Y. Kitaoka
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CLASS NUMBERS OF QUADRATIC FORMS OVER REAL QUADRATIC FIELDS

YOSHIYUKI KITAOKA

Let k be an algebraic number field, let K be a Galois extension of k of finite degree, and let O_K, O_k be the maximal orders of K, k, respectively. We consider the conjugate operation: for a given quadratic lattice M over O_K equipped with a bilinear form B and for an automorphism $\sigma \in G(K/k)$, we define a new quadratic lattice M^σ over O_K. Here M^σ has the same underlying abelian group as M, but a new O_K-action $a * v = \sigma(a)v$ $(a \in O_K, v \in M)$; the new bilinear form B^σ on M^σ is defined by $B^\sigma(u, v) = \sigma^{-1}(B(u, v))$ $(u, v \in M)$. Then the O_K-linearity of B^σ is checked as follows:

$$\begin{aligned} B^\sigma(a * u, v) &= \sigma^{-1}(B(\sigma(a)u, v)) \\ &= aB^\sigma(u, v) \qquad (a \in O_K, u, v \in M) . \end{aligned}$$

If M has an O_K-basis, i.e., $M = O_K[v_1, v_2, \cdots, v_n]$, then M^σ is a quadratic lattice corresponding to the matrix $(\sigma^{-1}(B(v_i, v_j)))$. In this paper we say that a quadratic lattice M is symmetric if M^σ is isometric to M for any σ in $G(K/k)$. There are some tools to know class numbers of positive definite quadratic forms over the ring $\boldsymbol{Z}$ of rational integers, and they are effective in principle in case of definite quadratic lattices over the maximal order of an algebraic number field. But they do not seem to be useful to know the class numbers of symmetric quadratic lattices apart from the cases of small class numbers. By using the theory of quaternions we prove

THEOREM. *Let K be a real quadratic field $\boldsymbol{Q}(\sqrt{q})$ where q is a rational prime* $\equiv 1 \pmod 4$, *and let V be a quaternary quadratic space over K with bilinear form B and quadratic form Q $(Q(x) = B(x, x))$ which satisfies*

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(i) *the discriminant* dV *of* V *is a square, that is,* $\det(B(v_i, v_j))$ *is a square, where* $\{v_i\}$ *is a basis of* V,

(ii) V *is positive definite at each infinite prime of* K,

(iii) *there is a lattice* M *over the maximal order* O_K *of* K *in* V *such that* M *is a unimodular quadratic lattice at each finite prime of* K, *and* $Q(x) \equiv 0 \pmod 2$ *for any* $x \in M$.

Furthermore, let G *denote the genus of the quadratic lattice* M. *Then, for* $q \neq 5$, *the class number of isometry classes of quadratic lattices in* G *is*

$$\tfrac{1}{2}H(H+1), \text{ where } H \text{ is } B_{2,\chi}/2^4\cdot 3 + h(\sqrt{-q})/8 + h(\sqrt{-3q})/6 ,$$

and the class number of isometry classes of symmetric)* *quadratic lattices in* G *is*

$$\frac{1}{2}\left\{\left(q+3-4\left(\frac{3}{q}\right)\right)\Big/24 + \left(1-\left(\frac{2}{q}\right)\right)\Big/4\right\}^2 + B_{2,\chi}/2^5\cdot 3 + h(\sqrt{-q})/16 + h(\sqrt{-3q})/12 ,$$

where $B_{2,\chi}$ *is a generalized Bernoulli number with* $\chi(n) = \left(\frac{n}{q}\right)$, $\left(\frac{n}{q}\right)$ *stands for the quadratic residue symbol, and* $h(\sqrt{-m})$ *is the class number of an imaginary quadratic field* $\mathbf{Q}(\sqrt{-m})$. *If* $q = 5$, *then both class numbers are one.*

Remark 1. Theorem in case of $q = 5$ is proved by Maass [3].

Remark 2. Every quadratic lattice in the genus G in Theorem has an O_K-basis (appendix). Hence G can be regarded as a set of matrices A in $SL(4, O_K)$ such that diagonal entries are divisible by 2, and $A, \sigma A$ are positive definite, where σ is a non-trivial automorphism of K.

Remark 3. Since a conjugate quadratic lattice L^σ of L is not unique up to rotations, it seems to be difficult to consider our problems within the category of rotations in general. However there are some exceptional cases which can be treated as follows:

Let K/k be a Galois extension and V_0 be a quadratic space over k, and put $V = K \otimes_k V_0$. For $\sigma \in G(K/k)$ and an element $v = \sum a_i v_i$, where $a_i \in K$ and $\{v_i\}$ is a basis of V_0 over k, we define $\sigma(v)$ by $\sum \sigma(a_i) v_i$.

