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# A CLASSIFICATION OF IRREDUCIBLE PREHOMOGENEOUS VECTOR SPACES AND THEIR RELATIVE INVARIANTS

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## Introduction

Let $G$ be a connected linear algebraic group, and $\rho$ a rational representation of $G$ on a finite-dimensional vector space $V$, all defined over the complex number field $\boldsymbol{C}$.

We call such a triplet $(G, \rho, V)$ a *prehomogeneous vector space* if $V$ has a Zariski-dense $G$-orbit. The main purpose of this paper is to classify all prehomogeneous vector spaces when $\rho$ is irreducible, and to investigate their relative invariants and the regularity.

This paper consists of the following eight sections.

§1. Preliminaries
§2. Castling transforms
§3. Classification of reduced triplets $(G, \rho, V)$ satisfying $\dim G \geqq \dim V$
§4. Relative invariants and the regularity
§5. The prehomogeneity and relative invariants of reduced triplets obtained in §3
§6. The semi-simple case
§7. Table of reduced irreducible prehomogeneous vector spaces
§8. Prehomogeneous vector spaces with finitely many orbits

We now make a brief survey of this paper. For the convenience of the reader, we shall review, at the beginning of §1, basic facts about complex simple Lie algebras, especially their irreducible representations and their classification. Then we shall construct a simple Lie algebra of each type and calculate its representation degrees which will be used in §3. We shall introduce in §2 an important notion of castling transform, which is an irreducible prehomogeneous vector space obtained from

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a given one by a certain process. It will be shown that each prehomogeneous vector space is obtained by successive castling transforms starting from a *reduced* one, which is characterized by the property that it has the smallest dimension in a "fixed tree" of those transforms.

Our solution of the classification problem consists of the explicit description of the process of castling transform and the table of reduced irreducible prehomogeneous vector spaces which will be given in § 7.

Obviously a triplet $(G, \rho, V)$ is a prehomogeneous vector space only when $\dim G \geq \dim V$, and hence as the first step, we classify in § 3 the reduced triplets satisfying such a condition. To investigate these triplets, we develop in § 4 a general theory about relative invariants and the regularity of prehomogeneous vector spaces. By using the results of § 4, we investigate in § 5 the reduced triplets obtained in § 3, especially we determine their prehomogeneity. By a well-known theorem of E. Cartan, if $\rho$ is irreducible, the Lie algebra $\mathfrak{g}$ of $\rho(G)$ is reductive with center at most one-dimensional. We have assumed in § 3 and § 5 that the center of $\mathfrak{g}$ is one-dimensional. The remaining case will be discussed in § 6. In the last section, we consider an irreducible triplet with finitely many orbits. It will be shown that such a triplet is a reduced irreducible prehomogeneous vector space with few exceptions.

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## § 1. Preliminaries

First of all, we shall review the basic facts about complex simple Lie algebras, especially their irreducible representations and their classification. We denote by $\mathfrak{gl}(V)$ the Lie algebra of all linear transformations of a vector space $V$. Similarly we denote by $\mathfrak{gl}(n)$ the Lie algebra of all $n \times n$ matrices.

The following two theorems give us a principle to solve the classification problem in the irreducible case.

THEOREM 1 (E. Cartan). *Let $d\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of a Lie algebra $\mathfrak{g}$ on $V$ over $\boldsymbol{C}$. Then the image $d\rho(\mathfrak{g})$ is reductive with center at most one-dimensional, i.e., a semi-simple Lie subalgebra of $\mathfrak{gl}(V)$ or a direct sum of scalar multiplications ($\cong \mathfrak{gl}(1)$) of $V$ and a semi-simple Lie subalgebra of $\mathfrak{gl}(V)$ (see* [9]).

THEOREM 2 (I. Schur). *Let* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ *be a Lie algebra of a direct sum of Lie algebras* $\mathfrak{g}_i (1 \leq i \leq \ell)$, *and* $d\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ *an irreducible representation of* $\mathfrak{g}$ *on the complex vector space* $V$. *Then there exist irreducible representations* $d\rho_i: \mathfrak{g}_i \to \mathfrak{gl}(V_i)$ *of* $\mathfrak{g}_i$ *on* $V_i (1 \leq i \leq \ell)$ *such that* (1) $V \cong V_1 \otimes \cdots \otimes V_\ell$, (2) $d\rho = d\rho_1 \otimes \cdots \otimes d\rho_\ell$, *i.e.*,

$$d\rho(X)v = \sum_{i=1}^{\ell} v_1 \otimes \cdots \otimes d\rho_i(X_i)v_i \otimes \cdots \otimes v_\ell$$

*for* $X = (X_1, \cdots, X_\ell) \in \mathfrak{g}$, $v = (v_1 \otimes \cdots \otimes v_\ell) \in V$ (*see* [1]).

DEFINITION 3. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\boldsymbol{C}$. Then there exists a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ satisfying the two conditions: (1) $\mathfrak{h}$ is a maximal abelian subalgebra, i.e., $[X, Y] = 0$ for any $X, Y \in \mathfrak{h}$, and any element $X$ of $\mathfrak{g}$ satisfying $[X, Y] = 0$ for all $Y \in \mathfrak{h}$ belongs to $\mathfrak{h}$. (2) For any $H \in \mathfrak{h}$, the linear endomorphism $\mathrm{ad}(H)$ of $\mathfrak{g}$ is diagonalizable (i.e., semi-simple). In this case, $\mathfrak{h}$ is called a *Cartan subalgebra.* Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Cartan subalgebras of $\mathfrak{g}$. Then there exists an automorphism $L$ of $\mathfrak{g}$ satisfying $\mathfrak{h}_1 = L\ \mathfrak{h}_2$. Hence the dimension of a Cartan subalgebra depends only on $\mathfrak{g}$ which is called the *rank* of $\mathfrak{g}$.

DEFINITION 4. Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ its fixed Cartan subalgebra, $d\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on $V$. An element $\lambda$ of the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ is called a *weight* of $d\rho$ if $\mathfrak{g}_\lambda \neq \{0\}$, where $\mathfrak{g}_\lambda = \{x \in V \mid d\rho(H)x = \lambda(H)x$ for any $H \in \mathfrak{h}\}$. A non-zero element of $\mathfrak{g}_\lambda$ is called a *weight vector*. Let $d\rho': \mathfrak{g} \to \mathfrak{gl}(V')$ be another representation of $\mathfrak{g}$ on $V'$. Then $d\rho$ and $d\rho'$ are equivalent if and only if they have same weights. A non-zero weight of the adjoint representation is called a *root* of $\mathfrak{g}$(with respect to $\mathfrak{h}$). The totality $\Delta$ of roots is called the *root system* of $\mathfrak{g}$(w. r. t. $\mathfrak{h}$). If $\alpha \in \Delta$, then $-\alpha \in \Delta$, $\dim \mathfrak{g}_\alpha = 1$, $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ and hence $\dim \mathfrak{g} = \mathrm{rank}\, \mathfrak{g} + \#(\Delta)$.

DEFINITION 5. Let $\mathfrak{g}$ be any Lie algebra. The symmetric bilinear form $B$ of $\mathfrak{g}$ defined by $B(X, Y) = \mathrm{Tr}\,\mathrm{ad}\, X \circ \mathrm{ad}\, Y$ is called *the Killing form* of $\mathfrak{g}$. This is non-degenerate if and only if $\mathfrak{g}$ is semi-simple.

DEFINITIOM 6. Let $\mathfrak{g}$ be a semi-simple Lie algebra of rank $\ell$. Then the restriction $B|_{\mathfrak{h}}$ of the Killing form $B$ to the fixed Cartan subalgebra $\mathfrak{h}$ is also non-degenerate. Hence, for each root $\alpha$, there exists uniquely an element $H_\alpha$ of $\mathfrak{h}$ satisfying $\alpha(H) = B(H, H_\alpha)$ for any $H \in \mathfrak{h}$. Let $\mathfrak{h}_0$ be the vector space over the rational number field $\boldsymbol{Q}$ spanned by $H_\alpha$ for all $\alpha \in \Delta$. Then $\dim \mathfrak{h}_0 = \ell$, and the restriction $B|_{\mathfrak{h}_0}$ of $B$ to $\mathfrak{h}_0$ is a $\boldsymbol{Q}$-valued positive definite symmetric bilinear form. Let $\mathfrak{h}_0^*$ be the dual vector space of $\mathfrak{h}_0$ over $\boldsymbol{Q}$. For each $\lambda \in \mathfrak{h}_0^*$, there exists uniquely an element $H_\lambda$ of $\mathfrak{h}_0$ satisfying $\lambda(H) = B(H, H_\lambda)$ for any $H \in \mathfrak{h}_0$. Note that $\Delta \subset \mathfrak{h}_0^*$ and for each $\alpha \in \Delta$, $H_\alpha$ is the same as the previous definition. We can define a positive definite inner product $(\lambda, \mu)$ on $\mathfrak{h}_0^*$ by $(\lambda, \mu) = B(H_\lambda, H_\mu) = \lambda(H_\mu) = \mu(H_\lambda)$.

DEFINITION 7. Fix a basis $H_1, \cdots, H_\ell$ of $\mathfrak{h}_0$ over $\boldsymbol{Q}$. An element $\lambda$ of $\mathfrak{h}_0^*$ is called *positive* if $\lambda(H_1) = \cdots = \lambda(H_{k-1}) = 0$, $\lambda(H_k) > 0$ for some $k = 1, \cdots, \ell$. We can define a *lexicographical order* in $\mathfrak{h}_0^*$. Namely, $\lambda > \mu$ implies that $\lambda - \mu$ is positive for $\lambda, \mu \in \mathfrak{h}_0^*$.

DEFINITION 8. The totality of positive roots will be denoted by $\Delta_+$. A positive root is called *simple* if it is not a sum of two positive roots. A subset $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ of $\Delta$ is called a *fundamental root system* if any root $\alpha$ is written uniquely as $\alpha = m_1\alpha_1 + \cdots + m_\ell\alpha_\ell$ where all $m_i$ are non-negative integers or all $m_i$ are non-positive integers. There exist just $\ell$ simple roots $\alpha_1, \cdots, \alpha_\ell$ and they form a fundamental root system. Conversely, a fundamental root system is the totality of simple roots under some lexicographical order in $\mathfrak{h}_0^*$.

DEFINITION 9. Let $\mathfrak{n}_+$ be a vector subspace of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ for all $\alpha \in \Delta_+$. Let $d\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{g}$ on $V$. Then there exists uniquely, up to constant, a non-zero element $x$ of $V$ such that $d\rho(\nu)x = 0$ for any $\nu \in \mathfrak{n}_+$. For such $x \in V$, there exists $\Lambda \in \mathfrak{h}^*$ such that $d\rho(H)x = \Lambda(H)\cdot x$ for any $H \in \mathfrak{h}$. Moreover, this $\Lambda$ is an element of $\mathfrak{h}_0^*$ and a *dominant integral form*, i.e., $\dfrac{2(\Lambda, \alpha)}{(\alpha, \alpha)}$ is a non-negative integer for any $\alpha \in \Delta_+$. We say that $x$ is a *highest weight vector* and $\Lambda$ is the *highest weight* of $d\rho$.

THEOREM 10. *Let $\Lambda$ be any dominant integral form of $\mathfrak{h}$. Then there exists an irreducible representation $d\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$, with the highest weight $\Lambda$. This gives a one-to-one correspondence between the equivalence classes of irreducible representations of $\mathfrak{g}$ and dominant integral forms. Sometimes we shall denote $d\rho$ by $\Lambda$.*

DEFINITION 11. Let $\alpha_1, \cdots, \alpha_\ell$ be the simple roots w.r.t. $(\mathfrak{g}, \mathfrak{h}, \Delta_+)$. Then there exist dominant integral forms $\Lambda_1, \cdots, \Lambda_\ell$ uniquely such that

$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$ $(i, j = 1, \cdots, \ell)$. These $\Lambda_1, \cdots, \Lambda_\ell$ are called *fundamental dominant weights*, and the corresponding irreducible representations $d\rho_i : \mathfrak{g} \to \mathfrak{gl}(V_i)$ $(1 \le i \le \ell)$ are called the *fundamental irreducible representations* of $\mathfrak{g}$. Any dominant integral form $\Lambda$ is of the form $\Lambda = \sum_{i=1}^{\ell} m_i \Lambda_i$ where each $m_i$ is a non-negative integer. Let $v_i$ be the highest weight vector of $d\rho_i (1 \le i \le \ell)$, and let $V$ be the least $\mathfrak{g}$-invariant subspace of $V_1 \overbrace{\otimes \cdots \otimes}^{m_1} V_1 \otimes \cdots \otimes V_\ell \overbrace{\otimes \cdots \otimes}^{m_\ell} V_\ell$ containing $v = v_1 \overbrace{\otimes \cdots \otimes}^{m_1} v_1 \otimes \cdots \otimes v_\ell \overbrace{\otimes \cdots \otimes}^{m_\ell} v_\ell$. Then the restriction $d\rho$ of $d\rho_1 \overbrace{\otimes \cdots \otimes}^{m_1} d\rho_1 \otimes \cdots \otimes d\rho_\ell \overbrace{\otimes \cdots \otimes}^{m_\ell} d\rho_\ell$ to $V$ is an irreducible representation of $\mathfrak{g}$ with the highest weight $\Lambda = \sum m_i \Lambda_i$. In view of Theorem 10, the dimension of $V$ depends only on the corresponding dominant integral form $\Lambda$, and hence we denote it by $d(\Lambda)$ which is called a representation degree of $\Lambda$ (or $d\rho$).

THEOREM 12 (Weyl's dimension formula).

$$d(\Lambda) = \prod_{\alpha \in \Delta_+} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)} \quad \textit{where} \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \,.$$

COROLLARY 13. *Let $\Lambda = \sum_{i=1}^{\ell} m_i \Lambda_i$ and $\Lambda' = \sum_{i=1}^{\ell} m'_i \Lambda_i$ be dominant integral forms such that $m_i \geqq m'_i$ for each $i = 1, \cdots, \ell$ and $\Lambda \neq \Lambda'$. Then $d(\Lambda) > d(\Lambda')$.*

*Proof.* For any positive root $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$ $(n_i \geqq 0)$, $(\Lambda, \alpha) = \sum_{i,j} m_i n_j (\Lambda_i, \alpha_j) = \frac{1}{2} \sum_i m_i n_i (\alpha_i, \alpha_i) > \frac{1}{2} \sum_i m'_i n_i (\alpha_i, \alpha_i) = (\Lambda', \alpha)$ and this implies $d(\Lambda) > d(\Lambda')$. Q.E.D.

Now, we shall review the classification of simple Lie algebras over $\boldsymbol{C}$.

DEFINITION 14. Let $R^\ell$ be an $\ell$-dimensional vector space over the real number field $\boldsymbol{R}$ with a positive definite inner product $( \, , \, )$. Define the length $\|\alpha\|$ of $\alpha \in R^\ell$ as $\|\alpha\| = \sqrt{(\alpha, \alpha)}$ and denote the angle of two vectors $\alpha_i, \alpha_j \in R^\ell$ by $\widehat{\alpha_i \alpha_j}$.

A subset $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ of $R^\ell$ is called an *irreducible admissible system* if the following three conditions are satisfied: (1) $\alpha_1, \cdots, \alpha_\ell$ are linearly independent.

(2) $-\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ is a non-negative integer if $i \neq j$.

(3) There is *no* decomposition of $\Pi$ such that $\Pi = \Pi_1 \cup \Pi_2$ and $\Pi_1 \perp \Pi_2$,

where $\Pi_1 \perp \Pi_2$ implies that $(\alpha, \beta) = 0$ for any $\alpha \in \Pi_1$ and $\beta \in \Pi_2$.

LEMMA 15. *Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be an irreducible admissible system of $R^\ell$. If $\alpha_i, \alpha_j \in \Pi$, $\alpha_i \neq \alpha_j$ and $\|\alpha_i\| \geqq \|\alpha_j\|$, then they satisfy one of the following conditions.*

(1) $\widehat{\alpha_i\alpha_j} = \dfrac{\pi}{2}$ *i.e.* $(\alpha_i, \alpha_j) = 0$

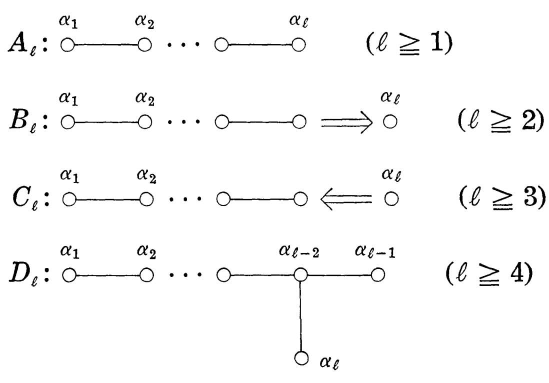
(2) $\widehat{\alpha_i\alpha_j} = \dfrac{2}{3}\pi$ *and* $\|\alpha_i\| = \|\alpha_j\|$

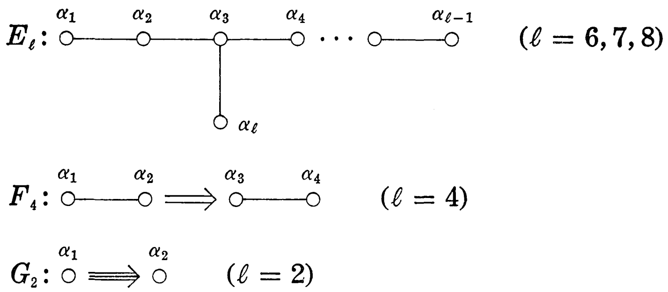
(3) $\widehat{\alpha_i\alpha_j} = \dfrac{3}{4}\pi$ *and* $\|\alpha_i\| = \sqrt{2}\,\|\alpha_j\|$

(4) $\widehat{\alpha_i\alpha_j} = \dfrac{5}{6}\pi$ *and* $\|\alpha_i\| = \sqrt{3}\,\|\alpha_j\|$

DEFINITION 16. Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be an irreducible admissible system in $R^\ell$. To each vector $\alpha_i \in \Delta$, associate a vertex and connect the two vertices associated to $\alpha_i$ and $\alpha_j$ if and only if $(\alpha_i, \alpha_j) \neq 0$, i.e., $\widehat{\alpha_i\alpha_j} \neq \dfrac{\pi}{2}$. We connect vertices corresponding to $\alpha_i$ and $\alpha_j$ with a single, double, or triple line according to whether $\widehat{\alpha_i\alpha_j} = \frac{2}{3}\pi, \frac{3}{4}\pi, \frac{5}{6}\pi$ respectively (see Lemma 15). The arrows point from a longer to a shorter vector, when the lengths are different. Thus we obtain a connected diagram which is called the *Dynkin diagram* of $\Pi$.

LEMMA 17. *Let $\Pi$ be an irreducible admissible system of $R^\ell$. Then its Dynkin diagram is one of the following diagrams.*






DEFINITION 18. Let $\mathfrak{g}$ be a semi-simple Lie algebra of rank $\ell$ over $C$, and let $\mathfrak{h}$ be a fixed Cartan subalgebra. We can extend the inner product of $\mathfrak{h}_0^*$ (see Definition 6) to a positive definite inner product of $\ell$-dimensional vector space $R^\ell = \mathfrak{h}_0^* \otimes_Q R$ over $R$. Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be a fundamental root system of $\mathfrak{g}$. Then, $\mathfrak{g}$ is simple if and only if $\Pi$ satisfies the third condition in Definition 14, i.e., there is no decomposition such that $\Pi = \Pi_1 \cup \Pi_2$ and $\Pi_1 \perp \Pi_2$. Moreover, if $\mathfrak{g}$ is a simple Lie algebra, $\Pi$ is an irreducible admissible system of $R^\ell$, and we can get its Dynkin diagram by Definition 16. This diagram depends only on $\mathfrak{g}$, and we can call it *the Dynkin diagram of a simple Lie algebra* $\mathfrak{g}$.

THEOREM 19 (Classification of simple Lie algebras). *Two simple Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic over $C$ if and only if they have the same Dynkin diagrams. Thus Lemma 17 says that a simple Lie algebra over $C$ is isomorphic to one of $A_\ell(\ell \geqq 1)$, $B_\ell(\ell \geqq 2)$ $C_\ell(\ell \geqq 3)$, $D_\ell(\ell \geqq 4)$, $E_\ell(\ell = 6, 7, 8)$, $F_4, G_2$.*

DEFINITION 20. The simple Lie algebras of type $A_\ell, B_\ell, C_\ell, D_\ell$ are called *classical Lie algebras*, and those of type $E_\ell, F_4, G_2$ are called *exceptional Lie algebras.*

We shall construct the all types of simple Lie algebras in Theorem 19 and calculate their representation degrees which will be used in § 3.

EXAMPLE 21. Let $\mathfrak{g} = \mathfrak{sl}(n, C)$ be a subalgebra $\{X \in \mathfrak{gl}(n, C) \mid \mathrm{Tr}\, X = 0\}$ of $\mathfrak{gl}(n, C)$. Then the Killing form $B$ of $\mathfrak{g}$ is given by $B(X, Y) = \mathrm{Tr}\, \mathrm{ad}\, X \circ \mathrm{ad}\, Y = 2n\, \mathrm{Tr}\, XY$ for any $X, Y \in \mathfrak{g}$. Since this is non-degenerate, $\mathfrak{g}$ is a semi-simple Lie algebra. We may take as a Cartan subalgebra $\mathfrak{h}$ the totality of diagonal matrices of trace zero. Let $E_{ij}$ be a matrix unit with $(i, j)$-element 1, all remaining entries zero. Denote an element $\lambda$ of $\mathfrak{h}^*$ by $\sum_{i=1}^n a_i \lambda_i (a_i \in C)$ if $\lambda(H) = \sum_{i=1}^n a_i \lambda_i$ for $H = \sum_{i=1}^n \lambda_i E_{ii} \in \mathfrak{h}$. Then the root system $\Delta$ of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$ is given by $\Delta = \{\lambda_i - \lambda_j \mid i \neq j, i, j = 1, \cdots, n\}$ and $\mathfrak{g}_{\lambda_i - \lambda_j} = \boldsymbol{C} \cdot E_{ij}$. We have $\dim \mathfrak{g} = n^2 - 1$, $\operatorname{rank} \mathfrak{g} = n - 1$. Put $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i = 1, \cdots, n-1$. Since $\lambda_i - \lambda_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ $(i < j)$ and $\lambda_i - \lambda_j = -(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{i-1})$ $(i > j)$, $\Pi = \{\alpha_1, \cdots, \alpha_{n-1}\}$ is a fundamental root system. For each root $\lambda_i - \lambda_j$, we shall calculate $H_{\lambda_i - \lambda_j}$. Put $H_{\lambda_i - \lambda_j} = \sum_{k=1}^n \mu_k E_{kk}$. Then, by the definition, $B(H, H_{\lambda_i - \lambda_j}) = 2n \sum \lambda_k \mu_k = \lambda_i - \lambda_j$ for any $H = \sum_{k=1}^n \lambda_k E_{kk} \in \mathfrak{h}$. Thus we get that $\mu_i = \dfrac{1}{2n}$, $u_j = -\dfrac{1}{2n}$, and $\mu_k = 0$ for $k \neq i, j$, i.e., $H_{\lambda_i - \lambda_j} = \dfrac{1}{2n}(E_{ii} - E_{jj})$. The fundamental root system $\Pi = \{\alpha_1, \cdots, \alpha_{n-1}\}$ is the totality of simple roots under the lexicographical order in $\mathfrak{h}_0^*$ defined by a basis $H_{\lambda_1 - \lambda_n}, \cdots, H_{\lambda_{n-1} - \lambda_n}$ of $\mathfrak{h}_0$. Moreover, by the definition,

$$(\alpha_i, \alpha_j) = B(H_{\alpha_i}, H_{\alpha_j}) = \begin{cases} 0 & |i - j| \geqq 2 \\ -\dfrac{1}{2n} & |i - j| = 1 \\ \dfrac{1}{n} & i = j \end{cases}$$

and this shows that $\Pi$ is an irreducible admissible system and its Dynkin diagram is of type $A_{n-1}$. Thus, $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$ is a simple Lie algebra of type $A_{n-1}$: $A_{n-1} = \mathfrak{sl}(n, \boldsymbol{C})$.

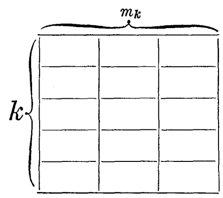
We shall determine the fundamental dominant weights $\Lambda_i = \sum_{k=1}^n m_{ik} \lambda_k$ for $i = 1, \cdots, n-1$. We may assume that $m_{in} = 0$ because $\sum_{i=1}^n \lambda_i = 0$. Since $\dfrac{2(\Lambda_i, \alpha_j)}{(\alpha_\alpha, \alpha_j)} = m_{i,j} - m_{i,j+1} = \delta_{ij}$ for $i, j = 1, \cdots, n-1$, and $m_{in} = 0$, we get that $m_{ij} = 1$ for $j \leq i$ and $m_{ij} = 0$ for $j > i$, i.e., $\Lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for $i = 1, \cdots, n-1$.

As $\Delta_+ = \{\lambda_i - \lambda_j \mid i < j\}$, $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha = \sum_{i<j} \boldsymbol{C} E_{ij}$ is the totality of upper triangular matrices with diagonal elements 0 (see Definition 9). Let $V_1$ be a $n$-dimensional vector space over $\boldsymbol{C}$ spanned by $u_1, \cdots, u_n$. Define a representation $d\rho_1$ of $\mathfrak{g}$ by $(u_1, \cdots, u_n) \mapsto (u_1, \cdots, u_n)A$ for any $A \in \mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$. Then $d\rho_1$ is the fundamental irreducible representation of $\mathfrak{g}$ with the highest weight $\Lambda_1$ and $u_1$ is the highest weight vector since $d\rho_1(\nu) u_1 = 0$ for any $\nu \in \mathfrak{n}_+$ and $d\rho_1(H) u_1 = \lambda_1 H$ for any $H = \sum \lambda_i E_{ii} \in \mathfrak{h}$. In general, let $V_k (1 \leq k \leq n-1)$ be a $\dbinom{n}{k}$-dimensional vector space over $\boldsymbol{C}$ spanned by exterior tensor products $u_{i_1} \wedge \cdots \wedge u_{i_k} (1 \leq i_1 < \cdots < i_k \leq n)$.

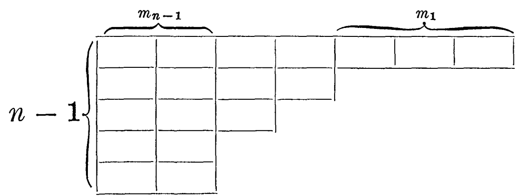
Define a representation $d\rho_k$ of $\mathfrak{g}$ by $d\rho_k(A)(u_{i_1} \wedge \cdots \wedge u_{i_k}) = \sum_{j=1}^{k} u_{i_1} \wedge \cdots \wedge d\rho_1(A)u_{i_j} \wedge \cdots \wedge u_{i_k}$. Then $d\rho_k$ is the fundamental irreducible representation of $\mathfrak{g}$ with the highest weight $\Lambda_k = \lambda_1 + \cdots + \lambda_k$ since $d\rho_k(\nu)(u_1 \wedge \cdots \wedge u_k) = 0$ for any $\nu \in \mathfrak{n}_+$ and $d\rho(H)u_1 \wedge \cdots \wedge u_k = (\lambda_1 + \cdots + \lambda_k)u_1 \wedge \cdots \wedge u_k$ for any $H = \sum_{i=1}^{n} \lambda_i E_{ii} \in \mathfrak{h}$. In particular, we obtain that $d(\Lambda_k) = \binom{n}{k}$ for $1 \leq k \leq n$. In view of Definition 11, any irreducible representation space of $\mathfrak{sl}(n, \boldsymbol{C})$ is obtained from a tensor product of $V_1$.

Although one can use the Weyl's dimension formula to calculate $d(\Lambda) = \dim V$, there is a simple method for $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$ which is also obtained from the Weyl's dimension formula. We shall introduce this method.

DEFINITION 22. To a dominant integral weight $\Lambda = m_k \Lambda_k (1 \leq k \leq n-1)$, we shall attach the diagram




In general, to $\Lambda = m_1 \Lambda_1 + \cdots + m_{n-1}\Lambda_{n-1}$ we shall attach the diagram




which is called the *Young diagram* (in detail, see [1]).

| | | | | | |
|---|---|---|---|---|---|
| $n$ | $n+1$ | $n+2$ | $n+3$ | $n+4$ | $n+5$ |
| $n-1$ | $n$ | $n+1$ | $n+2$ | | |
| $n-2$ | $n-1$ | $n$ | | | |
| $n-3$ | $n-2$ | | | | |

$(\Lambda = 2\Lambda_1 + \Lambda_2 + \Lambda_3 + 2\Lambda_4)$

Figure I.

Write down the number $n$ in each diagonal and to the right direction the increasing numbers, and to the down direction the decreasing numbers one by one like in Figure 1. Then, multiply all of them, which we shall denote by $d_0(\Lambda)$.

| | | | | | |
|---|---|---|---|---|---|
| 9 | 8 | 6 | 4 | 2 | 1 |
| 6 | 5 | 3 | 1 | | |
| 4 | 3 | 1 | | | |
| 2 | 1 | | | | |

Figure II.

Write down the number of the hook's length, i.e., the number of squares of right and down side including itself, like in Figure II. Then multiply all of them which we shall denote by $d_\infty(\Lambda)$.

THEOREM 23. *Let $\Lambda$ be a dominant integral form of $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$. Then the dimension $d(\Lambda)$ is given by*

$$d(\Lambda) = \frac{d_0(\Lambda)}{d_\infty(\Lambda)}\,.$$

EXAMPLE 24. (1) Let $d\rho$ be an irreducible representation of $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$ with the highest weight $\Lambda = \sum_{i=1}^{n-1} m_i\Lambda_i$. Then the highest weight $\Lambda'$ of the contragredient representation $d\rho^*$ of $d\rho$ is given by $\Lambda' = \sum_{i=1}^{n-1} m_{n-i}\Lambda_i$. Thus we obtain that $d(\sum_{i=1}^{n-1} m_i\Lambda_i) = d(\sum_{i=1}^{n-1} m_{n-i}\Lambda_i)$.

(2) $\quad d(\Lambda_1 + \Lambda_\nu) = \dfrac{(n+1)!\,\nu}{(n-\nu)!\,(\nu+1)!} \qquad (1 \leq \nu \leq n-1)$

| | |
|---|---|
| $n$ | $n+1$ |
| $n-1$ | |
| $\vdots$ | |
| $n-\nu+1$ | |

| | |
|---|---|
| $\nu+1$ | $1$ |
| $\nu-1$ | |
| $\vdots$ | |
| $1$ | |

Figure III.

