

CURVES IN GRASSMANN VARIETIES

AIGLI PAPANTONOPOULOU

§ 0. Introduction

The following question was the main motivation of this paper: which are the necessary and sufficient conditions for a non-singular subvariety of a Grassmann variety to have an ample normal bundle. Knowing that a non-singular subvariety of a Grassmann variety has an ample normal bundle we can apply on it several well-known theorems.

- a) A vanishing theorem on formal schemes (Hartshorne [6], Theorem 4.1.).
- b) A theorem on meromorphic functions (Hartshorne [7], Chapter 6.).
- c) Results on the cohomological dimension of a projective variety minus a subvariety (Hartshorne [7], Chapter 7.).

Let E be the universal subbundle on the Grassmannian $G(r, n)$, Q its universal quotient bundle, Y a closed subscheme of $G(r, n)$, and C a non-singular curve in $G(r, n)$. In this paper we prove the following results.

- 1) Necessary and sufficient conditions for $\check{E}|_Y$ to be ample.
- 2) Necessary and sufficient conditions for $Q|_Y$ to be ample.
- 3) Necessary and sufficient conditions for $T_G|_Y$ to be ample, where T_G is the tangent bundle of $G = G(r, n)$.
- 4) Necessary and sufficient conditions for $N_{C/G}$ to be ample, where $N_{C/G}$ is the normal bundle of a non-singular curve C in $G = G(r, n)$.
- 5) Sufficient conditions for $N_{Y/G}$ to be ample, where in this case Y is a non-singular subvariety of $G = G(r, n)$. We also give examples of non-singular subvarieties of a $G(r, n)$ with ample normal bundle which do not satisfy conditions 5) to show that these are not necessary conditions.

The paper is organized as follows: § 1 contains the basic definitions

and facts about Grassmann varieties, Schubert cycles and ample vector bundles which will be needed. §2 contains the proofs of the main results of this paper. In §3 several examples are given to illustrate our results. Also the different applications are explained. In the appendix we outline a proof of the fact that Ω_G^1 , the sheaf of 1-differentials on $G = G(r, n)$ can be expressed as $E \otimes \check{Q}$.

I would like to thank my advisor Robin Hartshorne for introducing me in these questions and for his invaluable help.

CONTENTS

- § 0. Introduction
- § 1. Definitions and elementary properties
- § 2. Curves with ample normal bundle
- § 3. Applications and examples
- Appendix
- Bibliography

§ 1. Definitions and elementary properties

Let $G = G(r, n)$ be the Grassmannian of P^r spaces in a fixed P^n space. The fixed P^n induces a trivial bundle O_G^{n+1} on G of rank $n + 1$, and the P^r subspaces induce a subbundle E of O_G^{n+1} of rank $r + 1$. So on G we have the following canonical short exact sequence

$$0 \longrightarrow E \longrightarrow O_G^{n+1} \longrightarrow Q \longrightarrow 0 .$$

Q is a vector bundle on G of rank $n - r$, called the canonical quotient bundle on G , and E is called the canonical subbundle on G .

$G = G(r, n)$ is uniquely determined by the following universal property of the canonical quotient bundle Q . For all schemes X , for all locally free sheaves Q_0 of rank $n - r$ on X and for all $n + 1$ global sections $s_0, \dots, s_n \in \Gamma(X, Q_0)$ which generate the stalks of Q_0 at every point $p \in X$, there exists a unique morphism $f: X \rightarrow G$ such that:

- a) $f^*Q = Q_0$ and
- b) $s_i = f^*(x_i)$ for $i = 0, \dots, n$ where $x_i \in \Gamma(G, Q)$ is the image of 1 in the i^{th} place of O_G^{n+1} under the canonical map

$$O_G^{n+1} \longrightarrow Q \longrightarrow 0 .$$

DEFINITION 1.1. Let $0 \leq a_0 < \dots < a_r \leq n$ be $r + 1$ integers and let

$A_0 \subset \dots \subset A_r \subset \mathbf{P}^n$ be $r + 1$ linear subspaces such that $\dim A_i = a_i$. Let $\Omega(A_0, \dots, A_r)$ be the subspace of $G(r, n)$ parametrizing all $\mathbf{P}^r \subset \mathbf{P}^n$ such that $\mathbf{P}^r \cap A_i \neq \emptyset$ and $\dim \mathbf{P}^r \cap A_i \geq i$ for $i = 0, \dots, r$. We know from [9] that these subspaces are actually subvarieties of $G(r, n)$ and that they depend, up to rational equivalence, only on the integers a_0, \dots, a_r and not on the choice of the linear subspaces A_i . We call them Schubert subvarieties or Schubert cycles of $G(r, n)$.

We are interested in the following two Schubert cycles.

