J. R. Choike
Nagoya Math. J.
Vol. 67 (1977), 1-13

ON THE ASYMPTOTIC BOUNDARY BEHAVIOR OF FUNCTIONS ANALYTIC IN THE UNIT DISK

JAMES R. CHOIKE

1. Introduction

In [8] a necessary and sufficient condition was given for determining the equivalence of two asymptotic boundary paths for an analytic function $w = f(p)$ on a Riemann surface F. In this paper we give a necessary and sufficient condition for determining the nonequivalence of two asymptotic boundary paths for $f(z)$ analytic in $|z| < R$, $0 < R \leq +\infty$. We shall, also, illustrate some applications of the main result and examine a class of functions introduced by Valiron.

2. Transcendental singularities and the main result

Let $w = f(z)$ be meromorphic in $|z| < R$, $0 < R \leq +\infty$. Let $z = \phi(w)$ denote the inverse function of $w = f(z)$ with domain the Riemann surface Φ. We shall write $Q(w, w_0)$ to denote a functional element with center $w = w_0$ for $z = \phi(w)$. The notation $Q(w; w_0)$ plays a dual role in this paper, representing not only a functional element of $z = \phi(w)$ at $w = w_0$, but also a point on the Riemann surface Φ lying above $w = w_0$.

Let

$$\Lambda: q(t) = Q(w; w(t)), \qquad 0 \leq t < 1,$$

with $\lim_{t \to 1} w(t) = \omega$, be a curve on the Riemann surface Φ. Then the curve Λ defines a transcendental singularity Ω of $z = \phi(w)$ on Φ, with projection $w = \omega$, if i) for every positive number $\delta, \delta < 1$, the system of functional elements $Q(w; w(t))$, $0 \leq t \leq \delta$, defines an analytic continuation (possibly, of algebraic character), but ii) for any functional element $Q(w; \omega)$, rational or algebraic, with center at $w = \omega$, the system $Q(w; w(t))$, $0 \leq t \leq 1$, where $w(1) = \omega$, never defines an analytic continuation.

Received October 18, 1974.

Let $r > 0$. Suppose that

$$\Lambda: q(t) = Q(w; w(t)), \qquad 0 \leq t < 1,$$

with $\lim_{t \to 1} w(t) = \omega$, defines a transcendental singularity Ω on Φ. Let t_r be the last value for t, $0 \leq t < 1$, such that $|w(t_r) - \omega| = r$, counting from $t = 0$. Then by an r-neighborhood of Ω, denoted by $U_r(\Omega)$, we mean all points $Q(w; c)$ of Φ such that $|c - \omega| < r$ and $Q(w; c)$ is an analytic continuation (possibly, of algebraic character) of $Q(w; w(t_r))$ along a curve lying inside the disk $|w - \omega| < r$. If the transcendental singularity lies above the point $w = \infty$, then the circle we use to define $U_r(\Omega)$ is $|w| = r$ and the disk used is $|w| > r$. The r-neighborhoods have the following properties: i) for $r_1 < r_2$, $U_{r_1}(\Omega) \subseteq U_{r_2}(\Omega)$, and ii) $\bigcap_{r>0} U_r(\Omega) = \emptyset$ (see Choike [8] for the proof of ii)). Two transcendental singularities Ω_1 and Ω_2 are said to be equal if for all $r > 0$, $U_r(\Omega_1) \cap U_r(\Omega_2) \neq \emptyset$.

The importance of transcendental singularities stems from the result of Iversen [11, p. 13], later generalized by Noshiro [13, p. 49–53] which states that there is a one-to-one correspondence between the asymptotic boundary paths of an analytic function $w = f(z)$ and the transcendental singularities of $z = \phi(w)$, the inverse function of $w = f(z)$. In view of this result, we shall say that two asymptotic boundary paths L_1 and L_2 for $w = f(z)$ meromorphic in $|z| < R$, $0 < R \leq +\infty$, are equivalent if L_1 and L_2 both correspond (in the sense of the Iversen-Noshiro theorem) to the same transcendental singularity Ω on the Riemann surface Φ of the inverse $z = \phi(w)$, and, we shall indicate this equivalence by the notation $[L_1] = [L_2]$. In [8, p. 32–35] it was shown that $[L_1] = [L_2]$ if and only if there exists a sequence $\{c_n\}$ of disjoint arcs in $|z| < R$ joining L_1 to L_2 such that $\{c_n\}$ converges to $|z| = R$ as $n \to +\infty$, and $w = f(z)$ converges to the value ω uniformly on c_n as $n \to +\infty$ (ω, of course, is the asymptotic value for $w = f(z)$ on L_1 and L_2).