From the Young diagram in Figure III and Definition 22, we obtain that $d_0(\Lambda) = (n+1)n\ (n-1) \cdots (n-\nu+1) = \dfrac{(n+1)!}{(n-\nu)!}$ and $d_\infty(\Lambda) = (\nu-1)!\,(\nu+1) = \dfrac{(\nu+1)!}{\nu}$. Hence by Theorem 23, we obtain our result.

In particular,

(i) $d(2\Lambda_1) = \frac{1}{2}n(n+1)$,

(ii) $d(\Lambda_1 + \Lambda_2) = \frac{1}{3}n(n^2-1)$,

(iii) $d(\Lambda_1 + \Lambda_{n-2}) = \frac{1}{2}n(n+1)(n-2)$,

(iv) $d(\Lambda_1 + \Lambda_{n-1}) = n^2 - 1$

(v) $d(\Lambda_3 + \Lambda_1) = \frac{1}{8}n(n-2)(n^2-1)$

(3) $$d(\Lambda_2 + \Lambda_\nu) = \frac{(n+1)!\,n(\nu-1)}{(n-\nu)!\,(\nu+1)!\,2} \qquad (2 \leq \nu \leq n-1)$$

$$\begin{array}{|c|c|}\hline n & n+1 \\ \hline n-1 & n \\ \hline n-2 & \\ \cline{1-1} \vdots & \\ \cline{1-1} n-\nu+1 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline \nu+1 & 2 \\ \hline \nu & 1 \\ \hline \nu-2 & \\ \cline{1-1} \vdots & \\ \cline{1-1} 1 & \\ \cline{1-1} \end{array}$$

Figure IV.

From the Young diagram in Figure IV and Definition 22, we obtain that $d_0(\Lambda) = (n+1)n^2(n-1)\cdots(n-\nu+1) = \dfrac{(n+1)!\,n}{(n-\nu)!}$ and $d_\infty(\Lambda) = 2(\nu+1)\nu(\nu-2)! = \dfrac{2(\nu+1)!}{(\nu-1)}$. Hence by Theorem 23 we obtain our result. In particular,

(i) $d(2\Lambda_2) = \frac{1}{12}n^2(n^2-1)$

(ii) $d(\Lambda_2 + \Lambda_{n-2}) = \frac{1}{4}n^2(n+1)(n-3)$

(iii) $d(\Lambda_3 + \Lambda_2) = \frac{1}{24}n^2(n^2-1)(n-2)$

(4) $$d(\Lambda_3 + \Lambda_\nu) = \frac{(n+1)!\,n(n-1)(\nu-2)}{(n-\nu)!\,(\nu+1)!\,6} \qquad (3 \leq \nu \leq n-1)$$

$$\begin{array}{|c|c|}\hline n & n+1 \\ \hline n-1 & n \\ \hline n-2 & n-1 \\ \hline n-3 & \\ \cline{1-1} \vdots & \\ \cline{1-1} n-\nu+1 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline \nu+1 & 3 \\ \hline \nu & 2 \\ \hline \nu-1 & 1 \\ \hline \nu-3 & \\ \cline{1-1} \vdots & \\ \cline{1-1} 1 & \\ \cline{1-1} \end{array}$$

Figure V.

From the Young diagram in Figure V and Definition 22, we obtain that $d_0(\Lambda) = (n+1)n^2(n-1)^2(n-2)\cdots(n-\nu+1) = \dfrac{(n+1)!\,n(n-1)}{(n-\nu)!}$ and

$d_\infty(\Lambda) = 3\cdot 2\cdot(\nu+1)\nu(\nu-1)(\nu-3)! = \dfrac{(\nu+1)!\,6}{(\nu-2)}$. Hence by Theorem 23 we obtain our result. In particular,

(i) $d(2\Lambda_3) = \frac{1}{144}(n+1)n^2(n-1)^2(n-2)$

(ii) $d(\Lambda_3 + \Lambda_{n-1}) = \frac{1}{6}n(n^2-1)(n-3)$

(iii) $d(\Lambda_3 + \Lambda_{\nu+1})/d(\Lambda_3 + \Lambda_\nu) = \dfrac{(\nu-1)(n-\nu)}{(\nu^2-4)} \qquad (\nu \geqq 3)$

(5) $d(2\Lambda_1 + \Lambda_{n-1}) = \frac{1}{2}n(n-1)(n+2)$

| | | |
|---|---|---|
| $n$ | $n+1$ | $n+2$ |
| $n-1$ | | |
| $\vdots$ | | |
| $2$ | | |

| | | |
|---|---|---|
| $n+1$ | $2$ | $1$ |
| $n-2$ | | |
| $\vdots$ | | |
| $1$ | | |

Figure VI.

From the Young diagram in Figure VI and Definition 22, we obtain that $d_0(\Lambda) = (n+2)!$ and $d_\infty(\Lambda) = 2(n+1)\cdot(n-2)!$. Thus we obtain our result by Theorem 23.

(6) $d(m\Lambda_1) = \dfrac{n(n+1)\cdots(n+m-1)}{m!} = \dfrac{(n+m-1)!}{m!\,(n-1)!}$

| | | | |
|---|---|---|---|
| $n$ | $n+1$ | $\cdots$ | $n+m-1$ |

| | | | |
|---|---|---|---|
| $m$ | $m-1$ | $\cdots$ | $1$ |

Figure VII.

From the Young diagram in Figure VII and Theorem 23, we obtain our result. We shall construct the representation $d\rho$ with the highest weight $m\Lambda_1$. Let $V_1$ be a $n$-dimensional vector space over $\boldsymbol{C}$ spanned by $u_1, \cdots, u_n$. Define a representation $d\rho_1$ of $\mathfrak{g}$ by $(u_1, \cdots, u_n) \mapsto (u_1, \cdots, u_n)A$ for any $A \in \mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$. Then as we saw in Example 21, this is a representation with the highest weight $\Lambda_1$. Let $V$ be a $\dfrac{(n+m-1)!}{m!\,(n-1)!}$-dimensional vector space of complex homogeneous polynomials of degree $m$ in variables $u_1, \cdots, u_n$. Define a representation $d\rho$ on $V$ by $d\rho(A)(u_{i_1} \cdots u_{i_m}) = \sum_{j=1}^m u_{i_1} \cdots (d\rho_1(A)u_{i_j}) \cdots u_{i_m}$. Then $d\rho$ is an irreducible representation with the highest weight $m\Lambda_1$ and the highest weight vector $u_1^m$.

(7) If $m = 2$, i.e., $\Lambda = 2\Lambda_1 (= \square\!\square)$, we can construct a representation $d\rho$ in another way. Namely, let $V$ be the totality of $n \times n$ symmetric matrices and define $d\rho$ by $d\rho(A)X = AX + X^tA$ for any $A \in \mathfrak{g} = \mathfrak{sl}(n, C)$, $X \in V$. Then $d\rho(\nu)E_{11} = 0$ for any $\nu \in \mathfrak{n}_+$ and $d\rho(H)E_{11} = 2\lambda_1 E_{11}$ for any $H = \sum_{i=1}^n \lambda_i E_{ii} \in \mathfrak{h}$. Thus $d\rho$ is an irreducible representation of $\mathfrak{sl}(n, C)$ with the highest weight $2\Lambda_1$. If we take $V$ as the totality of $n \times n$ skew-symmetric matrices and define $d\rho$ by $d\rho(A)X = AX + X^tA$ for any $A \in \mathfrak{g} = \mathfrak{sl}(n, C)$, $X \in V$, then $d\rho$ is an irreducible representation with the highest weight $\Lambda_2 = \lambda_1 + \lambda_2 \left(= \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$.

EXAMPLE 25. Let $\mathfrak{sp}(n, C) = \{A \in \mathfrak{gl}(2n, C) \mid {}^tAJ + JA = 0\}$ be a Lie subalgebra of $\mathfrak{gl}(2n, C)$ where $J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$. Then by the definition, an element $A$ of $\mathfrak{gl}(2n, C)$ is in $\mathfrak{g} = \mathfrak{sp}(n, C)$ if and only if $A$ is of the form:

$$A = \left(\begin{array}{c|c} X & Y \\ \hline Z & -{}^tX \end{array}\right), \qquad {}^tY = Y, {}^tZ = Z, X, Y, Z \in M(n, C)\,. \tag{1.1}$$

In particular, we have $\dim \mathfrak{sp}(n, C) = n(2n + 1)$. The Killing form $B$ of $\mathfrak{g} = \mathfrak{sp}(n, C)$ is given by $B(U, V) = \operatorname{Tr} \operatorname{ad} U \operatorname{ad} V = (2n + 2) \operatorname{Tr} UV$ for any $U$, $V \in \mathfrak{g}$ and as this is non-degenerate, $\mathfrak{g} = \mathfrak{sp}(n, C)$ is semi-simple. Define elements $H(\lambda_1, \cdots, \lambda_n), E_{\pm\lambda_i \pm \lambda_j}$ of $\mathfrak{g}$ as follows.

$$H(\lambda_1, \cdots, \lambda_n) = \left(\begin{array}{c|c} \begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{matrix} & 0 \\ \hline 0 & \begin{matrix}-\lambda_1 & & \\ & \ddots & \\ & & -\lambda_n\end{matrix} \end{array}\right), \quad E_{\lambda_i + \lambda_j} = \left(\begin{array}{c|c} 0 & 0 \\ \hline E_{ij} + E_{ji} & 0 \end{array}\right)$$

$$E_{-\lambda_i - \lambda_j} = \left(\begin{array}{c|c} 0 & E_{ij} + E_{ji} \\ \hline 0 & 0 \end{array}\right), \quad E_{\lambda_i - \lambda_j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & -E_{ji} \end{array}\right). \tag{1.2}$$

Then the totality $\mathfrak{h}$ of $H(\lambda_1, \cdots, \lambda_n)$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{sp}(n, C)$ and as $\operatorname{ad}(H)E_{\pm\lambda_i \pm \lambda_j} = (\pm\lambda_i \pm \lambda_j)E_{\pm\lambda_i \pm \lambda_j}$, $\Delta = \{\pm\lambda_i \pm \lambda_j \mid i, j = 1, \cdots, n\}$ is the root system of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$. Put $\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3, \cdots, \alpha_{n-1} = \lambda_{n-1} - \lambda_n, \alpha_n = 2\lambda_n$. Then $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is a fundamental root system because $\lambda_i - \lambda_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $i < j$, $\lambda_i - \lambda_j = -(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{i-1})$ for $i > j$, and $\pm(\lambda_i + \lambda_j) = \pm((\lambda_i - \lambda_n) + (\lambda_j - \lambda_n) + \alpha_n)$. As

$$H_{\alpha_i} = \frac{1}{4(n+1)} H(0, \cdots, \overset{\overset{i}{\vee}}{1}, \overset{\overset{i+1}{\vee}}{-1}, \cdots, 0)$$

for $1 \leq i \leq n-1$ and $H_{\alpha_n} = \dfrac{1}{4(n+1)} H(0, \cdots, 0, 2)$, we obtain that $(\alpha_i, \alpha_j) = 0$ for $|i-j| \geqq 2$, $(\alpha_i, \alpha_j) = -\dfrac{1}{4(n+1)}$ for $|i-j| = 1$ and $1 \leq i$, $j \leq n-1$, $(\alpha_{n-1}, \alpha_n) = -\dfrac{1}{2(n+1)}$, $(\alpha_i, \alpha_i) = \dfrac{1}{2(n+1)}$ for $i = 1, \cdots, n-1$, and $(\alpha_n, \alpha_n) = \dfrac{1}{n+1}$. This shows that $\mathfrak{g} = \mathfrak{sp}(n, \boldsymbol{C})$ is a simple Lie algebra of type $C_n$. The fundamental dominant weights are $\Lambda_1 = \lambda_1, \Lambda_2 = \lambda_1 + \lambda_2, \cdots, \Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Let $V_1$ be a $2n$-dimensional vector space over $\boldsymbol{C}$ spanned by $u_1, \cdots, u_{2n}$. For $1 \leq k \leq n$, put $T^k(V_1) = \overbrace{V_1 \otimes \cdots \otimes V_1}^{k}$ and define a map $\varphi\colon T^k(V_1) \to T^{k-2}(V_1)$ by $\varphi(u_1 \otimes \cdots \otimes u_k) = \varepsilon(u_1, u_2) u_3 \otimes \cdots \otimes u_k$ where $\varepsilon$ is a skew-symmetric bilinear form on $V_1$ such that $\varepsilon(u_i, u_{i+n}) = 1$ $(1 \leq i \leq n)$ and $\varepsilon(u_i, u_j) = 0$ $(i < j$ and $j \neq i + n)$. Let $V_k = \Lambda^k(V_1)$ be a vector space over $\boldsymbol{C}$ spanned by exterior products $u_{i_1} \wedge \cdots \wedge u_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq 2n)$. Then $V_k$ is a subspace of $T^k(V_1)$ and $\varphi(V_k) = V_{k-2}$. Hence $\dim V_k \cap \operatorname{Ker} \varphi = \binom{2n}{k} - \binom{2n}{k-2}$. As we saw in Example 21, $V_k$ is a representation space of $\mathfrak{sl}(2n, \boldsymbol{C})$. Since $\mathfrak{sp}(n, \boldsymbol{C})$ is a Lie subalgebra of $\mathfrak{sl}(2n, \boldsymbol{C})$, $V_k$ can be considered as a representation space of $\mathfrak{sp}(n, \boldsymbol{C})$. This is not irreducible, but the subspace $V_k \cap \operatorname{Ker} \varphi$ is an irreducible representation space of $\mathfrak{sp}(n, \boldsymbol{C})$ with the highest weight $\Lambda_k = \lambda_1 + \cdots + \lambda_k$, i.e., a fundamental representation of $\mathfrak{sp}(n, \boldsymbol{C})$. Thus we obtain that $d(\Lambda_\nu) = \binom{2n}{\nu} - \binom{2n}{\nu-2}$ for $1 \leq \nu \leq n$.

EXAMPLE 26. We shall calculate the representation degrees of $\mathfrak{sp}(n, \boldsymbol{C})$ for some cases which will be used in § 3. Define the lexicographical order in $\mathfrak{h}_0^*$ such that $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ in Example 25 are simple roots. Then $\Delta_+ = \{2\lambda_1, \cdots, 2\lambda_n, \lambda_i \pm \lambda_j \ (1 \leq i < j \leq n)\}$ and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \sum_{k=1}^{n} (n+1-k)\lambda_k .$$

The inner product in $\mathfrak{h}_0^*$ is given by

$$\left(\sum_{i=1}^{n} m_i\lambda_i,\ \sum_{j=1}^{n} n_j\lambda_j\right) = \frac{1}{4(n+1)} \sum_{k=1}^{n} m_k n_k .$$

Define another product $\{\ ,\ \}$ by $\{\sum m_i\lambda_i, \sum n_j\lambda_j\} = \sum m_k n_k$. Then the Weyl's dimension formula (Theorem 12) says that

$$d(\Lambda) = \prod_{\alpha \in \Delta_+} \frac{(\Lambda+\rho, \alpha)}{(\rho, \alpha)} = \prod_{\alpha \in \Delta_+} \frac{\{\Lambda+\rho, \alpha\}}{\{\rho, \alpha\}} .$$

Assume that $\Lambda = m_1\lambda_1 + m_2\lambda_2$ with $m_1 \geqq m_2 \geqq 0$. Then $(\Lambda, 2\lambda_i) = 0$ $(3 \leq i \leq n)$, $(\Lambda, \lambda_i \pm \lambda_j) = 0$ $(3 \leq i < j \leq n)$ and hence we obtain that

$$\begin{aligned} d(\Lambda) = {} & \frac{\{\Lambda+\rho, 2\lambda_1\}\{\Lambda+\rho, 2\lambda_2\}\{\Lambda+\rho, \lambda_1+\lambda_2\}\{\Lambda+\rho, \lambda_1-\lambda_2\}}{\{\rho, 2\lambda_1\}\{\rho, 2\lambda_2\}\{\rho, \lambda_1+\lambda_2\}\{\rho, \lambda_1-\lambda_2\}} \\ & \times \prod_{i=1}^{2} \prod_{j=3}^{n} \frac{\{\Lambda+\rho, \lambda_i+\lambda_j\}\{\Lambda+\rho, \lambda_i-\lambda_j\}}{\{\rho, \lambda_i+\lambda_j\}\{\rho, \lambda_i-\lambda_j\}} . \end{aligned}$$

Here $\{\rho, \lambda_i+\lambda_j\} = 2n+2-i-j$ $(1 \leq i \leq j \leq n)$, $\{\rho, \lambda_i-\lambda_j\} = j-i$ $(1 \leq i < j \leq n)$, $\{\Lambda, 2\lambda_i\} = 2m_i (i=1,2)$, $\{\Lambda, \lambda_1 \pm \lambda_2\} = m_1 \pm m_2$, and $\{\Lambda, \lambda_i \pm \lambda_j\} = m_i (i=1,2, j=3,\cdots,n)$. Thus we obtain that

$$\begin{aligned} & d(m_1\lambda_1 + m_2\lambda_2) \\ & = \frac{(2n-2+m_1)!\,(2n-3+m_2)!\,(2n-1+m_1+m_2)(1+m_1-m_2)}{(2n-1)!\,(2n-3)!\,(m_1+1)!\,m_2!} . \end{aligned}$$

In particular, we get

(1) $d(m\Lambda_1) = \dfrac{(2n-1+m)!}{(2n-1)!\,m!}$ (i.e., $m_1 = m, m_2 = 0$)

 (i) $d(\Lambda_1) = 2n$

 (ii) $d(2\Lambda_1) = n(2n+1) = \dim \mathfrak{sp}(n, \boldsymbol{C})$

 (iii) $d(3\Lambda_1) = \frac{2}{3}n(n+1)(2n+1)$

(2) $d(\Lambda_1 + \Lambda_2) = \frac{8}{3}n(n^2-1)$ (i.e., $m_1 = 2, m_2 = 1$)

(3) $d(2\Lambda_2) = \frac{1}{3}n(n-1)(2n-1)(2n+3)$ (i.e., $m_1 = m_2 = 2$)

(4) $d(\Lambda_2) = (n-1)(2n+1) = \binom{2n}{2} - \binom{2n}{0}$ (i.e., $m_1 = m_2 = 1$)

This is a special case of $d(\Lambda_\nu) = \binom{2n}{\nu} - \binom{2n}{\nu-2}$ in Example 25.

Now assume that $n = 3$ and $\Lambda = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3$ with $m_1 \geqq m_2 \geqq$

$m_3 \geqq 0$. Then $\Delta_+ = \{2\lambda_1, 2\lambda_2, 2\lambda_3, \lambda_1 \pm \lambda_2, \lambda_1 \pm \lambda_3, \lambda_2 \pm \lambda_3\}$ and $\rho = 3\lambda_1 + 2\lambda_2 + \lambda_3$. Since $\{\rho, \lambda_i + \lambda_j\} = 8 - i - j$, $\{\rho, \lambda_i - \lambda_j\} = j - i$, $\{\Lambda, \lambda_i \pm \lambda_j\} = m_i \pm m_j$, we obtain that $d(\Lambda) = \prod_{\alpha \in \Delta_+} \dfrac{\{\Lambda + \rho, \alpha\}}{\{\rho, \alpha\}} = \dfrac{1}{5760}(6 + 2m_1)(4 + 2m_2)(2 + 2m_3)$ $(5 + m_1 + m_2)(4 + m_1 + m_3)(3 + m_2 + m_3)(1 + m_1 - m_2)(1 + m_2 - m_3)$ $(2 + m_1 - m_3)$. In particular,

(5) $d(\Lambda_1 + \Lambda_3) = 70$ $(n = 3, m_1 = 2, m_2 = m_3 = 1)$

(6) $d(\Lambda_2 + \Lambda_3) = 126$ $(n = 3, m_1 = m_2 = 2, m_3 = 1)$

(7) $d(2\Lambda_3) = 84$ $(n = 3, m_1 = m_2 = m_3 = 2)$.

EXAMPLE 27. Let $\mathfrak{o}(n, \boldsymbol{C}) = \{X \in \mathfrak{gl}(n, \boldsymbol{C}) \mid {}^tX + X = 0\}$ be a Lie subalgebra of $\mathfrak{gl}(n, \boldsymbol{C})$. The Killing form $B$ of $\mathfrak{o}(n, \boldsymbol{C})$ is given by $B(X, Y) = \operatorname{Tr} \operatorname{ad} X \operatorname{ad} Y = (n-2) \operatorname{Tr} XY$. If an element $X$ of $\mathfrak{o}(n, \boldsymbol{C})$ satisfies the condition $B(X, A) = 0$ for any $A \in \mathfrak{o}(n, \boldsymbol{C})$, then $X = {}^tX$ since $\operatorname{Tr}(X - {}^tX)Z = \operatorname{Tr} X(Z - {}^tZ) = 0$ for any $Z \in \mathfrak{gl}(n, \boldsymbol{C})$. On the other hand, as $X \in \mathfrak{o}(n, \boldsymbol{C})$ implies $X = -{}^tX$, and we get that $X = 0$. This shows that $B$ is non-degenerate and hence $\mathfrak{o}(n, \boldsymbol{C})$ is semi-simple. First, we shall consider the case of $n = 2m + 1$. Put

$$(1.3) \qquad K = \left(\begin{array}{c|cc} 1 & 0 & \\ \hline & 0 & I_m \\ 0 & & \\ \hline & I_m & 0 \end{array}\right), \qquad T = \left(\begin{array}{c|cc} 1 & 0 & \\ \hline & \frac{1}{\sqrt{2}} I_m & \frac{\sqrt{-1}}{\sqrt{2}} I_m \\ 0 & & \\ \hline & \frac{1}{\sqrt{2}} I_m & -\frac{\sqrt{-1}}{\sqrt{2}} I_m \end{array}\right).$$

Let $\mathfrak{g} = \{A \in \mathfrak{gl}(2m+1, \boldsymbol{C}) \mid {}^tAK + KA = 0\}$ be a Lie subalgebra of $\mathfrak{gl}(2m+1, \boldsymbol{C})$. Since $T^{-1}\mathfrak{g}T = \mathfrak{o}(2m+1, \boldsymbol{C})$, $\mathfrak{g}$ is isomorphic to $\mathfrak{o}(2m+1, \boldsymbol{C})$ over $\boldsymbol{C}$ and sometimes we denote $\mathfrak{g}$ also by $\mathfrak{o}(2m+1, \boldsymbol{C})$. $\mathfrak{g}$ is the totality of elements of $\mathfrak{gl}(2m+1, \boldsymbol{C})$ of the form:

$$(1.4) \qquad \left(\begin{array}{c|c|c} 0 & a_1 \cdots a_m & b_1 \cdots b_m \\ \hline -b_1 & & \\ \vdots & X & Y \\ -b_m & & \\ \hline -a_1 & & \\ \vdots & Z & -{}^tX \\ -a_m & & \end{array}\right) \quad \text{with } {}^tY = -Y, \ {}^tZ = -Z.$$

Denote by $H = H(\lambda_1, \cdots, \lambda_m)$ the element of $\mathfrak{g}$ such that $X$ is a diagonal matrix with diagonal elements $\lambda_1, \cdots, \lambda_m$ and $Y = Z = 0$, $a_i = b_i = 0$ $(1 \leq i \leq m)$ in (1.4). The totality $\mathfrak{h}$ of such elements is a Cartan subalgebra of $\mathfrak{g}$, and $\Delta = \{\pm\lambda_i, \pm\lambda_j \pm \lambda_k \ (j < k)\}$ is the root system of $\mathfrak{g}$. w. r. t. $\mathfrak{h}$. Put $\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3, \cdots, \alpha_{m-1} = \lambda_{m-1} - \lambda_m, \alpha_m = \lambda_m$. Then $\pi = \{\alpha_1, \cdots, \alpha_m\}$ is a fundamental root system because $\lambda_i - \lambda_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $i < j$, $\lambda_i - \lambda_j = -(\alpha_j + \alpha_{j+1} + \cdots + \alpha_i)$ for $i > j$, $\pm\lambda_i = \pm((\lambda_i - \lambda_m) + \lambda_m)$, and $\pm(\lambda_i + \lambda_j) = \pm((\lambda_i - \lambda_m) + (\lambda_j - \lambda_m) + 2\lambda_m)$. As

$$H_{\alpha_i} = \frac{1}{2(2m-1)} H(0, \cdots, \overset{\overset{i}{\vee}}{1}, \overset{\overset{i+1}{\vee}}{-1}, 0, \cdots, 0)$$

for $1 \leq i \leq m-1$ and $H_{\alpha_m} = \frac{1}{2(2m-1)} H(0, \cdots, 0, 1)$, we obtain that $(\alpha_i, \alpha_j) = 0$ for $|i - j| \geqq 2$, $(\alpha_i, \alpha_j) = -\frac{1}{2(2m-1)}$ for $|i - j| = 1$, $(\alpha_i, \alpha_i) = \frac{1}{2m-1}$ for $1 \leq i \leq m-1$ and $(\alpha_m, \alpha_m) = \frac{1}{2(2m-1)}$. This shows that $\mathfrak{g} = \mathfrak{o}(2m+1, \boldsymbol{C})$ is a simple Lie algebra of type $B_m$. The fundamental dominant weights are $\Lambda_1 = \lambda_1, \Lambda_2 = \lambda_1 + \lambda_2, \cdots, \Lambda_{m-1} = \lambda_1 + \cdots + \lambda_{m-1}$, $\Lambda_m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_m)$. If $m = 1$, then $\Lambda_1 = \frac{1}{2}\lambda_1$. Let $V_1$ be a $n(= 2m + 1)$-dimensional vector space over $\boldsymbol{C}$ spanned by $u_1, \cdots, u_n$. For $1 \leq k \leq m-1$, let $V_k$ be a $\binom{n}{k}$-dimensional vector space over $\boldsymbol{C}$ spanned by exterior products $u_{i_1} \wedge \cdots \wedge u_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n)$. Then as we saw in Example 21, $V_k$ is an irreducible representation space of $\mathfrak{sl}(n, \boldsymbol{C})$ and its restriction to $\mathfrak{o}(n, \boldsymbol{C})$ is still irreducible. Thus $V_k$ is an irreducible representation space of $\mathfrak{o}(n, \boldsymbol{C})$ with the highest weight $\Lambda_k$ $(1 \leq k \leq m-1)$. Hence $d(\Lambda_k) = \binom{n}{k} = \binom{2m+1}{k}$ for $1 \leq k \leq m-1$.

Now we shall calculate $d(\Lambda_m)$. The inner product in $\mathfrak{h}_0^*$ is

$$(\sum m_i\lambda_i, \sum n_j\lambda_j) = \frac{1}{2(2m-1)} \sum_{k=1}^{m} m_k n_k \, .$$

Define the inner product $\{\ ,\ \}$ by $\{\sum m_i\lambda_i, \sum n_j\lambda_j\} = \sum m_k n_k$. Under the lexicographical order in $\mathfrak{h}_0^*$ such that $\Pi = \{\alpha_1, \cdots, \alpha_m\}$ are simple roots, $\Delta_+ = \{\lambda_1, \cdots, \lambda_m, \lambda_i \pm \lambda_j \ (1 \leq i < j \leq m)\}$ and $\rho = \sum_{k=1}^{m} (m - k + \frac{1}{2})\lambda_k$. Since $\Lambda_m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_m)$, $\{\Lambda_m, \lambda_i - \lambda_j\} = 0$, $\{\Lambda_m, \lambda_i + \lambda_j\} = 1$, and $\{\Lambda_m, \lambda_i\} = \frac{1}{2}$, the Weyl's dimension formula says that

$$d(\Lambda_m) = \prod_{\alpha\in\Delta_+} \frac{(\Lambda_m+\rho,\alpha)}{(\rho,\alpha)} = \prod_{i=1}^{m} \frac{\{\Lambda_m+\rho,\lambda_i\}}{\{\rho,\lambda_i\}} \prod_{1\le i<j\le m} \frac{\{\Lambda_m+\rho,\lambda_i+\lambda_j\}}{\{\rho,\lambda_i+\lambda_j\}}$$
$$= \prod_{i=1}^{m} \frac{(m-i+1)}{(m-i+\frac{1}{2})} \prod_{1\le i<j\le m} \frac{(2m+2-i-j)}{(2m+1-i-j)} .$$

Here

$$\prod_{1\le i<j\le m} \frac{(2m+2-i-j)}{(2m+1-i-j)}$$
$$= \frac{\prod_{i=1}^{m-1}(2m+2-i-(i+1))\prod_{2\le j-i}(2m+2-i-j)}{\prod_{i=1}^{m-1}(2m+1-i-m)\prod_{1\le i<j<m}(2m+1-i-j)}$$
$$= \prod_{i=1}^{m-1} \frac{(2m+2-i-(i+1))}{(2m+1-i-m)} = 2^{m-1}\prod_{i=1}^{m-1}\frac{(m-i+\frac{1}{2})}{(m-i+1)} ,$$

and hence we get $d(\Lambda_m) = 2^m = 2^{(n-1)/2}$. This $2^m$-dimensional representation of $\mathfrak{g} = \mathfrak{o}(2m+1, C)$ is called the *spin representation*. We shall construct the spin representation in § 5. Finally, we shall calculate $d(s\Lambda_1)$.

By the Weyl's dimension formula,

$$d(s\Lambda_1) = \frac{(s\Lambda_1+\rho,\lambda_1)}{(\rho,\lambda_1)} \prod_{j=2}^{m} \frac{(s\Lambda_1+\rho,\lambda_1+\lambda_j)(s\Lambda_1+\rho,\lambda_1-\lambda_j)}{(\rho,\lambda_1+\lambda_j)(\rho,\lambda_1-\lambda_j)}$$
$$= \frac{(m-\frac{1}{2}+s)}{(m-\frac{1}{2})} \prod_{j=2}^{m} \frac{(2m-j+s)(j-1+s)}{(2m-j)(j-1)}$$
$$= \frac{(2m+s-2)!\,(2m+2s-1)}{(2m-1)!\,s!} .$$

Thus we obtain the following results.

(1) $d(\Lambda_\nu) = \binom{n}{\nu}$ for $1 \le \nu \le m-1$ $(n = 2m+1)$

(2) $d(\Lambda_m) = 2^m = 2^{(n-1)/2}$ $(n = 2m+1)$

(3) $d(s\Lambda_1) = \dfrac{(2m+s-2)!\,(2m+2s-1)}{(2m-1)!\,s!}$, in particular

(3′) $d(2\Lambda_1) = m(2m+3) = \frac{1}{2}(n-1)(n+2)$ $(n = 2m+1)$.