1) Let Z_1 be the Schubert cycle of $G(r, n)$ parametrizing all $\mathbf{P}^r \subset \mathbf{P}^n$ which are contained in a fixed \mathbf{P}^{n-1} in \mathbf{P}^n . This is the Schubert cycle which satisfies the condition $(n - r - 1, \dots, n - 2, n - 1)$ and it is isomorphic to $G(r, n - 1)$. The fixed \mathbf{P}^{n-1} space induces a trivial subbundle of $O_G^{n+1}|_{Z_1}$ of rank n , namely $O_G^n|_{Z_1}$, and $E|_{Z_1}$ is a subbundle of $O_G^n|_{Z_1}$. In other words, the canonical short exact sequence,

$$0 \longrightarrow E|_{Z_1} \longrightarrow O_G^{n+1}|_{Z_1} \longrightarrow Q|_{Z_1} \longrightarrow 0$$

factors through

$$0 \longrightarrow E|_{Z_1} \longrightarrow O_G^n|_{Z_1} \longrightarrow Q' \longrightarrow 0$$

where Q' is the canonical quotient bundle of Z_1 , when Z_1 is viewed as a $G(r, n - 1)$ and $Q|_{Z_1} = Q' \oplus O_{Z_1}$ is the vector bundle on Z_1 of rank $n - r$, which induces the embedding of $Z_1 \hookrightarrow G(r, n)$.

2) Let Z_2 be the Schubert cycle of $G(r, n)$ parametrizing all $\mathbf{P}^r \subset \mathbf{P}^n$ such that $p \in \mathbf{P}^r \subset \mathbf{P}^n$, for some fixed point p in \mathbf{P}^n . This is a Schubert cycle satisfying the condition $(0, n - r + 1, \dots, n - 1, n)$. The fixed point p induces a trivial subbundle of $E|_{Z_2}$ of rank 1. In other words, we have on Z_2 the following short exact sequence,

$$0 \longrightarrow O_{Z_2} \longrightarrow E|_{Z_2} \longrightarrow E' \longrightarrow 0$$

where E' is a vector bundle on Z_2 of rank r . Thus on Z_2 we have the short exact sequence,

$$0 \longrightarrow E' \longrightarrow O_G^n|_{Z_2} \longrightarrow Q|_{Z_2} \longrightarrow 0.$$

By the Principle of Duality we can consider $G(r, n)$ equivalent to $G(n - r - 1, n)$. On $G(n - r - 1, n)$ we will have the dual short exact sequence,

$$0 \longrightarrow \check{Q} \longrightarrow O_G^{n+1} \longrightarrow \check{E} \longrightarrow 0.$$

If we let $d = n - r - 1$, we can see from [9] that Z_2 as a Schubert cycle now of $G(d, n)$ it satisfies the condition $(n - d - 1, \dots, n - 2, n - 1)$.

DEFINITION 1.2. A vector bundle E on a scheme X is ample if for every coherent sheaf F on X there exists an integer $n_0 > 0$ such that for all integers $n \geq n_0$ the sheaf $F \otimes S^n(E)$, (where $S^n(E)$ is the n^{th} symmetric product of E), is generated as an O_X -module by its global sections.

Throughout this paper we will be using basic properties of ample vector bundles which are proved in (Hartshorne [5], 2 and 4).

DEFINITION 1.3. Let L be a line bundle on a non-singular curve C . We define the degree of L to be the degree of D , where D is the corresponding divisor. If C is singular we define $\deg L = \deg L \otimes O_{\tilde{C}}$, where \tilde{C} is the normalization of C . If E is a vector bundle of rank r then we define $\deg E = \deg \wedge^r E$.

The following results of Hartshorne and Gieseker will be very useful.

THEOREM 1.4 (Hartshorne [8], Theorem 2.4.). *Let X be a complete non-singular curve over k , $\text{char } k = 0$, and E a vector bundle on X . Then E is ample if and only if every quotient bundle of E has positive degree.*

THEOREM 1.5 (Gieseker [4], Prop. 2.1.). *Let X be a complete scheme over a field k . Let E be a vector bundle on X which is generated by its global sections. Then E is ample if and only if for every curve $C \subseteq X$, and for every quotient line bundle L of $E|_C$, $\deg L > 0$.*

Throughout the paper we are working over a field k of characteristic 0.

§2. Curves with ample normal bundle

Our main theorem can be stated as follows.

THEOREM 2.1. *Let $C \subset G(r, n)$ be a non-singular curve. Let $N_{C/G}$ be the normal bundle of C in $G(r, n)$. Then $N_{C/G}$ is not ample if and only if C lies in some Z_3 , where Z_3 is the Schubert cycle parametrizing $\{\mathbf{P}^r \subset \mathbf{P}^n \mid \text{some fixed point } p \in \mathbf{P}^r \subset \text{some fixed } \mathbf{P}^{n-1} \subset \mathbf{P}^n\}$.*

The proof of the theorem will follow immediately from the following four propositions. By Z_1 and Z_2 we denote the two Schubert cycles as defined in §1.

PROPOSITION 2.2. *Let Y be a subvariety of $G(r, n)$. Then $\check{E}|_Y$ is not ample if and only if there is a curve C in Y which lies in some Z_2 .*

PROPOSITION 2.3. *Let Y be a subvariety of $G(r, n)$. Then $Q|_Y$ is not ample if and only if there is a curve C in Y which lies in some Z_1 .*

PROPOSITION 2.4. *Let Y be a subvariety of $G(r, n)$. Then $T_G|_Y$ is not ample if and only if there is a curve C in Y which lies in some Z_3 .*

PROPOSITION 2.5. *Let C be a non-singular curve in $G(r, n)$. Then $N_{C/G}$ is ample if and only if $T_G|_C$ is ample.*

We will need the following lemmas.