The result that we state next establishes a criterion for determining the non-equivalence of two asymptotic boundary paths for $w = f(z)$. It also generalizes a result of Bieberbach [6].

THEOREM 1. *Let*

$$L_1: z = z_1(t), \qquad 0 \leq t < 1,$$
$$L_2: z = z_2(t), \qquad 0 \leq t < 1,$$

be asymptotic boundary paths for $w = f(z)$ *in* $|z| < R$, $0 < R \leq +\infty$, *with asymptotic value* α. *Suppose* L_1 *and* L_2 *have the same initial point, but are otherwise disjoint in* $|z| < R$. *Let* D_1 *and* D_2 *denote the simply-connected regions of* $|z| < R$ *formed by* L_1 *and* L_2. *If* $[L_1] \neq [L_2]$, *then*

i) *if* $\alpha = \infty$, *there exists* $B > 0$ *such that for every* $\varepsilon > 0$ *there exist boundary paths* L_1' *in* D_1 *and* L_2' *in* D_2 *such that* $|f(z)| = B + \varepsilon$ *for all* $z \in L_1' \cup L_2'$;

ii) *if* $\alpha \neq \infty$, *there exists* $b > 0$ *such that for every* ε, $0 < \varepsilon < b$, *there exist boundary paths* L_1' *in* D_1 *and* L_2' *in* D_2 *such that* $|f(z) - \alpha| = b - \varepsilon$ *for all* $z \in L_1' \cup L_2'$.

3. Preliminary lemmas

Suppose that the boundary paths L_1 and L_2 are free of poles of $w = f(z)$. For any $t_0, 0 < t_0 < 1$, let

$$A_j(t_0) = \{z_j(t) \mid 0 \leq t \leq t_0\} ,$$
$$L_j - A_j(t_0) = \{z_j(t) \mid t_0 < t < 1\}$$

for $j = 1, 2$. Let $A_0(t_0) = A_1(t_0) \cup A_2(t_0)$. Let

$$M(t_0) = \max |f(z)| \qquad \text{for } z \in A_0(t_0) .$$

Since L_1 and L_2 contain no poles for $f(z)$, we have $M(t_0) < +\infty$ for t_0, $0 \leq t_0 < 1$. Let B be a positive number such that $B > M(t_0)$. We define the following non-empty open set in D_1:

$$G = \{z \in D_1 \,|\, |f(z)| < B\} .$$

It is clear that there exists a component G_0 of G such that $A_0(t_0) \subseteq Fr(G_0)$, where $Fr(A)$ denotes the frontier set of the set A. In general, the component G_0 is not simply-connected. It follows rather directly from the maximum principle that each hole of G_0 contains at least one pole of $f(z)$.

Let G_0^* be the open set in D_1 which consists of the open component G_0 of G plus the closure of the holes of G_0 in D_1. Then G_0^* is a simply-connected region containing G_0 with $Fr(G_0^*) \subseteq Fr(G_0)$. In the following lemmas we will examine the frontier of G_0^*.

LEMMA 1. $Fr(G_0^*) \cap \{|z| < R\}$ *is locally connected at each of its points.*

Proof. The part of $Fr(G_0^*)$ which lies within $|z| < R$ is made up of portions of L_1, of L_2, and level curves of $f(z)$. Since the level curves of $f(z)$ are analytic curves [17, p. 17], the lemma now follows.

LEMMA 2. *Every point of $Fr(G_0^*) \cap \{|z| < R\}$ is an accessible point from G_0^*.*

Proof. If $Fr(G_0^*) \cap \{|z| = R\} = \emptyset$, then $Fr(G_0^*)$ is a bounded continuum which is locally connected according to Lemma 1. Thus, in this case every point of $Fr(G_0^*)$ is accessible from G_0^* [18, p. 112].