EXAMPLE 28. We shall consider the Lie algebra $\mathfrak{o}(n, C)$ with $n = 2m$. Put

$$(1.5) \qquad K = \left(\begin{array}{c|c} 0 & I_m \\ \hline I_m & 0 \end{array}\right), \qquad T = \left(\begin{array}{c|c} \frac{1}{\sqrt{2}}I_m & \frac{\sqrt{-1}}{\sqrt{2}}I_m \\ \hline \frac{1}{\sqrt{2}}I_m & -\frac{\sqrt{-1}}{\sqrt{2}}I_m \end{array}\right) .$$

$\mathfrak{g} = \{X \in \mathfrak{gl}(2m, \boldsymbol{C}) \,|\, {}^tAK + KA = 0\}$. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{o}(2m, \boldsymbol{C})$ over $\boldsymbol{C}$ since $\mathfrak{o}(2m, \boldsymbol{C}) = T^{-1}\mathfrak{g}T$. One can easily check that

$$(1.6) \qquad \mathfrak{g} = \left\{\left(\begin{array}{c|c} X & Y \\ \hline Z & -{}^tX \end{array}\right) \,\middle|\, {}^tY = -Y, {}^tZ = -Z\right\}$$

and the totality $\mathfrak{h}$ of elements $H = H(\lambda_1, \cdots, \lambda_{n1})$ of $\mathfrak{g}$ such that $X$ is a diagonal matrix with diagonal elements $\lambda_1, \cdots, \lambda_m$ and $Y = Z = 0$ in (1.6) is a Cartan subalgebra of $\mathfrak{g}$. The root system $\Delta$ of $\mathfrak{g}$ w. r. t. $\mathfrak{h}$ is given by $\Delta = \{\lambda_i \pm \lambda_k \ (i \neq k)\}$ and $\Pi = \{\alpha_1 = \lambda_1 - \lambda_2, \cdots, \alpha_{m-1} = \lambda_{m-1} - \lambda_m, \alpha_m = \lambda_{m-1} + \lambda_m\}$ is a fundamental root system because $\lambda_i - \lambda_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $i < j$, $\lambda_i - \lambda_j = -(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{i-1})$ if $i > j$, and $\pm(\lambda_i + \lambda_j) = \pm((\lambda_i - \lambda_{m-1}) + (\lambda_j - \lambda_m) + (\lambda_{m-1} + \lambda_m))$ for $i < j$. As $H_{\alpha_i} = \frac{1}{2(2m-2)} H(0, \cdots, \overset{\overset{i}{\vee}}{1}, \overset{\overset{i+1}{\vee}}{-1}, \cdots, 0)$ $(1 \leq i \leq m-1)$ and $H_{\alpha_m} = \frac{1}{2(2m-2)} H(0, \cdots, 0, 1, 1)$, we obtain that

$$(1.7) \qquad (\alpha_i, \alpha_j) = \begin{cases} 0 & (|i - j| \geqq 2, i \leq m-1, j \leq m-1) \\ \dfrac{-1}{2(2m-2)} (i = m-2, j = m) & \\ \dfrac{-1}{2(2m-2)} (|i - j| = 1, i \leq m-1, j \leq m-1) & \\ 0 & (i \neq m-2, j = m) \\ \dfrac{1}{(2m-2)} & (i = j)\,. \end{cases}$$

If $m = 2$, then $\Pi = \{\alpha_1, \alpha_2\}$, $\{\alpha_1\} \perp \{\alpha_2\}$ and this implies that $\mathfrak{o}(4, \boldsymbol{C})$ is not simple. In fact, $\mathfrak{o}(4, \boldsymbol{C}) \cong \mathfrak{sl}(2, \boldsymbol{C}) \oplus \mathfrak{sl}(2, \boldsymbol{C})$. If $m \geqq 3$, this shows that $\mathfrak{g}(\cong \mathfrak{o}(2m, \boldsymbol{C}))$ is a simple Lie algebra of type $D_m$. The fundamental dominant weights are $\Lambda_\nu = \lambda_1 + \cdots + \lambda_\nu$ $(1 \leq \nu \leq m-2)$, $\Lambda_{m-1} = \frac{1}{2}(\lambda_1 + \cdots + \lambda_{m-1} - \lambda_m)$, and $\Lambda_m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_m)$. Similarly as in Example 27, we can calculate the following results.

(1) $d(\Lambda_\nu) = \binom{n}{\nu} = \binom{2m}{\nu}$ for $1 \leq \nu \leq m-2$ $(n = 2m)$

(2) $d(\Lambda_{m-1}) = d(\Lambda_m) = 2^{m-1} = 2^{n/2-1}$ $(n = 2m)$

(3) $d(2\Lambda_1) = (2m-1)(m+1) = \frac{1}{2}(n-1)(n+2)$ $(n = 2m)$.

Here $\Lambda_2$ is the adjoint representation (see Example 24 (7)).

The irreducible representation $d\rho_\nu$ with the highest weight $\Lambda_\nu(\nu = m-1$, or $m)$ is called the *even* (*resp. odd*) *half-spin representation* of

$\mathfrak{g} = \mathfrak{o}(2m, \boldsymbol{C})$ if $\nu$ is even (resp. odd). Although $d\rho_{m-1}$ and $d\rho_m$ are not equivalent, there exists an outer automorphism $\sigma$ of $\mathfrak{g}$ such that $d\rho_{m-1} \circ \sigma$ and $d\rho_m$ are equivalent. The restrictions of $d\rho_{m-1}$ and $d\rho_m$ to $\mathfrak{o}(2m-1, \boldsymbol{C})$ are equivalent and it is the spin representation of $\mathfrak{o}(2m-1, \boldsymbol{C})$. In the case of $m = 4$, we have $d(\Lambda_1) = d(\Lambda_3) = d(\Lambda_4) = 8$, and moreover there exist outer automorphisms $\sigma_3, \sigma_4$ such that $d\rho_1$, $d\rho_3 \circ \sigma_3$, $d\rho_4 \circ \sigma_4$ are equivalent, where $d\rho_1$ denotes the standard representation of $\mathfrak{o}(8, \boldsymbol{C})$ with the highest weight $\Lambda_1$ (see (5.30) in § 5). The weights of $d\rho_1, d\rho_3, d\rho_4$ are $\{\pm\lambda_i,\ i = 1, 2, 3, 4\}$, $\{\pm\Lambda'_i, i = 1, 2, 3, 4\}$, $\{\pm\Lambda^*_i, i = 1, 2, 3, 4\}$ respectively, where $\Lambda'_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - \lambda_i$, $\Lambda^*_1 = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$, $\Lambda^*_2 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)$, $\Lambda^*_3 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)$, $\Lambda^*_4 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)$. We shall construct the even half-spin representation in § 5.

DEFINITION 29. Let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ on $V$. Let $x$ be an element of $V$ and let $\mathfrak{g}_x = \{A \in \mathfrak{g} \mid d\rho(A)x = 0\}$ be a subset of $\mathfrak{g}$. Then $\mathfrak{g}_x$ is a subalgebra of $\mathfrak{g}$ and is called the *isotropy subalgebra* of $\mathfrak{g}$ at $x$.

EXAMPLE 30. Let $\mathfrak{g}$ be the totality of elements of $\mathfrak{gl}(7, \boldsymbol{C})$ of the form:

$$(1.8) \qquad A = \left(\begin{array}{c|ccc|ccc} 0 & 2d & 2e & 2f & 2a & 2b & 2c \\ \hline a & & & & 0 & f & -e \\ b & & X & & -f & 0 & d \\ c & & & & e & -d & 0 \\ \hline d & 0 & -c & b & & & \\ e & c & 0 & -a & & -{}^tX & \\ f & -b & a & 0 & & & \end{array}\right) \qquad \text{with } X \in \mathfrak{sl}(3, \boldsymbol{C}) .$$

Then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(7, \boldsymbol{C})$. In fact, this is the isotropy subalgebra of $\mathfrak{gl}(7, \boldsymbol{C})$ at $x = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$ under the irreducible representation with the highest weight $\Lambda_3\left(= \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array}\right)$ (see Example 21 and (8) in § 5).

The Killing form $B$ of $\mathfrak{g}$ is $B(A, A') = \operatorname{Tr} \operatorname{ad} A \operatorname{ad} A' = 24(ad' + a'd + be' + b'e + cf' + c'f) + 8 \operatorname{Tr} XX'$, and as this is non-degenerate, $\mathfrak{g}$ is a semi-simple Lie algebra. The totality $\mathfrak{h}$ of elements $H = H(\lambda_1, \lambda_2)$ of $\mathfrak{g}$ such that $X$ is a diagonal matrix with diagonal element $\lambda_1, \lambda_2, \lambda_3 = -\lambda_1 - \lambda_2$, all remaining entries zero in (1.8), is a Cartan subalgebra of $\mathfrak{g}$.

The root system $\Delta$ of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$ is given by

$$\Delta = \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_1\pm\lambda_2, \pm(\lambda_1 + 2\lambda_2), \pm(2\lambda_1 + \lambda_2)\} .$$

A subset $\Pi = \{\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2\}$ of $\Delta$ is a fundamental root system since $\pm\lambda_1 = \pm(\alpha_1 + \alpha_2), \pm\lambda_2 = \pm\alpha_2, \pm(\lambda_1 + \lambda_2) = \pm(\alpha_1 + 2\alpha_2), \pm(\lambda_1 - \lambda_2) = \pm\alpha_1, \pm(\lambda_1 + 2\lambda_2) = \pm(\alpha_1 + 3\alpha_2), \pm(2\lambda_1 + \lambda_2) = \pm(2\alpha_1 + 3\alpha_2)$. Since $B(H, H') = 8(2\lambda_1 + \lambda_2)\lambda_1' + 8(\lambda_1 + 2\lambda_2)\lambda_2'$, we obtain that

$$H_{m_1\lambda_1+m_2\lambda_2} = H\left(\frac{2m_1 - m_2}{24}, \frac{2m_2 - m_1}{24}\right)$$

and $(m_1\lambda_1 + m_2\lambda_2, n_1\lambda_1 + n_2\lambda_2) = \frac{1}{12}(m_1n_1 + m_2n_2) - \frac{1}{24}(m_1n_2 + m_2n_1)$, in particular, $(\alpha_1, \alpha_1) = \frac{1}{4}, (\alpha_1, \alpha_2) = -\frac{1}{8}, (\alpha_2, \alpha_2) = \frac{1}{12}$, i.e., $\|\alpha_1\| = \sqrt{3}\|\alpha_2\|, \widehat{\alpha_1\alpha_2} = \frac{5}{6}\pi$. This shows that the Dynkin diagram of $\mathfrak{g}$ is $\overset{\alpha_1}{\circ} \Rrightarrow \overset{\alpha_2}{\circ}$ (see Lemma 17), and hence $\mathfrak{g}$ is a simple Lie algebra of type $G_2$.

We sometimes denote $\mathfrak{g}$ by $\mathfrak{g}_2$ or $(\mathfrak{g}_2)$. Now we shall calculate the fundamental dominant weights $\Lambda_1 = m_1\lambda_1 + m_2\lambda_2$ and $\Lambda_2 = n_1\lambda_1 + n_2\lambda_2$. Since $\dfrac{2(\Lambda_1, \alpha_1)}{(\alpha_1, \alpha_1)} = m_1 - m_2 = 1$ and $(\Lambda_1, \alpha_2) = \dfrac{1}{24}(2m_2 - m_1) = 0$, we obtain that $m_1 = 2, m_2 = 1$, i.e., $\Lambda_1 = 2\lambda_1 + \lambda_2(=2\alpha_1 + 3\alpha_2)$. Similarly as $(\Lambda_2, \alpha_1) = \dfrac{1}{8}(n_1 - n_2) = 0$ and $\dfrac{2(\Lambda_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 2n_2 - n_1 = 1$, we obtain that $n_1 = n_2 = 1$, i.e., $\Lambda_2 = \lambda_1 + \lambda_2$ $(=\alpha_1 + 2\alpha_2)$. Define the lexicographical order in $\mathfrak{h}_0^*$ such that $\Pi = \{\alpha_1, \alpha_2\}$ are simple roots. Then $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = 3\alpha_1 + 5\alpha_2$. For a dominant integral form $\Lambda = m_1\Lambda_1 + m_2\Lambda_2$, since $(\Lambda + \rho, \alpha_1) = \dfrac{m_1 + 1}{8}, (\Lambda + \rho, \alpha_2) = \dfrac{m_2 + 1}{24}, (\rho, \alpha_1) = \dfrac{1}{8}$, and $(\rho, \alpha_2) = \dfrac{1}{24}$, the Weyl's dimension formula says that $d(\Lambda) = \prod_{\alpha\in\Delta_+} \dfrac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)} = \dfrac{1}{120}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)(2m_1 + m_2 + 3)(3m_1 + m_2 + 4)(3m_1 + 2m_2 + 5)$. In particular, $d(\Lambda_2) = 7, d(\Lambda_1) = 14, d(2\Lambda_2) = 27, d(\Lambda_1 + \Lambda_2) = 64, d(2\Lambda_1) = d(3\Lambda_2) = 77, d(2\Lambda_1 + \Lambda_2) = 189$. In view of Corollary 13, 14 $(= \dim(\mathfrak{g}_2))$-dimensional representation is only $\Lambda_1$, and hence $\Lambda_1$ must be the adjoint representation of $(\mathfrak{g}_2)$. The identity map of $\mathfrak{g}$ into $\mathfrak{gl}(7, \boldsymbol{C})$ is $\Lambda_2$. (We denote the representation corresponding to $\Lambda$, by $\Lambda$. See Theorem 10.) The derivation algebra of the Cayley numbers is also $(\mathfrak{g}_2)$ (see Example 41).

Now we shall construct the exceptional Lie algebras of type $F_4$ and $E_6$.

DEFINITION 31. Let $Q = \boldsymbol{C}\cdot 1 + \boldsymbol{C}\cdot e_1 + \boldsymbol{C}\cdot e_2 + \boldsymbol{C}\cdot e_1e_2 (e_1^2 = e_2^2 = -1, e_1e_2 = -e_2e_1)$ be the *quaternion algebra* over $\boldsymbol{C}$. For a $Q$-module $\mathfrak{C} = Q + Qe$, define the multiplication by $(q + re)\cdot(s + te) = (qs - \bar{t}r) + (tq + r\bar{s})e$ where $q, r, s, t \in Q$ and $\bar{s}, \bar{t}$ are conjugates of $s, t$ respectively. Thus we obtain a non-associative algebra $\mathfrak{C}$ of dimension 8, called the *Cayley algebra* over $\boldsymbol{C}$. The conjugate $\bar{x}$ of $x = q + re (q, r \in Q)$ is defined by $\bar{x} = \bar{q} - re$. Then $\overline{xy} = \bar{y}\cdot\bar{x}$. *The exceptional simple Jordan algebra* $\mathscr{J}$ over $\boldsymbol{C}$ is the non-associative algebra of dimension 27 whose elements are $3 \times 3$ Hermitian matrices with elements in the Cayley algebra $\mathfrak{C}$, multiplication being defined by $X \circ Y = \frac{1}{2}(XY + YX)$ where $XY$ is the ordinary matrix product.

$$\mathscr{J} = \left\{ X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \middle| \begin{matrix} \xi_1, \xi_2, \xi_3 \in \boldsymbol{C} \\ x_1, x_2, x_3 \in \mathfrak{C} \end{matrix} \right\} \tag{1.9}$$

We write the trace $\xi_1 + \xi_2 + \xi_3 = \mathrm{Tr}\, X$. The derivation algebra $\mathscr{D}$ of $\mathscr{J}$ is the Lie algebra of endomorphisms D of $\mathscr{J}$ satisfying $D(X \circ Y) = DX \circ Y + X \circ DY$.

We shall see that $\mathscr{D}$ is a simple Lie algebra of type $F_4$. By a right multiplication $R_Y$ is meant the endomorphism $X \mapsto X \circ Y$ for every $X$ in $\mathscr{J}$. Then $[R_X, R_Y] = R_X \circ R_Y - R_Y \circ R_X$ is a derivation of $\mathscr{J}$ and $[D, R_X] = R_{DX}$ for $D \in \mathscr{D}, X \in \mathscr{J}$. Let $\mathfrak{o}(8, \boldsymbol{C}) = \{X \in \mathfrak{gl}(8, \boldsymbol{C}) \mid {}^tX + X = 0\}$ be the simple Lie algebra $D_4$. We regard the elements of $\mathfrak{o}(8, \boldsymbol{C})$ as endomorphisms of the Cayley algebra $\mathfrak{C}$. This algebra is equipped with a trace function $\mathrm{tr}\, x = x + \bar{x}$ satisfying $\mathrm{tr}\, xy = \mathrm{tr}\, yx, \mathrm{tr}\, x(yz) = \mathrm{tr}\, (xy)z (= \mathrm{tr}\, xyz)$. Also, $\mathrm{tr}\, xy$ is a non-degenerate bilinear form. An endomorphism $U$ of $\mathfrak{C}$ is in $\mathfrak{o}(8, \boldsymbol{C})$ if and only if $U$ leaves the norm form $x\bar{x}$ invariant:

$$(Ux)\bar{x} + x(\overline{Ux}) = 0 \,. \tag{1.10}$$

PROPOSITION 32 (Principle of Triality). *For $U$ in $\mathfrak{o}(8, \boldsymbol{C})$, there exist unique $U', U''$ in $\mathfrak{o}(8, \boldsymbol{C})$ such that*

$$\mathrm{tr}\, (Ux)yz + \mathrm{tr}\, x(U'y)z + \mathrm{tr}\, xy(U''z) = 0 \tag{1.11}$$

*for all $x, y, z$ in $\mathfrak{C}$. These $U'$ and $U''$ are the inequivalent half-spin representations of $U$ in $D_4$ (see Example* 28).

Associated with the exceptional simple Jordan algebra $\mathscr{J}$ are the bilinear form $\mathrm{Tr}\, X \circ Y$ and the trilinear form

$$\phi(X, Y, Z) = \operatorname{Tr}(X \circ Y) \circ Z = \operatorname{Tr} X \circ (Y \circ Z) .$$

PROPOSITION 33. *An endomorphism $D$ of $\mathscr{J}$ is a derivation if and only if $D$ leaves both* $\operatorname{Tr} X \circ Y$ *and* $\phi(X, Y, Z)$ *invariant, i.e.,*

(i) $\operatorname{Tr} DX \circ Y + \operatorname{Tr} X \circ DY = 0$

(ii) $\phi(DX, Y, Z) + \phi(X, DY, Z) + \phi(X, Y, DZ) = 0.$

DEFINITION 34.

$$E_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$(a)_1 = \begin{pmatrix} 0 & & \\ & 0 & a \\ & \bar{a} & 0 \end{pmatrix}, \quad (a)_2 = \begin{pmatrix} 0 & & \bar{a} \\ & 0 & \\ a & & 0 \end{pmatrix}, \quad (a)_3 = \begin{pmatrix} 0 & a & \\ \bar{a} & 0 & \\ & & 0 \end{pmatrix}$$

for $a \in \mathfrak{C}$.

$$\overline{A}_i = \{(a_i) \in \mathscr{J} \mid a \in \mathfrak{C}\} \qquad (1 \leq i \leq 3)$$
$$\mathscr{D}_0 = \{A \in \mathscr{D} \mid A E_i = 0 \qquad \text{for } 1 \leq i \leq 3\}$$

PROPOSITION 35. $\mathscr{D}_0 \cong \mathfrak{o}(8, \boldsymbol{C})$, $\dim \mathscr{D}_0 = 28$.

*Proof.* Since $E_i \circ (a)_i = 0$ and $(a)_i = 2E_j \circ (a)_i (j \neq i)$, we obtain that $D(a)_i = 2E_j \circ D(a)_i$ for $D \in \mathscr{D}_0$, $j \neq i$. This implies that $\mathscr{D}_0 \overline{A}_i \subset \overline{A}_i$ for $1 \leq i \leq 3$, and let $U, U', U''$ be the restriction of $D$ to $\overline{A}_1, \overline{A}_2, \overline{A}_3$ respectively for each $D \in \mathscr{D}_0$. As $D$ leaves $\operatorname{Tr} \begin{pmatrix} 0 & & \\ & 0 & x_1 \\ & \bar{x}_1 & 0 \end{pmatrix}^2 = 2x_1\bar{x}_1$ invariant, $U$ is an element of $\mathfrak{o}(8, \boldsymbol{C})$ and so do $U'$ and $U''$ similarly. Since $D$ leaves the trilinear form $\phi(X, Y, Z)$ invariant, $U, U', U''$ satisfy the principle of triality, i.e., $U', U''$ are two inequivalent half-spin representations of $U$. Thus we get

$$X \mapsto DX = \begin{pmatrix} 0 & U''x_3 & \overline{U'x_2} \\ \overline{U''x_3} & 0 & Ux_1 \\ U'x_2 & \overline{Ux_1} & 0 \end{pmatrix} \quad \text{for } D \in \mathscr{D}_0 . \tag{1.12}$$

Conversely, as a linear transformation of this type leaves both $\operatorname{Tr} X \circ Y$ and $\phi(X, Y, Z)$ invariant, it is a derivation of $\mathscr{J}$ by Proposition 33, and hence we obtain our assertion. Q.E.D.

DEFINITION 36. $(a)'_1 = [R_{E_2}, R_{(a)_1}]$, $(a)'_2 = [R_{E_1}, R_{(a)_2}]$, $(a)'_3 = [R_{E_1}, R_{(a)_3}]$ for $a \in \mathfrak{C}$. $\mathfrak{J}_i = \{(a)'_i \in \mathscr{D} \mid a \in \mathfrak{C}\}$ $(1 \le i \le 3)$

PROPOSITION 37. $\mathscr{D} = \mathscr{D}_0 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \mathfrak{J}_3$, $\dim \mathscr{D} = 52$.

*Proof.* Let $D$ be any derivation of $\mathscr{J}$. As $E_i \circ E_i = E_i$, we get $2E_i \circ (DE_i) = DE_i$ for $1 \le i \le 3$ and hence there exist $a, b, c, d, e, f, g \in \mathfrak{C}$ satisfying $DE_1 = (a)_3 - (b)_2$, $DE_2 = (c)_1 - (d)_3$ and $DE_3 = (f)_2 - (g)_3$. On the other hand, we get $E_1 \circ (DE_2) + (DE_1) \circ E_2 = 0$ from $E_1 \circ E_2 = 0$. This implies $(a)_3 = (d)_3$. Thus we get $DE_1 = (a)_3 - (b)_2$, $DE_2 = -(a)_3 + (c)_1$ and $DE_3 = (b)_2 - (c)_1$. Put $T = 4(a)'_3 - 4(b)'_2 + 4(c)'_1$. Then $T$ is a derivation satisfying $TE_1 = (a)_3 - (b)_2$, $TE_2 = -(a)_3 + (c)_1$, $TE_3 = (b)_2 - (c)_1$ and hence $(D - T)E_i = 0$ $(1 \le i \le 3)$, i.e., $D - T \in \mathscr{D}_0$. This implies that $\mathscr{D} = \mathscr{D}_0 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \mathfrak{J}_3$ and $\dim \mathscr{D} = 52$ since $\dim \mathscr{D}_0 = 28$, $\dim \mathfrak{J}_i = 8$ $(1 \le i \le 3)$. Q.E.D.

EXAMPLE 38. We shall study the derivation Lie algebra $\mathfrak{g} = \mathscr{D}$ of the exceptional simple Jordan algebra $\mathscr{J}$. Let $D$ be an element of $\mathscr{D}_0$ of the form in (1.12). Then $ad(D)\cdot(a)'_1 = [D, [R_{E_2}, R_{(a)_1}]] = [R_{E_2}, [D, R_{(a)_1}]] + [[D, R_{E_2}], R_{(a)_1}] = [R_{E_2}, R_{D(a)_1}] + [R_{DE_2}, R_{(a)_1}] = [R_{E_2}, R_{(Ua)_1}] = (Ua)'_1$. Similarly we have $ad(D)\cdot(a)'_2 = (U'a)'_2$, $ad(D)\cdot(a)'_3 = (U''a)'_3$. Since $U', U''$ are two inequivalent half-spin representations of $U$ in $\mathfrak{o}(8, \boldsymbol{C})$, the adjoint representation of $\mathfrak{g}$ induces a representation of $\mathscr{D}_0 \cong \mathfrak{o}(8, \boldsymbol{C})$ on $\mathfrak{g}$ which is the sum of the adjoint representation $\Lambda_2$ on $\mathscr{D}_0$, the standard representation $\Lambda_1$ on $\mathfrak{J}_1$, and two inequivalent half-spin representations $\Lambda_3, \Lambda_4$ on $\mathfrak{J}_2, \mathfrak{J}_3$ (see Example 28). Hence a Cartan subalgebra $\mathfrak{h}$ of $\mathscr{D}_0 \cong \mathfrak{o}(8, \boldsymbol{C})$ is a Cartan subalgebra of $\mathfrak{g}$ and the root system $\Delta$ is the sum of the weights of $\Lambda_2, \Lambda_1, \Lambda_3, \Lambda_4$, i.e., $\Delta = \{\pm\lambda_i \pm \lambda_j, i < j = 1, 2, 3, 4\,;\ \pm\lambda_i, \pm\Lambda'_i, \pm\Lambda^*_i, i = 1, 2, 3, 4\}$ where $\Lambda'_i, \Lambda^*_i$ are defined as in Example 28. Put $\alpha_1 = \lambda_2 - \lambda_3$, $\alpha_2 = \lambda_3 - \lambda_4$, $\alpha_3 = \lambda_4$, $\alpha_4 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$. Then, $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a fundamental root system, and under the lexicographical order of $\mathfrak{h}_0^*$ such that $\alpha_i (1 \le i \le 4)$ are simple roots, $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$ and $\rho = \sum_{\alpha \in \Delta_+} \alpha = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4$. Let $B$ be the Killing form of $\mathfrak{g}$. Then for $D = D_0 + (a)'_1 + (b)'_2 + (c)'_3 \in \mathfrak{g}$, $D_0 \in \mathscr{D}_0$, $B(D, D) = \frac{3}{2}B'(D_0, D_0) - \frac{9}{2}(a\bar{a} + b\bar{b} + c\bar{c})$ where $B'$ is the Killing form of $\mathscr{D}_0 \cong \mathfrak{o}(8, \boldsymbol{C})$ (see p. 111

[5]). As $B$ is non-degenerate, $\mathfrak{g}$ is semi-simple. Since $B(H, H) = 18\sum_{i=1}^{4}\lambda_i^2$ for $H = H(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathfrak{h}$, we get

$$H_{\Sigma m_i \lambda_i} = H\left(\frac{m_1}{18}, \frac{m_2}{18}, \frac{m_3}{18}, \frac{m_4}{18}\right),$$

$(\sum m_i\lambda_i, \sum n_j\lambda_j) = \frac{1}{18}\sum_{i=1}^{4} m_i n_i$ and in particular $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = \frac{1}{9}$, $(\alpha_3, \alpha_3) = (\alpha_4, \alpha_4) = \frac{1}{18}$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -\frac{1}{18}$, $(\alpha_3, \alpha_4) = -\frac{1}{36}$, $(\alpha_1, \alpha_3) = (\alpha_1, \alpha_4) = (\alpha_2, \alpha_4) = 0$. This shows that the Dynkin diagram of $\mathfrak{g}$ is $\overset{\alpha_1}{\circ}\!-\!\!\!-\!\!\!-\overset{\alpha_2}{\circ} \Longrightarrow \overset{\alpha_3}{\circ}\!-\!\!\!-\!\!\!-\overset{\alpha_4}{\circ}$ and $\mathfrak{g} = \mathscr{D}$ is the simple Lie algebra of type $F_4$. Put $\Lambda_1 = 18(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4)$, $\Lambda_2 = 18(3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4)$, $\Lambda_3 = 18(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4)$, and $\Lambda_4 = 18(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$. Then since $\dfrac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$ for $1 \leq i, j \leq 4$, $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are the fundamental dominant weights. Let $\Lambda = \sum_{i=1}^{4} m_i\Lambda_i$ ($m_i \geqq 0$, integer) be any dominant integral form. Then the Weyl's dimension formula says that $d(\Lambda) = \prod_{\alpha \in \Delta_+} \dfrac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)}$ $= \dfrac{1}{2^{15}\cdot 3^7\cdot 5^4\cdot 7^2\cdot 11}(m_1 + 1)(m_2 + 1)(m_3 + 1)(m_4 + 1)(m_1 + m_2 + 2)(2m_2 + m_3 + 3)(m_3 + m_4 + 2)(m_2 + m_3 + 2)(2m_1 + 2m_2 + m_3 + 5)(2m_2 + m_3 + m_4 + 4)(2m_1 + 2m_2 + m_3 + m_4 + 6)(m_1 + m_2 + m_3 + 3)(2m_2 + 2m_3 + m_4 + 5)(m_2 + m_3 + m_4 + 3)(m_1 + 2m_2 + m_3 + 4)(2m_1 + 2m_2 + 2m_3 + m_4 + 7)(m_1 + m_2 + m_3 + m_4 + 4)(2m_1 + 4m_2 + 2m_3 + m_4 + 9)(m_1 + 2m_2 + m_3 + m_4 + 5)(2m_1 + 4m_2 + 3m_3 + m_4 + 10)(2m_1 + 4m_2 + 3m_3 + 2m_4 + 11)(m_1 + 2m_2 + 2m_3 + m_4 + 6)(m_1 + 3m_2 + 2m_3 + m_4 + 7)(2m_1 + 3m_2 + 2m_3 + m_4 + 8)$. For example, $d(\Lambda_4) = 26$, $d(\Lambda_1) = 52$, $d(\Lambda_3) = 273$, $d(2\Lambda_4) = 324$, $d(2\Lambda_1) = d(\Lambda_1 + \Lambda_4) = 1053$, $d(\Lambda_2) = 1274$, $d(3\Lambda_4) = 2652$, $d(2\Lambda_3) = 19448$, $d(2\Lambda_2) = 226746$, etc. Let $\mathscr{J}_0 = \{X \in \mathscr{J} \mid \operatorname{Tr} X = 0\}$ be a 26-dimensional subspace of $\mathscr{J}$ over $\boldsymbol{C}$. Then, $\mathscr{J}_0$ is an invariant subspace of the derivation algebra $F_4 = \mathscr{D}$ of $\mathscr{J}$, i.e., $\mathscr{J}_0$ is an irreducible representation space of $\Lambda_4$. Obviously, $\Lambda_1$ is the adjoint representation of $F_4$.