LEMMA 2.6. *Let Y be a subvariety of $G(r, n)$. Then there exists a splitting $Q|_Y = Q_0 \oplus O_Y$ if and only if Y lies in some $Z_1 = G(r, n - 1) \subset G(r, n)$. Then $Q|_Y$ embeds $Y \hookrightarrow G(r, n)$ and Q_0 embeds $Y \hookrightarrow Z_1$.*

Proof of lemma. a) Assume Y lies in some $Z_1 \subset G(r, n)$. Then $Q|_Y = (Q|_{Z_1})|_Y = Q'|_Y \oplus O_Y$, where Q' is the canonical quotient bundle of $Z_1 = G(r, n - 1)$.

b) Suppose there exists a splitting of $Q|_Y$, and $Q|_Y = Q_0 \oplus O_Y$. Q_0 is a vector bundle on Y of rank $n - r - 1$, generated by n global sections. Therefore, Q_0 induces a unique morphism,

$$f: Y \longrightarrow Z_1 = G(r, n - 1)$$

such that $f^*(Q') = Q_0$, where Q' is the canonical quotient of Z_1 , and $f^*(x_i) = s_i$ for all n global sections $s_i \in \Gamma(Y, Q_0)$ generating the stalks of Q_0 at every point p of Y . Let

$$g: Z_1 \hookrightarrow G(r, n)$$

be the embedding induced by $Q' \oplus O_{Z_1}$. By the universal property of the canonical quotient bundle Q on $G(r, n)$ we know that $g^*(Q) = Q' \oplus O_{Z_1}$ and $g^*(y_i) = x_i$, where $y_0, \dots, y_n \in \Gamma(G, Q)$ and $x_0, \dots, x_{n-1}, 1 \in \Gamma(Z_1, Q' \oplus O_{Z_1})$. Thus

$$f^*(g^*(Q)) = Q_0 \oplus O_Y$$

and

$$f^*(g^*(y_i)) = s_i,$$

where $s_0, \dots, s_{n-1}, 1$ are in $\Gamma(Y, Q_0 \oplus O_Y)$, so by uniqueness $g \circ f$ is the

embedding induced by $Q_0 \oplus O_Y$, $Y \hookrightarrow G(r, n)$. Therefore,

$$f: Y \hookrightarrow Z_1$$

is an embedding.

q.e.d.

By the Principle of Duality and its application on Z_2 that we explained in §1, the following lemma is equivalent to Lemma 2.6.

LEMMA 2.7. *Let Y be a subvariety of $G(r, n)$. Then there exists a splitting of $E|_Y = E_0 \oplus O_Y$ if and only if Y lies in some $Z_2 \subset G(r, n)$.*

We can now prove our four propositions.

Proof of Proposition 2.2. Suppose that there is a curve C in Y which lies in some Z_2 . By Lemma 2.7. there is a splitting $\check{E}|_C = \check{E}_0 \oplus O_C$. Hence $\check{E}|_C$ is not ample and that implies that $\check{E}|_Y$ is not ample.

Now suppose that $\check{E}|_Y$ is not ample. Then by Theorem 1.5. there exists a curve C in Y and a quotient line bundle E' of $\check{E}|_C$ on C

$$\check{E}|_C \longrightarrow E' \longrightarrow 0$$

such that $\deg E' \leq 0$.

CLAIM. $E' = O_C$. We break the proof of the claim into two cases.

Case 1. Assume C is a non-singular curve. E' is a line bundle on C generated by its global sections. Therefore, E' corresponds to an effective divisor D on C such that $D = \sum_i n_i p_i$, where $n_i \geq 0$ and $\sum_i n_i = \deg E' \leq 0$. Therefore $\sum_i n_i = 0$ and $D = 0$, the zero divisor. Hence $E' = O_C$.

Case 2. Let C be any reduced, irreducible curve not necessarily non-singular. E' is generated by its global sections; hence there exists $0 \neq s \in \Gamma(C, E')$ which induces an injective map $O_C \rightarrow E'$, and E' has no torsion. Thus we have the short exact sequence,

$$1) \quad 0 \longrightarrow O_C \longrightarrow E' \longrightarrow F \longrightarrow 0,$$

where F is the quotient sheaf. Let \tilde{C} be the normalization of C . Tensoring the short exact sequence 1) by $O_{\tilde{C}}$ we get

$$O_{\tilde{C}} \longrightarrow E' \otimes O_{\tilde{C}} \longrightarrow F \otimes O_{\tilde{C}} \longrightarrow 0.$$

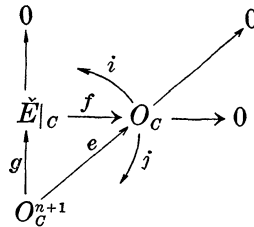
\tilde{C} is non-singular and by definition $\deg E' \otimes O_{\tilde{C}} = \deg E' \leq 0$. Therefore by the non-singular case $E' \otimes O_{\tilde{C}} = O_{\tilde{C}}$. This implies that $F \otimes O_{\tilde{C}} = 0$.