*) We consider the rational number field $\mathbf{Q}$ as k in the introduction.

Then, for a given quadratic lattice L in V and $\sigma \in G(K/k)$, we have $L^\sigma \cong \sigma^{-1}(L)$. Suppose that two lattices L, M are isometric by a rotation φ, $\varphi(L) = M$. Put $\varphi_\sigma = \sigma^{-1}\varphi\sigma$; then $\varphi_\sigma(\sigma^{-1}L) = \sigma^{-1}M$. Next we show that φ_σ is a rotation. Let $\varphi(v_1, \cdots, v_n) = (v_1, \cdots, v_n)T$. Then $\varphi_\sigma(v_1, \cdots, v_n) = (v_1, \cdots, v_n)\sigma^{-1}(T)$ implies $\det \varphi_\sigma = \det \sigma^{-1}(T) = \sigma^{-1} \det T = \sigma^{-1} \det \varphi = 1$, and $(B(\varphi_\sigma(v_i), \varphi_\sigma(v_j))) = \sigma^{-1}({}^tT)(B(v_i, v_j))\sigma^{-1}(T) = \sigma^{-1}({}^tT(B(v_i, v_j))T) = \sigma^{-1}(B(v_i, v_j)) = (B(v_i, v_j))$, where B is the bilinear form associated with V_0. Thus φ_σ is a rotation. Hence by taking $\sigma^{-1}(L)$ as a realization in V of L^σ we can consider our problems in the category of rotations.

§1. In this section we summarize our necessities without proofs from the theory of Tamagawa which was lectured in the Summer Institute at Tokyo in 1970 (for details and more see [6]).

Let q be a prime $\equiv 1 \pmod 4$, and D_0 be a quaternion algebra over the rational number field $\boldsymbol{Q}$ which is ramified at and only at q and ∞, and K be $\boldsymbol{Q}(\sqrt{q})$. We denote the maximal order of K by O_K. D denotes $K \otimes_{\boldsymbol{Q}} D_0$; then D is a quaternion algebra over K which is ramified at two infinite primes only. Moreover we denote the non-trivial automorphism of K by the bar, $x \to \bar{x}$, and the main involution of D_0 by the star, $x \to x^*$. These two linear mappings are cannonically extended to D and the idele group $D_A^\times$ of D, and we denote them by the bar and the star again. For an O_K-module $\tilde{M}$ in D we denote the $\mathfrak{p}$-adic closure of $\tilde{M}$ in $D_\mathfrak{p} = D \otimes_K K_\mathfrak{p}$ by $\tilde{M}_\mathfrak{p}$. The two linear mappings are locally as follows:

Let $\mathfrak{p}$ be a prime of K ($\mathfrak{p} \nmid \infty$); then $D_\mathfrak{p}$ is isomorphic to $M_2(K_\mathfrak{p})$ and the main involution $*$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad (a, b, c, d \in K_\mathfrak{p}) .$$

Let p be a rational prime.

1) In case that p splits in K, $(p) = \mathfrak{p}_1\mathfrak{p}_2$, $\mathfrak{p}_1 \neq \mathfrak{p}_2$, we have $D_{\mathfrak{p}_1} \oplus D_{\mathfrak{p}_2} = M_2(\boldsymbol{Q}_p) \oplus M_2(\boldsymbol{Q}_p)$ and the non-trivial automorphism of K operates as the permutation on it.

If p does not split in K and $\mathfrak{p} | p$, then $K_\mathfrak{p}$ is a quadratic extension of $\boldsymbol{Q}_p$ and the non-trivial automorphism of K induces one of $K_\mathfrak{p}$, and it operates on $D_\mathfrak{p} \cong M_2(K_\mathfrak{p})$:

2) $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ if $p \neq q$,

3) $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \bar{d} & e^{-1}\bar{c} \\ e\bar{b} & \bar{a} \end{pmatrix}$ if $p = q$,

where $a, b, c, d \in K_{\mathfrak{p}}$, and e is a unit of $\boldsymbol{Q}_p$ which is not the norm of an element of $K_{\mathfrak{p}}$.

Now it is obvious that there is a maximal order $\mathfrak{O}$ in D such that $\bar{\mathfrak{O}} = \mathfrak{O}^* = \mathfrak{O}$ and $\mathfrak{O}_{\mathfrak{p}} = M_2(O_{K_{\mathfrak{p}}})$ by the correspondence $D_{\mathfrak{p}} \cong M_2(K_{\mathfrak{p}})$, where $O_{K_{\mathfrak{p}}}$ is the maximal order of $K_{\mathfrak{p}}$. We fix it hereafter.