EXAMPLE 39. Let $\mathfrak{g}$ be the Lie algebra spanned by the derivations $(=F_4)$ of $\mathscr{J}$ and the right multiplications of elements $Y$ of trace 0. If $X$ and $Y$ are in $\mathscr{J}$, then $[R_X, R_Y]$ is in $\mathscr{D}(=F_4)$; moreover, if $D \in \mathscr{D}$, then $[D, R_Y] = R_{DY}$ and $\operatorname{Tr} DY = 0$. It follows that $\mathfrak{g} = \mathscr{D} + \{R_Y\}$, $\operatorname{Tr} Y = 0$. Since $D(1) = 0$ for every derivation $D$ of $\mathscr{J}$, $D + R_Y = 0$ implies $Y = 0$, $D = 0$; thus $\mathfrak{g}$ is of dimension 78, and the adjoint representation of $\mathfrak{g}$

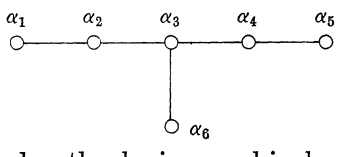
induces a representation of $\mathscr{D}$ ($=F_4$) which is the sum of the adjoint representation $\Lambda_1$ and of the representation $\Lambda_4$ whose space is the set $\mathscr{R}$ of right multiplications of elements in $\mathscr{J}_0$ Since $\mathscr{D}$ and $\mathscr{R}$ yield irreducible representation spaces of $\mathscr{D}$ of distinct dimensions, the only possible ideals are $\{0\}$, $\mathscr{D}, \mathscr{R}$ and $\mathscr{D} + \mathscr{R} = \mathfrak{g}$; but $\mathscr{D}$ and $\mathscr{R}$ are obviously not ideals, which proves that $\mathfrak{g}$ is simple (these are quoted from [4]). Let $B$ be the Killing form of $\mathfrak{g}$. Then

$$B(R_a + D, R_a + D) = 12 \operatorname{Tr} a \circ a + \frac{4}{3} B'(D, D)$$

where $B'$ is the Killing form of $F_4 = \mathscr{D}$. Let $\mathfrak{h}'$ be a Cartan subalgebra of $F_4 = \mathscr{D}$ and let $H_1, H_2, H_3, H_4$ be its basis satisfying $H(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_{i=1}^4 \lambda_i H_i$ for any $H(\lambda_1, \cdots, \lambda_4) \in \mathfrak{h}'$. Set $H_5 = R_{E_1}, H_6 = R_{E_2}, H_7 = R_{E_3}$ where $E_i$ are as in Definition 34. Then $\mathfrak{h} = \{\sum_{i=1}^7 \lambda_i H_i \mid \lambda_5 + \lambda_6 + \lambda_7 = 0\}$ is a Cartan subalgebra of $\mathfrak{g}$. The root system $\Delta$ of $\mathfrak{g}$ w. r. t. $\mathfrak{h}$ is given by $\Delta = \{\pm\lambda_i \pm \lambda_j, i < j = 1, 2, 3, 4;\ \pm\lambda_i \pm \frac{1}{2}(\lambda_5 - \lambda_7);\ \pm\Lambda'_i \pm \frac{1}{2}(\lambda_5 - \lambda_6);\ \pm\Lambda^*_i \pm \frac{1}{2}(\lambda_6 - \lambda_7),\ i = 1, 2, 3, 4\}$ where $\Lambda'_i, \Lambda^*_i$ are defined as in Example 28. Put $\alpha_1 = -\Lambda'_1 + \frac{1}{2}(\lambda_5 - \lambda_6)$, $\alpha_2 = \lambda_4 - \frac{1}{2}(\lambda_5 - \lambda_7)$, $\alpha_3 = \lambda_3 - \lambda_4$, $\alpha_4 = \lambda_4 + \frac{1}{2}(\lambda_5 - \lambda_7)$, $\alpha_5 = -\Lambda'_1 - \frac{1}{2}(\lambda_5 - \lambda_6)$, $\alpha_6 = \lambda_2 - \lambda_3$. Then $\Pi = \{\alpha_1, \cdots, \alpha_6\}$ is a fundamental root system. Since

$$B\left(\sum \lambda_i H_i, \sum \lambda'_i H_i\right) = 24 \sum_{i=1}^{4} \lambda_i \lambda'_i + 12 \sum_{i=5}^{7} \lambda_i \lambda'_i ,$$

we obtain that $(\sum m_i \lambda_i, \sum n_j \lambda_j) = \frac{1}{24} \sum_{i=1}^4 m_i n_i + \frac{1}{12} \sum_{i=5}^7 m_i n_i$, in particular $(\alpha_i, \alpha_i) = \frac{1}{12}$ for $1 \leq i \leq 6$, $(\alpha_i, \alpha_{i+1}) = -\frac{1}{24}$ for $1 \leq i \leq 4$, and $(\alpha_i, \alpha_j) = 0$ for $|i - j| \geqq 2$ except for $(\alpha_3, \alpha_6) = -\frac{1}{24}$. This shows that the Dynkin diagram of $\mathfrak{g}$ is




i.e., $\mathfrak{g}$ is the simple Lie algebra of type $E_6$. Under the lexicographical order in $\mathfrak{h}_0^*$ such that $\Pi = \{\alpha_1, \cdots, \alpha_6\}$ are simple roots, all positive roots $\Delta_+$ is given by $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6;\ \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_3 + \alpha_6;\ \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_6;\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6;\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6;\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6, \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6;\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6;\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 +$

$\alpha_6$; $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$; $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6\}$. Hence we obtain that $\rho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 15\alpha_4 + 8\alpha_5 + 11\alpha_6$, and $(\rho, \alpha_i) = \frac{1}{24}$ for $1 \leq i \leq 6$. Let $\Lambda_1, \cdots, \Lambda_6$ be the fundamental weights, i.e., $\dfrac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$ for $i, j, = 1, \cdots, 6$. Let $\Lambda = \sum_{i=1}^{6} m_i \Lambda_i$ ($m_i \geqq 0$, integer) be any dominant integral form of $\mathfrak{g} = E_6$. Since $(\Lambda + \rho, \alpha_i) = \frac{1}{2} m_i(\alpha_i, \alpha_i) + \frac{1}{24} = \frac{1}{24}(m_i + 1)$ for $1 \leq i \leq 6$, the Weyl's dimension formula says that

$$d(\Lambda) = \prod_{\alpha \in \Delta_+} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)} = \prod_{\sum n_i \alpha_i \in \Delta_+} \frac{(\sum_{i=1}^{6} n_i(m_i + 1))}{(\sum_{i=1}^{6} n_i)} .$$

For example, $d(\Lambda_1) = d(\Lambda_5) = 27$, $d(\Lambda_6) = 78$, $d(2\Lambda_1) = d(\Lambda_2) = d(\Lambda_4) = 351$, $d(\Lambda_3) = 2925$, $d(2\Lambda_6) = 2430$, etc. Here $\Lambda_1$ and $\Lambda_5$ are contragredient of each other and so do $\Lambda_2$ and $\Lambda_4$. Let $d\rho_i$ be the fundamental irreducible representations of $E_6$ ($1 \leq i \leq 6$). Then there exists an outer automorphism $\sigma$ of $E_6$ such that $d\rho_5$ and $d\rho_1 \circ \sigma$ are equivalent. The representation space of $\Lambda_1$ (and $\Lambda_5$) is the exceptional simple Jordan algebra $\mathscr{J}$. $\Lambda_6$ is the adjoint representation.

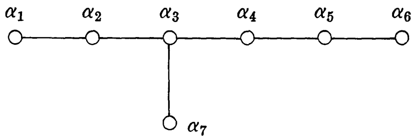
Let $N(X) = \det X = \xi_1\xi_2\xi_3 + \operatorname{tr} x_1x_2x_3 - \xi_1 x_1 \bar{x}_1 - \xi_2 x_2 \bar{x}_2 - \xi_3 x_3 \bar{x}_3$ be the determinant of $X$ of the form (1.9) in $\mathscr{J}$. Then $E_6$ leaves $N(X)$ invariant. Let $N(X, Y, Z)$ be the trilinear form obtained by polarizing $N(X)$. Then the Lie algebra $E_6$ can be characterized as the set of linear transformations $L$ of $\mathscr{J}$ such that $N(LX, Y, Z) + N(X, LY, Z) + N(X, Y, LZ) = 0$.

EXAMPLE 40. We shall construct the Lie algebra of type $E_7$. Let $\bar{\mathscr{E}}_6(\mathscr{J})$ be the Lie algebra spanned by the derivations $\mathscr{D}(=F_4)$ of $\mathscr{J}$ and and the right multiplications of elements $Y$, but not necessarily of trace 0, i.e., $\bar{\mathscr{E}}_6(\mathscr{J}) = E_6 \oplus \boldsymbol{C}R_I$ where $I$ is the unit matrix in $\mathscr{J}$. Let $\bar{\mathscr{J}}$ be a vector space isomorphic to $\mathscr{J}$ under the mapping $X \mapsto \bar{X}$. We shall define the structure of a Lie algebra in a vector space $\mathfrak{g} = \mathscr{J} \oplus \bar{\mathscr{J}} \oplus \bar{\mathscr{E}}_6(\mathscr{J})$ of dimension 133 over $\boldsymbol{C}$ such that $\bar{\mathscr{E}}_6(\mathscr{J})$ is a subalgebra and

$$\begin{aligned}
&[X, Y] = 0 = [\bar{X}, \bar{Y}] \qquad \text{for } X, Y \in \mathscr{J} \\
&[X, \bar{Y}] = 2R_{X \circ Y} + 2[R_Y, R_X] \\
&[L, X] = LX \qquad \text{for } L \in \bar{\mathscr{E}}_6(\mathscr{J}), X \in \mathscr{J} \\
&[L, \bar{X}] = \overline{\bar{L}X} \qquad \text{where } \bar{L} = -R_Y + D \text{ if} \\
&\qquad L = R_Y + D, D \in \mathscr{D}(=F_4) .
\end{aligned} \tag{1.13}$$

The Killing form $B$ of $\mathfrak{g}$ is given by $B(A, A') = 2B'(D, D') + 18T(Z, Z')$

$-36(T(X, Y') + T(Y, X'))$ for $A = (X, \bar{Y}, R_Z + D)$, $A' = (X', \bar{Y}', R_{Z'} + D') \in \mathfrak{g}$ where $B'$ is the Killing form of $F_4$ and $T(X, Y) = \mathrm{Tr}\, X \circ Y$ for $X, Y \in \mathcal{J}$. Since $B$ is non-degenerate, the Lie algebra $\mathfrak{g}$ is semi-simple. Let $\mathfrak{h} = \{\sum_{i=1}^{7} \lambda_i H_i \mid \lambda_1, \cdots, \lambda_7 \in \boldsymbol{C}\}$ be the subalgebra of $\mathscr{E}_6(\mathcal{J})$ where $\{H_1, \cdots, H_4\}$ is a basis of a Cartan subalgebra of $F_4$ and $H_5 = R_{E_1}$, $H_6 = R_{E_2}$, $H_7 = R_{E_3}$ (see Example 39), but here we don't assume that $\lambda_5 + \lambda_6 + \lambda_7 = 0$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and the root system $\Delta$ of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$ is given by $\Delta = \{\pm\lambda_i \pm \lambda_j,\ i < j = 1, 2, 3, 4,\ \pm\lambda_i \pm \frac{1}{2}(\lambda_5 - \lambda_7),\ \pm\Lambda_i' \pm \frac{1}{2}(\lambda_5 - \lambda_6),\ \pm\Lambda_i^* \pm \frac{1}{2}(\lambda_6 - \lambda_7),\ \pm\lambda_i \pm \frac{1}{2}(\lambda_5 + \lambda_7),\ \pm\Lambda_i' \pm \frac{1}{2}(\lambda_5 + \lambda_6),\ \pm\Lambda_i^* \pm \frac{1}{2}(\lambda_6 + \lambda_7),\ i = 1, 2, 3, 4,\ \pm\lambda_5, \pm\lambda_6, \pm\lambda_7\}$ where $\Lambda_i', \Lambda_i^*$ are defined as in Example 28. Put $\alpha_1 = \lambda_2 - \lambda_3$, $\alpha_2 = \lambda_3 - \lambda_4$, $\alpha_3 = \lambda_4 - \frac{1}{2}(\lambda_5 + \lambda_7)$, $\alpha_4 = \lambda_5$, $\alpha_5 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6)$, $\alpha_6 = \lambda_6$, $\alpha_7 = \lambda_7$. Then $\Pi = \{\alpha_1, \cdots, \alpha_7\}$ is a fundamental root system of $\mathfrak{g}$. Since $B(\sum \lambda_i H_i, \sum \lambda_i' H_i) = 36 \sum_{i=1}^{4} \lambda_i \lambda_i' + 18 \sum_{i=5}^{7} \lambda_i \lambda_i'$, we obtain the inner product $(\sum m_i \lambda_i, \sum n_j \lambda_j) = \frac{1}{36} \sum_{i=1}^{4} m_i n_i + \frac{1}{18} \sum_{i=5}^{7} m_i n_i$. Hence $(\alpha_i, \alpha_i) = \frac{1}{18}$ for $1 \leq i \leq 7$, $(\alpha_3, \alpha_7) = (\alpha_i, \alpha_{i+1}) = -\frac{1}{36}(1 \leq i \leq 5)$, and $(\alpha_i, \alpha_j) = 0$ if $|i - j| \geqq 2$ except $(\alpha_3, \alpha_7) = (\alpha_7, \alpha_3)$. This shows that the Dynkin diagram of $\mathfrak{g}$ is




, i.e., $\mathfrak{g}$ is the simple Lie algebra of type $E_7$. Since we have $\Delta$ and $\Pi$, we can easily determine 63-positive roots $\Delta_+$ and we get $2\rho = \sum_{\alpha \in \Delta_+} \alpha = 34\alpha_1 + 66\alpha_2 + 96\alpha_3 + 75\alpha_4 + 52\alpha_5 + 27\alpha_6 + 49\alpha_7$ and $(\rho, \alpha_i) = \frac{1}{36}(1 \leq i \leq 7)$. Let $\Lambda_1, \cdots, \Lambda_7$ be the fundamental weights of $\mathfrak{g} = E_7$, i.e., $\dfrac{(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}\ i, j = 1, \cdots, 7$, and let $\Lambda = \sum_{i=1}^{7} m_i \Lambda_i$ ($m_i \geqq 0$, integer) be any dominant integral form. Then $(\Lambda + \rho, \alpha_i) = \dfrac{m_i}{2}(\alpha_i, \alpha_i) + (\rho, \alpha_i) = \dfrac{1}{36}(m_i + 1)$, and hence $(\Lambda + \rho, \sum n_i \alpha_i) + \dfrac{1}{36} \sum_{i=1}^{7} n_i(m_i + 1)$. The Weyl's dimension formula says that

$$d(\Lambda) = \prod_{\alpha \in \Delta_+} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)} = \prod_{\sum n_i \alpha_i \in \Delta_+} \frac{\sum_{i=1}^{7} n_i(m_i + 1)}{(\sum_{i=1}^{7} n_i)} .$$

For example, $d(\Lambda_1) = 133$, $d(\Lambda_2) = 8645$, $d(\Lambda_3) = 365750$, $d(\Lambda_4) = 27664$, $d(\Lambda_5) = 1539$, $d(\Lambda_6) = 56$, $d(\Lambda_7) = 912$, $d(2\Lambda_1) = 7371$, $d(2\Lambda_6) = 1463$, $d(\Lambda_1 + \Lambda_6) = 3920$, etc. Obviously $\Lambda_1$ is the adjoint representation of $\mathfrak{g} = E_7$. We shall construct the representation $\Lambda_6$ of degree 56. For this purpose, we shall define the Freudenthal product $\times$ in $\mathcal{J}$ by $a \times b = a \circ b - \frac{1}{2}\mathrm{Tr}\,(a)b - \frac{1}{2}\mathrm{Tr}\,(b)a + \frac{1}{2}[\mathrm{Tr}\,(a) \cdot \mathrm{Tr}\,(b) - \mathrm{Tr}\, a \circ b] \cdot 1$ for $a, b \in \mathcal{J}$. Let $A^*$

denote the adjoint of a linear transformation $A$ in $\mathscr{J}$ relative to the trace form $T(a, b) = \operatorname{Tr} a \circ b$, i.e., $T(Aa, b) = T(a, A^*b)$. Then we know that $R_a$ is self-adjoint.

We now define $\mathfrak{M} = \boldsymbol{C} \oplus \boldsymbol{C} \oplus \mathscr{J} \oplus \mathscr{J}$ a vector space direct sum of two copies of $\boldsymbol{C}$ and two copies of $\mathscr{J}$, so $\dim \mathfrak{M} = 56$. We write the elements of $\mathfrak{M}$ as $X = (\xi, \eta, x, y)$ where $\xi, \eta \in \boldsymbol{C}, x, y \in \mathscr{J}$, and we define an action of $\mathfrak{g} = \mathscr{J} \oplus \bar{\mathscr{J}} \oplus \bar{\mathscr{E}}_6(\mathscr{J})$ by the following formulas:

$$
\begin{aligned}
[a, X] &= (T(a, y), 0, \eta a, 2a \times x) \qquad \text{for } a \in \mathscr{J} \\
[\bar{a}, X] &= (0, -T(a, x), -2a \times y, -\xi a) \qquad \text{for } \bar{a} \in \bar{\mathscr{J}} \\
[2R_I, X] &= (3\xi, -3\eta, -x, y) \\
[L, X] &= (0, 0, Lx, -L^*y) \qquad \text{for } L \in \mathscr{E}_6(\mathscr{J})\,.
\end{aligned}
\tag{1.14}
$$

Here, $\mathscr{E}_6(\mathscr{J})$ is a subalgebra of $\bar{\mathscr{E}}_6(\mathscr{J})$ of elements of the form $R_a + D$, $\operatorname{Tr} a = 0, D$ a derivation. Then the action of $\mathfrak{g}$ on $\mathfrak{M}$ thus defined gives an irreducible representation of $\mathfrak{g}$ on $\mathfrak{M}$ of the highest weight $\Lambda_6$. Define a non-degenerate skew bilinear form $\{\ ,\ \}$ and a quartic form $q$ on $\mathfrak{M}$ by

$$\{X_1, X_2\} = \xi_1\eta_2 - \xi_2\eta_1 + T(x_1, y_2) - T(x_2, y_1) \tag{1.15}$$

$$q(X) = T(x^\#, y^\#) - \xi N(x) - \eta N(y) - \tfrac{1}{4}(T(x, y) - \xi\eta)^2 \tag{1.16}$$

where $a^\# = a^2 - \operatorname{Tr}(a)\cdot a + \frac{1}{2}\{(\operatorname{Tr} a)^2 - \operatorname{Tr}(a^2)\}\cdot 1$, and $N(x) = \det x$ (see Example 39), $T(x, y) = \operatorname{Tr} x \circ y$, $X = (\xi, \eta, x, y)$, $X_i = (\xi_i, \eta_i, x_i, y_i)$ for $i = 1, 2$.

It can be shown that if $q(X_1, X_2, X_3, X_4)$ is the symmetric 4-linear form obtained by polarizing $q$, then the Lie of algebra linear transformations in $\mathfrak{M}$ corresponding to $\mathfrak{g}$ can be characterized as the set of linear transformations $A$ of $\mathfrak{M}$ such that $q(AX_1, X_2, X_3, X_4) + q(X_1, AX_2, X_3, X_4) + q(X_1, X_2, AX_3, X_4) + q(X_1, X_2, X_3, AX_4) = 0$. Also direct verification using (1.14) shows that $\{AX_1, X_2\} + \{X_1, AX_2\} = 0$ for $A \in \mathfrak{g}$.

EXAMPLE 41. We shall construct the Lie algebra of type $E_8$. Only the fact that the least dimensional representation of $E_8$ is the adjoint representation will be used later. For this purpose, we consider first the derivation algebra $\operatorname{Der}(\mathfrak{C})$ of the Cayley algebra $\mathfrak{C}$ (see Definition 31). Put $f_1 = \dfrac{1 + \sqrt{-1}\, e_1}{2}$, $f_2 = \dfrac{1 - \sqrt{-1}\, e_1}{2} = 1 - f_1$. Then $f_i^2 = f_i$ $(i = 1, 2)$, $f_1 f_2 = f_2 f_1 = 0$, $f_1 + f_2 = 1$, and $f_1 = \bar{f}_2$, $f_2 = \bar{f}_1$. Since $f_i \mathfrak{C} f_i = \boldsymbol{C} f_i$ $(i = 1, 2)$, we obtain the Peirce decomposition $\mathfrak{C} = (f_1 + f_2)\mathfrak{C}(f_1 + f_2) = \boldsymbol{C} f_1 \oplus \boldsymbol{C} f_2 \oplus f_1\mathfrak{C} f_2 \oplus f_2 \mathfrak{C} f_1$ of $\mathfrak{C}$, where $f_1 \mathfrak{C} f_2$ and $f_2 \mathfrak{C} f_1$ are three-dimensional.

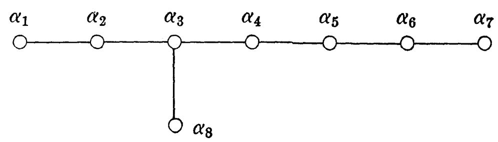
Let $\mathfrak{D}_0$ be the subalgebra of Der($\mathfrak{C}$) of derivations mapping $f_1$ (and hence $f_2$) into 0. Such a $D$ maps the Peirce components $f_i\mathfrak{C}f_j$ into themselves. The representations of $\mathfrak{D}_0$ in the space $f_i\mathfrak{C}f_j$ $(i \neq j)$ can be considered as the standard representation of $\mathscr{D}_0 \cong \mathfrak{sl}(3, \boldsymbol{C})$ and its contragredient. Let $D$ be any derivation in $\mathfrak{C}$. We have $Df_i = f_i(Df_i) + (Df_i)f_i$ since $f_i^2 = f_i$ $(i = 1, 2)$ and hence, together with $Df_1 = -Df_2$, $Df_1 \in f_1\mathfrak{C}f_2 + f_2\mathfrak{C}f_1$ so that we have $Df_1 = a_{12} - b_{21}$, $Df_2 = -a_{12} + b_{21}$ where $a_{12} \in f_1\mathfrak{C}f_2$, $b_{21} \in f_2\mathfrak{C}f_1$. Put $D_{a,b} = (ab)_R - (ab)_L - 3[a_L b_R]$ for $a, b \in \mathfrak{C}$ where we write $a_R$ for $X \to Xa$ and $a_L$ for $X \to aX$. Then $D_{a,b}$ is a derivation of $\mathfrak{C}$ and $D_{f_1,a_{12}} \cdot f_1 = a_{12}$, $D_{f_2,b_{21}} f_2 = b_{21}$ for $a_{12} \in f_1\mathfrak{C}f_2$, $b_{21} \in f_2\mathfrak{C}f_1$. Thus we obtain that $E = D - D_{f_1,a_{12}} - D_{f_2,b_{21}} \in \mathscr{D}_0$, i.e., $D = E + D_{f_1,a_{12}} + D_{f_2,b_{21}}$. Since this expression is unique, we have $Der(\mathfrak{C}) = \mathscr{D}_0 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2$, $\mathfrak{J}_1 = \{D_{f_2,b_{21}} | b_{21} \in f_2\mathfrak{C}f_1\}$, $\mathfrak{J}_2 = \{D_{f_1,a_{12}} | a_{12} \in f_1\mathfrak{C}f_2\}$ and $\dim Der(\mathfrak{C}) = 8 + 3 + 3 = 14$. Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{D}_0 \cong \mathfrak{sl}(3, \boldsymbol{C})$. Then $\mathscr{H}$ acts diagonally in $Der(\mathfrak{C})$ and has weights of the adjoint representation, the standard representation and the contragredient of this. It follows that $\mathscr{H}$ is a Cartan subalgebra of $Der(\mathfrak{C})$ and the root system $\varDelta$ of $Der(\mathfrak{C})$ w.r.t. $\mathscr{H}$ is given by

$$\varDelta = \{\pm\lambda_1, \pm\lambda_2, \pm(\lambda_1 + \lambda_2), \pm(\lambda_1 - \lambda_2), \pm(\lambda_1 + 2\lambda_2), \pm(2\lambda_1 + \lambda_2)\} .$$

Squaring these and adding we obtain $\langle h, h \rangle_{Der(\mathfrak{C})} = 16(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)$ for $h = h(\lambda_1, \lambda_2) \in \mathscr{H}$ where $\langle\ ,\ \rangle_{Der(\mathfrak{C})}$ is the Killing form of $Der(\mathfrak{C})$. Thus we obtain that $Der(\mathfrak{C})$ is a simple Lie algebra of type $G_2$, i.e., $Der(\mathfrak{C}) = \mathfrak{g}_2$ (see Example 30), and $\mathfrak{C}_0 = \{a \in \mathfrak{C} | \operatorname{tr} a = a + \bar{a} = 0\}$ is a 7-dimensional irreducible representation space of $\mathfrak{g}_2 = Der(\mathfrak{C})$. Let $\mathscr{J}$ be the exceptional simple Jordan algebra and $\mathscr{J}_0 = \{X \in \mathscr{J} | \operatorname{Tr} X = 0\}$. As we saw in Example 38, $\mathscr{J}_0$ is a 26-dimensional irreducible representation space of $F_4 = Der(\mathscr{J})$. We are now ready to define Lie algebra $\mathfrak{g} = Der(\mathfrak{C}) \oplus \mathfrak{C}_0 \otimes \mathscr{J}_0 \oplus Der(\mathscr{J})$ of dimension $14 + 7 \times 26 + 52 = 248$. We require $Der(\mathfrak{C}) \oplus Der(\mathscr{J})$ the Lie algebra direct sum of $Der(\mathfrak{C})$ and $Der(\mathscr{J})$ to be a subalgebra of $\mathfrak{g}$. We define that $[a \otimes x, D + E] = (Da) \otimes x + a \otimes (Ex)$ for $D \in Der(\mathfrak{C})$, $E \in Der(\mathscr{J})$, $a \in \mathfrak{C}_0$, $x \in \mathscr{J}_0$. Finally we require $[a \otimes x, b \otimes y] = \frac{1}{12}T(x, y)D_{a,b} + (a*b) \otimes (x*y) + \frac{1}{2}t(a, b)[R_X, R_Y]$, $a, b \in \mathfrak{L}_0$, $x, y \in \mathscr{J}_0$ where $a*b = ab - \frac{1}{2}t(a, b)\cdot 1$, $x*y = x \circ y - \frac{1}{3}T(x, y)\cdot 1$, $t(a, b) = \operatorname{tr} ab$, $T(x, y) = \operatorname{Tr} x \circ y$. Then one can check that this defines the Lie algebra structure in $\mathfrak{g}$ (see [5]). The Killing form $B$ of $\mathfrak{g}$ is given by $B(D + a \otimes x + E, D' + b \otimes y + E') = \frac{15}{2}\langle D, D' \rangle_{Der(\mathfrak{C})} + 15\ t(a, b)T(X, Y) + \frac{10}{3}\langle E, E' \rangle_{Der(\mathscr{J})}$, $D, D' \in Der(\mathfrak{C})$, $E, E'$

$\in Der(\mathscr{J})$, $a, b \in \mathfrak{C}_0$, $x, y \in \mathscr{J}_0$ where $\langle\ ,\ \rangle_{Der(\mathfrak{C})}$ and $\langle\ ,\ \rangle_{Der(\mathscr{J})}$ is the Killing form of $\mathfrak{g}_2 = Der(\mathfrak{C})$ and $F_4 = Der(\mathscr{J})$ respectively. Let $\mathfrak{M}_0 = (\sum_{i=1}^{3} CE_i) \cap \mathscr{J}_0$, $\mathfrak{n}_0 = C(f_1 - f_2)$ and put $\mathfrak{h} = \mathfrak{h}_{\mathfrak{g}_2} \oplus \mathfrak{n}_0 \otimes \mathfrak{M}_0 \oplus \mathfrak{h}_{F_4}$ where $\mathfrak{h}_{\mathfrak{g}_2}, \mathfrak{h}_{F_4}$ are Cartan subalgebras of $\mathfrak{g}_2 = Der(\mathfrak{C})$ and $F_4 = Der(\mathscr{J})$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\{H_1, H_2, H_3, H_4\}$ be a basis for $\mathfrak{h}_{F_4}$ such that $H(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_{i=1}^{4} \lambda_i H_i$ in Example 38, $\{H_5, H_6\}$ a basis for $\mathfrak{h}_{\mathfrak{g}_2}$ as in Example 30, and put $H_7 = (f_1 - f_2) \otimes (E_1 - E_3)$, $H_8 = (f_1 - f_2) \otimes (E_2 - E_3)$. Then $\{H_1, \cdots, H_8\}$ is a basis of $\mathfrak{h}$, and if we denote an element $\sum \lambda_i H_i \mapsto \sum a_i \lambda_i$ of $\mathfrak{h}^*$ by $\sum a_i \lambda_i$, the root system $\Delta$ of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$ is given by $\Delta = \Big\{ \pm\lambda_i \pm \lambda_j\ i < j = 1, 2, 3, 4;\ \pm(\lambda_5 - \lambda_6), \pm(\lambda_5 + 2\lambda_6), \pm(2\lambda_5 + \lambda_6);\ \pm\lambda_i \pm \dfrac{2\lambda_7 + \lambda_8}{2}, \pm\Lambda_i' \pm \dfrac{\lambda_7 - \lambda_8}{2}, \pm\Lambda_i^* \pm \dfrac{\lambda_7 + 2\lambda_8}{2}\ i = 1, 2, 3, 4;\ \pm(\lambda_5 - \lambda_7), \pm(\lambda_5 - \lambda_8), \pm(\lambda_5 + \lambda_7 + \lambda_8), \pm(\lambda_6 - \lambda_7), \pm(\lambda_6 - \lambda_8), \pm(\lambda_6 + \lambda_7 + \lambda_8), \pm(\lambda_5 + \lambda_6 + \lambda_7), \pm(\lambda_5 + \lambda_6 + \lambda_8), \pm(\lambda_5 + \lambda_6 - \lambda_7 - \lambda_8);\ \pm\lambda_i \pm \Big(\lambda_5 + \dfrac{\lambda_8}{2}\Big), \pm\lambda_i \pm \Big(\lambda_6 + \dfrac{\lambda_8}{2}\Big), \pm\lambda_i \pm \Big(\lambda_5 + \lambda_6 - \dfrac{\lambda_8}{2}\Big), \pm\Lambda_i' \pm \Big(\lambda_5 - \dfrac{\lambda_7 + \lambda_8}{2}\Big), \pm\Lambda_i' \pm \Big(\lambda_6 - \dfrac{\lambda_7 + \lambda_8}{2}\Big), \pm\Lambda_i' \pm \Big(\lambda_5 + \lambda_6 + \dfrac{\lambda_7 + \lambda_8}{2}\Big), \pm\Lambda_i^* \pm \Big(\lambda_5 + \dfrac{\lambda_7}{2}\Big), \pm\Lambda_i^* \pm \Big(\lambda_6 + \dfrac{\lambda_7}{2}\Big), \pm\Lambda_i^* \pm \Big(\lambda_5 + \lambda_6 - \dfrac{\lambda_7}{2}\Big)\ i = 1, 2, 3, 4 \Big\}$ where $\Lambda_1', \Lambda_i^*$ are as in Example 28. (See [5], but there are mistakes about the roots of $E_8$ in p. $102 \sim 103$ of [5].)