Suppose, on the other hand, that $Fr(G_0^*) \cap \{|z| = R\} \neq \emptyset$. Let $z_0 \in Fr(G_0^*)$, $|z_0| < R$. Let $r' = |z_0| + 1$ if $R = +\infty$, and $r' = \frac{1}{2}(|z_0| + R)$ if $R < +\infty$. Denote by H_n, $n = 1, 2, 3, \cdots$ the components of the nonempty open set $H = G_0^* \cap \{|z| < r'\}$. Each component H_n is simply-connected (any hole for H_n is a hole for G_0^*) and the frontier $Fr(H_n)$ of H_n is locally connected being composed of portions of $Fr(G_0^*)$ and closed arcs of $\{|z| = r'\}$. We, also, remark that $Fr(H_n)$ must have some points of $\{|z| = r'\}$, otherwise we contradict the fact that G_0^* is connected. From this it follows that z_0 is a frontier point of some component H_{n_0}. If not, then we can find a sequence of continua lying in $Fr(G_0^*)$ which converges to a non-degenerate arc of $Fr(G_0^*)$ containing z_0 and a point of $\{|z| = r'\}$. This, however, contradicts Lemma 1. Thus, z_0 is an accessible point from H_{n_0}. Hence, z_0 is an accessible point from G_0^*.

LEMMA 3. *If on every curve L lying, except for its endpoints in D_1, joining a point of $L_1 - A_1(t_0)$ to a point of $L_2 - A_2(t_0)$, there exist points at which $|f(z)| < B$, then $Fr(G_0^*) \cap \{|z| = R\} \neq \emptyset$.*

Proof. Suppose, on the contrary, that $Fr(G_0^*) \cap \{|z| = R\} = \emptyset$. Then we have that, for some positive $r < R$, $Fr(G_0^*) \subseteq \{|z| < r\}$, and, according to Lemma 2, each point of $Fr(G_0^*)$ is accessible from G_0^*.

Let us map the simply-connected region G_0^* onto the disk $|\zeta| < 1$ by a conformal map $\zeta = \zeta(z)$. Denote by $z = z(\zeta)$ the inverse map of $\zeta = \zeta(z)$. By Carathéodory's extension of the Riemann mapping theorem there exists a one-to-one correspondence between points $e^{i\theta}$ of $|\zeta| = 1$ and prime ends $P = P(e^{i\theta})$ of G_0^* such that the cluster set of $z(\zeta)$ at $e^{i\theta}$ equals the impression of the prime end $P(e^{i\theta})$ [9, p. 173]. Since there can be at most one accessible point in the impression of a prime end [9, p. 177], it follows that each impression of G_0^* consists of precisely one point of $Fr(G_0^*)$. Thus, $z = z(\zeta)$ may be extended continuously to the closed disk $|\zeta| \leq 1$.

Since $|f(z_1(t_0))| < B$ and $|f(z_2(t_0))| < B$, there are points $a_1 \in Fr(G_0^*) \cap (L_1 - A_1(t_0))$ and $a_2 \in Fr(G_0^*) \cap (L_2 - A_2(t_0))$. Let $e^{i\theta_1} = \zeta(a_1)$ and $e^{i\theta_2} = \zeta(a_2)$. Denote by β the arc of $|\zeta| = 1$ joining $e^{i\theta_1}$ and $e^{i\theta_2}$ which contains no points of $\zeta(A_0(t_0))$. We may assume, without loss of generality, that $\theta_1 < \theta_2$ and $\beta = \{e^{i\theta} | \theta_1 \leq \theta \leq \theta_2\}$. Let

$$T_1 = \beta \cap \zeta(L_1 \cap Fr(G_0^*))$$

and

$$T_2 = \beta \cap \zeta(L_2 \cap Fr(G_0^*)) .$$

Since $G_0^* \subseteq \{|z| < r\}$, $L_1 \cap Fr(G_0^*)$ and $L_2 \cap Fr(G_0^*)$ are closed relative to $Fr(G_0^*)$. Hence, T_1 and T_2 are closed sets on $|\zeta| = 1$. Furthermore, if $e^{i\phi_1} \in T_1$ and $e^{i\phi_2} \in T_2$, we must have that $\phi_1 < \phi_2$. Otherwise, the image of the interval $[\theta_1, \theta_2]$ under $z = z(e^{i\theta})$ would be a curve in D_1 starting at $a_1 \in L_1 - A_1(t_0)$ proceeding to $L_2 - A_2(t_0)$ at $z(e^{i\phi_1})$ then back to $L_1 - A_1(t_0)$ at $z(e^{i\phi_1})$ and finally terminating at $a_2 \in L_2 - A_2(t_0)$. But such a boundary curve for $Fr(G_0^*)$ contradicts the connectivity of G_0^*.