The morphism $\tilde{C} \rightarrow C$ is finite and surjective, therefore $F = 0$ also. Hence $E' = O_C$.

Going now back to the proof of Prop. 2.2., we have shown that on C we have a surjective map

$$f: \check{E}|_C \longrightarrow O_C \longrightarrow 0 .$$

We actually have the following diagram on C



$e = f \circ g$ but e splits, i.e. there exists a map j such that $e \circ j = \text{id}_{O_C}$. Now if we let $i = g \circ j$ we get

$$f \circ i = f \circ g \circ j = e \circ j = \text{id}_{O_C} .$$

Therefore the map f splits and $\check{E}|_C = E_0 \oplus O_C$, where E_0 is some vector bundle on C of rank r . Thus by Lemma 2.7. C lies in some Z_2 . q.e.d.

Proof of Proposition 2.3. Suppose that there is a curve C in Y which lies in some Z_1 . Then by Lemma 2.6. there exists a splitting of $Q|_C$, $Q|_C = Q_0 \oplus O_C$. Thus $Q|_C$ is not ample. Hence $Q|_Y$ is not ample.

Now suppose that $Q|_Y$ is not ample. Then by Theorem 1.5. there exists a curve C in Y and a quotient bundle Q' of $Q|_C$ on C

$$Q|_C \longrightarrow Q' \longrightarrow 0$$

such that $\text{deg } Q' \leq 0$. $Q|_C$ is generated by its global sections and hence Q' is also, so we can go through the same steps as in the proof of Proposition 2.2. and show that $Q|_C = Q_0 \oplus O_C$, where Q_0 is some vector bundle on C . Thus by Lemma 2.6. C lies in some Z_1 .

With the two propositions we just proved and with the fact that $T_G = Q \otimes \check{E}$ (see Appendix), where T_G is the tangent bundle on $G = G(r, n)$, we can prove immediately the next proposition.

Proof of Proposition 2.4. Assume that $T_G|_Y$ is not ample. Hence

by Theorems 1.4. and 1.5. there exists a curve C in Y such that $T_G|_C = Q|_C \otimes \check{E}|_C$ is not ample. Since both $Q|_C$ and $\check{E}|_C$ are generated by global sections and their tensor product $Q|_C \otimes \check{E}|_C$ is not ample, we get from (Hartshorne [5], 2) that neither $Q|_C$ nor $\check{E}|_C$ is ample. Hence by Propositions 2.2. and 2.3. C lies in some $Z_3 = Z_1 \cap Z_2$.

Now assume that C lies in some $Z_3 = Z_1 \cap Z_2$, where C is a curve in Y . By Lemmas 2.6. and 2.7. we have the following splittings

$$\begin{aligned} Q|_C &= Q_0 \oplus O_C & \text{and} \\ \check{E}|_C &= \check{E}_0 \oplus O_C . \end{aligned}$$

Thus

$$T_G|_C = Q|_C \otimes \check{E}|_C = (Q_0 \otimes \check{E}_0) \oplus Q_0 \oplus \check{E}_0 \oplus O_C$$

and hence $T_G|_C$ is not ample. Therefore $T_G|_Y$ is not ample. q.e.d.

Proof of Proposition 2.5. Assume that $N_{C/G}$ is not ample. Then from the short exact sequence

$$0 \longrightarrow T_C \longrightarrow T_G|_C \longrightarrow N_{C/G} \longrightarrow 0$$

we have that $T_G|_C$ is not ample.

Now assume that $T_G|_C$ is not ample. Hence by Proposition 2.4. C lies in some Z_3 . The cycle Z_3 is isomorphic to the Grassmannian $G(r-1, n-2)$. So let Q_Z and E_Z be its canonical quotient bundle and its canonical subbundle, respectively. Let T_Z be its tangent bundle. Hence $T_Z = Q_Z \otimes \check{E}_Z$. We have on Z_3 the short exact sequence

$$0 \longrightarrow T_Z \longrightarrow T_G|_Z \longrightarrow N_{Z/G} \longrightarrow 0$$

and furthermore

$$\begin{aligned} T_G|_Z &= Q|_Z \otimes \check{E}|_Z = (Q_Z \oplus O_Z) \otimes (\check{E}_Z \oplus O_Z) \\ &= T_Z \oplus Q_Z \oplus \check{E}_Z \oplus O_Z . \end{aligned}$$

Therefore

$$\begin{aligned} N_{Z/G} &= Q_Z \oplus \check{E}_Z \oplus O_Z & \text{and} \\ N_{Z/G}|_C &= (Q_Z \oplus \check{E}_Z)|_C \oplus O_C . \end{aligned}$$

Hence $N_{Z/G}|_C$ is not ample. Finally from the short exact sequence

$$0 \longrightarrow N_{C/Z} \longrightarrow N_{C/G} \longrightarrow N_{Z/G}|_C \longrightarrow 0$$

we get that $N_{C/G}$ is not ample.

q.e.d.

Remark. One can easily show that any Schubert cycle in $G(r, n)$ which is isomorphic to a lower Grassmannian does not have an ample normal bundle. In particular if Z is a cycle of type Z_1 , then its normal bundle is not ample and it is actually isomorphic to $\check{E}|_Z$.