Put $H = \{x \in D\,;\, \bar{x} = x^*\}$ and $H_0 = H \cap \mathfrak{O}$, and we consider the quaternion algebra D as a quadratic space with $Q(x) = 2n(x) = 2xx^*$ over K; then for $x \in H_0$ $Q(x)$ is a rational number since $Q(x) = 2xx^* = 2x\bar{x}$. Hence we can regard H_0 as a quaternary positive definite quadratic lattice over the ring $\boldsymbol{Z}$ of rational integers.

If p splits in K, $(p) = \mathfrak{p}_1\mathfrak{p}_2$ $(\mathfrak{p}_1 \neq \mathfrak{p}_2)$, then the closure of H_0 in $D_{\mathfrak{p}_1} \oplus D_{\mathfrak{p}_2} \cong M_2(\boldsymbol{Q}_p) \oplus M_2(\boldsymbol{Q}_p)$ is $\left\{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \oplus \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix};\ a_i, b_i, c_i, d_i \in \boldsymbol{Z}_p,\ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}^* = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right\}$. If $(p) = \mathfrak{p}$ is a prime in K, then the closure of H_0 in $D_{\mathfrak{p}}$ is $\left\{\begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix};\ a, b, c \in O_{K_{\mathfrak{p}}},\ \bar{b} = -b, \bar{c} = -c\right\}$.

If $p = q$, then the closure of H_0 in $D_{\mathfrak{p}}$ $(\mathfrak{p} \mid q)$ is $\left\{\begin{pmatrix} a & b \\ -e\bar{b} & d \end{pmatrix};\ b \in O_{K_{\mathfrak{p}}},\ a, d \in \boldsymbol{Z}_q\right\}$. Hence the norm of H_0 is $2\boldsymbol{Z}$ and the discriminant $|B(x_i, x_j)|$ is q, where $\{x_i\}$ is a $\boldsymbol{Z}$-basis of H_0 and $B(x_i, x_j) = \mathrm{tr}\,(x_i x_j^*) = x_i x_j^* + x_j x_i^*$.

Denote the idele group of K, $\boldsymbol{Q}$ by $K_A^\times$, $\boldsymbol{Q}_A^\times$ respectively and put $N = \{(A, c) \in (D_A^\times, \boldsymbol{Q}_A^\times)\,;\, cA\mathfrak{O}\bar{A}^* = \mathfrak{O}\}$, where $cA\mathfrak{O}\bar{A}^*$ means $D \bigcap_{\mathfrak{p}} c_{\mathfrak{p}} A_{\mathfrak{p}} \mathfrak{O}_{\mathfrak{p}} (\bar{A}^*)_{\mathfrak{p}}$ ($\boldsymbol{Q}_A^\times$ is embedded in $D_A^\times$). Then $D^\times \times \boldsymbol{Q}^\times \backslash D_A^\times \times \boldsymbol{Q}_A^\times / N$ is bijectively corresponding to the equivalence classes of ideals of form $cA\mathfrak{O}\bar{A}^* (c \in \boldsymbol{Q}_A^\times$, $A \in D_A^\times)$ where the equivalence relation is defined as follows: $\mathfrak{M}$, $\mathfrak{N}$ are equivalent if and only if $\mathfrak{M} = bB\mathfrak{N}\bar{B}^*$, where $b \in \boldsymbol{Q}^\times$, $B \in D^\times$. Since D is unramified for any finite prime of K and $\mathfrak{O}$ is a maximal order, $A_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}}B_{\mathfrak{p}} = \mathfrak{O}_{\mathfrak{p}}$ $(A_{\mathfrak{p}}, B_{\mathfrak{p}} \in D_{\mathfrak{p}}^\times)$ implies $A_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}} = a_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}}$, $\mathfrak{O}_{\mathfrak{p}}B_{\mathfrak{p}} = a_{\mathfrak{p}}^{-1}\mathfrak{O}_{\mathfrak{p}}$ for any finite prime $\mathfrak{p}$, where $a_{\mathfrak{p}}$ is in $K_{\mathfrak{p}}$. Hence we get $N = Z(U \times U_{\boldsymbol{Q}})$, where Z is $\{(a, N_{K_A^\times/\boldsymbol{Q}_A^\times} a^{-1})\,;\, a \in K_A^\times\}$, U is the group of unit ideles of $\mathfrak{O}$, and $U_{\boldsymbol{Q}}$ is the group of unit ideles of $\boldsymbol{Q}$. Hence the number of double cosets $D^\times \times \boldsymbol{Q}^\times \backslash D_A^\times \times \boldsymbol{Q}_A^\times / N$ is equal to $\sharp\{D^\times \backslash D_A^\times / K_A^\times U\}$. Let A, $\mathfrak{a}$ and α be an element of $D_A^\times$, an ideal of K and an element of $D^\times$ respectively. Then $\mathfrak{O}A\mathfrak{a} = \mathfrak{O}A\alpha$ implies $\mathfrak{a}^2 = (n(\alpha))$ and so $\mathfrak{a}$ is a principlal ideal since the class number of K is odd. Hence we have $\sharp\{D^\times \times \boldsymbol{Q}^\times \backslash D_A^\times \times \boldsymbol{Q}_A^\times / N\} =$