Let
$$\theta_1^* = \sup\{\theta | e^{i\theta} \in T_1\} , \quad \text{and}$$
$$\theta_2^* = \inf\{\theta | e^{i\theta} \in T_2\} .$$

Since $e^{i\theta_1^*} \in T_1$ and $e^{i\theta_2^*} \in T_2$, we have $\theta_1^* < \theta_2^*$. Let L be the curve in $\bar{D}_1$ defined by $z = z(e^{i\theta})$, $\theta_1^* \leq \theta \leq \theta_2^*$. Clearly, $z(e^{i\theta_1^*})$ is a point of $L_1 - A_1(t_0)$ and $z(e^{i\theta_2^*})$ is a point of $L_2 - A_2(t_0)$. From the choice of θ_1^* and θ_2^*, we, also, have that $z(e^{i\theta})$, $\theta_1^* < \theta < \theta_2^*$, is a curve lying in D_1 on the boundary $Fr(G_0^*)$ of G_0^*. But this implies that $|f(z(e^{i\theta}))| = B$ for $\theta_1^* \leq \theta \leq \theta_2^*$. This, however, contradicts our hypothesis. Therefore, $Fr(G_0^*) \cap \{|z| = R\} \neq \emptyset$.

4. Proof of Theorem 1

Let Ω_1 and Ω_2 be the transcendental singularities on the Riemann surface Φ of the inverse function $z = \phi(w)$ which correspond to the boundary paths L_1 and L_2, respectively. Since we assume $[L_1] \neq [L_2]$, it follows that $\Omega_1 \neq \Omega_2$. We point out that we may alter the boundary paths L_1 and L_2 so that, while we still preserve the hypothesis of Theorem 1, we may additionally assume that L_1 and L_2 do not contain α-points of $f(z)$.

First, we assume that $\alpha = \infty$. Since $\Omega_1 \neq \Omega_2$, there exists a positive number r_1 such that

$$(1) \qquad U_{r_1}(\Omega_1) \cap U_{r_1}(\Omega_2) = \emptyset$$

(see section 2). Since $f(z) \to \infty$ on L_1 and L_2 as $|z| \to R$, there exists $t_0 = t_0(r_1)$ such that $0 < t_0 < 1$ and $|f(z_j(t_0))| > r_1$ for all $t > t_0$, $j = 1, 2$.

Let

$$(2) \qquad M = \max |f(z)| \qquad \text{for } z \in A_0(t_0)\ , \qquad \text{and}$$

$$(3) \qquad B = \max (r_1, M)\ .$$

Since $f(z)$ has no poles on L_1 and L_2, $M < +\infty$. Let $\varepsilon > 0$. We define the non-empty open set G in D_1 as follows:

$$G = \{z \in D_1 \,|\, |f(z)| < B + \varepsilon\}\ .$$

Let G_0 be the open component of G such that $A_0(t_0) \subseteq Fr(G_0)$. As in section 2, G_0^* denotes the simply-connected domain in D_1 which consists of G_0 plus the closure of the holes of G_0.

Let L be any curve lying, except for the endpoints, in D_1 joining a point of $L_1 - A_1(t_0)$ to a point $L_2 - A_2(t_0)$. Because of the selection of t_0, the image of L under $f(z)$ is a path on the Riemann surface Φ joining a point of $U_{r_1}(\Omega_1)$ to a point of $U_{r_1}(\Omega_2)$. By (1), it follows that there exist points $z \in L$ such that $|f(z)| \leq r_1 < B + \varepsilon$. Hence, applying Lemma 3, we have $Fr(G_0^*) \cap \{|z| = R\} \neq \emptyset$.