For non-singular subvarieties of $G(r, n)$ of dimension higher than 1, we have the following sufficient conditions.

PROPOSITION 2.8. *Let Y be a non-singular subvariety of $G(r, n)$. Assume that Y does not contain any curve C which lies in a Z_3 . Then the normal bundle $N_{Y/G}$ is ample.*

Proof. $N_{Y/G}$ is generated by its global sections. Hence if $N_{Y/G}$ is not ample, then there is a curve C in Y such that $N_{Y/G}|_C$ is not ample. Thus from the short exact sequence

$$0 \longrightarrow T_Y|_C \longrightarrow T_G|_C \longrightarrow N_{Y/G}|_C \longrightarrow 0$$

we get that $T_G|_C$ is not ample. Therefore by Proposition 2.4. C lies in some Z_3 , and this contradicts our hypothesis.

q.e.d.

For non-singular subvarieties of codimension 1 in $G(r, n)$ we can show that their normal bundle is always ample.

PROPOSITION 2.9. *Let Y be a non-singular subvariety of $G(r, n)$ with $\text{cod}_G Y = 1$. Then $N_{Y/G}$ is ample.*

Proof. A subvariety Y of codimension 1 can be seen as an effective divisor on $G = G(r, n)$. Therefore it corresponds to a line bundle L . Since $\text{Pic}(G) \cong Z$, $L = O_G(v)$. Y is an effective divisor. This implies that $v > 0$. We have the short exact sequence

$$0 \longrightarrow O_G(-v) \longrightarrow O_G \longrightarrow O_Y \longrightarrow 0,$$

where $O_G(-v)$ is the sheaf of ideals defining Y in G . Therefore

$$\begin{aligned} \check{N}_{Y/G} &= O_G(-v) \otimes O_Y = O_Y(-v) & \text{and} \\ N_{Y/G} &= O_Y(v) & \text{where } v > 0. \end{aligned}$$

Hence $N_{Y/G}$ is ample.

q.e.d.

§3. Applications and examples

Let $G = G(1, 3)$, the Grassmannian parametrizing P^1 spaces in a fixed P^3 . Then $\dim G = 4$ and G is a quadric in P^5 .

The Schubert cycles of G can be described as follows.

- a) $Z_0 = G \cap H$, H a hyperplane in P^5 , $\dim Z_0 = 3$,
 $Z_0 = \{P^1 \subset P^3 \mid P^1 \cap \text{some fixed } P^1 \neq \emptyset\}$.
- b) $Z_1 = \{P^1 \subset P^3 \mid P^1 \subset \text{some fixed } P^2 \subset P^3\} \cong P^2$.
 $Z_2 = \{P^1 \subset P^3 \mid \text{some fixed point } p \in P^1 \subset P^3\} \cong P^2$.
- c) $Z_3 = \{P^1 \subset P^3 \mid \text{some fixed point } p \in P^1 \subset \text{some fixed } P^2 \subset P^3\} \cong P^1$.
- d) $p = \text{any point}$.

So according to our Theorem 2.1., the only curves in $G(1, 3)$ that do not have an ample normal bundle are the Schubert cycles of the type $Z_3 \cong P^1$. Using Grothendieck's theorem on the decomposition of vector bundles on P^1 , one can actually calculate the normal bundle of a curve C in $G(1, 3)$ of the type Z_3 and show that $N_{C/G} = O_C(1) \oplus O_C(1) \oplus O_C$.

By §2, the non-singular subvarieties of G of dimension 3 always have an ample normal bundle, the non-singular surfaces in G which do not contain a cycle of type Z_3 have an ample normal bundle, while the surfaces which are cycles of type Z_1 or Z_2 do not. The case left unanswered is the case of a non-singular surface in G which is not a Schubert cycle but which contains a curve of type Z_3 . With the following example we show that such a surface in G could have an ample normal bundle. Hence we show that the conditions in Proposition 2.8. for a non-singular subvariety Y in a $G(r, n)$ to have an ample normal bundle are sufficient but not necessary conditions.

EXAMPLE 3.1. Let $Y = G(1, 3) \cap \text{some general linear } P^3 \subset P^5$. Y is a proper intersection of $G(1, 3)$ and P^3 in P^5 . Therefore

$$N_{Y/G} \cong N_{P^3/P^5}|_Y$$

which is ample. It is easy to see that Y contains a cycle of type Z_3 . $G(1, 3) \subset P^5$ is a quadric defined by the equation $x_0x_1 + x_2x_3 + x_4x_5 = 0$, $(x_0, x_1, x_2, x_3, x_4, x_5)$ are the homogeneous coordinates of P^5 . Let P^3 be given by $(x_0 = x_2 = 0)$. Let C be given by $(x_0 = x_2 = x_4 = x_5 = 0)$. Then C is a cycle of type Z_3 and lies in $G(1, 3) \cap P^3 = Y$.

Therefore Y is an example of a subvariety of a Grassmannian such that $T_{G|_Y}$ is not ample though $N_{Y/G}$ is ample. Hence Proposition

2.5. can not be extended to subvarieties of a $G(r, n)$ of dimension greater than 1.

Another application of our main theorem is the following corollary.