$h(D)/h(K)$, where $h(D)$, $h(K)$ are the class numbers of D and K respectively.

Let a be an element of $K_A^\times$; then a defines cannonically an ideal of K. We denote the ideal by $\mathrm{id}(a)$. From the oddness of the class number of K and the fact that the norm of a fundamental unit of K is -1 follows that there is an element $a \in K_A^\times$ for a given $A \in D_A^\times$ such that $\mathrm{id}(n(Aa)) = \mathrm{id}(a^2 n(A))$ is a principal ideal (x) with totally positive $x \in K$. Then there is an element α in D such that $n(\alpha) = x$. Thus $cA\mathfrak{O}\bar{A}^*(c \in \boldsymbol{Q}_A^\times)$ is equivalent to $A_1\mathfrak{O}\bar{A}_1^*$, where $A_1 = \alpha^{-1}aA$, and $\mathrm{id}(n(A_1))$ is the maximal order O_K of K. Now we consider $H \cap A_1\mathfrak{O}\bar{A}_1^* = A_1(H \cap \mathfrak{O})\bar{A}_1^*$ as a quadratic lattice over $\boldsymbol{Z}$ with quadratic form $Q(x) = 2n(x)$. Then $H \cap A_1\mathfrak{O}\bar{A}_1^*$ is in the genus of H_0. A main result of Tamagawa is as follows:

The above correspondence gives a bijection from the equivalence classes of $cA\mathfrak{O}\bar{A}^*(c \in \boldsymbol{Q}_A^\times, A \in D_A^\times)$ to the equivalence classes in the narrow sense, namely, by the group of rotations, of even quaternary positive definite quadratic lattices with discriminant q.

§**2.** Keeping all in §1, let L be an O_K-lattice of D; then L is by definition a normal ideal if and only if the right or left order of L is a maximal order; then $L = A\mathfrak{O}B = D \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}}B_{\mathfrak{p}}(A, B \in D_A^\times)$, where $\mathfrak{O}$ is the maximal order of D in §1 satisfying $\mathfrak{O} = \mathfrak{O}^* = \bar{\mathfrak{O}}$ and $\mathfrak{O} \cap H$ is an even quaternary positive definite quadratic lattice with discriminant q. Two normal ideals L, M are said to be equivalent, $L \sim M$, if there exist α, $\beta \in D^\times$ such that $M = \alpha L \beta$. This equivalence relation is different from one in §1. Let G be the genus of the maximal order $\mathfrak{O}$ with quadratic form $Q(x) = 2n(x)$, that is, G consists of quaternary positive definite unimodular quadratic lattices N over O_K such that $Q(x) \equiv 0 \pmod 2$ for each x in N and the quadratic space $K \otimes_{O_K} N$ is of discriminant 1, and so it is the same genus as G in Theorem. Regarding D as a quadratic space over K with quadratic form $Q(x) = 2n(x)$, the rotations of D are all the mappings of the form $x \mapsto \alpha x \beta$, where $\alpha, \beta \in D$ and $n(\alpha\beta) = 1$, and a non-rotational isometry is given by $x \mapsto x^*$.

LEMMA 1. *The class number of isometry classes, by the group of rotations, of quadratic lattices in G is equal to*

$$h(K)^{-1} \times \textit{the class number of normal ideals} = h(D)^2/h(K)^2 ,$$

where $h(D)$, $h(K)$ is the class number of D, K, respectively.