Put $\alpha_1 = \frac{1}{2}\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_7 - \lambda_8\} - \lambda_5 - \lambda_6$, $\alpha_2 = \lambda_6 + \lambda_7 + \lambda_8$, $\alpha_3$ $\lambda_5 - \lambda_6$, $\alpha_4 = \lambda_6 - \lambda_8$, $\alpha_5 = \lambda_4 - \lambda_5 - \lambda_6 + \dfrac{\lambda_8}{2}$, $\alpha_6 = \lambda_3 - \lambda_4$, $\alpha_7 = \lambda_2 - \lambda_3$, $\alpha_8 = \lambda_6 - \lambda_7$. Then $\Pi = \{\alpha_1, \cdots, \alpha_8\}$ is a fundamental root system. For example, $\lambda_1 + \lambda_2 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8$. Since $B(\sum \lambda_i H_i, \sum \lambda_i H_i) = 60(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) + 120(\lambda_5^2 + \lambda_5\lambda_6 + \lambda_6^2) + 60(\lambda_7^2 + \lambda_7\lambda_8 + \lambda_8^2)$, we obtain that $(\sum m_i\lambda_i, \sum m_i\lambda_i) = \frac{1}{60}(m_1^2 + m_2^2 + m_3^2 + m_4^2) + \frac{1}{90}(m_5^2 - m_5m_6 + m_6^2) + \frac{1}{45}(m_7^2 - m_7m_8 + m_8^2)$. In particular, $(\alpha_i, \alpha_i) = \frac{1}{30}$ for $1 \leq i \leq 8$, $(\alpha_i, \alpha_{i+1}) = -\frac{1}{60}$ for $1 \leq i \leq 6$, $(\alpha_i, \alpha_j) = 0$ for $|i - j| \geqq 2$ except for $(\alpha_3, \alpha_8) = -\frac{1}{60}$. This shows that the Dynkin diagram of $\mathfrak{g}$ is




, i.e., $\mathfrak{g}$ is a simple Lie algebra of type $E_8$. Since we have $\Delta$ and $\Pi$, we have 120 positive roots $\Delta_+$ and

$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = 46\alpha_1 + 91\alpha_2 + 135\alpha_3 + 110\alpha_4 + 84\alpha_5 + 57\alpha_6 + 29\alpha_7 + 68\alpha_8$. Let $\Lambda_1, \cdots, \Lambda_8$ be the fundamental weights of $E_8$, i.e., $\dfrac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$, $i, j = 1, \cdots, 8$. Let $\Lambda = \sum_{i=1}^{8} m_i\Lambda_i$ ($m_i \geqq 0$, integer) be any dominant integral form. Then

$$(\Lambda + \rho, \alpha_i) = \frac{m_i}{2}(\alpha_i, \alpha_i) + (\rho, \alpha_i) = \frac{1}{60}(m_i + 1)\ ,$$

$(\rho, \alpha_i) = \frac{1}{60}$ for $1 \le i \le 8$. The Weyl's dimension formula says that

$$d(\Lambda) = \prod_{\alpha\in\Delta_+}\frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)} = \prod_{\sum_{i=1}^{8} n_i\alpha_i\in\Delta_+}\frac{\sum_{i=1}^{8} n_i(m_i + 1)}{\sum_{i=1}^{8} n_i}\ .$$

For example $d(\Lambda_1) = 3875$, $d(\Lambda_2) = 6696000$, $d(\Lambda_3) = 6899079264$, $d(\Lambda_4) = 146325270$, $d(\Lambda_5) = 2450240$, $d(\Lambda_6) = 30380$, $d(\Lambda_7) = 248$, $d(\Lambda_8) = 147250$, $d(2\Lambda_7) = 27000$, $d(3\Lambda_7) = 1763125$, etc. This shows that the least dimensional irreducible representation of $E_8$ is the adjoint representation $\Lambda_7$.

PROPOSITION 42. *Let* $\mathfrak{g}$ *be a simple Lie algebra over* $\boldsymbol{C}$ *and let* $d\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ *be any representation of* $\mathfrak{g}$ *on* $V$ *with* $g = \dim\mathfrak{g}$ *and* $d = \dim V > 1$. *Then we have* $g \le \frac{1}{2}d(d + 1)$ *except when* $\mathfrak{g} \cong \mathfrak{sl}(d, \boldsymbol{C})$.

*Proof.* If $\mathfrak{g}$ is of type $A_{n-1}$ ($n \geqq 2$), we have $d(\Lambda) \geqq d(\Lambda_2) = d(\Lambda_{n-2})$ $\frac{1}{2}n(n - 1) \geqq 2(n - 1)$ ($n \geqq 4$) and $d(\Lambda) \geqq d(2\Lambda_1) = \frac{1}{2}n(n + 1) \geqq 2(n - 1)$ ($n = 2, 3$) for any $\Lambda \neq \Lambda_1, \Lambda_{n-1}$, and hence, $\frac{1}{2}d(d + 1) \geqq (n - 1)(2n - 1) \geqq n^2 - 1 = g$ for $d = d(\Lambda), \Lambda \neq \Lambda_1, \Lambda_{n-1}$. If $\Lambda = \Lambda_1$ or $\Lambda = \Lambda_{n-1}$, we have $d = d(\Lambda_1) = d(\Lambda_{n-1}) = n$ and hence $\mathfrak{g} \cong \mathfrak{sl}(d, \boldsymbol{C})$. If $\mathfrak{g}$ is of type $C_n$, we have $d(\Lambda) \geqq d(\Lambda_1) = 2n$ for any $\Lambda$, and hence $\frac{1}{2}d(d + 1) \geqq n(2n + 1) = g$. If $\mathfrak{g} = \mathfrak{o}(n, \boldsymbol{C})$ ($n \geqq 5$), we have $d(\Lambda) \geqq d(\Lambda_1) = n$ for any $\Lambda$ and hence $\frac{1}{2}d(d + 1) \geqq \frac{1}{2}n(n + 1) \geqq \frac{1}{2}n(n - 1) = g$. If $\mathfrak{g}$ is of type $G_2$ (resp. $F_4, E_6, E_7, E_8$), we have seen that the least representation degree is 7 (resp. 26, 27, 56, 248) and $g = \dim\mathfrak{g}$ is 14 (resp. 52, 78, 133, 248), hence $\frac{1}{2}d(d + 1) \geqq g$. Q.E.D.

*Remark* 43. If $d = 2$, then $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{C})$, $d\rho = \Lambda_1$. If $d = 3$, then $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{C})$, $d\rho = 2\Lambda_1$, or $\mathfrak{g} = \mathfrak{sl}(3, \boldsymbol{C})$, $d\rho = \Lambda_1$ or its contragredient $\Lambda_2$.

DEFINITION 44. Let $V$ be a $n$-dimensional vector space over $\boldsymbol{C}$. Then all non-singular endomorphisms of $V$ form a group $GL(V)$. By fixing a basis of $V$, we may identity $GL(V)$ with the group $GL(n, \boldsymbol{C})$ of all non-singular $n \times n$ matrices, which is called *the general linear group*. A subgroup $G$ of $GL(n, \boldsymbol{C})$ is called a *linear algebraic group* if there exists a finite number of polynomials $P_1, \cdots, P_\mu$ on $M(n, \boldsymbol{C})$ such that the group $G$ is the intersection of $GL(n, \boldsymbol{C})$ and the common zeros of these polynomials. In this case, the tangent space $\mathfrak{g}$ of $G$ at the unit matrix $I_n$, i.e.,

$$\mathfrak{g} = \left\{\xi \in M(n, \boldsymbol{C}) \,\middle|\, \sum_{i,j} \xi_{ij}\left(\frac{\partial P_k}{\partial x_{ij}}(I_n)\right) = 0 \qquad \text{for } 1 \leq k \leq \mu\right\}$$

is a Lie subalgebra of $\mathfrak{gl}(n, \boldsymbol{C})$, which is called *the Lie algebra of* $G$. For example, *the special linear group* $SL(n, \boldsymbol{C}) = \{A \in GL(n, \boldsymbol{C}) \mid \det A - 1 = 0\}$ is a linear algebraic group and its Lie algebra $\mathfrak{g}$ is, by definition,

$$\mathfrak{g} = \left\{\xi \in M(n, \boldsymbol{C}) \,\middle|\, \sum_{i,j} \xi_{ij}\left(\frac{\det X - 1}{2x_{ij}}(I_n)\right) = \sum_{i=1}^{n} \xi_{ii} = 0\right\},$$

i.e., $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$ (see Example 21). *The orthogonal group* $O(n, \boldsymbol{C}) = \{A \in M(n, \boldsymbol{C}) \mid {}^tAA = I_n\}$ and *the special orthogonal group* $SO(n, \boldsymbol{C}) = \{A \in O(n, \boldsymbol{C}) \mid \det A = 1\}$ are linear algebraic groups and their Lie algebras are the same as $\mathfrak{o}(n, \boldsymbol{C})$ (see Example 27). The Lie algebra of *the symplectic group* $Sp(n, \boldsymbol{C}) = \{A \in GL(2n, \boldsymbol{C}) \mid {}^tAJA = J\}$ is a simple Lie algebra $\mathfrak{sp}(n, \boldsymbol{C})$ of type $C_n$, where $J$ is defined as in Example 25. Here $SL(n, \boldsymbol{C})$ and $Sp(n, \boldsymbol{C})$ are connected and simply connected, $SO(n, \boldsymbol{C})$ is connected but not simply connected, and $O(n, \boldsymbol{C})$ is not connected. We shall construct the spin group $Spin(n, \boldsymbol{C})$ in § 5, which is connected and simply connected and whose Lie algebra is isomorphic to $\mathfrak{o}(n, \boldsymbol{C})$.

In general, the Lie algebras of two linear algebraic groups $G_1$ and $G_2$ are isomorphic if and only if $G_1$ and $G_2$ are locally isomorphic, and in this case we write $G_1 \sim G_2$. We say that a connected linear algebraic group is *almost simple* when its Lie algebra $\mathfrak{g}$ is simple. Note that an almost simple algebraic group might have the center of finite numbers.

Let $\rho: G \to GL(V)$ be a representation of a linear algebraic group $G$ on $V$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Then $\exp tX (t \in \boldsymbol{C}, X \in \mathfrak{g})$ is in $G$ and there exists a representation $d\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ on $V$ defined by $d\rho(X) = \lim_{t \to 0} \dfrac{1}{t}(\rho(\exp tX) - 1)$, i.e., $\rho(\exp tX) = \exp t\, d\rho(X)$ for $X \in \mathfrak{g}$. This representation $d\rho$ is called the *infinitesimal* (or differential) *representation* of $\rho$. Assume that $G$ is connected. Then $\rho$ is irreducible if and only if $d\rho$ is irreducible. Moreover, two representations of $G$ are equivalent if and only if their infinitesimal representations of $\mathfrak{g}$ are equivalent. Conversely, if $G$ is connected and simply connected, for any given representation $d\rho$ of $\mathfrak{g}$, there exists a representation $\rho$ of $G$ uniquely such that its infinitesimal representation is $d\rho$. For example, there is no representation of $SO(n, \boldsymbol{C})$ corresponding to the (half-) spin representation of $\mathfrak{o}(n, \boldsymbol{C})$ (see Example 27, 28), and we have to consider the spin group $Spin(n, \boldsymbol{C})$.

EXAMPLE 45. Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{C}$. Then there exists, uniquely up to isomorphism, a connected and simply connected linear algebraic group $G$ with the Lie algebra $\mathfrak{g}$. For example, since the Dynkin diagrams of type $A_3$ and $D_3$ are the same as $\circ\!-\!\!-\!\!-\!\circ\!-\!\!-\!\!-\!\circ$, the corresponding connected and simply connected algebraic groups $SL(4, \boldsymbol{C})$ and $Spin(6, \boldsymbol{C})$ are isomorphic. Two inequivalent half-spin representations of $Spin(6, \boldsymbol{C})$ correspond to the standard representation $\Lambda_1$ and its contragredient representation $\Lambda_3$ of $SL(4, \boldsymbol{C})$. The representation $\Lambda_2\left(= \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right)$ of $SL(4, \boldsymbol{C})$ has a kernel $\{\pm 1\}$ and its image is $SO(6, \boldsymbol{C})$ since its image leaves the Pfaffian of $4 \times 4$-skew symmetric matrices invariant (see Definition 22, (7)). This fact corresponds to the exact sequence $1 \to \{\pm 1\} \to Spin(6, \boldsymbol{C}) \to SO(6, \boldsymbol{C}) \to 1$ (see (5.27)). Similarly the Dynkin diagrams of $B_2$ and $C_2$ are the same as $\circ\!\Longrightarrow\!\circ$, the corresponding connected and simply connected algebraic group $Spin(5, \boldsymbol{C})$ and $Sp(2, \boldsymbol{C})$ are isomorphic. The restriction of $\Lambda_1$ (and also of $\Lambda_3$) of $SL(4, \boldsymbol{C})$ to $Sp(2, \boldsymbol{C})$ is $\Lambda_1$ of $Sp(2, \boldsymbol{C})$ which is corresponding to the spin representation of $Spin(5, \boldsymbol{C})$. The representation $\Lambda_2\left(= \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array},\ d(\Lambda_2) = 5\right)$ of $Sp(2, \boldsymbol{C})$ has a kernel $\{\pm 1\}$ and its image is $SO(5, \boldsymbol{C})$. This fact is corresponding to the exact sequence $1 \to \{\pm 1\} \to Spin(5, \boldsymbol{C}) \to SO(5, \boldsymbol{C}) \to 1$. Since the Dynkin diagram of $\mathfrak{o}(4, \boldsymbol{C})$ and $\mathfrak{sl}(2, \boldsymbol{C}) \oplus \mathfrak{sl}(2, \boldsymbol{C})$ are the same as $\circ \Longrightarrow \circ$, we have the isomorphism $Spin(4, \boldsymbol{C}) \cong SL(2, \boldsymbol{C}) \times SL(2, \boldsymbol{C})$. The two representations $\Lambda_1 \otimes 1$, $1 \otimes \Lambda_1$ (i.e., $\square \otimes 1, 1 \otimes \square$) of $SL(2, \boldsymbol{C}) \times SL(2, \boldsymbol{C})$ are corresponding to the two inequivalent half-spin representations of $Spin(4, \boldsymbol{C})$. Let $V$ be all $2 \times 2$ matrices $M(2, \boldsymbol{C})$. Define $\rho\colon SL(2) \times SL(2) \to GL(V)$ by $X \mapsto AX^tB$ for $X \in V$, $(A, B) \in SL(2) \times SL(2)$. Then $\rho = \Lambda_1 \otimes \Lambda_1$ $(= \square \otimes \square)$ and its kernel is $\{\pm 1\}$. The image is $SO(4, \boldsymbol{C})$ since it leaves $\det X (X \in V)$ invariant. This fact is corresponding to the exact sequence $1 \to \{\pm 1\} \to Spin(4, \boldsymbol{C}) \to SO(4, \boldsymbol{C}) \to 1$. Since the Dynkin diagrams of $A_1, B_1, C_1$ are the same as $\circ$, we have $SL(2, \boldsymbol{C}) \cong Spin(3, \boldsymbol{C}) \cong Sp(1, \boldsymbol{C})$. Under the isomorphism $Spin(4, \boldsymbol{C}) \cong SL(2) \times SL(2)$, the subgroup $\{(A, A) \in SL(2) \times SL(2) \mid A \in SL(2)\} \cong SL(2)$ corresponds to $Spin(3, \boldsymbol{C})$. Thus the restriction of $\Lambda_1 \otimes 1$ (and $1 \otimes \Lambda_1$) to that subgroup $\cong SL(2, \boldsymbol{C})$ is $\Lambda_1$ which corresponds to the spin representation of $Spin(3, \boldsymbol{C})$. The representation $2\Lambda_1$ of $SL(2, \boldsymbol{C})$ has a kernel $\{\pm 1\}$ and the image $SO(3, \boldsymbol{C})$. This fact is corresponding to the exact sequence $1 \to \{\pm 1\} \to Spin(3, \boldsymbol{C}) \to SO(3, \boldsymbol{C}) \to 1$.

DEFINITION 46. Let $\rho : SL(n, \boldsymbol{C}) \to GL(V)$ be an irreducible representation of $G$ on $V$ with the highest weight $\Lambda$. Then there exists canonically a representation $\rho' : GL(n, \boldsymbol{C}) \to GL(V)$ of $GL(n, \boldsymbol{C})$ such that the restriction of $\rho'$ to $SL(n, \boldsymbol{C})$ is $\rho$. In this case we say that the highest weight of $\rho'$ is $\Lambda$. We also apply the Young diagram to $GL(n, \boldsymbol{C})$.

## § 2. Castling transforms

DEFINITION 1. Let $G$ be a connected linear algebraic group, $V$ a finite dimensional vector space ($\dim V \geqq 1$), and $\rho$ a rational representation of $G$ on $V$, all defined over the complex number field $\boldsymbol{C}$. We call a triplet $(G, \rho, V)$ a *prehomogeneous vector space* (abbrev. P. V.) when there exists a proper algebraic subset $S$ of $V$ such that $V - S$ consists of a single $G$-orbit. In this case, points of $S$ (resp. $V - S$) are called *singular* (resp. *generic*) *points*. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the infinitesimal representation of $\rho$. For an element $x$ of $V$, the Lie algebra of the isotropy subgroup $G_x = \{g \in G \mid \rho(g)x = x\}$ of $G$ at $x$ is the isotropy subalgebra $\mathfrak{g}_x = \{A \in \mathfrak{g} \mid d\rho(A)x = 0\}$ of $\mathfrak{g}$ at $x$.

PROPOSITION 2. *The following conditions are equivalent.*

(1) *A triplet* $(G, \rho, V)$ *is a P. V.*

(2) *There exists an element* $x$ *of* $V$ *satisfying* $\dim G_x = \dim G - \dim V$, *i.e.*, $\dim \mathfrak{g}_x = \dim \mathfrak{g} - \dim V$.

*Proof.* (1) $\Rightarrow$ (2): Let $x$ be a generic point. Then we have $V - S = \rho(G) \cdot x \approx G / G_x$ and hence $\dim G - \dim G_x = \dim (V - S) = \dim V$.

(2) $\Rightarrow$ (1): In general, we have $\rho(G) \cdot x = \overline{\rho(G) \cdot x} - \overline{(\overline{\rho(G) \cdot x} - \rho(G) \cdot x)}$ where $\overline{\phantom{xx}}$ denotes the Zariski closure. The second condition implies that $\dim \overline{\rho(G) \cdot x} = \dim \rho(G) \cdot x = \dim V$. Since $V$ is irreducible, we have $V = \overline{\rho(G) \cdot x}$ and $\rho(G) \cdot x = V - S$ where $S = \overline{(\overline{\rho(G) \cdot x} - \rho(G) \cdot x)}$. Q.E.D.

Proposition 2 implies that the prehomogeneity of a triplet is an infinitesimal condition.

PROPOSITION 3. *Let* $(G, \rho, V)$ *be a triplet. Assume that there exists a non-constant rational function* $f(x)$ *satisfying* $f(\rho(g)x) = f(x)$ *for all* $g \in G$, $x \in V$. *Then this triplet* $(G, \rho, V)$ *is not a P. V.*

*Proof.* Assume that a triplet $(G, \rho, V)$ is a P. V. and let $x_0$ be a generic point. Then $f(x)$ is constant on the open orbit $\rho(G) \cdot x_0$. Since $f(x)$ is a rational function, it is constant on the Zariski closure of $\rho(G) \cdot x_0$, i.e., on $V$. Q.E.D.

These propositions will be used in § 5 to investigate the prehomogeneity of a given triplet.

DEFINITION 4. Two triplets $(G, \rho, V)$ and $(G', \rho', V')$ are called *equivalent* (or strongly equivalent) if there exist a rational isomorphism $\sigma: \rho(G) \to \rho'(G')$ and an isomorphism $\tau: V \to V'$, both defined over $\boldsymbol{C}$ such that the following diagram is commutative for all $g \in G$. This equivalence

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & V' \arrow[d, "\sigma\rho(g)"] \\ V \arrow[r, "\tau"'] & V' \end{tikzcd}$$

relation will be denoted by $(G, \rho, V) \cong (G', \rho', V')$. Note that we consider only the image $\rho(G)$, not $G$ itself. For example, we have $(SL(4, \boldsymbol{C}), \Lambda_2, V(6)) \cong (SO(6, \boldsymbol{C}), \Lambda_1, V(6))$ although $SL(4, \boldsymbol{C})$ and $SO(6, \boldsymbol{C})$ are not isomorphic (see Example 45, § 1). A triplet $(G, \rho, V)$ is called *irreducible* when $\rho$ is irreducible. In this case, by Theorem 1 in § 1, the Lie algebra $\mathfrak{g}$ of the image $\rho(G)$ is reductive, with center at most one-dimensional. In particular, a triplet $(G, \rho, V)$ is equivalent to its *dual* $(G, \rho^*, V^*)$ where $\rho^*$ is the contragredient representation of $\rho$ on the dual vector space $V^*$ of $V$. In the following, except § 4, we shall assume that a triplet is irreducible.

LEMMA 5. *Let* $\tilde{G}$ *be a connected algebraic group and let* $W, W'$ *be irreducible algebraic varieties on which* $\tilde{G}$ *acts. Let* $f: W \to W'$ *be a generically surjective (i.e.,* $\overline{f(W)} = W'$*),* $\tilde{G}$*-equivariant morphism (i.e., compatible with the action of* $\tilde{G}$*). Then the following conditions are equivalent:*

(i) $W$ *is* $\tilde{G}$*-prehomogeneous, i.e., it has a Zariski-dense* $\tilde{G}$*-orbit.*

(ii) $W'$ *is* $\tilde{G}$*-prehomogeneous, and for a point* $x'$ *of a Zariski-dense* *orbit, $f^{-1}(x')$ is $\tilde{G}_{x'}$-prehomogeneous, where $\tilde{G}_{x'}$ is the isotropy subgroup of $\tilde{G}$ at $x'$.*

DEFINITION 6. Let $(G, \rho, V)$ be a P. V. The isotropy subgroup $G_x$ of $G$ at a generic point $x \in V - S$ is called a *generic isotropy subgroup.* Note that all generic isotropy subgroups of $(G, \rho, V)$ are isomorphic to each other. Similarly, the isotropy subalgebra of a generic point is called a *generic isotropy subalgebra.* Since we consider everything over $\boldsymbol{C}$, we shall denote $GL(n, \boldsymbol{C})$ (resp. $SL(n, \boldsymbol{C})$, $O(n, \boldsymbol{C})$, $SO(n, \boldsymbol{C})$, $Spin(n, \boldsymbol{C})$, $Sp(n, \boldsymbol{C})$) by $GL(n)$ (resp. $SL(n)$, $O(n)$, $SO(n)$, $Spin(n)$, $Sp(n)$).

Let $G$ be a linear algebraic group, and let $\rho: G \to GL(V(m))$ be a faithful irreducible representation of $G$ on the $m$-dimensional vector space $V(m)$. Let $\rho^*: G \to GL(V(m)^*)$ be the contragredient representation of $\rho$ on the dual vector space $V(m)^*$ of $V(m)$, and let $n$ be a positive integer with $m > n \geqq 1$.

PROPOSITION 7. *A triplet $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a P. V. if and only if a triplet $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$ is a P. V., and in this case, their generic isotropy subgroups are isomorphic to each other.*

*Proof.* Identify $V = V(m) \otimes V(n)$ with $\overbrace{V(m) \oplus \cdots \oplus V(m)}^{n}$, and let $W$ be an algebraic variety whose points are vectors $v = (v_1, \cdots, v_n) \in V (v_i \in V(m))$ such that $v_1, \cdots, v_n$ are linearly independent in $V(m)$. Then the triplet $(G \times GL(n), \rho \times \Lambda_1, V(m) \otimes V(n))$ is a P. V. if and only if $W$ is $\tilde{G}$-prehomogeneous for $\tilde{G} = G \times GL(n)$. Let $W'$ be the Grassmann variety $\mathrm{Grass}_n(V(m))$ whose points are $n$-dimensional subspaces of $V(m)$. For an element $v = (v_1, \cdots, v_n)$ in $W$, let $f(v)$ be the $n$-dimensional subspace of $V(m)$ spanned by $v_1, \cdots, v_n$. Then $f: W \to W'$ is a surjective, $\tilde{G}$-equivariant morphism. By Lemma 5, $W$ is $\tilde{G}$-prehomogeneous if and only if $W' = \mathrm{Grass}_n(V(m))$ is $\tilde{G}$-prehomogeneous, i.e., $G$-prehomogeneous since $GL(n)$ acts on $\mathrm{Grass}_n(V(m))$ trivially, and it acts on each fibre homogeneously. As $\mathrm{Grass}_n(V(m))$ is $G$-prehomogeneous if and only if $\mathrm{Grass}_{m-n}(V(m)^*)$ is $G$-prehomogeneous, again by Lemma 5, that is so if and only if $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$ is a P. V., and thus we obtain our first assertion. Since each fibre is a principal homogeneous space of $GL(n)$, the generic isotropy subgroup of $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is isomorphic to that of $(G, \mathrm{Grass}_n(V(m)))$. As the generic isotropy subgroups of $(G, \mathrm{Grass}_n(V(m)))$ and $(G, \mathrm{Grass}_{m-n}(V(m)^*))$ are isomorphic, we get our second assertion. Q.E.D.

LEMMA 8. *Let $G$ be a connected semi-simple algebraic group and let $\rho: G \to GL(V)$ be an irreducible representation. Assume that the triplet $(G \times GL(1), \rho \otimes \square, V \otimes V(1))$ is a P.V. with the generic isotropy subgroup $H$. Then the triplet $(G, \rho, V)$ is a P.V., if and only if the connected component of $H$ is not contained in $G$.*

*Proof.* The triplet $(G, \rho, V)$ is a P.V. if and only if $\dim G - \dim G \cap H = \dim V\ (= \dim G \times GL(1) - \dim H)$, i.e., $\dim G \cap H = \dim H - 1$. Since $\dim G \cap H = \dim H - 1$ if and only if the connected component of $H$ is not contained in $G$, we obtain our assertion. Q.E.D.

PROPOSITION 9. *Let $G$ be a linear algebraic group and let $\rho: G \to GL(V(m))$ be a faithful irreducible representation of $G$ on the $m$-dimensional vector space $V(m)$. Let $n$ be a positive number with $m > n \geqq 1$. Then a triplet $(G \times SL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a P.V. if and only if $(G \times SL(m-n), \rho^* \otimes \Lambda_1, V^*(m) \otimes V(m-n))$ is a P.V., and in this case, their generic isotropy subgroups are isomorphic to each other.*

*Proof.* Note that $G$ is reductive with at most one-dimensional center by Theorem 1 in §1. When $G$ has the one-dimensional center, our assertion is the same as Proposition 7, and hence we may assume that $G$ is semi-simple. Assume that $(G \times SL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a P.V. with the generic isotropy subgroup $H$. Then $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a P.V. and its generic isotropy subgroup is isomorphic to $H \times GL(1)$ by Lemma 8. Then Proposition 8 says that $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$ is a P.V. with the generic isotropy subgroup $H' \cong H \times GL(1)$. Since $H' \cap (G \times SL(m-n)) \cong H$, $(G \times SL(m-n), \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$ is a P.V. by Lemma 8. Q.E.D.

This Proposition 9 is very important because it gives us a general method to obtain infinitely many new prehomogeneous vector spaces from a given prehomogeneous vector space. Let $(G, \rho, V)$ be any P.V. with $\dim V = m \geqq 2$. Then since $SL(1) = \{1\}$ and $V \otimes V(1) \cong V$, we have $(G, \rho, V) \cong (G \times SL(1), \rho \otimes \Lambda_1, V \otimes V(1))$ and by Proposition 9, we obtain a new P.V. $(G \times SL(m-1)), \rho^* \otimes \Lambda_1, V^* \otimes V(m-1))$. Since we may assume that $G$ is reductive, this P.V. is equivalent to $(G \times SL(m-1), \rho \otimes \Lambda_1, V \otimes V(m-1))$. Applying this procedure again to this new P.V., we obtain the second new P.V. $(G \times SL(m-1) \times SL(m^2-m-1),\ \rho \otimes \Lambda_1 \otimes \Lambda_1,\ V \otimes V(m-1) \otimes V(m^2-m-1))$. Now there are two ways to apply Proposition 9 to this second new P.V., namely we have two new P.V. $(G \times SL(m^2-m-1) \times SL(m^3-m^2-2m+1),\ \rho \otimes \Lambda_1 \otimes \Lambda_1,\ V \otimes V(m^2-m-1) \otimes V(m^3-m^2-2m+1))$ and $(G \times SL(m-1) \times SL(m^2-m-1) \times SL(m^4-2m^3+m-1),\ \rho \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V \otimes V(m-1) \otimes V(m^2-m-1) \otimes V(m^4-2m^3+m-1))$ where $m^4-2m^3+m-1 = m(m-1)(m^2-m-1)-1$. If $m \geqq 3$, these new P.V. are not equivalent to the original P.V. For example, a triplet $(SL(3), \Lambda_1, V(3))$ is obviously a P.V. and hence $(SL(3) \times SL(2) \times SL(5) \times SL(29),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(2) \otimes V(5) \otimes V(29))$ is a P.V., etc. Repeating this procedure, we can obtain infinitely many new P.V.'s. Although these prehomogeneous vector spaces obtained from a given P.V. are in general not equivalent, they have many common properties. For example, their generic isotropy subgroups are isomorphic to the original one. Thus we attain the concept of castling transforms or castling classes of prehomogeneous vector spaces. It is convenient to define these concepts among irreducible triplets.

DEFINITION 10. We say that two triplets $(G, \rho, V)$ and $(G', \rho', V')$ are *castling transforms* of each other when there exist a triplet $(\tilde{G}, \tilde{\rho}, V(m))$ and a positive number $n$ with $m > n \geqq 1$ such that

$$(G, \rho, V) \cong (\tilde{G} \times SL(n),\ \tilde{\rho} \otimes \Lambda_1,\ V(m) \otimes V(n))$$

and

$$(G', \rho', V') \cong (\tilde{G} \times SL(m-n),\ \tilde{\rho}^* \otimes \Lambda_1,\ V(m)^* \otimes V(m-n))$$

where $\tilde{\rho}^*$ is the contragredient representation of $\tilde{\rho}$ on the dual vector space $V(m)^*$ of $V(m)$. A triplet $(G, \rho, V)$ is called *reduced* if there is no castling transform $(G', \rho', V')$ of $(G, \rho, V)$ with $\dim V' < \dim V$.

DEFINITION 11. We say that two triplets $(G, \rho, V)$ and $(G', \rho', V')$ belong to the same *castling class* when one is obtained from the other by a finite number of castling transforms, and in this case we write $(G, \rho, V) \sim (G', \rho', V')$.