Let us map G_0^* onto $|\zeta| < 1$ by a conformal map $\zeta = \zeta(z)$. We denote the inverse map by $z = z(\zeta)$. If we analyze the prime ends of G_0^* we shall find that each prime end P whose impression is contained in $|z| < R$ consists of a single point. This follows directly from the two facts: i) the points of $Fr(G_0^*) \cap \{|z| < R\}$ are accessible from G_0^* (see Lemma 2), and ii) there can be at most one accessible point in the impression of a prime end [9, p. 177]. Hence, if we let $C(z(\zeta), e^{i\theta})$ denote the cluster set of $z(\zeta)$ at $e^{i\theta}$, we have that $C(z(\zeta), e^{i\theta})$ is a singleton for those points $e^{i\theta}$ of $|\zeta| = 1$ which correspond to prime ends whose impressions contain points of $|z| < R$.

Let

$$V = \{\theta \,|\, C(z(\zeta), e^{i\theta}) \subseteq Fr(G_0^*) \cap \{|z| < R\}\}\ .$$

A rather direct and elementary argument shows V to be an open subset of the interval $[0, 2\pi]$. Let (θ_1, θ_2) be an open component of V. Let $\theta_0 = \frac{1}{2}(\theta_1 + \theta_2)$. Let L be the curve in D_1 given by $z = z(e^{i\theta})$, $\theta_0 \leq \theta < \theta_2$.

Since $\theta_2 \notin V$, we have that $C(z(\zeta), e^{i\theta_2}) \subseteq \{|z| = R\}$. This implies that L is a boundary path in $|z| < R$. From the fact that $f(z) \to \infty$ on L_1 and L_2 as $|z| \to R$, there exists a positive number δ, $\delta < R$, such that $|f(z)| > B + \varepsilon$ for $z \in L_1 \cup L_2$, $\delta < |z| < R$. Choose θ', $\theta_0 < \theta' < \theta_2$, such that $\delta < |z(e^{i\theta})| < R$ for $\theta' \leq \theta < \theta_2$. Let L_1' be the boundary path in $|z| < R$ defined by $z = z(e^{i\theta})$, $\theta' \leq \theta < \theta_2$. By construction, L_1' is a boundary path in D_1 lying on $Fr(G_0^*)$. Thus, $|f(z(e^{i\theta}))| = B + \varepsilon$ for $\theta' \leq \theta < \theta_2$. This proves Theorem 1 for the case $\alpha = \infty$.

Suppose α is finite. The function

$$F(z) = \frac{1}{f(z) - \alpha}$$

has the property that $F(z) \to \infty$ on L_1 and L_2 as $|z| \to R$. Also, $[L_1] \neq [L_2]$ with respect to the inverse function of $F(z)$. Otherwise, there would exist a sequence of disjoint arcs $\{c_n\}$ joining L_1 to L_2 converging to an arc of $|z| = R$ (to $z = \infty$, if $R = +\infty$) on which $F(z) \to \infty$ uniformly on c_n as $n \to +\infty$ [8, p. 32]. On these same arcs, $f(z) \to \alpha$ uniformly with $n \to +\infty$. This contradicts our assumption that $[L_1] \neq [L_2]$ with respect to the inverse of $f(z)$. Apply part i) of Theorem 1 to $F(z)$ and set $b = 1/B$ where B is the positive number obtained in part i). This completes the proof of Theorem 1 since the above argument can also be applied to the region D_2.

Remark. If L_1 and L_2 have different asymptotic values α_1 and α_2, respectively, then the method of proof of Theorem 1 can be used to prove the following:

i) if $\alpha_1 = \infty$ and $\alpha_2 \neq \infty$, there exists $b > 0$ such that for every ε, $0 < \varepsilon < b$, there exist boundary paths L_1', L_1'' in D_1 and L_2', L_2'' in D_2 such that

$$|f(z)| = (b - \varepsilon)^{-1} \qquad \text{for all } z \in L_1' \cup L_2' ,$$

and

$$|f(z) - \alpha_2| = b - \varepsilon \qquad \text{for all } z \in L_1'' \cup L_2'' ,$$

ii) if α_1 and α_2 are both finite, there exists $b > 0$ such that for every $\varepsilon > 0$, $0 < \varepsilon < b$, there exist boundary paths L_1', L_1'' in D_1 and L_2', L_2'' in D_2 such that

$$|f(z) - \alpha_1| = b - \varepsilon \qquad \text{for all } z \in L_1' \cup L_2' ,$$

and

$$|f(z) - \alpha_2| = b - \varepsilon \qquad \text{for all } z \in L_1'' \cup L_2'' .$$