Proof. Let $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ be a normal ideal such that id $(n(AB))$ is a principal ideal (a); then we may assume a is totally positive and then there is an element $\alpha \in D$ such that $n(\alpha) = a$. Put $M = \alpha^{-1}L = \alpha^{-1}A\mathfrak{O}B$. Since id $(n(\alpha^{-1}AB))$ is the maximal order O_K of K, M is in the genus of $\mathfrak{O}$ as a quadratic lattice with quadratic form $2n(x)$, i.e., $M \in G$. This correspondence gives a bijection from the equivalence classes of normal ideals $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ such that id $(n(AB))$ is principal to the equivalence classes by rotations of quadratic lattices in G. It is obvious that the class number of normal ideals equals $h(K) \times$ the class number of normal ideals $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ such that id $(n(AB))$ is principal, since $h(K)$ is odd. For a normal ideal $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ we put $\varphi(L) = (\underline{A\mathfrak{O}A^{-1}}, \underline{B^{-1}\mathfrak{O}B})$, where the underline means the equivalence class of maximal orders, namely, $\underline{A\mathfrak{O}A^{-1}} = \{\alpha A\mathfrak{O}A^{-1}\alpha^{-1}; \alpha \in D^\times\}$. Then φ gives a bijection from the equivalence classes of normal ideals $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ such that id$(n(AB))$ is principal to the direct product of two copies of equivalence classes of maximal orders, noting $a^{-1}\mathfrak{O}a = \mathfrak{O}$ $(a \in K_A^\times)$. The number of equivalence classes of maximal orders is, by definition of equivalence, $\#\{\{A \in D_A^\times; A^{-1}\mathfrak{O}A = \mathfrak{O}\}\backslash D_A^\times/D^\times\}$ and it is $h(D)/h(K)$ since $\{A \in D_A^\times; A^{-1}\mathfrak{O}A = \mathfrak{O}\} = K_A^\times \times U$ as in §1. This completes the proof.

By the correspondence in the proof of Lemma 1 we regard a quadratic lattice L in G as a normal ideal $A\mathfrak{O}B$ $(A, B \in D_A^\times)$ such that id $(n(AB))$ is principal. Then for quadratic lattices L_1, L_2 in G corresponding normal ideals $A_1\mathfrak{O}B_1$, $A_2\mathfrak{O}B_2$, L_1, L_2 are rotationally isometric if and only if $A_1\mathfrak{O}B_1 \sim A_2\mathfrak{O}B_2$, and L_1 is isometric to L_2 if and only if $A_1\mathfrak{O}B_1 \sim A_2\mathfrak{O}B_2$ or $A_1\mathfrak{O}B_1 \sim (A_2\mathfrak{O}B_2)^*$.

Let L be a quadratic lattice; then L has an isometry which is not a rotation if and only if any quadratic lattice M which is isometric to L is always rotationally isometric to L. Hence denoting the number of isometry classes of quadratic lattices in G and the number of isometry classes by rotations of them by h and h^+ respectively, $2h - h^+$ equals the number of quadratic lattices in G which have a non-rotational isometry.

LEMMA 2. $2h - h^+ = h(D)/h(K)$.

Proof. The idea of the proof is essentially due to H. Hijikata. By the above remark it suffices to prove that the class number of normal ideals $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ such that $L \sim L^*$ and $\mathrm{id}\,(n(AB))$ is principal equals $h(D)/h(K)$. Put $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$. If $L^* = \alpha L \beta$ $(\alpha, \beta \in D)$, then $B^*\mathfrak{O}A^* = \alpha A\mathfrak{O}B\beta$. Hence $\varphi(L)$ (in the proof of Lemma 1) $= (\underline{A\mathfrak{O}A^{-1}}, \underline{B^{-1}\mathfrak{O}B}) = (\underline{\alpha^{-1}B^*\mathfrak{O}(\alpha^{-1}B^*)^{-1}}, \underline{B^{-1}\mathfrak{O}B}) = (\underline{B^{-1}\mathfrak{O}B}, \underline{B^{-1}\mathfrak{O}B})$. If, conversely, $B \in D_A^\times$ is given, then we put $A = n(B)^{-1}B^*$. Then $\varphi(A\mathfrak{O}B) = (\underline{B^{-1}\mathfrak{O}B}, \underline{B^{-1}\mathfrak{O}B})$ and $(A\mathfrak{O}B)^* = A\mathfrak{O}B$ and moreover $\mathrm{id}\,(n(AB))$ is principal. This completes the proof.