We can obtain the reduced triplet from any given one by a finite number of successive castling transforms for the dimension reasons. For any given triplet, such a reduced one is unique; namely,

PROPOSITION 12. *Each castling class contains one and, up to strong equivalence relation, only one reduced triplet.*

*Proof*. Suppose that a triplet $(G, \rho, V)$ has at least two castling transforms $(G', \rho', V')$ and $(G'', \rho'', V'')$, then we may assume without loss of generality that there exists a triplet $(\tilde{G}, \tilde{\rho}, V(m))$ such that

$$(G, \rho, V) \cong (\tilde{G} \times SL(n_1) \times SL(n_2),\ \tilde{\rho} \otimes \Lambda_1 \otimes \Lambda_1,\ V(m) \otimes V(n_1) \otimes V(n_2))$$

with $m \geqq 2$, and

$$\begin{aligned} (G', \rho', V') \cong (\tilde{G} \times SL(n_1) \times SL(mn_1 - n_2),& \\ \tilde{\rho}^* \otimes \Lambda_1^* \otimes \Lambda_1,&\ V(m)^* \otimes V(n_1)^* \otimes V(mn_1 - n_2)) \\ (G'', \rho'', V'') \cong (\tilde{G} \times SL(n_2) \times SL(mn_2 - n_1),& \\ \tilde{\rho}^* \otimes \Lambda_1^* \otimes \Lambda_1,&\ V(m)^* \otimes V(n_2)^* \otimes V(mn_2 - n_1)) \,. \end{aligned}$$

In this case $\dim V = mn_1n_2$, $\dim V' = mn_1(mn_1 - n_2)$ and $\dim V'' = mn_2(mn_2 - n_1)$. Assume that $\dim V' < \dim V$ and $\dim V'' < \dim V$. This implies that $mn_1 < 2n_2$ and $mn_2 < 2n_1$, and hence $m^2 < 2^2$, i.e., $m < 2$, a contradiction. This shows that if there exists a castling transform $(G', \rho', V')$ of $(G, \rho, V)$ satisfying $\dim V' < \dim V$, then it is unique. Since $\dim V < +\infty$, we obtain our assertion. Q.E.D.

Proposition 12 implies that a reduced triplet $(G, \rho, V)$ satisfies the condition that $\dim V \leq \dim V'$ whenever $(G, \rho, V) \sim (G', \rho', V')$. By a classification of irreducible prehomogeneous vector spaces, we mean the determination of all reduced irreducible prehomogeneous vector spaces up to strong equivalence relation. Note that in this paper we shall use essentially Lie algebras even if we use the terminology of groups for the convenience. Finally we shall show two propositions which are obtained from Lemma 5 as Proposition 7.

Let $G$ be a linear algebraic group and let $\rho: G \to GL(V(d))$ be a representation of $G$ on the $d$-dimensional vector space $V(d)$. By choosing a basis of $V(d)$, we may identify $V(d)$ with $\boldsymbol{C}^d$. Thus we may consider $\rho(g) (g \in G)$ as a $d \times d$ matrix. Define the vector space $\Lambda^2(V(d))$ (resp. $S^2(V(d))$) as the all $d \times d$ skew-symmetric (resp. symmetric) matrices. Define the representation $\rho_{\substack{\square\\\square}}$ (resp. $\rho_{\square\square}$) of $G$ on $\Lambda^2(V(d))$ (resp. $S^2(V(d))$) by $X \mapsto \rho(g)X{}^t\rho(g)$ for $X \in \Lambda^2(V(d))$ (resp. $X \in S^2(V(d))$), $g \in G$. Then $\rho_{\substack{\square\\\square}}, \rho_{\square\square}$ are up to equivalence uniquely determined by $\rho$.

PROPOSITION 13. *Assume that* $2n \geqq d$. *Then a triplet* $(Sp(n) \times G,$

*$\Lambda_1 \otimes \rho$, $V(2n) \otimes V(d))$ is a P.V. if and only if the triplet $(G, \rho_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, \Lambda_2(V(d)))$ is a P.V.*

*Proof.* Let $\langle\ ,\ \rangle$ be a skew-symmetric bilinear form on $V(2n) \times V(2n)$ which is left invariant under the action of $Sp(n)$. Identify $V(2n) \otimes V(d)$ with $W = \overbrace{V(2n) \oplus \cdots \oplus V(2n)}^{d}$ and for an element $v = (v_1, \cdots, v_d \in W(v_i \in V(2n))$, let $f(v)$ be a $d \times d$ skew-symmetric matrix with $(i, j)$-element $\langle v_i, v_j \rangle$ $(i, j = 1, \cdots, 2n)$. Then $f(v)$ is an element of $W' = \Lambda^2(V(d))$ and $f : W \to W'$ is a generically surjective, $\tilde{G}$-equivariant morphism for $\tilde{G} = Sp(n) \times G$. By Lemma 5, $(Sp(n) \times G,\ \Lambda_1 \otimes \rho,\ V(2n) \otimes V(d))$ is a P.V. if and only if $W'$ is $\tilde{G}$-prehomogeneous, i.e., $G$-prehomogeneous since each generic fibre is $Sp(n)$-prehomogeneous and $Sp(n)$ acts trivially on $W' = \Lambda^2(V(d))$. Q.E.D.

PROPOSITION 14. *Assume that $n \geqq d$. Then a triplet $(SO(n) \times G$, $\Lambda_1 \otimes \rho$, $V(n) \otimes V(d))$ is a P.V. if and only if the triplet $(G, \rho_{\square\square}, S^2(V(d)))$ is a P.V.*

*Proof.* Let $\langle\ ,\ \rangle$ be a symmetric bilinear form on $V(n) \times V(n)$ which is left invariant under the action of $SO(n)$. Then the rest of a proof is the same as Proposition 13. Q.E.D.

Proposition 13 and Proposition 14 will be used in § 3.

## § 3. Classification of reduced triplets $(G, \rho, V)$ satisfying $\dim G \geq \dim V$

PROPOSITION 1. *If a triplet $(G, \rho, V)$ is a P.V., then we have*

$$\dim G \geq \dim V\ .$$

*Proof.* By Proposition 2 in § 2 we have $\dim G - \dim V = \dim G_x \geq 0$ for some $x$ in $V$. Q.E.D.

According to this proposition we shall determine in this section all the irreducible reduced triplets $(G, \rho, V)$ satisfying $\dim G \geq \dim V$. By Theorem 1 in § 1, the Lie algebra $\mathfrak{g}$ of $\rho(G)$ is reductive with at most one-dimensional center. We shall consider in § 6 the case when $\mathfrak{g}$ is semi-simple. In this section we shall consider the case that the center of $\mathfrak{g}$ is of one dimension. Then by Theorem 2 in § 1, we may assume that a triplet $(G, \rho, V)$ is of the form: $G = GL(1) \times G_1 \times \cdots \times G_k$, $\rho = \square \otimes \rho_1 \otimes \cdots \otimes \rho_k$, $V = V(1) \otimes V(d_1) \otimes \cdots \otimes V(d_k)$ with $d_1 \geq d_2 \geq \cdots \geq d_k \geq 2$, where each $G_i$ is a connected almost simple algebraic group, $\rho_i$ is an irreducible representation of $G_i$ on the $d_i$-dimensional vector space $V(d_i)$ $(1 \leq i \leq k)$, and $\square$ is the standard representation of $GL(1)$ on the one-dimensional vector space $V(1)$.

Put $g_i = \dim G_i$ $(1 \leq i \leq k)$. These notations such as $G_i, \rho_i, d_i, g_i, k$ will be used throughout this section. We shall denote by $(G_2)$ the exceptional simple algebraic group of type $G_2$ of dimension 14 to distinguish it from the second group $G_2$.

Proposition 1 implies:

$$1 + g_1 + \cdots + g_k \geq d_1 d_2 \cdots d_k \,. \tag{3.1}$$

We shall induce some inequalities from (3.1).

LEMMA 2. *Let $n$ be a natural number, and let $a, c$ be any real numbers satisfying $a \leq ca^{n-1} - a$. Then*

$$\sum_{i=1}^{n} x_i^2 - c \prod_{i=1}^{n} x_i \leq na^2 - ca^n$$

*holds for any real numbers $x_\nu$ with $a \leq x_\nu \leq ca^{n-1} - a$ $(\nu = 1, \cdots, n)$.*

*Proof.* Let $M$ be the maximum value of

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^2 - c \prod_{i=1}^{n} x_i$$

on the closed interval $[a, b]^n$, where $b = ca^{n-1} - a$. Since $f$ is quadratic in each variable with a positive leading coefficient, the maximum is attained at the boundary points. Hence if $M_\mu$ $(1 \leq \mu \leq n)$ denotes the value of $f$ at those points where $x_i = a$ for $\mu$ distinct indices $i$ and $x_j = b$ for $n - \mu$ distinct indices $j$, then $M = \max_{1 \leq \mu \leq n} M_\mu$. We have $M_n = na^2 - ca^n$, and $M_n - M_\mu = -(n - \mu)(b^2 - a^2) + ca^\mu(b^{n-\mu} - a^{n-\mu})$. If $a = b$, then $M_\mu = M_n$ for all $\mu$. On the other hand, if $a < b$, then we have $(M_n - M_\mu)/(b - a) \geq -(n - \mu)(b + a) + (n - \mu)ca^{n-1} = (n - \mu)(ca^{n-1} - a - b) = 0$, and hence $M_\mu \leq M_n$ for all $\mu$. This implies that $M = M_n = na^2 - ca^n$.

Q.E.D.

PROPOSITION 3. *Assume that a triplet $(G, \rho, V)$ is a P.V. with $2^{k-2}d_1 - 2 \geq d_2$. Then we have*

$$1 + g_1 \geq 2^{k-1}d_1 - 3(k - 1) \,.$$

*Proof.* Since the image $\rho_i(G_i)$ of the simple algebraic group $G_i$ is contained in $SL(d_i)$, we have $d_i^2 - 1 \geq g_i$ for $2 \leq i \leq k$, and hence from (3.1), we get

$$1 + g_1 \geq (k-1) - (d_2^2 + \cdots + d_k^2 - d_1 d_2 \cdots d_k) . \tag{3.2}$$

Putting $x_1 = d_2, \cdots, x_n = d_k$, $n = k - 1$, $c = d_1$, $a = 2$ in Lemma 2, we have the inequality

$$\begin{aligned} & d_2^2 + \cdots + d_k^2 - d_1 d_2 \cdots d_k \\ & \quad \leq (k-1)2^2 - d_1 \cdot 2^{k-1} \qquad (2 \leq d_i \leq 2^{k-2} d_1 - 2) . \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we obtain the desired inequality. Q.E.D.

PROPOSITION 4. *If a triplet $(G, \rho, V)$ is a P.V. with $k \geq 3$, then the following inequality holds.*

$$1 + g_1 \geq 2^{k_0 - 1} d_1 - 3(k_0 - 1) \qquad \textit{for } k \geq k_0 \geq 3 .$$

*In particular, we have*

$$1 + g_1 \geq 4d_1 - 6 . \tag{3.4}$$

*Proof.* Since $k \geq 3$, we have $2^{k-2} d_1 - 2 \geq d_2$, i.e., the assumption of Proposition 3 is satisfied. Put $f(k) = 2^{k-1} d_1 - 3(k-1)$. Then we have $f(k) - f(k_0) = (2^{k-1} - 2^{k_0 - 1}) d_1 - 3(k - k_0) \geq 8(2^{k-k_0} - 1) - 3(k - k_0) \geq 0$ for $k \geq k_0 \geq 3$, and hence $1 + g_1 \geq f(k) \geq f(k_0) \geq f(3) = 4d_1 - 6$. Q.E.D.

DEFINITION 5. Let $G$ be a semi-simple algebraic group, and let $\rho : G \to GL(V(n))$ be an irreducible $n$-dimensional representation of $G$. Then a triplet $(G \times GL(m), \rho \otimes \Lambda_1, V(n) \otimes V(m))$ is an irreducible P.V. for $n \leq m$, because the natural action of $GL(m)$ on $V(n) \oplus \cdots \oplus V(n)$ ($m$-copies) for $m \geq n$, yields a P.V. This triplet is called a trivial P.V. It is reduced except for $G = SL(n)$ with $m > n > m/2$, $\rho = \Lambda_1$. When $G = SL(n)$ with $m > n > m/2$, $\rho = \Lambda_1$, its castling transform $(SL(m-n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(m-n) \otimes V(m))$ is a reduced trivial P.V.

From now on, we shall consider case by case according to the type of the Lie algebra $\mathfrak{g}_1$ of $G_1$.

I) *The case for $G_1 \sim SL(n)$ (i.e., $\mathfrak{g}_1 = \mathfrak{sl}(n, \boldsymbol{C})$)*

We may assume that $G_1 = SL(n)$. First of all, we shall determine the irreducible representations $\rho : SL(n) \to GL(V)$ satisfying $\dim V \leq n^2$. We denote $\rho$ by $\Lambda$ when the highest weight of the infinitesimal representation $d\rho$ of $\rho$ is $\Lambda$. Let $\Lambda_1, \cdots, \Lambda_{n-1}$ be the fundamental dominant weights (see Example 21 in §1 and let $\Lambda = \sum_{i=1}^{n-1} m_i \Lambda_i$ ($m_i \geq 0$, integer $1 \leq i \leq n-1$) be a dominant integral form satisfying $d(\Lambda) \leq n^2$.

LEMMA 6. (1) $1 \leq m_1 \leq 3$ $(n = 2)$.

(2) $0 \leq m_1 + m_{n-1} \leq 2$ $(n \geq 3)$.

(3) *$m_2, m_{n-2} = 0$ or 1, and if $m_2 = 1$ (resp. $m_{n-2} = 1$), then $m_{n-2} = 0$ (resp. $m_2 = 0$) and $m_1 = m_{n-1} = 0$ $(n \geq 4)$.*

(4) *$m_3, m_{n-3} = 0$ or 1, and if $m_3 = 1$ or $m_{n-3} = 1$, then $m_j = 0$ for all other $j$ $(8 \geq n \geq 6)$.*

(5) $m_4 = 0$ $(n = 8)$.

(6) *$m_j = 0$ for $3 \leq j \leq n - 3$ $(n \geq 9)$.*

*Proof.* To prove this lemma, we shall use Corollary 13 in §1 and the results of Example 24 in §1.

(1) When $n = 2$, we have $\Lambda = m_1\Lambda_1$ and $d(m_1\Lambda_1) = m_1 + 1$. Hence $d(\Lambda) = m_1 + 1 \leq 4 = n^2$ if and only if $m_1 \leq 3$. Since $\Lambda \neq 0$, we have $1 \leq m_1$.

(2) Since $d(3\Lambda_1) = d(3\Lambda_{n-1}) = \frac{1}{6}n(n+1)(n+2) > n^2$ and $d(2\Lambda_1 + \Lambda_{n-1}) = d(\Lambda_1 + 2\Lambda_{n-1}) = \frac{1}{2}n(n-1)(n+2) > n^2$ if $n \geq 3$, we have $0 \leq m_1$, $m_{n-1} \leq 2$ and $m_{n-1} = 0$ (resp. $m_1 = 0$) if $m_1 = 2$ (resp. $m_{n-1} = 2$), i.e., $0 \leq m_1 + m_2 \leq 2$ in view of Corollary 13 in §1.

(3) Since $d(2\Lambda_2) = d(2\Lambda_{n-2}) = \frac{1}{12}n^2(n^2-1) > n^2$ if $n \geq 4$, we have $m_2, m_{n-2} = 0$ or 1. Since $d(\Lambda_1 + \Lambda_2) = d(\Lambda_{n-1} + \Lambda_{n-2}) = \frac{1}{3}n(n^2-1) > n^2$, $d(\Lambda_1 \Lambda_{n-2}) = d(\Lambda_{n-1} + \Lambda_2) = \frac{1}{2}n(n+1)(n-2) > n^2$, and $d(\Lambda_2 + \Lambda_{n-2}) = \frac{1}{4}n^2(n+1)(n-3) > n^2$ if $n \geq 4$, $m_2 = 1$ (resp. $m_{n-2} = 1$) implies that $m_{n-2} = 0$ (resp. $m_2 = 0$) and $m_1 = m_{n-1} = 0$ in view of Corollary 13 in §1.

(4) It is sufficient to show that $d(\Lambda_3 + \Lambda_\nu) > n^2$ for $1 \leq \nu \leq n-1$, $6 \leq n \leq 8$. For $\nu = 1, 2$, we have $d(\Lambda_3 + \Lambda_1) = \frac{1}{8}n(n^2-1)(n-2) > n^2$ and $d(\Lambda_3 + \Lambda_2) = \frac{1}{24}n^2(n^2-1)(n-2) > n^2$ $(6 \leq n \leq 8)$. Since $d(\Lambda_3 + \Lambda_{\nu+1})/d(\Lambda_3 + \Lambda_\nu) = (\nu-1)(n-\nu)/(\nu^2-4) \geq 1$ for $3 \leq \nu \leq \frac{1}{4}(n+1+\sqrt{n^2-6n+33})$, we have $d(\Lambda_3 + \Lambda_\nu) \geq d(2\Lambda_3) = \frac{1}{144}n^2(n-1)^2(n+1)(n-2) > n^2$ for $4 \leq \nu \leq \frac{1}{4}(n+5+\sqrt{n^2-6n+33})$. Similarly if $\frac{1}{4}(n+5+\sqrt{n^2-6n+33}) < \nu \leq n-1$, we have $d(\Lambda_3 + \Lambda_\nu) \geq d(\Lambda_3 + \Lambda_{n-1}) = \frac{1}{6}n(n^2-1)(n-3) > n^2$ (see Example 24, (4) in §1).

(5) Since $d(\Lambda_4) = \binom{8}{4} = 70 > 64 = n^2$, *we* have $m_4 = 0$ $(n = 8)$.

(6) Since $d(\Lambda_j) = \binom{n}{j} \geq \binom{n}{3} = d(\Lambda_3) = \frac{1}{6}n(n-1)(n-2) > n^2$, we have $m_j = 0$ for $3 \leq j \leq n-3$, $n \geq 9$. Q.E.D.

PROPOSITION 7. *Let* $\rho: SL(n) \to GL(V)$ *be an irreducible representation satisfying* $\dim V \leq n^2$. *Then,*

i) $\rho$ *is one of*: $\Lambda_1, 2\Lambda_1, 3\Lambda_1$ $(n = 2)$.

ii) $\rho$ *is one of*: $\Lambda_1, \Lambda_{n-1}$; $\Lambda_2, \Lambda_{n-2}$; $2\Lambda_1, 2\Lambda_{n-1}$; $\Lambda_1 + \Lambda_{n-1}$ $(n \geq 9$ *or* $5 \geq n \geq 3)$.

iii) $\rho$ *is one of*: $\Lambda_1, \Lambda_{n-1}$; $\Lambda_2, \Lambda_{n-2}$; $2\Lambda_1, 2\Lambda_{n-1}$; $\Lambda_1 + \Lambda_{n-1}$; $\Lambda_3, \Lambda_{n-3}$ $(8 \geq n \geq 6)$.

*Proof.* i) is from (1) in Lemma 6. By (5) and (6) in Lemma 6, we may assume that $\Lambda$ is of the form $\Lambda = m_1\Lambda_1 + m_2\Lambda_2 + m_3\Lambda_3 + m_{n-3}\Lambda_{n-3} + m_{n-2}\Lambda_{n-2} + m_{n-1}\Lambda_{n-1}$. Assume that $6 \leq n \leq 8$. If $m_3 = 1$ (resp. $m_{n-3} = 1$), we have $\Lambda = \Lambda_3$ (resp. $\Lambda = \Lambda_{n-3}$) by (4). If $m_3 = m_{n-3} = 0$, we have $\Lambda = m_1\Lambda_1 + m_2\Lambda_2 + m_{n-2}\Lambda_{n-2} + m_{n-1}\Lambda_{n-1}$. By (6), $\Lambda$ is always of this form if $n \geq 9$, or $n \leq 5$. Assume that $n \geq 4$. If $m_2 = 1$ (resp. $m_{n-2} = 1$), we have $\Lambda = \Lambda_2$ (resp. $\Lambda = \Lambda_{n-2}$) by (3). If $m_2 = m_{n-2} = 0$, we have $\Lambda = m_1\Lambda_1 + m_{n-1}\Lambda_{n-1}$. If $n = 3$, $\Lambda$ is always of this form. Assume that $n \geq 3$. Since $\Lambda \neq 0$ and by (2) in Lemma 6, we have $1 \leq m_1 + m_{n-1} \leq 2$, i.e., $\Lambda = \Lambda_1$, $\Lambda_{n-1}$, $2\Lambda_1$, $\Lambda_1 + \Lambda_{n-1}$, $2\Lambda_{n-1}$. Q.E.D.

COROLLARY 8. *Let* $\rho: SL(n) \to GL(V)$ *be an irreducible representation of* $SL(n)$ *with* $n \geq 3$. *Assume that* $\rho \neq \Lambda_1, \Lambda_{n-1}, \Lambda_2, \Lambda_{n-2}$. *Then if we put* $d = \dim V$, *we have* $d \geq \frac{1}{2}n(n+1)$ $(n \neq 6)$ *and* $d \geq 20$ $(n = 6)$.

*Proof.* Since $d(\Lambda_1 + \Lambda_{n-1}) = n^2 - 1 > d(2\Lambda_1) = d(2\Lambda_{n-1}) = \frac{1}{2}n(n+1)$ $(n \geq 3)$, we have $d \geq d(2\Lambda_1) = \frac{1}{2}n(n+1)$ for $n \geq 9$ or $5 \geq n \geq 3$ by Proposition 7, ii). Since $d(\Lambda_3) = d(\Lambda_{n-3}) = \frac{1}{6}n(n-1)(n-2)$, we have $d(\Lambda_3) > d(2\Lambda_1)$ for $n = 7, 8$ and $d(2\Lambda_1) = 21 > d(\Lambda_3) = 20$ for $n = 6$. By Proposition 7, iii), we obtain our assertion. Q.E.D.

PROPOSITION 9. *Let* $(G, \rho, V)$ *be a reduced triplet with* $k = 1$, $G_1 \sim SL(n)$ *satisfying* $\dim G \geq \dim V$. *Then it is equivalent to one of the following triplets.*

(1) $(GL(n), \Lambda_1, V(n))$ $(n \geq 1)$.
(2) $(GL(n), 2\Lambda_1, V(\frac{1}{2}n(n+1)))$ $(n \geq 2)$.
(3) $(GL(n), \Lambda_2, V(\frac{1}{2}n(n-1)))$ $(n \geq 4)$.
(4) $(GL(n), \Lambda_1 + \Lambda_{n-1}, V(n^2 - 1))$ $(n \geq 3)$.
(5) $(GL(n), \Lambda_3, V(\frac{1}{6}n(n-1)(n-2)))$ $(n = 6, 7, 8)$.
(6) $(GL(2), 3\Lambda_1, V(4))$.

*Proof*. A triplet $(G, \rho, V)$ with $k=1$ is always reduced. Since the contragredient representation $\Lambda^*$ of $\Lambda = \sum_{i=1}^{n-1} m_i \Lambda_i$ is $\Lambda^* = \sum_{i=1}^{n-1} m_{n-i}\Lambda_i$ for $SL(n)$, and $(GL(n), \Lambda, V) \simeq (GL(n), \Lambda^*, V^*)$, we have our assertion from Proposition 7. Q.E.D.

Now we shall prove the following proposition.

PROPOSITION 10. *Let $(G, \rho, V)$ be a reduced triplet with $G_1 \sim SL(n)$ satisfying* $\dim G \geq \dim V$. *Assume that it is not a trivial P.V. Then we have $1 \leq k \leq 3$. Moreover, when $k=3$, it is equivalent to the triplet $(SL(n) \times SL(n) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(n) \otimes V(2))$.*

*Proof*. Assume that $d_1 = n$, i.e., $\rho_1 = \Lambda_1$ or $\Lambda_{n-1}$. Then (3.1) implies that $n^2 + g_2 + \cdots + g_k \geq nd_2 \cdots d_k$, i.e., $g_2 + \cdots + g_k \geq n(d_2 \cdots d_k - n)$. If $n \geq d_2 \cdots d_k$, then it is a trivial P.V. (see Definition 5), and if $d_2 \cdots d_k > n > \frac{1}{2} d_2 \cdots d_k$, it is not reduced. Therefore we may assume that $\frac{1}{2} d_2 \cdots d_k \geq n \geq d_2$. In this case, we have $n(d_2 \cdots d_k - n) - d_2(d_2 \cdots d_k - d_2) = (n - d_2)(d_2 \cdots d_k - n - d_2) \geq (n - d_2)(d_2 \cdots d_k - \frac{1}{2} d_2 \cdots d_k - d_2) = \frac{1}{2} d_2 (n - d_2)(d_3 \cdots d_k - 2) \geq 0$ for $k \geq 3$. Now assume that $k \geq 4$. Then we have $2^{k-2} \geq k$, and hence $(k-1)(d_2^2 - 1) \geq (d_2^2 - 1) + \cdots + (d_k^2 - 1) \geq g_2 + \cdots + g_k \geq n(d_2 \cdots d_k - n) \geq d_2(d_2 \cdots d_k - d_2) \geq d_2^2(2^{k-2} - 1) \geq (k-1)d_2^2$, i.e., a contradiction. Thus we have $1 \leq k \leq 3$. Assume that $k=3$. Then we have $(d_2^2 - 1) + (d_3^2 - 1) \geq d_2(d_2 d_3 - d_2) = d_2^2(d_3 - 1)$, and hence $d_3^2 - 2 \geq d_2^2(d_3 - 2) \geq d_3^2(d_3 - 2)$, i.e., $d_3^3 - 3d_3^2 + 2 = (d_3 - 1)(d_3^2 - 2d_3 - 2) \leq 0$. Together with $d_3 \geq 2$, we have $d_3 = 2$, and hence $G_3 = SL(2)$, $g_3 = 3$, $\rho_3 = \Lambda_1$ (see Remark 43 in §1). By the assumption that $d_2 = \frac{1}{2} d_2 d_3 \geq n \geq d_2$, we have $d_2 = n$. Since $n^2 + g_2 + g_3 = n^2 + g_2 + 3 \geq nd_2 d_3 = 2n^2$, we have $g_2 \geq n^2 - 3$. Assume that $G_2 \neq SL(n)$. Then by Proposition 42 in §1, we have $g_2 \leq \frac{1}{2} n(n+1)$ and hence $n^2 - 3 \leq \frac{1}{2} n(n+1)$, i.e., $n \leq 3$. Again by Remark 43 in §1, we have $n = 3$, $G_2 = SL(2)$ and $\rho_2 = 2\Lambda_1$. In this case, however, the dimension of $G$ is less than that of $V$ since $\dim G = n^2 + g_2 + g_3 = 9 + 3 + 3 = 15$ and $\dim V = nd_2 d_3 = 18$. Thus we have $G_2 = SL(n)$, $\rho_2 = \Lambda_1$, $d_2 = n$, and hence we obtain our assertion when $d_1 = n$. Now assume that $d_1 \neq n$, i.e., $\rho_1 \neq \Lambda_1, \Lambda_{n-1}$. Then we have $d_1 \geq d(\Lambda_2) = \frac{1}{2} n(n-1)$ $(n \geq 4)$ and $d_1 \geq d(2\Lambda_1) = \frac{1}{2} n(n+1)$ $(n = 2, 3)$. Assume that $k \geq 3$. Then by (3.4), we have $n^2 \geq 4d_1 - 6 \geq 2n(n-1) - 6$ $(n \geq 4)$, i.e., $n \leq 2$, and $n^2 \geq 4d_1 - 6 \geq 2n(n+1) - 6$ $(n = 2, 3)$, i.e., $4 \geq 6$ $(n = 2)$, $9 \geq 18$ $(n = 3)$. This is a contradiction and hence we have $1 \leq k \leq 2$. Q.E.D.

Finally, we shall consider the case for $k=2$.

PROPOSITION 11. *Let $(G, \rho, V)$ be a reduced triplet with $k=2$, $G_1 \sim SL(n)$ satisfying* $\dim G \geq \dim V$. *Assume that $\rho_1 \neq \Lambda_2, \Lambda_{n-2}$ and it is not a trivial P.V. Then it is equivalent to the triplet $(SL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \times V(2))$.*

*Proof.* If $\rho_1 = \Lambda_1$ or $\Lambda_{n-1}$, then it is a trivial P.V., and hence we may assume that $\rho_1 \neq \Lambda_1, \Lambda_{n-1}, \Lambda_2, \Lambda_{n-2}$. First we shall consider the case when $d_1 - 2 \geq d_2 \geq 2$. Then by Proposition 3, we have $n^2 = 1 + g_1 \geq 2d_1 - 3$, in particular, $n \geq 3$ since $d_1 \geq 4$. By Corollary 8, we have $d_1 \geq \frac{1}{2}n(n+1)$ $(n \neq 6)$ and $d_1 \geq 20$ $(n=6)$ and hence $n^2 \geq n(n+1) - 3$ $(n \neq 6)$, i.e., $n=3$, and $n^2 \geq 37$ if $n=6$, i.e., $n \neq 6$. Thus we get $n = 3$, and $9 = 1 + g_1 \geq 2d_1 - 3$, i.e., $6 \geq d_1 (\geq 4)$. Since $6 \geq d(\Lambda) \geq 4$ implies $\Lambda = 2\Lambda_1$ (or its dual $2\Lambda_2$) for $SL(3)$, we have $\rho_1 = 2\Lambda_1$, $d_1 = 6$. If $G_2 = SL(d_2)$, we may assume that $3 \geq d_2 \geq 2$ since it is not reduced in the case of $6 > d_2 > 3$. Since $9 + (d_2^2 - 1) \geq 6d_2$ with $3 \geq d_2 \geq 2$ implies that $d_2 = 2$, we have a triplet $(SL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$. If $G_2 \neq SL(d_2)$, we have $\frac{1}{2}d_2(d_2+1) \geq g_2 \geq 6d_2 - 9$ $(d_1 - 2 = 4 \geq d_2 \geq 2)$ by Proposition 42 in § 1 and (3.1), and hence $d_2 = 2$. This implies that $G_2 = SL(d_2)$ by Remark 43 in § 1, i.e., a contradiction.