COROLLARY. *Let $w = f(z)$ be meromorphic in $|z| < R$, $0 < R \leq +\infty$. Let L_1 and L_2 be disjoint boundary paths in $|z| < R$ such that $f(z) \to \alpha$, finite or infinite, on L_1 and L_2 as $|z| \to R$. Then* $[L_1] \neq [L_2]$ if and only *if there exists a positive number b and there exist paths L_1' in D_1 and L_2' in D_2 such that $|f(z) - \alpha| = b$ for all $z \in L_1' \cup L_2'$.*

The next corollary is originally due to Heins [10]. It follows as a direct application of Theorem 1 and Bagemihl's ambiguous point theorem [1] (see, also, [9, p. 85]).

COROLLARY. *Let $f(z)$ be meromorphic in $|z| < 1$. The number of points $e^{i\theta}$ of $|z| = 1$ which have the property that they are terminal points of two or more non-equivalent asymptotic paths is at most countable.*

Proof. Let L_1 and L_2 be non-equivalent asymptotic paths for $f(z)$ with asymptotic values α_1 and α_2, respectively, which terminate at $e^{i\theta}$. If $\alpha_1 \neq \alpha_2$, then $e^{i\theta}$ is an ambiguous point of $f(z)$. If $\alpha_1 = \alpha_2$, then by Theorem 1, since $[L_1] \neq [L_2]$ we can find two boundary paths L_1' and L_1'' between L_1 and L_2 terminating at $e^{i\theta}$ having the property that the cluster set of $f(z)$ along L_1' is disjoint from the cluster set of $f(z)$ along L_1''. Thus, in this case, $e^{i\theta}$ is again an ambiguous point. By Bagemihl's theorem the number of such points is at most countable.

Another application of Theorem 1, originally due to Bieberbach, is found in the next corollary. Bieberbach's original proof was based on a special case of Theorem 1 and the Denjoy-Carleman theorem. Since Ahlfors subsequently improved this result, we think it appropriate to update Bieberbach's result.

COROLLARY. *If $f(z)$ is an entire function of order ρ, then $f(z)$ has at most 2ρ non-equivalent asymptotic paths with ∞ as the asymptotic value.*

Proof. We refer the reader to [6, p. 39–40] for the proof.

5. Some special classes of functions in the unit disk

A boundary path $S: z = s(t)$, $0 \leq t < 1$, of $|z| < 1$ shall be called a spiral path if $\arg s(t) \to +\infty$ or $\arg s(t) \to -\infty$ as $t \to 1$. The end $E(L)$ of a boundary path L is the set of limit points of L on $|z| = 1$. The end of a spiral path is clearly the circumference $|z| = 1$. Just as clear is the fact that a non-spiral path can also have $|z| = 1$ as its end. We shall say that a function $f(z)$, holomorphic and nonconstant in $|z| < 1$, belongs to the class (S) if it possesses an asymptotic value, finite or infinite, on a spiral path. We shall say that $f(z)$, holomorphic in $|z| < 1$, belongs to the class (V) if $f(z)$ is unbounded in $|z| < 1$ yet remains bounded on a spiral path. Functions of class (V) were first introduced by Valiron [15], and they have been studied extensively by Bagemihl and Seidel [2], [3] and Seidel [14].

Valiron [16] showed that $(V) \subseteq (S)$ by virtue of $f(z) \in (V)$ possessing ∞ as a spiral asymptotic value. We would like to begin this section with an elementary proof of this result of Valiron.

The proof that follows was first announced in [7].