From Lemma 1 and 2 we have $h = \frac{1}{2}(h(D)^2/h(K)^2 + h(D)/h(K))$, and $h(D) = 1$ $q = 5$. If $q > 5$, then $h(D)/h(K)$ is $B_{2,\chi}/2^4\cdot 3 + h(\sqrt{-q})/8 + h(\sqrt{-3q})/6$, where $B_{2,\chi}$ is a generalized Bernoulli number with $\chi(n) = \left(\frac{n}{q}\right)$ (the quadratic residue symbol) and $h(\sqrt{-m})$ denotes the class number of an imaginary quadratic field $Q(\sqrt{-m})$ ([1], [5], and [2] combining with § 1). This completes the proof of the former part of Theorem. Hereafter we calculate the class number $\bar{h}$ of quadratic lattices L in G such that L is isometric to L^σ where σ is a non-trivial automorphism of K and L^σ is defined in the introduction. Here we introduce a new equivalence relation $\approx$ for normal ideals:

$$L \approx M \text{ if and only if } L \sim M \quad \text{or} \quad L \sim M^* .$$

Then $\bar{h}$ is the class number by the new equivalence $\approx$ of normal ideals $L = A\mathfrak{O}B$ such that $L \approx \bar{L}$ and $\mathrm{id}\,(n(AB))$ is principal. Let $L = A\mathfrak{O}B$ stand for normal ideals such that $\mathrm{id}\,(n(AB))$ is principal; then we have

$$\begin{aligned}
2\#\{\{L; L \sim \bar{L}\}/\approx\} &= \#\{\{L; L \sim \bar{L}\}/\sim\} + \#\{\{L; L \sim \bar{L} \sim L^*\}/\sim\} , \\
\#\{\{L: L \not\sim \bar{L}, L \sim \bar{L}^*\}/\approx\} &= \#\{\{L; L \sim \bar{L}^*\}/\approx\} - \#\{\{L; L \sim \bar{L}^* \sim \bar{L}\}/\approx\} \\
&= \#\{\{L; L \sim \bar{L}^*\}/\approx\} - \#\{\{L; L \sim \bar{L}^* \sim \bar{L}/\sim\} , \\
2\#\{\{L; L \sim \bar{L}^*\}/\approx\} &= \#\{\{L; L \sim \bar{L}^*\}/\sim\} + \#\{\{L; L \sim \bar{L}^* \sim L^*\}/\sim\} .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\bar{h} &= \#\{\{L; L \sim \bar{L}\}/\approx\} + \#\{\{L; L \not\sim \bar{L}, L \sim \bar{L}^*\}/\approx\} \\
&= \tfrac{1}{2}\#\{\{L; L \sim \bar{L}\}/\sim\} + \tfrac{1}{2}\#\{\{L; L \sim \bar{L}^*\}/\sim\} .
\end{aligned}$$

Denote $\#\{\{L; L \sim \bar{L}\}/\sim\}$, $\#\{\{L; L \sim \bar{L}^*\}/\sim\}$ by h_1, h_2 respectively.

LEMMA 3. *h_1 is the square of the number h_0 of equivalence classes of maximal orders $A\mathfrak{O}A^{-1}$ $(A \in D_A^\times)$ such that* $\underline{\bar{A}\mathfrak{O}\bar{A}^{-1}} = \underline{A\mathfrak{O}A^{-1}}$.

Proof. Let $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ be a normal ideal such that $\bar{L} = \alpha L\beta$, $\alpha, \beta \in D$; then $\bar{A}\mathfrak{O}\bar{B} = \alpha A\mathfrak{O}B\beta$ implies $\varphi(L) = (\underline{A\mathfrak{O}A^{-1}}, \underline{B^{-1}\mathfrak{O}B}) = (\underline{\bar{A}\mathfrak{O}\bar{A}^{-1}}, \underline{\bar{B}^{-1}\mathfrak{O}\bar{B}})$. If, conversely, $\underline{\bar{A}\mathfrak{O}\bar{A}^{-1}} = \underline{A\mathfrak{O}A^{-1}}$, $\underline{\bar{B}^{-1}\mathfrak{O}\bar{B}} = \underline{B^{-1}\mathfrak{O}B}$ hold for $A, B \in D_A^\times$, then taking $A_1, B_1 \in D_A^\times$ such that $\mathrm{id}\,(n(A_1)) = \mathrm{id}\,(n(B_1)) = O_K$ and $\underline{A_1\mathfrak{O}\bar{A}_1^{-1}} = \underline{A\mathfrak{O}A^{-1}}$, $\underline{\bar{B}_1^{-1}\mathfrak{O}B_1} = \underline{B^{-1}\mathfrak{O}B}$, we have $\varphi(L) = (\underline{A\mathfrak{O}A^{-1}}, \underline{B^{-1}\mathfrak{O}B})$ for $L = A_1\mathfrak{O}B_1$, and $\varphi(L) = \varphi(\bar{L})$. Hence we get $L \sim \bar{L}$. This completes the proof.