Next, we shall consider the case when $d_2 = d_1$ or $d_2 = d_1 - 1$. Assume that $G_2 = SL(d_2)$. In this case it is a trivial P.V. if $d_2 = d_1$, and it is not reduced if $d_2 = d_1 - 1$ since it belongs to the same castling class as a triplet $(GL(n), \rho_1, V(d_1))$. Assume that $G_2 \neq SL(d_2)$. Then we have $n^2 + \frac{1}{2}d_2(d_2+1) \geq d_1 d_2$ by Proposition 42 in § 1. Since $d_2 = d_1$ or $d_2 = d_1 - 1$, we have

$$n^2 \geq \tfrac{1}{2}d_1(d_1 - 1)\ . \tag{3.5}$$

On the other hand, by Corollary 8 we have

$$d_1 \geq \tfrac{1}{2}n(n+1)\ (n \neq 6)\ , \qquad d_1 \geq 20\ (n=6)\ . \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{gathered} n^2 \geq \tfrac{1}{4}n(n+1)\{\tfrac{1}{2}n(n+1) - 1\}\ (n \neq 6) \quad \text{and} \\ 36 \geq 190 \qquad \text{if } n = 6\ . \end{gathered} \tag{3.7}$$

Since (3.7) holds only for $n=2$, we have $d_1 = 3$ from (3.5) and (3.6). As we have assumed that $G_2 \neq SL(d_2)$ with $d_2 = d_1$ or $d_2 = d_1 - 1$, we have $G_2 = SL(2)$, $d_2 = 3$, $\rho_2 = 2\Lambda_1$ by Remark 43 in §1. In this case, however, the dimension of $G$ is less than that of $V$ since $\dim G = 1 + 3 + 3 = 7$ and $\dim V = 3 \times 3 = 9$. Q.E.D.

PROPOSITION 12. *Let $(G, \rho, V)$ be a reduced triplet with $k = 2$, $G_1 \sim SL(n)$ satisfying $\dim G \geq \dim V$. Assume that $\rho_1 = \Lambda_2$ or $\Lambda_{n-2}$ and it is not a trivial P.V. Then it is equivalent to one of the following triplets.*

(1) $(SL(n) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(\frac{1}{2}n(n-1)) \otimes V(2))$ $(n \geq 4)$.
(2) $(SL(4) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(6) \otimes V(3))$.
(3) $(SL(5) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$.
(4) $(SL(5) \times GL(4),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(4))$.
(5) $(SL(4) \times GL(2),\ \Lambda_2 \otimes 2\Lambda_1,\ V(6) \otimes V(3))$.
(6) $(GL(4) \times Sp(2),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(4))$.
(7) $(GL(4) \times Sp(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(6))$.

*Proof.* We may assume that $\rho_1 = \Lambda_2$ and $n \geq 4$. First we consider the case when $G_2 = SL(d_2)$, i.e., $\rho_2 = \Lambda_1$ or its dual. If $d_2 = 2$, then $\dim G = n^2 + 3 > \dim V = n(n-1)$ and it is reduced since $d_1 = \frac{1}{2}n(n-1) \geq 6$, i.e., $d_1 - d_2 > d_2$. Thus we obtain (1), and we may assume that $d_2 \geq 3$. If $\frac{1}{2}n(n-1) = d_2$, it is a trivial P.V., and if $\frac{1}{2}n(n-1) > d_2 > \frac{1}{4}n(n-1)$, it is not reduced. Therefore, we may assume

$$\frac{1}{4}n(n-1) \geq d_2 \geq 3\ . \tag{3.8}$$

Thus if $n = 4$, we have $d_2 = 3$, i.e., (2). Since (3.1) implies that $n^2 + d_2^2 - 1 > \frac{1}{2}n(n-1)d_2$ $(n \geq 5)$, together with (3.8) we obtain

$$\frac{1}{4}[n(n-1) - \sqrt{n^2(n-1)^2 - 16(n^2-1)}] \geq d_2 \geq 3 \quad (n \geq 5)\ . \tag{3.9}$$

Thus if $n = 5$, we have $d_2 = 3$ or $4$, i.e., (3) or (4). If $n \geq 6$, then (3.9) has no solution $d_2$. Now we shall assume that $G_2 \neq SL(d_2)$. In this case we have $d_2 \geq 3$ by Remark 43 in §1. By Proposition 42 in §1 and (3.1), we have $n^2 + \frac{1}{2}d_2(d_2+1) \geq \frac{1}{2}n(n-1)d_2$, i.e.,

$$d_2^2 - (n^2 - n - 1)d_2 + 2n^2 \geq 0\ . \tag{3.10}$$

On the other hand, we have

$$d_1 = \frac{n(n-1)}{2} \geq d_2 \geq 3\ . \tag{3.11}$$

Now we shall prove that there is no solution $d_2$ which satisfies both (3.10) and (3.11) if $n \geq 6$. Let $d_2$ be a solution of (3.10) satisfying $d_2 \geq \frac{1}{2}(n^2 - n - 1 + \sqrt{(n^2 - n - 1)^2 - 8n^2})$. Then by (3.11), we have $n(n-1) \geq n^2 - n - 1 + \sqrt{(n^2 - n - 1)^2 - 8n^2}$, i.e., $1 \geq \sqrt{(n^2 - n - 1)^2 - 8n^2}$. Since $n \geq 6$, we have $1 \geq (n^2 - n - 1)^2 - 8n^2 \geq (n^2 - 2n)^2 - 8n^2 = n^2\{(n-2)^2 - 8\} \geq 8n^2$, i.e., a contradiction. Hence we have

$$(3.12) \qquad 3 \leq d_2 \leq \tfrac{1}{2}f(n) \quad \text{where } f(x) = x^2 - x - 1 - \sqrt{(x^2 - x - 1)^2 - 8x^2}\ .$$

Since $\frac{1}{2}f(6) < 3$, it is sufficient to show that $f(x)$ $(x \geq 6)$ is monotone decreasing, i.e.,

$$\frac{\partial f}{\partial x}(x) \leq 0 \qquad (x \geq 6)\ .$$

Since

$$\begin{aligned} &\frac{\partial f}{\partial x}(x) \cdot \sqrt{(x^2 - x - 1)^2 - 8x^2} \\ &\quad = (2x-1)\sqrt{(x^2 - x - 1)^2 - 8x^2} \\ &\qquad - (2x-1)(x^2 - x - 1) + 8x \leq 0 \end{aligned}$$

if and only if $2x^2 + 3x + 2 \geq 0$ $(x \geq 6)$, we have $(\partial f / \partial x)(x) \leq 0$, and hence $n \leq 5$. If $n = 5$, we get $d_2 = 3$ from (3.10) and (3.11). Since $G_2 \neq SL(d_2)$, we have $g_2 = 3$ by Remark 43 in § 1. In this case, however, we have $\dim G = 28 < \dim V = 30$. Finally, assume that $n = 4$. Then from (3.1), (3.11) and Proposition 42 in § 1, we have

$$(3.13) \qquad \tfrac{1}{2}d_2(d_2 + 1) \geq g_2 \geq 6d_2 - 16 \qquad (3 \leq d_2 \leq 6)\ .$$

If $d_2 = 3$, we have $6 \geq g_2 \geq 2$, and hence by Remark 43 in § 1, $G_2 = SL(2)$, $g_2 = 3$, i.e., (5).

If $d_2 = 4$, then $10 \geq g_2 \geq 8$. Note that in the case of $d_2 = 4$, $G_2$ must be one of $SL(2), SL(4), Sp(2)$, and hence we have $G_2 = Sp(2)$, $g_2 = 10$, i.e., (6).

If $d_2 = 5$, then $15 \geq g_2 \geq 14$ by (3.13). Note that $d_2 = 5$ only if $G_2 = SL(2)$, $SL(5)$ and $Sp(2)$. Hence we have $d_2 \neq 5$.

If $d_2 = 6$, then we have $21 \geq g_2 \geq 20$. Note that $d_2 = 6$ only if $G_2 = SL(2), SL(3), SL(4), SL(6)$ and $Sp(3)$. Hence we have $G_2 = Sp(3)$, $g_2 = 21$, i.e., (7). Q.E.D.

II) *The case for* $G_1 \sim Sp(n)$

We may assume that $G_1 = Sp(n)$ and $n \geq 2$ since $Sp(1) \simeq SL(2)$ (see

Example 45 in § 1). First of all, we shall consider the irreducible representations $\rho: Sp(n) \to GL(V)$ satisfying $\dim V \leq 1 + \dim Sp(n) = n(2n+1) + 1$. Let $\Lambda_1, \cdots, \Lambda_n$ be the fundamental dominant weights of $Sp(n)$, and let $\Lambda = \sum_{i=1}^{n} m_i \Lambda_i$ be a dominant integral form satisfying $d(\Lambda) \leq n(2n+1)+1$.

LEMMA 13. (1) $0 \leq m_1 \leq 2$ $(n \geq 2)$.
(2) $m_2 = 0$ *or* 1, *and if* $m_2 = 1$, *then* $m_1 = 0$ $(n \geq 2)$.
(3) $m_3 = 0$ *or* 1, *and if* $m_3 = 1$, *then* $m_1 = m_2 = 0$ $(n = 3)$.
(4) $m_\nu = 0$ *for* $3 \leq \nu \leq n$ $(n \geq 4)$.

*Proof.* (1): Since $d(3\Lambda_1) = \frac{2}{3}n(n+1)(2n+1) > n(2n+1)+1$ $(n \geq 2)$, we obtain (1) in view of Corollary 13 in § 1 (see (1) in Example 26 in § 1). (2): Since $d(2\Lambda_2) = \frac{1}{3}n(n-1)(2n-1)(2n+3) > n(2n+1)$ $(n \geq 2)$, we have $0 \leq m_2 \leq 1$, and by $d(\Lambda_1 + \Lambda_2) = \frac{8}{3}n(n^2-1) > n(2n+1)+1$ $(n \geq 2)$, we obtain (2) (see (2), (3) in Example 26). (3): Since $d(2\Lambda_3) = 84$, $d(\Lambda_1 + \Lambda_3) = 70$, $d(\Lambda_2 + \Lambda_3) = 126$, we have $d(\Lambda_3 + \Lambda_\nu) > \dim Sp(3) + 1 = 22$ $(1 \leq \nu \leq 3,\ n = 3)$ and hence we obtain (3) (see Example 26). Note that $d(\Lambda_3) = 14 < 22 = \dim Sp(3) + 1$ $(n = 3)$. (4): Since $d(\Lambda_\nu) = \binom{2n}{\nu} - \binom{2n}{\nu-2}$, we have $d(\Lambda_3) = \frac{2}{3}n(n-2)(2n+1) > n(2n+1)+1$ $(n \geq 4)$. Next we shall show that $d(\Lambda_n) = [2(2n+1)!/n!(n+2)!] > n(2n+1)+1$ $(n \geq 4)$. Put $c_n = 2n(n+1)/d(\Lambda_n) = n(n+1)!(n+2)!/(2n+1)!$. Since $c_4 < 1$ and $c_{n+1}/c_n = (n+2)(n+3)/2n(2n+3) < 1$ $(n \geq 2)$, we have $c_n < 1$ and hence $d(\Lambda_n) > n(2n+1)$ for $n \geq 4$. Finally, we shall show that $d(\Lambda_\nu) \geq \min(d(\Lambda_3), d(\Lambda_n))$ for $3 \leq \nu \leq n$ $(n \geq 4)$. Since $d(\Lambda_\nu) - d(\Lambda_{\nu-1}) = 2(2n+1)!(2n-2\nu+3+\sqrt{2n+3})\{\frac{1}{2}(2n+3-\sqrt{2n+3}) - \nu\}/\nu!(2n-\nu+3)!$, $d(\Lambda_\nu)$ is monotone increasing if $1 \leq \nu \leq \frac{1}{2}(2n+3-\sqrt{2n+3})$ and monotone decreasing if $\frac{1}{2}(2n+3-\sqrt{2n+3}) \leq \nu \leq n$. This shows that $d(\Lambda_\nu) \geq \min(d(\Lambda_3), (d(\Lambda_\nu))$ $(3 \leq \nu \leq n)$ and hence $d(\Lambda_\nu) > n(2n+1)+1$. In view of Corollary 13 in § 1, we obtain our assertion. Q.E.D.

PROPOSITION 14. *Let* $(G, \rho, V)$ *be a reduced triplet with* $k = 1$ *and* $G_1 \sim Sp(n)$ *satisfying* $\dim V \leq \dim G$. *Then it is equivalent to one of the following triplets.*

(1) $(GL(1) \times Sp(n),\ \square \otimes \Lambda_1,\ V(1) \otimes V(2n))$.
(2) $(GL(1) \times Sp(n),\ \square \otimes \Lambda_2,\ V(1) \otimes V((n-1)(2n+1)))$.
(3) $(GL(1) \times Sp(n),\ \square \otimes 2\Lambda_1,\ V(1) \otimes V(n(2n+1)))$.
(4) $(GL(1) \times Sp(3),\ \square \otimes \Lambda_3,\ V(1) \otimes V(14))$.

*Proof.* Since $\dim G = n(2n+1)+1$, we have $d(\Lambda) \leq n(2n+1)+1$ for $\rho_1 = \Lambda$. By Lemma 13, $\Lambda$ must be one of $\Lambda_1, 2\Lambda_1, \Lambda_2$ $(n \geq 2)$ and $\Lambda_3$ $(n=3)$. Since a triplet with $k=1$ is always reduced, we have our assertion. Q.E.D.

PROPOSITION 15. *Let $(G, \rho, V)$ be a reduced triplet with $G_1 \sim Sp(n)$ satisfying* $\dim V \leq \dim G$. *Assume that $\rho_1 \neq \Lambda_1$. Then we have $1 \leq k \leq 2$. Moreover, if $k=2$, it is equivalent to one of the following triplets.*

(1) $(Sp(2) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(5) \otimes V(2))$.

(2) $(Sp(2) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(5) \otimes V(3))$.

(3) $(GL(1) \times Sp(2) \times Sp(2),\ \square \otimes \Lambda_2 \otimes \Lambda_1,\ V(1) \otimes V(5) \otimes V(4))$.

*Proof.* Assume that $k \geq 3$. Since $d(2\Lambda_1) > d(\Lambda_2)$, we have $d_1 = d(\Lambda) \geq d(\Lambda_2) = (n-1)(2n+1)$ for $\Lambda \neq \Lambda_1$ in view of Proposition 14. Then together with (3.4), we have $1 + n(2n+1) \geq 4(n-1)(2n+1) - 6$, i.e., $(6n-11)(n+1) \leq 0$. This is a contradiction since $n \geq 2$, and hence we have $1 \leq k \leq 2$. Now assume that $k=2$. If $d_1 - 2 \geq d_2 \geq 2$, we have $1 + n(2n+1) \geq 2d_1 - 3 \geq 2(n-1)(2n+1) - 3$ by Proposition 3. This implies that $n=2$ and $7 \geq d_1 \geq 5$. Thus we have $\rho_1 = \Lambda_2$, $d_1 = 5$, and hence $d_1 - 2 = 3 \geq d_2 \geq 2$. By Remark 43 in § 1, we obtain (1), (2) and a triplet $(Sp(2) \times GL(2), \Lambda_2 \otimes \Lambda_2, V(5) \otimes V(3))$. This latter triplet, however, does not satisfy $\dim V \leq \dim G$ since $\dim V = 15$ and $\dim G = 14$. Assume that $d_2 = d_1$ or $d_1 - 1$. In this case we have $d_1 d_2 - \frac{1}{2}d_2(d_2+1) = \frac{1}{2}d_1(d_1-1)$. Therefore, if $G_2 \neq SL(d_2)$, together with Proposition 42 in § 1 and (3.1), we have $1 + n(2n+1) \geq \frac{1}{2}d_1(d_1-1) \geq \frac{1}{2}(n-1)(2n+1)\{(n-1)(2n+1)-1\}$, and hence $n=2$, $d_1=5$, i.e., $\rho_1 = \Lambda_2$. If $d_2 = d_1 = 5$, then $G_2$ must be $SL(2)$ or $Sp(2)$ $(G_2 \neq SL(5)$ by assumption). However, in both cases we have $\dim G < \dim V$. If $d_2 = d_1 - 1 = 4$, then $G_2$ is again $SL(2)$ or $Sp(2)$. If $G_2 = SL(2)$, we have $\dim G < \dim V$, and if $G_2 = Sp(2)$, we obtain (3). Finally, if $G_2 = SL(d_2)$, it is a trivial *P.V.* when $d_1 = d_2$, and it is not reduced when $d_2 = d_1 - 1$. Q.E.D.

We shall consider the case of $\rho_1 = \Lambda_1$.

LEMMA 16. *Let $(G, \rho, V)$ be a triplet with $G_1 \neq SL(d_1)$ satisfying* $\dim V \leq \dim G$. *Then we have $d_1 \geq d_2 \cdots d_k$.*

*Proof.* Assume that $d_2 \cdots d_k > d_1 \geq d_2$ and hence $k \geq 3$. By Proposition 42 in § 1 and (3.1) we have $1 + \frac{1}{2}d_1(d_1+1) + g_2 + \cdots + g_k \geq d_1 d_2 \cdots d_k$, i.e., $g_2 + \cdots + g_k \geq \frac{1}{2}d_1(2d_2 \cdots d_k - 1 - d_1) - 1 = \frac{1}{2}d_1(A - d_1) - 1$

where $A = 2d_2 \cdots d_k - 1$. Since $A/2 > d_1 \geq d_2$ and $x(A - x)$ is a monotone increasing function for $x \leq A/2$, we have $d_1(A - d_1) \geq d_2(A - d_2)$. Assume that $k \geq 4$. Then we have $2^{k-2} \geq k$, and hence $(k-1)(d_2^2 - 1) \geq (d_2^2 - 1) + \cdots + (d_k^2 - 1) \geq g_2 + \cdots + g_k \geq \frac{1}{2}d_1(A - d_1) - 1 \geq \frac{1}{2}d_2(A - d_2) - 1 = d_2^2(d_3 \cdots d_k - 1) + \frac{1}{2}d_2(d_2 - 1) - 1 \geq d_2^2(2^{k-2} - 1) \geq (k-1)d_2^2$. This is a contradiction, and hence we have $k = 3$. In this case we have $d_3 = 2$ since $2(d_2^2 - 1) \geq d_2^2(d_3 - 1) + \frac{1}{2}d_2(d_2 - 1) - 1$ and $d_3 \geq 3$ implies $0 \geq \frac{1}{2}d_2(d_2 - 1) + 1$. Therefore, we have $(d_2^2 - 1) + (2^2 - 1) \geq d_2^2(2 - 1) + \frac{1}{2}d_2(d_2 - 1) - 1$, i.e., $d_2(d_2 - 1) \leq 6$. This implies $d_2 = 2$ or $d_2 = 3$. By Remark 43 in §1 and $G_1 \neq SL(d_1)$, we have $d_1 \geq 3$, and $d_1 = 3$ implies $g_1 = 3$. Hence if $d_2 = 2$, then $d_2 d_3 = 4 > d_1 \geq 2$ and we have $d_1 = 3$, $g_1 = 3$. In this case, however, we have $10 = 1 + g_1 + g_2 + g_3 \geq d_1 d_2 d_3 = 12$, i.e., a contradiction. Hence we have $d_2 = 3$, and $d_2 d_3 = 6 > d_1 \geq d_2 = 3$. Assume that $d_1 = 3$. Then $15 = 1 + 3 + 8 + 3 \geq 1 + g_1 + g_2 + g_3 \geq d_1 d_2 d_3 = 18$, i.e., a contradiction. Since $d_1 = 4$ or $5$ does not satisfy the inequality $1 + d_1(d_1 + 1)/2 + 8 + 3 \geq 1 + g_1 + g_2 + g_3 \geq d_1 d_2 d_3 = 6d_1$, there is no triplet such as $G_1 \neq SL(d_1)$, $\dim V \leq \dim G$ and $d_2 \leq d_1 < d_2 \cdots d_k$. Hence we have $d_1 \geq d_2 \cdots d_k$. Q.E.D.

LEMMA 17. *Let $(G, \rho, V)$ be a triplet satisfying* $\dim G \geq \dim \Lambda^2(V) = \frac{1}{2}d(d-1)$ *where* $d = \dim V$. *Then it is equivalent to one of the following triplets.*

(1) $(GL(d),\ \Lambda_1,\ V(d))$.

(2) $(GL(1) \times Sp(m),\ \square \otimes \Lambda_1,\ V(1) \otimes V(d))$ $(d = 2m)$.

(3) $(GL(1) \times SO(d),\ \square \otimes \Lambda_1,\ V(1) \otimes V(d))$.

*Moreover, if* $\dim G \geq S^2(V) = \frac{1}{2}d(d+1)$, *it is equivalent to* (1) *or* (2).

*Proof*. First assume that $k \geq 2$. In this case, we may assume that $G = \tilde{G}_1 \times \tilde{G}_2$, $V = V(n_1) \otimes V(n_2)$, $d = n_1 n_2$, $n_1 \geq n_2 \geq 2$, where $\tilde{G}_1$ and $\tilde{G}_2$ are not necessarily simple. If $n_2 \geq 3$, then we have $2n_1^2 - 1 \geq 1 + (n_1^2 - 1) + (n_2^2 - 1) \geq \dim G \geq \frac{1}{2}d(d-1) = \frac{1}{2}n_1 n_2(n_1 n_2 - 1) \geq \frac{9}{2}n_1^2 - \frac{3}{2}n_1 \geq 3n_1^2$. This is a contradiction, and hence we have $n_2 = 2$. Then we have $n_1 = 2$ since $n_1^2 + 3 \geq \dim G \geq \frac{1}{2}d(d-1) = 2n_1^2 - n_1$, i.e., $n_1(n_1 - 1) \leq 3$. In this case we have $(SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2)) \simeq (GL(1) \times SO(4), \square \otimes \Lambda_1, V(1) \otimes V(4))$ (see Example 51 in §1), i.e., (3) for $d = 4$. Next, assume that $k = 1$. Since we have seen in §1 that the least representation degree $d$ of $(G_2)$ (resp. $F_4, E_6, E_7, E_8$) is 7 (resp. 26, 27, 56, 248) while the dimension $g$ of $(G_2)$ (resp. $F_4, E_6, E_7, E_8$) is 14 (resp. 52, 78, 133, 248), we have $g <$

$\frac{1}{2}d(d-1)$ and hence $G_1$ is not an exceptional algebraic group. If $G_1 = SL(n)$, i.e., $G = GL(n)$ we have $\rho = \Lambda_1$ $(n \geq 1)$, i.e., (1), $\rho = 2\Lambda_1$ $(n = 2)$, and $\rho = \Lambda_2$ $(n = 4)$ in view of Proposition 9. Note that $(GL(2), 2\Lambda_1, V(3)) \simeq (GL(1) \times SO(3), \square \otimes \Lambda_1, V(1) \otimes V(3))$, i.e., (3) for $d = 3$, and $(GL(4), \Lambda_2, V(6)) \simeq (GL(1) \times SO(6), \square \otimes \Lambda_1, V(1) \otimes V(6))$, i.e., (3) for $d = 6$ (see Example 45 in § 1). If $G_1 = Sp(n)$, we have $\rho_1 = \Lambda_1$, i.e., (2) and $\rho_1 = \Lambda_2$ $(n = 2)$ in view of Proposition 14. Note that $(GL(1) \times Sp(2), \square \otimes \Lambda_2, V(1) \otimes V(5)) \simeq (GL(1) \times SO(5), \square \otimes \Lambda_1, V(1) \otimes V(5))$, i.e., (3) for $d = 5$. If $G_1 = SO(n)$, $\rho_1 = \Lambda_1$, i.e., (3) since in this case $\dim G = 1 + \frac{1}{2}n(n-1) \geq \frac{1}{2}d(d-1)$, i.e., $d = n$. The second assertion is now obvious. Q.E.D.

PROPOSITION 18. *Let $(G, \rho, V)$ be a reduced triplet with $k \geq 2$, $G_1 \sim Sp(n)$, $\rho_1 = \Lambda_1$. Then it is not a P.V. unless it is a trivial P.V. or equivalent to one of the following triplets.*

(1) $(Sp(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(m))$ $(n \geq m \geq 2)$.

(2) $(GL(1) \times Sp(n) \times Sp(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(2m))$ $(n \geq m \geq 2)$.

(3) $(GL(1) \times Sp(n) \times SO(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(m))$ $(2n \geq m \geq 3)$.

*Proof.* From Proposition 13 in § 2, Lemma 16 and Lemma 17, our assertion is obvious. Q.E.D.

III. *The case for $G_1 \sim SO(n)$*

Since $SO(n)$ is not simply connected, we have to consider its covering group $Spin(n)$ when $\rho_1$ is a (half-) spin representation. However, we need not consider $Spin(8)$ since $(SO(8), \Lambda_1, V(8)) \simeq (Spin(8)$, half-spin rep. $V(8))$ (see Example 28 in § 1). Note that if $\Lambda_{m-1}$ and $\Lambda_m$ are two inequivalent half-spin representations of $Spin(2m)$, we have $(Spin(2m), \Lambda_{m-1}, V(2^{m-1})) \simeq (Spin(2m), \Lambda_m, V(2^{m-1}))$. We denote it by $(Spin(2m)$, half-spin rep. $V(2^{m-1}))$. Since we have seen in Example 45 in § 1 that $Spin(6) \simeq SL(4)$, $Spin(5) \simeq Sp(2)$, $Spin(4) \simeq SL(2) \times SL(2)$, $Spin(3) \simeq SL(2)$, we may assume that $n \geq 7$.

PROPOSITION 19. *Let $(G, \rho, V)$ be a reduced triplet with $G_1 \sim SO(n)$, $\rho_1 = \Lambda_1$ Then it is not a P.V. unless it is a trivial P.V. or equivalent to one of the following triplets.*

(1) $(SO(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ $(n \geq 7,\ n/2 \geq m \geq 1)$.

(2) $(GL(1) \times SO(n) \times Sp(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(n) \otimes V(2m))$ $(n \geq 7,\ n \geq 2m \geq 4)$.

*Proof.* From Proposition 14 in § 2, Lemma 16 and Lemma 17, our assertion is obvious. Q.E.D.

PROPOSITION 20. *Let $(G, \rho, V)$ be a reduced triplet with $G_1 \sim SO(n)$, $\rho_1 \neq \Lambda_1$, satisfying* $\dim V \leq \dim G$. *Then we have $1 \leq k \leq 2$. Moreover, if $k = 1$, it is equivalent to one of the following triplets.*

(1) $(GL(1) \times Spin(7),\ \square \otimes$ *spin rep.*, $V(1) \otimes V(8))$.
(2) $(GL(1) \times Spin(9).\ \square \otimes$ *spin rep.*, $V(1) \otimes V(16))$.
(3) $(GL(1) \times Spin(10),\ \square \otimes$ *half-spin rep.*, $V(1) \otimes V(16))$.
(4) $(GL(1) \times Spin(11),\ \square \otimes$ *spin rep.*, $V(1) \otimes V(32))$.
(5) $(GL(1) \times Spin(12),\ \square \otimes$ *half-spin rep.*, $V(1) \otimes V(32))$.
(6) $(GL(1) \times Spin(13),\ \square \otimes$ *spin rep.*, $V(1) \otimes V(64))$.
(7) $(GL(1) \times Spin(14),\ \square \otimes$ *half-spin rep.*, $V(1) \otimes V(64))$.
(8) $(GL(1) \times SO(n),\ \square \otimes \Lambda_2,\ V(1) \otimes V(\frac{1}{2}n(n-1))\ (n \geq 7)$.

*Proof.* First we shall show that $d_1 \geq \frac{1}{2}n(n-1)$ $(n \geq 15)$ and $d_1 \geq 2^{[(n-1)/2]}$ $(14 \geq n \geq 7)$ where $[\alpha]$ is an integer satisfying $1 > \alpha - [\alpha] \geq 0$. As we have seen in Example 27 and Example 28 in § 1, $d(\Lambda_\nu) = \binom{n}{\nu}$ for $1 \leq \nu \leq [(n-3)/2]$ and $d(\Lambda_\nu) = 2^{[(n-1)/2]}$ for $\nu = (n-1)/2$ when $n$ is odd, and for $\nu = n/2,\ n/2 - 1$, when $n$ is even. Since $d(2\Lambda_1) = \frac{1}{2}(n-1)(n+2) > d(\Lambda_2) = \frac{1}{2}n(n-1)$, we have $d_1 = d(\Lambda) \geq \min\{\frac{1}{2}n(n-1), 2^{[(n-1)/2]}\}$ for any $\Lambda \neq \Lambda_1$. Since $2^{[(n-1)/2]} > \frac{1}{2}n(n-1)$ $(n \geq 15)$ and $\frac{1}{2}n(n-1) > 2^{[(n-1)/2]}$ $(14 \geq n \geq 7)$, we have $d_1 \geq \frac{1}{2}n(n-1)$ $(n \geq 15)$ and $d_1 \geq 2^{[(n-1)/2]}$ $(14 \geq n \geq 7)$. Here if $n = 8$, we may assume that $d_1 \geq d(\Lambda_2) = 28$ since $(SO(8), \Lambda_1, V(8)) \simeq (Spin(8)$, half-spin rep., $V(8))$. Assume that $k \geq 3$. Then together with (3.4), if $n \geq 15$ or $n = 8$, we have $1 + \frac{1}{2}n(n-1) \geq 2n(n-1) - 6$, i.e., $n(n-1) \leq \frac{14}{3}$. This is a contradiction. If $14 \geq n \geq 7$ and $n \neq 8$, we have $1 + \frac{1}{2}n(n-1) \geq 4 \cdot 2^{[(n-1)/2]} - 6$. This is also a contradiction and hence we have $1 \leq k \leq 2$. Since $2^{[(n-1)/2]} > \frac{1}{2}n(n-1) + 1$ $(n \geq 15)$, and $d(2\Lambda_1) > \frac{1}{2}n(n-1) + 1$ $(n \geq 7)$, we have $d(\Lambda) \leq \frac{1}{2}n(n-1) + 1$ if and only if $\Lambda$ is a (half-) spin representation for $7 \leq n \leq 14$, $n \neq 8$, or $\Lambda = \Lambda_2$. Hence we obtain our assertion. Q.E.D.

PROPOSITION 21. *Let $(G, \rho, V)$ be a reduced triplet with $k = 2$, $G_1 \sim SO(n)$, $\rho_1 \neq \Lambda_1$ satisfying* $\dim V \leq \dim G$. *Then it is a trivial P.V. or equivalent to one of the following triplets.*

(1) $(Spin(7) \times GL(d)$, *spin rep.* $\otimes \Lambda_1,\ V(8) \otimes V(d))\ (2 \leq d \leq 4)$.
(2) $(Spin(7) \times GL(2)$, *spin rep.* $\otimes 2\Lambda_1,\ V(8) \otimes V(3))$.

(3) $(GL(1) \times Spin(7) \times Sp(2),\ \square \otimes spin\ rep. \otimes \Lambda_1,\ V(1) \otimes V(8) \otimes V(4))$.
(4) $(Spin(9) \times GL(2),\ spin\ rep. \otimes \Lambda_1,\ V(16) \otimes V(2))$.
(5) $(Spin(10) \times GL(d),\ half\text{-}spin\ rep. \otimes \Lambda_1,\ V(16) \otimes V(d))$ $(2 \leq d \leq 3)$.
(6) $(Spin(10) \times GL(2),\ half\text{-}spin\ rep. \otimes 2\Lambda_1,\ V(16) \otimes V(3))$.
(7) $(Spin(12) \times GL(2),\ half\text{-}spin\ rep. \otimes \Lambda_1,\ V(32) \otimes V(2))$.