COROLLARY 3.2. *Let $G = G(r, n)$. Then T_G is ample if and only if G is a trivial Grassmannian, i.e. $G \cong \mathbf{P}^n$.*

Proof. Assume $G = G(r, n)$ is a non-trivial Grassmannian, i.e. $r \neq 0$, $r \neq n - 1$. Therefore if $Z \subset G(r, n)$ is a cycle of type Z_3 , then $\dim Z \geq 1$. Hence it contains at least a curve. Hence $T_{G|Z}$ is not ample, which implies that T_G is not ample.

Now assume that $G = \mathbf{P}^n$. On \mathbf{P}^n we have the following short exact sequence

$$0 \longrightarrow E \longrightarrow O_{\mathbf{P}^n}^{n+1} \longrightarrow O_{\mathbf{P}^n}(1) \longrightarrow 0 .$$

where $\text{rank } E = n$ and $O_{\mathbf{P}^n}(1) = Q$. Tensoring the short exact sequence by \check{Q} and then dualizing we get

$$0 \longrightarrow O_{\mathbf{P}^n} \longrightarrow O_{\mathbf{P}^n}^{n+1}(1) \longrightarrow \check{E} \otimes Q \longrightarrow 0 .$$

Hence $T_{\mathbf{P}^n} = \check{E} \otimes Q$ is ample.

To subvarieties of $G(r, n)$ with ample normal bundle we can apply the following well-known results.

THEOREM 3.3 (Hartshorne [6], Theorem 4.1.). *Let X be a non-singular projective variety over a field k , $\text{char } k = 0$, and let Y be a non-singular subvariety of X of dimension d . Let \hat{X} be the formal completion of X along Y . Assume that $N_{Y/X}$ is ample. Then $H^i(\hat{X}, \hat{F})$ is a finite-dimensional k -vector space for all locally free sheaves F on X and all $i < d$.*

DEFINITION 3.4. Let S be a noetherian scheme of finite Krull dimension. We define the cohomological dimension of S , written $cd(S)$, to be the smallest integer $n \geq 0$ such that $H^i(S, F) = 0$ for all $i > n$ and for all coherent sheaves F on S .

DEFINITION 3.5. Let S be a scheme of finite type over k . We define the integer $q(S)$ to be the smallest integer $n \geq -1$ such that $H^i(S, F)$ is a finite-dimensional k -vector space for all $i > n$, and for all coherent sheaves F on S .

DEFINITION 3.6. Let S be as above. We define the integer $p(S)$ to

be the largest integer n (or ∞) such that $H^i(S, F)$ is a finite-dimensional k -vector space for all $i < n$, and for all locally free sheaves F on S .

THEOREM 3.7 (Hartshorne [7], Corollary 5.5.). *Let X, Y and d be as in Theorem 3.3. Then*

- a) $p(U) = q(U) = n - d - 1$, where $U = X - Y$, $n = \dim X$,
- b) *If $O_X(1)$ is an ample line bundle on X , then $H^i(U, F(m)) = 0$ for all $i \geq n - d$, $m \geq 0$, and for all coherent sheaves F on $U = X - Y$.*

THEOREM 3.8 (Hartshorne [7], Corollary 5.4.). *Let X and Y be as above and assume further that X is connected and that $d \geq 1$. Then*

$$H^0(\hat{X}, O_{\hat{X}}) = k ,$$

i.e., there are no non-constant holomorphic functions on X .

Also from (Hartshorne [7], Corollary 6.8.) we get that any subvariety of a Grassmannian with ample normal bundle is $G(2, n)$, i.e. the field of meromorphic functions $K(\hat{G})$ on the formal completion of G along the subvariety Y is a finite algebraic extension of $K(G)$.

Appendix

In this paper we have been using the following well-known fact.

THEOREM A.1. *Let $G = G(r, n)$, E the canonical subbundle on G and Q the canonical quotient bundle on G . Then the sheaf of 1-differentials on G can be expressed as follows,*

$$\Omega_G^1 = E \otimes \check{Q} .$$

We will outline here a proof of this fact, since we were not able to find a reference for it.

We will recall first some basic properties of the associated projective r -space bundle over a projective variety X and state the theorems that are needed for the proof.

DEFINITION A.2. Let X be a projective variety, E a vector bundle on X of rank r . We let $P(E)$ be the associated projective $(r - 1)$ -space bundle: the fibre over a point $x \in X$ is a projective space of hyperplanes of the vector space E_x .

Let $p: P(E) \rightarrow X$ be the projection map. Then the bundle E lifts to a bundle $p^*(E)$ on $P(E)$. There is a canonical quotient line bundle of

$p^*(E)$, L_E , which to each point $y \in \mathbf{P}(E)$ it associates the 1-dimensional vector space $E/E_{p(y)}$, where $E_{p(y)}$ is the hyperplane corresponding to the point y . Then $p_*L_E = E$ and $p_*\check{L}_E = 0$.

THEOREM A.3 ([3], SGA 1, II, Thm. 4.3.). *Let X and Y be non-singular projective varieties over a field k . Let $p: X \rightarrow Y$ be a smooth morphism. Then there is a short exact sequence of differentials*

$$0 \longrightarrow p^*\Omega_{Y/k}^1 \longrightarrow \Omega_{X/k}^1 \longrightarrow \Omega_{X/Y}^1 \longrightarrow 0,$$

where $\Omega_{X/Y}^1$ is the sheaf of relative differentials.

