+$ is $C^1$-close to $\mathscr{F}|V_+$ if $\delta$ is small enough. We get $\mathscr{F}'_-$ on $V_-$ similarly. On small neighborhoods of $F, \mathscr{F}'_+$ and $\mathscr{F}'_-$ are product foliations. Let $U = V_+ \cup V_-$. Then, we get $\mathscr{F}'$ on $U$ by $\mathscr{F}'|V_\sigma = \mathscr{F}'_\sigma$, $\sigma = \pm$. We can take $\mathscr{F}'$ in any neighborhood $N$ of $\mathscr{F}|U$ in $\mathrm{Fol}_F^1(U)$. By the assumption $\mathscr{F}'$ is not locally equivalent to $\mathscr{F}|U$.

Next, we assume that there is a neighborhood $V$ of $F$ such that $\mathscr{F}|V$ is a product foliation. Then, $V$ is leaf preservingly diffeomorphic to $F \times [-\varepsilon, \varepsilon]$. Consider that $V = F \times [-\varepsilon, \varepsilon]$ and $F = F \times 0$. Let $U = F \times (-\varepsilon/2, \varepsilon/2)$. Let $\alpha_i$ be a generator of $\boldsymbol{Z}_{(i)}$. Then, the holonomy map $f_i : [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$ of $\alpha_i$ is the identity map. Let $g_i$ be the perturbation of $f_i$ such that $g_i = f_i$ for $i > 1$ and that $|g_1(t)| < |t|$ and $|g_1(\pm\varepsilon)| > \varepsilon/2$. Let $\mathscr{F}_g$ and $X_g$ be as above defined from $g_i$ and $F_* \times [-\varepsilon, \varepsilon]$. Then, we can consider that $U \subset X_g \subset V$ and that $\mathscr{F}_g|U$ is close to $\mathscr{F}|U$ if $g_1$ is close enough to $f_1$. Any leaf of $\mathscr{F}_g$ is asymptotic to $F$, but any leaf of $\mathscr{F}|V$ is not asymptotic. Hence, $\mathscr{F}_g|U$ is not locally equivalent to $\mathscr{F}|V$. This completes the proof of Theorem 5.

## REFERENCES

[ 1 ] M. Hirsch, Stability of compact leaves of foliations, Dynamical Systems, ed. M. Peixoto, Academic Press, N.Y., 1973, 135–153.
[ 2 ] H. Levine and M. Shub, Stability of foliations, Trans. A.M.S., **184** (1973), 419–437.
[ 3 ] H. Nakatsuka, On representations of homology classes, Proc. Japan Acad., **48** (1972), 360–364.
[ 4 ] T. Nishimori, Compact leaves with abelian holonomy, Tōhoku Math. J. The Second Series, **27** (1975), 259–272.
[ 5 ] S. Sterenberg, Local $C^n$ transformation of the real line, Duke Math. J., **24** (1957), 97–102.

*Department of Mathematics*
*College of General Education*
*Nagoya University*