LEMMA 4. *Let L be a boundary path in $|z| < 1$ whose end $E(L)$ is $|z| = 1$. Then the simply-connected region $\{|z| < 1\} - L$ has exactly one prime end P whose impression $I(P)$ is $|z| = 1$.*

Proof. Suppose $\{|z| < 1\} - L$ has two prime ends P_1 and P_2 whose impressions $I(P_1)$ and $I(P_2)$ equal the circumference $|z| = 1$. Let $\zeta = \zeta(z)$ be a one-to-one conformal map of the simply-connected region $\{|z| < 1\} - L$ onto the disk $|\zeta| < 1$. Denote the inverse map of $\zeta = \zeta(z)$ by $z = z(\zeta)$. By Carathéodory's extension of the Riemann mapping theorem [9, p. 173] the boundary path L is mapped by $\zeta = \zeta(z)$ onto an arc γ of $|\zeta| = 1$ with the initial point of L corresponding to an interior point of γ. Also, since $P_1 \neq P_2$, $\zeta(P_1) = e^{i\theta_1}$ and $\zeta(P_2) = e^{i\theta_2}$ with $\theta_1 \neq \theta_2$.

The points $e^{i\theta_1}$ and $e^{i\theta_2}$ separate $|\zeta| = 1$ into two distinct arcs. One of these arcs is disjoint from γ. Let us denote this arc by γ_0. By Fatou's theorem [9, p. 17] and the Riesz theorem [9, p. 22], there exist $e^{i\theta_1^*}, e^{i\theta_2^*} \in \gamma_0$, $\theta_1^* \neq \theta_2^*$, such that

$$z(e^{i\theta_1^*}) = \lim_{r\to 1} z(re^{i\theta_1^*}) , \qquad z(e^{i\theta_2^*}) = \lim_{r\to 1} z(re^{i\theta_2^*})$$

exist, and $z(e^{i\theta_1^*}) \neq z(e^{i\theta_2^*})$. Since $e^{i\theta_1^*}, e^{i\theta_2^*} \notin \gamma$, $|z(e^{i\theta_1^*})| = |z(e^{i\theta_2^*})| = 1$. Thus, $z(e^{i\theta_1^*})$ and $z(e^{i\theta_2^*})$ are accessible points of $\{|z| < 1\} - L$. This implies that the impression of P_1 and P_2 has two accessible points. But this is impossible [9, p. 177]. Thus, $\{|z|<1\}-L$ has exactly one prime end P whose impression $I(P)$ is $|z|=1$.

LEMMA 5. *Let $f(z)$ be holomorphic in $|z|<1$ and continuous for $|z|\leq 1$ with the exception of the point $z=+1$. Suppose also that $|f(e^{i\theta})| \leq M<+\infty$ for $\theta\neq 0$, and that there exists z_0 such that $|z_0|<1$ and $|f(z_0)|>M$. Then there exists a boundary path L' in $|z|<1$ terminating at $z=+1$ with $f(z)\to\infty$ on L' as $|z|\to 1$.*

Proof. By the Phragmén-Lindelöf theorem [12 p. 43–44], we have that the properties $|f(e^{i\theta})|\leq M<+\infty$ for $\theta\neq 0$ and $|f(z_0)|>M$ for $|z_0|<1$ imply $f(z)$ is unbounded in $|z|<1$. Then, $\infty\in C(f,1)-C_B(f,1)$, where $C(f,1)$ is the cluster set of $f(z)$ at $z=1$ [9, p. 1] and $C_B(f,1)$ is the boundary cluster set of $f(z)$ at $z=1$ [9, p. 81]. Since ∞ is an omitted value of $f(z)$ in $|z|<1$, by the Gross-Iversen theorem [9, p. 101], ∞ is an asymptotic value along a boundary path L' terminating at $z=1$.

THEOREM 2. (*Valiron's theorem*). *Let $f(z)$ be an unbounded holomorphic function in $|z|<1$ that is bounded on a boundary path L whose end is $|z|=1$. Then there exists a boundary path L^* along which $f(z)\to\infty$ as $|z|\to 1$. In particular, if L is a spiral, then L^* is also a spiral.*

Proof. We map the simply-connected region $\{|z|<1\}-L$ in a one-to-one conformal manner onto the unit disk $|\zeta|<1$ by a map $\zeta=\zeta(z)$ in such a way that the prime end P of $\{|z|<1\}-L$, whose impression is $|z|=1$, corresponds to $\zeta=1$.