THEOREM A.4 (Manin [11], Prop. 17.12). *Let $X = \mathbf{P}(E)$ be the associated projective $(r-1)$ -space bundle over Y , where E is a vector bundle of rank r on Y . Let $p: \mathbf{P}(E) \rightarrow Y$ be the projection map. Then there is a short exact sequence*

$$0 \longrightarrow \Omega_{X/Y}^1 \longrightarrow p^*(E) \otimes \check{L}_E \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where L_E is the canonical quotient line bundle of $p^*(E)$.

THEOREM A.5 (Manin [11], Prop. 7.7.). *Let X and Y be as above. Then*

a) $R^i p_*(p^*(H) \otimes F) = H \otimes R^i p_* F$ for all $i \geq 0$, for all vector bundles H on Y and for all vector bundles F on X .

b) $R^0 p_*(\mathcal{O}_X) = \mathcal{O}_Y$ and

$R^i p_*(\mathcal{O}_X(n)) = 0$ for all $i > 0$ and for all $n \geq 0$.

Proof of Theorem A.1. We want to show that

$$\Omega_G^1 = E \otimes \check{Q},$$

where $G = G(r, n)$ is any Grassmannian and E, Q are the canonical subbundle and quotient bundle on G , respectively. We have seen that the theorem is true for $r = n - 1$. So we are going to use descending induction on r and the incidence correspondence. Let $G_{r, r-1}$ be the subspace of

$$G(r, n) \times G(r-1, n)$$

parametrizing all chains $(P^{r-1} \subset P^r)$ of subspaces of a fixed P^n . $G_{r, r-1}$ is called the incidence correspondence.

Let p, q be the restrictions to $G_{r, r-1}$ of the projections of $G(r, n) \times G(r-1, n)$ onto the first and second factor, respectively. Let $G = G_{r, r-1}$,

$G' = G(r, n)$ and $G'' = G(r - 1, n)$. So we have the following diagram

$$\begin{array}{ccc} & G = G_{r,r-1} & \\ p \swarrow & & \searrow q \\ G' = G(r, n) & & G'' = G(r - 1, n) \end{array}$$

We also have the following canonical short exact sequences

$$0 \longrightarrow E' \longrightarrow O_{G'}^{n+1} \longrightarrow Q' \longrightarrow 0 \quad \text{on } G',$$

and

$$0 \longrightarrow E'' \longrightarrow O_{G''}^{n+1} \longrightarrow Q'' \longrightarrow 0 \quad \text{on } G''.$$

- CLAIM. a) $G = P(E')$,
b) $G = P(\check{Q}'')$.

Proof.

$$\begin{array}{c} \text{a) } G = G_{r,r-1} \\ \downarrow p \\ G' = G(r, n) \end{array}$$

Each point $y \in G$ corresponds to a chain $(P^{r-1} \subset P^r)$, so it corresponds to a hyperplane of the vector space $E'_{p(y)}$. Thus $G = P(E')$ over G . Furthermore, on $P(E')$ we have the canonical quotient line bundle, $L_{E'}$. The kernel of the map

$$p^*(E') \longrightarrow L_{E'} \longrightarrow 0$$

corresponds to hyperplanes and each hyperplane corresponds to a P^{r-1} . Therefore, the kernel is exactly $q^*(E'')$. So on $P(E')$ we have the short exact sequence

$$0 \longrightarrow q^*(E'') \longrightarrow p^*(E') \longrightarrow L_{E'} \longrightarrow 0.$$

$$\begin{array}{c} \text{b) } G = G_{r,r-1} \\ \downarrow q \\ G'' = G(r - 1, n) \end{array}$$

Each point $y \in G$ corresponds to a chain $(P^{r-1} \subset P^r)$, hence to a line in the vector space $Q''_{q(y)}$. So each point $y \in G$ corresponds to a hyperplane of the vector space $\check{Q}''_{q(y)}$. Hence $G = P(\check{Q}'')$. Furthermore,

on $P(\check{Q}'')$ we have the canonical line bundle, $M_{Q''}$. The kernel of the map

$$q^*(\check{Q}'') \longrightarrow M_{Q''} \longrightarrow 0$$

associates to each point $y \in G$ a hyperplane of the vector space $\check{Q}'_{q(y)}$. Hence the kernel is $p^*(\check{Q}')$. So on $P(\check{Q}'')$ we have the short exact sequence

$$0 \longrightarrow p^*(\check{Q}') \longrightarrow q^*(\check{Q}'') \longrightarrow M_{Q''} \longrightarrow 0 .$$

From the following diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ & & L_{E'} & & & & 0 \\ & & \uparrow & & & & \uparrow \\ 0 & \longrightarrow & p^*(E') & \longrightarrow & p^*(O_{G'}^{n+1}) & \longrightarrow & p^*(Q') \longrightarrow 0 \\ & & \uparrow & & \wr & & \uparrow \\ 0 & \longrightarrow & q^*(E'') & \longrightarrow & q^*(O_{G''}^{n+1}) & \longrightarrow & q^*(Q'') \longrightarrow 0 \\ & & \uparrow & & & & \uparrow \\ & & 0 & & & & L_{E'} \\ & & & & & & \uparrow \\ & & & & & & 0 \end{array}$$

we can see that $M_{Q''} = \check{L}_{E'}$.