LEMMA 4. $h_0 = \left(q + 3 - 4\left(\frac{3}{q}\right)\right)/24 + \left(1 - \left(\frac{2}{q}\right)\right)/4$, *where* $\left(\frac{n}{q}\right)$ *stands for the quadratic residue symbol.*

Proof. Let $A\mathfrak{O}A^{-1}$ $(A \in D_A^\times)$ be a maximal order. Then the equivalence class of $A\mathfrak{O}\bar{A}^*$ where the equivalence relation is one defined in § 1 is uniquely determined by $\underline{A\mathfrak{O}A^{-1}}$. The correspondence is bijective from equivalence classes of maximal orders to the equivalence classes of ideals of form $cA\mathfrak{O}\bar{A}^*$ $(c \in \boldsymbol{Q}_A^\times, A \in D_A^\times)$. Let $A\mathfrak{O}A^{-1}$ $(A \in D_A^\times)$ be a maximal order such that $\underline{A\mathfrak{O}A^{-1}} = \underline{\bar{A}\mathfrak{O}\bar{A}^{-1}}$. We may assume that $\mathrm{id}\,(n(A)) = O_K$ without changing the class of the given maximal order $A\mathfrak{O}A^{-1}$. $\underline{A\mathfrak{O}A^{-1}} = \underline{\bar{A}\mathfrak{O}\bar{A}^{-1}}$ implies $A\mathfrak{O} = a\beta\bar{A}\mathfrak{O}$ $(a \in K_A^\times, \beta \in D)$. Since $\mathrm{id}\,(n(A)) = \mathrm{id}\,(a^2 n(\beta)n(\bar{A})) = O_K$, $\mathrm{id}\,(a)$ is principal. Hence we have $A\mathfrak{O} = \gamma\bar{A}\mathfrak{O}$ $(\gamma \in D)$ and $\mathrm{id}\,(n(\gamma)) = O_K$. Now we define a linear mapping η by $\eta(x) = \gamma x^*\bar{\gamma}^*$ for $x \in D$. Then $\eta(A\mathfrak{O}\bar{A}^*) = \gamma\bar{A}\mathfrak{O}A^*\bar{\gamma}^* = A\mathfrak{O}\bar{A}^*$, and $\eta(A\mathfrak{O}\bar{A}^* \cap H) = A\mathfrak{O}\bar{A}^* \cap H$. Moreover $n(\eta(x)) = n(\gamma x^*\bar{\gamma}^*) = N_{K/Q}(n(\gamma))n(x) = n(x)$ since $n(\gamma)$ is a totally positive unit of K. Therefore an even positive definite quadratic lattice $A\mathfrak{O}\bar{A}^* \cap H$ with discriminant q has a non-trivial isometry. Then there exists an element e in $A\mathfrak{O}\bar{A}^* \cap H$ with $n(e) = 1$ by 2.5 in [2]. Conversely assume that $A\mathfrak{O}\bar{A}^* \cap H$ has an element e with $n(e) = 1$, where $\mathrm{id}\,(n(A)) = O_K$. Put $e = A_\mathfrak{p} e_\mathfrak{p} (\bar{A}^*)_\mathfrak{p}$ $(e_\mathfrak{p} \in \mathfrak{O}_\mathfrak{p})$; then $n(e_\mathfrak{p})$ is a unit since $n(e) = n(A_\mathfrak{p})n(e_\mathfrak{p})n((\bar{A}^*)_\mathfrak{p}) = 1$ and $n(A_\mathfrak{p}), n((\bar{A}^*)_\mathfrak{p})$ are units of $K_\mathfrak{p}$ from our assumption. Hence we get $e_\mathfrak{p} \in \mathfrak{O}_\mathfrak{p}^\times$. Take an element f in $D_A^\times$ such that $f_\mathfrak{p} = e_\mathfrak{p}$ for any finite prime. Then we have $\underline{\bar{A}\mathfrak{O}\bar{A}^{-1}} = \underline{(f^{-1}A^{-1}e)^*\mathfrak{O}(f^{-1}A^{-1}e^{-1})^{-1*}} = \underline{e^{-1}A\mathfrak{O}A^{-1}e} = \underline{A\mathfrak{O}A^{-1}}$. By virtue of Tamagawa's bijection in § 1 and the above bijection h_0 is equal to the class number by rotations of even positive definite quaternary lattices with discriminant q which have an element with length 2, and it is $\left(q + 3 - 4\left(\frac{3}{q}\right)\right)/24 + \left(1 - \left(\frac{2}{q}\right)\right)/4$ (§ 1 in [2]) since for such quadratic lattices the equivalence by rotations is the same as the equivalence by isometries (a vector of length 2 gives a symmetry).

LEMMA 5. $h_2 = h(D)/h(K)$.