We consider the function $F(\zeta)=f(z(\zeta))$ in $|\zeta|<1$, where $z=z(\zeta)$ is the inverse map of $\zeta=\zeta(z)$. Then $F(\zeta)$ is holomorphic and unbounded in $|\zeta|<1$, and, since $f(z)$ is bounded, by assumption, on L, $F(e^{i\theta})$ is bounded for $\theta\neq 0$. Applying Lemma 5, there exists a path L' in $|\zeta|<1$ terminating at $\zeta=1$ on which $F(\zeta)\to\infty$ as $|\zeta|\to 1$. The image of L' under $z=z(\zeta)$ is a boundary path L^* in $|z|<1$ on which $f(z)\to\infty$ as $|z|\to 1$. Since L^* is disjoint from L, it is clear that L^* is a spiral if L is a spiral. This concludes the proof.

It is not true, in general, that the boundary path L^* in Valiron's theorem has for its end the circumference $|z|=1$. The following example illustrates this situation.

Let

$$f(z)=\exp\left(\frac{1+z}{1-z}\right).$$

The function $f(z)$ is holomorphic in $|z| < 1$ and has $z = \infty$ as an asymptotic value along the radial segment from 0 to 1. Let $D' = \{z \mid |z - \frac{1}{2}| < \frac{1}{2}\}$. Then it is easily verified that $|f(z)| \leq e$ for all values z, $z \notin D'$, $|z| < 1$. It is also easy to construct a path L in $\{|z| < 1\} - D'$ such that $E(L) = \{|z| = 1\}$.

THEOREM 3. *If $w = f(z) \in (S)$ and if the Riemann surface Φ of the inverse function $z = \phi(w)$ has at least two transcendental singularities with projection $w = \infty$, then $f(z) \in (V)$.*

Proof. This is a direct application of Theorem 1.

In connection with Theorem 3 we point out that annular functions (see [5]) are functions belonging to class $(S)\backslash(V)$ with the property that their inverse functions have exactly one transcendental singularity above $w = \infty$. In [16], Valiron offers a construction of functions in class (V) with the property that their inverses have precisely k transcendental singularities above $w = \infty$. The subtleties of Valiron's construction we feel prompts the need for another approach to the construction of such functions.

THEOREM 4. *Let $f(z) \in (S)$. Then every pair of non-equivalent finite spiral asymptotic paths for $f(z)$ is separated by spirals along which $f(z) \to \infty$ as $|z| \to 1$.*

Proof. Suppose S_1 and S_2 are non-equivalent finite asymptotic spiral paths in $|z| < 1$ for $f(z)$. We may suppose that S_1 and S_2 have a common initial point but are otherwise disjoint. They, then, divide $|z| < 1$ into two simply-connected regions D_1 and D_2. Map D_1 in a one-to-one conformal manner onto the disk $|\zeta| < 1$ by $\zeta = \zeta(z)$ in such a way that the prime end P of D_1, whose impression is $|z| = 1$, corresponds to $\zeta = 1$. The function $F(\zeta) = f(z(\zeta))$, where $z = z(\zeta)$ is the inverse of $\zeta = \zeta(z)$, is holomorphic in $|\zeta| < 1$ and continuous in $|\zeta| \leq 1$, $\zeta \neq 1$. If $F(\zeta)$ were bounded in $|\zeta| < 1$, then, by a well-known theorem of Lindelöf, $F(\zeta)$ would be continuous in the closed disk $|\zeta| \leq 1$. But this would imply $[S_1] = [S_2]$ which contradicts our hypothesis. Thus, $\infty \in C(F, 1) - C_B(F, 1)$ and ∞ is omitted by $F(\zeta)$ in $|\zeta| < 1$. Hence, by the Gross-Iversen theorem, ∞ is an asymptotic value along a path L terminating at $\zeta = 1$. The image of L under $z = z(\zeta)$ is a spiral in D_1 along which

$f(z) \to \infty$ as $|z| \to 1$. To complete the proof, we apply the above argument to the region D_2.

We remark that non-equivalent spirals with ∞ as an asymptotic value are not, in general, separated by finite spiral asymptotic paths. Imitating the pole-sweeping technique of Barth and Schneider, as found in [4], we can construct a function $f(z) \in (S)$ such that $f(z)$ is bounded on the spirals S_1 and S_2 and $f(z) \to \infty$ on the spirals S_1' in D_1 and S_2' in D_2 (D_1 and D_2 are the regions in $|z| < 1$ formed by S_1 and S_2), with ∞ the only asymptotic value of $f(z)$.

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Department of Mathematics
Oklahoma State University