So we have shown that on G we have the following short exact sequences and relations

$$1) \quad 0 \longrightarrow q^*(E'') \longrightarrow p^*(E') \longrightarrow L_{E'} \longrightarrow 0$$

$$2) \quad 0 \longrightarrow p^*(\check{Q}') \longrightarrow q^*(\check{Q}'') \longrightarrow M_{Q''} \longrightarrow 0$$

and

$$\begin{aligned} p_*L_{E'} &= p_*\check{M}_{Q''} = E' \\ p_*\check{L}_{E'} &= p_*M_{Q''} = 0 \\ q_*M_{Q''} &= q_*\check{L}_{E'} = \check{Q}'' \\ q_*\check{M}_{Q''} &= q_*L_{E'} = 0 . \end{aligned}$$

Applying Theorem A.3. on G'' we get

$$3) \quad 0 \longrightarrow q^*\Omega_{G''}^1 \longrightarrow \Omega_G^1 \longrightarrow \Omega_{G/G''}^1 \longrightarrow 0 .$$

Applying Theorem A.4.. on G'' we get

$$0 \longrightarrow \Omega_{G/G''}^1 \longrightarrow q^*(\check{Q}'') \otimes \check{M}_{Q''} \longrightarrow O_G \longrightarrow 0 .$$

From the short exact sequence 2) we get

$$4) \quad 0 \longrightarrow p^*(\check{Q}') \otimes \check{M}_{Q''} \longrightarrow q^*(\check{Q}'') \otimes \check{M}_{Q''} \longrightarrow O_G \longrightarrow 0 .$$

Therefore $\Omega_{G/G''}^1 = p^*(\check{Q}') \otimes \check{M}_{Q''}$ and 3) becomes

$$5) \quad 0 \longrightarrow q^*\Omega_{G''}^1 \longrightarrow \Omega_G^1 \longrightarrow p^*(\check{Q}') \otimes \check{M}_{Q''} \longrightarrow 0 .$$

Applying q_* to 5) we get

$$0 \longrightarrow \Omega_{G''}^1 \longrightarrow q_*\Omega_G^1 \longrightarrow q_*(p^*(\check{Q}') \otimes \check{M}_{Q''}) \longrightarrow \dots$$

The third term though is zero for the following reason

$$\begin{aligned} 0 \longrightarrow q_*(p^*(\check{Q}') \otimes \check{M}_{Q''}) &\longrightarrow \check{Q}'' \otimes q_*(\check{M}_{Q''}) \longrightarrow \dots \\ &\parallel \\ &0 \end{aligned}$$

Here we applied q_* to the short exact sequence 4) and used the projection formula. Therefore

$$6) \quad \Omega_{G''}^1 \cong q_*\Omega_G^1 .$$

We will use now our induction hypothesis. We assume that for $G' = G(r, n)$ $\Omega_{G'}^1 = E' \otimes \check{Q}'$ and we will show that for $G'' = G(r-1, n)$ $\Omega_{G''}^1 = E'' \otimes \check{Q}''$. By Theorem A.3. we have on G'

$$0 \longrightarrow p^*\Omega_{G'}^1 \longrightarrow \Omega_G^1 \longrightarrow \Omega_{G/G'}^1 \longrightarrow 0 ,$$

and by Theorem A.4. we have on G'

$$0 \longrightarrow \Omega_{G/G'}^1 \longrightarrow p^*(E') \otimes \check{L}_{E'} \longrightarrow O_G \longrightarrow 0 .$$

From the short exact sequence 1) we get

$$0 \longrightarrow q^*(E'') \otimes \check{L}_{E'} \longrightarrow p^*(E') \otimes \check{L}_{E'} \longrightarrow O_G \longrightarrow 0 .$$

Therefore

$$\Omega_{G/G'}^1 = q^*(E'') \otimes \check{L}_{E'} .$$

and on G' we have

$$7) \quad 0 \longrightarrow p^*\Omega_{G'}^1 \longrightarrow \Omega_G^1 \longrightarrow q^*(E'') \otimes \check{L}_{E'} \longrightarrow 0 .$$

Applying q_* to 7) we get the following long exact sequence

$$\begin{aligned} 8) \quad 0 \longrightarrow q_*p^*\Omega_{G'}^1 &\longrightarrow q_*\Omega_G^1 \xrightarrow{\tau} E'' \otimes \check{Q}'' \longrightarrow \\ &\longrightarrow R^1q_*p^*\Omega_{G'}^1 \xrightarrow{\chi} R^1q_*\Omega_G^1 \longrightarrow E'' \otimes R^1q_*\check{L}_{E'} \end{aligned}$$

Using the two short exact sequences 1) and 2) one can easily show that $q_* p^* \Omega_G^1 = 0$. Hence the map τ in the long exact sequence 8) is injective. Furthermore, using Theorem A.5. one can show that the map χ in 8) is also injective. Thus τ is an isomorphism and this completes the proof by 6). q.e.d.

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University of Pennsylvania