Proof. Let $L = A\mathfrak{O}B$ $(A, B \in D_A^\times)$ be a normal ideal such that $L = \alpha\bar{L}^*\beta$ $(\alpha, \beta \in D)$ and $\mathrm{id}\,(n(AB))$ is principal. Then $A\mathfrak{O}B = \alpha\bar{B}^*\mathfrak{O}\bar{A}^*\beta$ implies $\varphi(L) = (\underline{A\mathfrak{O}A^{-1}}, \underline{B^{-1}\mathfrak{O}B}) = (\underline{\bar{B}^*\mathfrak{O}\bar{B}^{*-1}}, \underline{B^{-1}\mathfrak{O}B}) = (\underline{\bar{B}^{-1}\mathfrak{O}\bar{B}}, \underline{B^{-1}\mathfrak{O}B})$. Conversely take an order $B^{-1}\mathfrak{O}B$ $(B \in D_A^\times)$; then there is some C in $D_A^\times$ such that $\underline{C^{-1}\mathfrak{O}C} = \underline{B^{-1}\mathfrak{O}B}$ and $\mathrm{id}\,(n(C))$ is principal. $L = \bar{C}^*\mathfrak{O}C$ satisfies $\bar{L}^* = L$ and $\varphi^*(L) = (\underline{\bar{B}^{-1}\mathfrak{O}\bar{B}}, \underline{B^{-1}\mathfrak{O}B})$. This completes the proof of Lemma 5 and of our Theorem.

Appendix

PROPOSITION. *Let k be an algebraic number field with the maximal order $\mathfrak{o}$, and V be a regular quadratic space over k with bilinear form B and we denote* $\det(B(v_i, v_j))$ *by dV, where $\{v_i\}$ is a basis of V over k. Then, a lattice L in V has an $\mathfrak{o}$-basis, i.e., $L = \mathfrak{o}u_1 + \cdots + \mathfrak{o}u_n$ if and only if there is an element a in $k^\times$ such that the discriminant $dL_\mathfrak{p}$ of $L_\mathfrak{p}$ is equal to a^2dV* $(\mathrm{mod}\ \mathfrak{o}_\mathfrak{p}^{\times 2})$ *for any prime $\mathfrak{p}$ in k.*

Proof. Suppose that L has an $\mathfrak{o}$-basis, $L = \mathfrak{o}u_1 + \cdots + \mathfrak{o}u_n$. We define a matrix A by $(u_1, \cdots, u_n) = (v_1, \cdots, v_n)A$, and put $a = |A|$. Then $dL_\mathfrak{p} = |(B(u_i, u_j))| = |A|^2|(B(v_i, v_j))| = a^2dV$. Conversely suppose that $dL_\mathfrak{p} = a^2dV$ for an element a in $k^\times$ and any prime $\mathfrak{p}$ in k. Put $L_0 = \mathfrak{o}v_1 + \cdots + \mathfrak{o}v_n$, $M = \mathfrak{o}av_1 + \mathfrak{o}v_2 + \cdots + \mathfrak{o}v_n$, and $L = \mathfrak{o}e_1 + \cdots + \mathfrak{o}e_{n-1} + \mathfrak{a}e_n$ where $\mathfrak{a}$ is an ideal in k; then $dM_\mathfrak{p} = dM = a^2dV$, and $dL_\mathfrak{p}\mathfrak{o}_\mathfrak{p} = |(B(e_i, e_j))|\,\mathfrak{a}_\mathfrak{p}^2 = a^2dV\mathfrak{o}_\mathfrak{p}$. Thus we have $a^2dV\,|(B(e_i, e_j))|^{-1}\,\mathfrak{o} = \mathfrak{a}^2$. Since $dV\,|(B(e_i, e_j))|^{-1}$ is a square in k, $\mathfrak{a}$ is principal. This completes the proof.

COROLLARY. *Keeping the notations of Proposition, we assume further that there is a lattice L in V such that $L_\mathfrak{p}$ is unimodular for any prime $\mathfrak{p}$ in k. Then, L has an $\mathfrak{o}$-basis if and only if dV is a unit of k up to a square of k.*

Proof. If L has an $\mathfrak{o}$-basis, then dL is a unit at any prime in k. Hence dL is a unit of k. If, conversely, dV is a unit, then $dL_\mathfrak{p}/dV$ is a square of unit of $k_\mathfrak{p}$. Hence $dL_\mathfrak{p} = dV$ ($dL_\mathfrak{p}$ is uniquely determined up to squares of units of $k_\mathfrak{p}$ by definition). Taking 1 as a in Proposition, we get Corollary.

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Nagoya University