*Proof*. First assume that $d_1 - 2 \geq d_2 \geq 2$. Then by Proposition 3, we have $1 + \frac{1}{2}n(n-1) \geq 2d_1 - 3$. If $\rho_1$ is not a (half-) spin representation, by the proof of Proposition 20, we have $d_1 \geq d(\Lambda_2) = \frac{1}{2}n(n-1)$ and hence $1 + \frac{1}{2}n(n-1) \geq n(n-1) - 3$, i.e., $n(n-1) \leq 8$ $(n \geq 7)$. This is a contradiction, and hence $\rho_1$ is a (half-) spin representation $(n \neq 8)$. In this case we have $1 + \frac{1}{2}n(n-1) \geq 2 \cdot 2^{[(n-1)/2]} - 3$ $(n \neq 8,\ n \geq 7)$ and hence we get $n = 7, 9$ (spin rep.) and $n = 10, 12$ (half-spin rep.). We shall consider each case. In the case of $n = 7$, we have $2 \leq d_2 \leq d_1 - 2 = 6$. If $G_2 = SL(d_2)$, we have $2 \leq d_2 \leq 4$, i.e., (1) since otherwise it is not reduced. If $G_2 \neq SL(d_2)$, by Proposition 42 in §1 and (3.1), we have $1 + 21 + \frac{1}{2}d_2(d_2+1) \geq 8d_2$ $(2 \leq d_2 \leq 6)$, hence $d_2 = 3$, $G_2 = SL(2)$, i.e., (2) or $d_2 = 4$, $G_2 = Sp(2)$, $\rho_2 = \Lambda_1$, i.e., (3) since otherwise the condition (3.1) or $G_2 \neq SL(d_2)$ is not satisfied. In the case of $n = 9$, we have $2 \leq d_2 \leq 14 = d_1 - 2$. If $G_2 \neq SL(d_2)$, we have $g_2 \leq \frac{1}{2}d_2(d_2+1)$ and $d_2 \geq 3$, and hence (3.1) implies $1 + 36 + \frac{1}{2}d_2(d_2+1) \geq 16d_2$, i.e., $(\frac{31}{2} - d_2)^2 \geq 166 + \frac{1}{4}$ $(14 \geq d_2 \geq 3)$. This is a contradiction and hence we have $G_2 = SL(d_2)$. In this case we have $2 \leq d_2 \leq 8$ since otherwise it is not reduced. Then (3.1) implies that $1 + 36 + (d_2^2 - 1) \geq 16d_2$ $(2 \leq d_2 \leq 8)$ and hence $d_2 = 2$, i.e., (4). In the case of $n = 10$, we have $2 \leq d_2 \leq 14 \leq d_1 - 2$. If $G_2 = SL(d_2)$, we have $2 \leq d_2 \leq 8$ since otherwise it is not reduced. Then (3.1) implies $1 + 45 + (d_2^2 - 1) \geq 16d_2$ $(2 \leq d_2 \leq 8)$ and hence $2 \leq d_2 \leq 3$, i.e., (5). If $G_2 \neq SL(d_2)$, we have $1 + 45 + \frac{1}{2}d_2(d_2+1) \geq 16d_2$ $(3 \leq d_2 \leq 14)$ and hence $d_2 = 3$, $G_2 = SL(2)$, $\rho_2 = 2\Lambda_1$, i.e., (6). In the case of $n = 12$, we have $2 \leq d_2 \leq 30$. If $G_2 \neq SL(d_2)$, we have $1 + 66 + \frac{1}{2}d_2(d_2+1) \geq 32d_2$ $(3 \leq d_2 \leq 30)$. This has no solution and hence $G_2 = SL(d_2)$. In this case we may assume that $2 \leq d_2 \leq 16$ since otherwise it is not reduced. Then by (3.1) we have $1 + 66 + (d_2^2 - 1) \geq 32d_2$ $(2 \leq d_2 \leq 16)$ and hence $d_2 = 2$, i.e., (7). Finally, we shall consider the case when $d_2 = d_1$ or $d_2 = d_1 - 1$. In this case we have $d_1d_2 - \frac{1}{2}d_2(d_2+1) = \frac{1}{2}d_1(d_1-1)$. Hence if $G_2 \neq SL(d_2)$, by (3.1) we have $1 + \frac{1}{2}n(n-1) \geq \frac{1}{2}d_1(d_1-1)$, i.e., $n = d_1$, $\rho_1 = \Lambda_1$. As our assumption is $\rho_1 \neq \Lambda_1$, we have $G_2 = SL(d_2)$. However, in this case it is a trivial P.V. if $d_2 = d_1$, and it is not reduced if $d_2 = d_1 - 1$. Q.E.D.

IV. *The case when $G_1$ is an exceptional algebraic group*

We shall denote by $(G_2)$ the exceptional simple algebraic group of dimension 14, of rank 2, to distinguish it from the second group $G_2$.

PROPOSITION 22. *Let $(G, \rho, V)$ be a reduced triplet satisfying* $\dim V \leq \dim G$ *with an exceptional simple algebraic group $G_1$. Then we have $1 \leq k \leq 2$. Moreover, if $k = 1$, it is equivalent to one of the following triplets.*

(1) $(GL(1) \times (G_2),\ \square \otimes \Lambda_2,\ V(1) \otimes V(7))$.
(2) $(GL(1) \times (G_2),\ \square \otimes \Lambda_1,\ V(1) \otimes V(14))$.
(3) $(GL(1) \times F_4,\ \square \otimes \Lambda_4,\ V(1) \otimes V(26))$.
(4) $(GL(1) \times F_4,\ \square \otimes \Lambda_1,\ V(1) \otimes V(52))$.
(5) $(GL(1) \times E_6,\ \square \otimes \Lambda_1,\ V(1) \otimes V(27))$.
(6) $(GL(1) \times E_6,\ \square \otimes \Lambda_6,\ V(1) \otimes V(78))$.
(7) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$.
(8) $(GL(1) \times E_7,\ \square \otimes \Lambda_1,\ V(1) \otimes V(133))$.
(9) $(GL(1) \times E_8,\ \square \otimes \Lambda_7,\ V(1) \otimes V(248))$.

*Proof.* If $k = 1$, we obtain our assertion from the calculation of representation degree in § 1 (see Example 30 for $(G_2)$, Example 38 for $F_4$, Example 39 for $E_6$, Example 40 for $E_7$ and Example 41 for $E_8$). Assume that $k \geq 3$.Then we have $g_1 \geq 4d_1 - 7$ by (3.4). However, as we see above, there is no solution of this and hence we have $1 \leq k \leq 2$. Q.E.D.

PROPOSITION 23. *Let $(G, \rho, V)$ be a reduced triplet satisfying* $\dim V \leq \dim G$ *with an exceptional simple algebraic group and $k = 2$. Then it is a trivial P.V. or it is equivalent to one of the following triplets.*

(1) $((G_2) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(2))$.
(2) $((G_2) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(3))$.
(3) $(F_4 \times GL(2),\ \Lambda_4 \otimes \Lambda_1,\ V(26) \otimes V(2))$.
(4) $(E_6 \times GL(2),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(2))$.
(5) $(E_6 \times GL(3),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(3))$.
(6) $(E_6 \times GL(2),\ \Lambda_1 \otimes 2\Lambda_1,\ V(27) \otimes V(3))$.
(7) $(E_7 \times GL(2),\ \Lambda_6 \otimes \Lambda_1,\ V(56) \otimes V(2))$.

*Proof.* First we shall show that $d_1 < g_1$. Assume that $d_1 \geq g_1$. In this case, (3.1) implies that $1 + d_1 + (d_2^2 - 1) \geq d_1 d_2$. Since $d_1 \geq 14$ and $d_2 \geq 2$, we have $d_2 \geq \frac{1}{2}(d_1 + \sqrt{d_1^2 - 4d_1}) \geq d_1 - 1$ and hence $d_2 = d_1$ or $d_2 = d_1 - 1$. Hence if $G_2 \neq SL(d_2)$, together with Proposition 42 in § 1, we have $1+d_1 \geq d_1d_2-\frac{1}{2}d_2(d_2+1)=\frac{1}{2}d_1(d_1-1)$, i.e., $d_1(d_1-3)\leq 2$ $(d_1\geq 14)$. This is a contradiction and hence $G_2=SL(d_2)$. In this case, however, it is a trivial P.V. when $d_2=d_1$ and it is not reduced if $d_2=d_1-1$. Hence we may assume that $d_1<g_1$. Assume that $G_1=(G_2)$. Then we have $d_1=7$ since $2\leq d_1<g_1=14$. If $G_2\neq SL(d_2)$, we have $1+14+\frac{1}{2}d_2(d_2+1)\geq 7d_2$ $(2\leq d_2\leq 7)$ and hence $d_2=3$, $G_2=SL(2)$. However, in this case we have $18=1+g_1+g_2\geq d_1d_2=21$, i.e., a contradiction. Hence $G_2=SL(d_2)$. Then we have $2\leq d_2\leq 3$, i.e., (1) and (2) since otherwise it is not reduced. Next assume that $G_1=F_4$. Then $d_1=26$ since $2\leq d_1<g_1=56$. If $G_2\neq SL(d_2)$, we have $1+52+\frac{1}{2}d_2(d_2+1)\geq 26d_2$ $(26\geq d_2\geq 3)$. There is no solution and hence $G_2=SL(d_2)$. In this case we have $2\leq d_2\leq 13$ since otherwise it is not reduced. Since $1+52+(d_2^2-1)\geq 26d_2$ $(13\geq d_2\geq 2)$, we have $d_2=2$, i.e., (3). Assume that $G_1=E_6$. Then $d_1=27$. If $G_2=SL(d_2)$, we may assume that $13\geq d_2\geq 2$ since otherwise it is not reduced. Then (3.1) implies that $1+78+(d_2^2-1)\geq 27d_2$ $(13\geq d_2\geq 2)$, and hence $2\leq d_2\leq 3$, i.e., (4) and (5). If $G_2\neq SL(d_2)$, we have $1+78+\frac{1}{2}d_2(d_2+1)\geq 27d_2$ $(27\geq d_2\geq 3)$ and hence $d_2=3$, $G_2=SL(2)$, i.e., (6). Assume that $G_1=E_7$. Then $d_1=56$. If $G_2\neq SL(d_2)$, we have $1+133+\frac{1}{2}d_2(d_2+1)\geq 56d_2$ $(56\geq d_2\geq 3)$. Since there is no solution we have $G_2=SL(d_2)$. We may assume that $2\leq d_2\leq 28$. By (3.1) we have $1+133+(d_2^2-1)\geq 56d_2$ $(28\geq d_2\geq 2)$ and hence $d_2=2$, i.e., (7). Finally assume that $G_1=E_8$. Then the least representation of $E_8$ is the adjoint representation (see Example 41 in § 1), we have $g_1=d_1<g_1$, i.e., a contradiction. Thus we obtain our assertion. Q.E.D.

THEOREM 24. *Let $(\tilde{G},\tilde{\rho},\tilde{V})$ be a reduced triplet and let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\tilde{\rho}(\tilde{G})$. Assume that the center of $\tilde{\mathfrak{g}}$ is one-dimensional. Then it is not a P.V. unless it is equivalent to one of the following reduced triplets.*

(1) *$(G\times GL(m), \rho\otimes\Lambda_1, V(n)\otimes V(m))$ where $\rho: G\to GL(V(n))$ is an $n$-dimensional irreducible representation of a semi-simple algebraic group $G\,(\neq SL(n))$ with $m\geq n\geq 3$.*

(2) *$(SL(n)\times GL(m),\ \Lambda_1\otimes\Lambda_1,\ V(n)\otimes V(m))$*
*$(m/2\geq n\geq 1$, or $n=m\geq 1)$.*

(3) *$(GL(1)\times G,\ \square\otimes$ adjoint rep., $V(1)\otimes V(n))$ where $G$ is an almost simple algebraic group of dimension $n\,(\geq 3)$.*

(4) *$(GL(n),\ 2\Lambda_1,\ V(\frac{1}{2}n(n+1)))$ $(n\geq 3)$.*

(5) *$(GL(n),\ \Lambda_2,\ V(\frac{1}{2}n(n-1)))$ $(n\geq 5)$.*

(6) $(GL(2),\ 3\Lambda_1,\ V(4))$.

(7) $(GL(6),\ \Lambda_3,\ V(20))$.

(8) $(GL(7),\ \Lambda_3,\ V(35))$.

(9) $(GL(8),\ \Lambda_3,\ V(56))$.

(10) $(SL(3)\times GL(2),\ 2\Lambda_1\otimes\Lambda_1,\ V(6)\otimes V(2))$.

(11) $(SL(n)\times GL(2),\ \Lambda_2\otimes\Lambda_1,\ V(\frac{1}{2}n(n-1))\otimes V(2))\ (n\geq 5)$.

(12) $(SL(5)\times GL(3),\ \Lambda_2\otimes\Lambda_1,\ V(10)\otimes V(3))$.

(13) $(SL(5)\times GL(4),\ \Lambda_2\otimes\Lambda_1,\ V(10)\otimes V(4))$.

(14) $(SL(n)\times SL(n)\times GL(2),\ \Lambda_1\otimes\Lambda_1\otimes\Lambda_1,\ V(n)\otimes V(n)\otimes V(2))$ $(n\geq 3)$.

(15) $(Sp(n)\times GL(m),\ \Lambda_1\otimes\Lambda_1,\ V(2n)\otimes V(m))\ (n\geq m\geq 1)$.

(16) $(GL(1)\times Sp(n)\times SO(m),\ \square\otimes\Lambda_1\otimes\Lambda_1,\ V(1)\otimes V(n)\otimes V(m))$ $(2n\geq m\geq 3)$.

(17) $(GL(1)\times Sp(n),\ \square\otimes\Lambda_2,\ V(1)\otimes V((n-1)(2n+1)))\ (n\geq 3)$.

(18) $(GL(1)\times Sp(n)\times Sp(m),\ \square\otimes\Lambda_1\otimes\Lambda_1,\ V(1)\otimes V(2n)\otimes V(2m))$ $(n\geq m\geq 2)$.

(19) $(GL(1)\times Sp(3),\ \square\otimes\Lambda_3,\ V(1)\otimes V(14))$.

(20) $(SO(n)\times GL(m),\ \Lambda_1\otimes\Lambda_1,\ V(n)\otimes V(m))\ (n\geq 3,\ n/2\geq m\geq 1)$.

(21) $(GL(1)\times SO(n)\times Sp(m),\ \square\otimes\Lambda_1\otimes\Lambda_1,\ V(1)\otimes V(n)\otimes V(2m))$ $(n>2m\geq 4)$.

(22) $(Spin(7)\times GL(d),\ spin\ rep.\otimes\Lambda_1,\ V(8)\otimes V(d))\ (1\leq d\leq 4)$.

(23) $(Spin(7)\times GL(2),\ spin\ rep.\otimes 2\Lambda_1,\ V(8)\otimes V(3))$.

(24) $(GL(1)\times Spin(7)\times Sp(2),\ \square\otimes spin\ rep.\otimes\Lambda_1,\ V(1)\otimes V(8)\otimes V(4))$.

(25) $(Spin(10)\times GL(d),\ half\text{-}spin\ rep.\otimes\Lambda_1,\ V(16)\otimes V(d))\ (1\leq d\leq 3)$.

(26) $(Spin(10)\times GL(2),\ half\text{-}spin\ rep.\otimes 2\Lambda_1,\ V(16)\otimes V(3))$.

(27) $(Spin(9)\times GL(d),\ spin\ rep.\otimes\Lambda_1,\ V(16)\otimes V(d))\ (1\leq d\leq 2)$.

(28) $(Spin(12)\times GL(d),\ half\text{-}spin\ rep.\otimes\Lambda_1,\ V(32)\otimes V(d))\ (1\leq d\leq 2)$.

(29) $(GL(1)\times Spin(11),\ \square\otimes spin\ rep.,\ V(1)\otimes V(32))$.

(30) $(GL(1)\times Spin(14),\ \square\otimes half\text{-}spin\ rep.,\ V(1)\otimes V(64))$.

(31) $(GL(1)\times Spin(13),\ \square\otimes spin\ rep.,\ V(1)\otimes V(64))$.

(32) $((G_2)\times GL(d),\ \Lambda_2\otimes\Lambda_1,\ V(7)\otimes V(d))\ (1\leq d\leq 3)$.

(33) $(F_4\times GL(d),\ \Lambda_4\otimes\Lambda_1,\ V(26)\otimes V(d))\ (1\leq d\leq 2)$.

(34) $(E_6\times GL(d),\ \Lambda_1\otimes\Lambda_1,\ V(27)\otimes V(d))\ (1\leq d\leq 3)$.

(35) $(E_6\times GL(2),\ \Lambda_1\otimes 2\Lambda_1,\ V(27)\otimes V(3))$.

(36) $(E_7\times GL(d),\ \Lambda_6\otimes\Lambda_1,\ V(56)\otimes V(d))\ (1\leq d\leq 2)$.

*Proof.* By Propositions $9, 12, 14, 15, 18\sim 23$, and Example 45 in § 1, we obtain our assertion. Note that the adjoint representation of $SL(n)$

$(n \geq 2)$ (resp. $Sp(n)$, $SO(n)$) is $\Lambda_1 + \Lambda_{n-1}$ (resp. $2\Lambda_1, \Lambda_2$). A triplet (5) in Proposition 12 is equivalent to the triplet $(GL(1) \times SO(6) \times SO(3), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(6) \otimes V(3))$. Hence it should be omitted in view of Proposition 14 in §2 and Lemma 17 in §3. Q.E.D.

## §4. Relative invariants and the regularity

Before going on to investigate the reduced triplets in Theorem 24 in §3, we shall prepare some general notions about prehomogeneous vector spaces. Since it is convenient to consider them in a general situation, we do not assume the irreducibility in this section. The results in this section are already published in Japanese (see [11]) except Propositions 15, 16, 18, and 23.

DEFINITION 1. Let $G$ be a connected linear algebraic group. A rational homomorphism $\chi: G \to \boldsymbol{C}^\times$ ($\boldsymbol{C}^\times = \boldsymbol{C} - \{0\}$) is called a *rational character* of $G$. The group of all rational characters of $G$ will be denoted by $X(G)$. Rational characters $\chi_1, \cdots, \chi_\ell$ are called *multiplicatively independent* if they generate a free abelian group of rank $\ell$ in $X(G)$.

DEFINITION 2. Let $(G, \rho, V)$ be a triplet where $\rho$ is not necessarily irreducible. A non-constant rational function $f(x)$ on $V$ is called a *relative invariant* of $(G, \rho, V)$ if there exists a rational character $\chi$ of $G$ satisfying $f(\rho(g)x) = \chi(g)f(x)$ for any $g \in G$ and $x \in V$. A relative invariant corresponding to the identity character $\chi = 1$, is called an *absolute invariant.*

PROPOSITION 3. *Let $(G, \rho, V)$ be a P. V. Then a relative invariant is, up to a constant multiple, uniquely determined by its corresponding character. In particular, any relative invariant is a homogeneous function.*

*Proof.* Let $f_1(x)$ and $f_2(x)$ be relative invariants corresponding to the same character. Then the quotient $f_1(x)/f_2(x)$ is an absolute invariant, and hence constant by Proposition 3 in §2. Let $f(x)$ be any relative invariant. Since $f_t(x) = f(tx)$ $(t \in \boldsymbol{C}^\times)$ is also a relative invariant with the same character as $f(x)$, we have $f(tx) = c \cdot f(x)$ for some $c \in \boldsymbol{C}$. This implies the homogeneity of $f(x)$. Q.E.D.

LEMMA 4. *Relative invariants $f_1, \cdots, f_\ell$ corresponding to multiplicatively independent characters $\chi_1, \cdots, \chi_\ell$, are algebraically independent.*

*Proof.* Assume that $f_1, \cdots, f_\ell$ are algebraically dependent. Then there exist monomials $U_1, \cdots, U_s$ of $f_1, \cdots, f_\ell$ such that they are linearly dependent and any $(s-1)$ of them are linearly independent. By the definition, the subspace $W$ of $\boldsymbol{C}^s$ defined by $W = \{(c_1, \cdots, c_s) \in \boldsymbol{C}^s \mid \sum_{i=1}^s c_i U_i = 0\}$ is one-dimensional. On the other hand, $U_i$ is a relative invariant corresponding to some character $\mu_i$ $(1 \leq i \leq s)$. Then $(c_1, \cdots, c_s) \in W$ implies $(c_1\mu_1(g), \cdots, c_s\mu_s(g)) \in W$ for any $g \in G$. Since $\dim W = 1$, we have $\mu_1 = \cdots = \mu_s$. This is a contradiction since multiplicative independence of $\chi_1, \cdots, \chi_\ell$ implies that $\mu_1, \cdots, \mu_s$ are different from each other. Q.E.D.

Let $(G, \rho, V)$ be a P. V. and let $S$ be its singular set, i.e., $V - S = \rho(G)\cdot x_0$ $(x_0 \in V - S)$. Let $S_1, \cdots, S_\ell$ be the irreducible components of $S$ with codimension one. Then we may assume that each $S_i$ is the zeros of some irreducible polynomial $f_i(x)$ $(1 \leq i \leq \ell)$: $S_i = \{x \in V \mid f_i(x) = 0\}$.

PROPOSITION 5. *These $f_1(x), \cdots, f_\ell(x)$ are algebraically independent relative invariants. Moreover, any relative invariant $f(x)$ is of the form $f(x) = cf_1(x)^{m_1} \cdots f_\ell(x)^{m_\ell}$ $(c \in \boldsymbol{C}, (m_1, \cdots, m_\ell) \in \boldsymbol{Z}^\ell)$.*

*Proof.* First we shall show that each $f_i(x)$ is a relative invariant $(1 \leq i \leq \ell)$. Since $G$ is connected and $S_i$ is irreducible, the Zariski closure $\overline{\rho(G)\cdot S_i}$ of $\rho(G)\cdot S_i = \{\rho(g)x \mid g \in G, x \in S_i\}$ is also irreducible and as $S_i \subset \overline{\rho(G)\cdot S_i} \subset S$, we have $\overline{\rho(G)\cdot S_i} = S_i$. In particular, we have $\rho(G)\cdot S_i = S_i$. Therefore, a polynomial $f_i(\rho(g)^{-1}x)$ coincides with $f_i(x)$ up to a constant multiple and hence there exists a character $\chi_i$ of $G$ satisfying $f_i(\rho(g)x) = \chi_i(g) f_i(x)$ for $g \in G$, $x \in V$. Since $f_1(x), \cdots, f_\ell(x)$ are irreducible and different from each other, corresponding characters $\chi_1, \cdots, \chi_\ell$ are multiplicatively independent. Hence $f_1(x), \cdots, f_\ell(x)$ are algebraically independent by Lemma 4. Finally, let $f(x)$ be any relative invariant. Since $G$ is connected, every prime divisor of $f(x)$ is also a relative invariant. Hence we may assume that $f(x)$ is an irreducible polynomial. Then the zeros of $f(x)$ must coincide with $S_i$ for some $i$ $(1 \leq i \leq \ell)$ since it is a $G$-invariant irreducible hypersurface. This implies that $f(x) = cf_i(x)$ for some $c \in \boldsymbol{C}$. Q.E.D.

COROLLARY 6. *Let $(G, \rho, V)$ be a P. V. and let $S$ be its singular set. Then there exists a relative invariant of $(G, \rho, V)$ if and only if $S$ has an irreducible component of codimension one.*

DEFINITION 7. A prehomogeneous vector space is called *regular* if there exists a relative invariant $f(x)$ such that the Hessian $H_f(x) = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$ of $f(x)$ is not identically zero.

PROPOSITION 8. *Let $(G, \rho, V)$ be a regular P. V. Then there exists a relative invariant corresponding to the character $\chi_0(g) = \det \rho(g)^2$ $(g \in G)$ where $\det \rho(g)$ denotes the determinant of $\rho(g)$ in $V$.*

*Proof.* Let $f(x)$ be a relative invariant satisfying $H_f(x) \neq 0$, and let $\chi$ be the character of $f(x)$. By choosing a basis of $V$, we may assume that $V \simeq \boldsymbol{C}^n$ and $G \subset GL(n, \boldsymbol{C})$ where $n = \dim V$. Then we have

$$\begin{aligned}\frac{\partial^2}{\partial x_i \partial x_j}(f(gx)) &= \frac{\partial}{\partial x_i} \sum_k \frac{\partial f}{\partial x_k}(gx) \cdot \frac{\partial \sum_\ell g_{k\ell} x_\ell}{\partial x_j} \\ &= \sum_k g_{kj} \frac{\partial}{\partial x_i} \cdot \frac{\partial f}{\partial x_k}(gx) \\ &= \sum_{k,\ell} g_{\ell i} \cdot \frac{\partial^2 f}{\partial x_\ell \partial x_k}(gx) \cdot g_{kj} ,\end{aligned}$$

i.e.,

$$\left(\frac{\partial^2}{\partial x_i \partial x_j}(f(gx))\right) = {}^t g\left(\frac{\partial^2 f}{\partial x_k \partial x_\ell}(gx)\right) g$$

for $g = (g_{ij}) \in G$. Since $f(gx) = \chi(g) f(x)$, we have

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(gx)\right) = \chi(g) \cdot {}^t g^{-1}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right) g^{-1}$$

and hence by taking the determinant, we have $H_f(gx) = \chi(g)^n \cdot \det(g)^{-2} \cdot H_f(x)$. This implies that $H_f(x)$ is a relative invariant corresponding to the character $\chi(g)^n \cdot \det \rho(g)^{-2}$. Hence the quotient $f^n(x)/H_f(x)$ is a relative invariant with the character $\chi_0(g) = \det \rho(g)^2$. Q.E.D.

Let $(G, \rho, V)$ be a P. V. and let $S$ be its singular set. Assume that this P. V. has a relative invariant $f(x)$. By choosing a basis, we may identify $V$ with $\boldsymbol{C}^n$. Moreover, by the inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ $(x, y \in \boldsymbol{C}^n)$, we identify the dual vector space $V^*$ of $V$ with $\boldsymbol{C}^n$. Then we can define a rational map $\varphi \colon V - S \to V^*$ by

$$\varphi(x) = \left(\frac{1}{f(x)} \cdot \frac{\partial f}{\partial x_1}(x), \cdots, \frac{1}{f(x)} \frac{\partial f}{\partial x_n}(x)\right) .$$

Sometimes we denote $\varphi$ by $\operatorname{grad} \log f$.

Let $\chi$ be the character of $f(x)$ and let $\delta\chi$ be its infinitesimal character, i.e., $\chi(\exp tA) = \exp t\delta\chi(A)$ for any $t \in \boldsymbol{C}$, $A \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$. Similarly, let $d\rho$ be the infinitesimal representation of $\rho$ (see Definition 44 in § 1).

PROPOSITION 9. *A rational map* $\varphi = \operatorname{grad}\log f$ *of* $V - S$ *into* $V^*$ *does not depend on the choice of coordinate systems and it satisfies the following conditions.*

(1) $\varphi(\rho(g)x) = \rho^*(g)\varphi(x)$ *for* $x \in V - S$, $g \in G$, *where* $\rho^*$ *is the contragredient representation of* $\rho$.

(2) $\langle d\rho(A)x, \varphi(x)\rangle = \delta\chi(A)$ *for* $A \in \mathfrak{g}$, $x \in V - S$.

*Proof.* We may assume that $G \subset GL(n, \boldsymbol{C})$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \boldsymbol{C})$. Since $\dfrac{\partial}{\partial x_i} f(gx) = \sum_k \dfrac{\partial f}{\partial x_k}(gx)\dfrac{\partial (gx)_k}{\partial x_i} = \sum_k \dfrac{\partial f}{\partial x_k}(gx) g_{ki}$ for $g = (g_{ij}) \in G$, we have $\operatorname{grad} f(gx) = \chi(g)^t g^{-1} \operatorname{grad} f(x)$ where $\operatorname{grad} f(x) = \left(\dfrac{\partial f}{\partial x_1}(x), \cdots, \dfrac{\partial f}{\partial x_n}(x)\right)$ and hence we obtain (1). By differentiation of the equality $f(gx) = \chi(g)f(x)$ for $g = \exp tA \in G$, we obtain (2), i.e., $\langle Ax, \operatorname{grad} f(x)\rangle = \delta\chi(A)f(x)$. Note that this condition (2) characterizes $\varphi = \operatorname{grad}\log f$ since $\{d\rho(A)x \mid A \in \mathfrak{g}\} = V$ for any $x \in V - S$. This shows that the definition of $\operatorname{grad}\log f$ does not depend on the choice of coordinate systems.

Q.E.D.

PROPOSITION 10. *The following conditions are equivalent.*

(1) $\operatorname{grad}\log f : V - S \to V^*$ *is generically surjective, i.e., the image is Zariski dense.*

(2) $H_{\log f(x)} \neq 0$ $(x \in V - S)$.

(3) $H_f(x) \neq 0$ $(x \in V - S, \deg f \geq 2)$; $\det\left(\dfrac{\partial f}{\partial x_i} \cdot \dfrac{\partial f}{\partial x_j}\right) \neq 0$ $(\deg f = 1)$.

*Proof.* For $x \in V - S$, let $(d\varphi)_x : T_x(V - S) \to T_{\varphi(x)}V^*$ be the differential map of $\varphi = \operatorname{grad}\log f$. By (1) in Proposition 9, $\varphi$ is generically surjective if $\det(d\varphi)_x \neq 0$ for some $x \in V - S$. Since

$$\det(d\varphi)_x = \det\left(\frac{\partial \varphi_i}{\partial x_j}(x)\right) = \det\left(\frac{\partial}{\partial x_j}\left(\frac{1}{f}\,\frac{\partial f}{\partial x_i}\right)(x)\right)$$
$$= \det\left(\frac{\partial^2 \log f}{\partial x_j \partial x_i}\right) = H_{\log f}\,,$$

the equivalence of (1) and (2) is obvious. Moreover, since

$$\det(d\varphi)_x = \det\left(\frac{\partial}{\partial x_j}\left(\frac{1}{f}\,\frac{\partial f}{\partial x_i}\right)(x)\right)$$
$$= \det\left(\frac{1}{f}\cdot\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{1}{f^2}\cdot\frac{\partial f}{\partial x_i}\,\frac{\partial f}{\partial x_j}\right),$$

if $\deg f = 1$, we have

$$\det(d\varphi)_x = (-1)^n f(x)^{-2n}\cdot\det\left(\frac{\partial f}{\partial x_i}\,\frac{\partial f}{\partial x_j}\right),$$

i.e., (3) for $\deg f = 1$. Assume that $r = \deg f(x) \geq 2$. Then the Euler's identity says that

$$\sum_{i=1}^{n} x_i\,{}^t\!\left(\frac{1}{f}\cdot\frac{\partial^2 f}{\partial x_1 \partial x_i}, \ldots, \frac{1}{f}\cdot\frac{\partial^2 f}{\partial x_n \partial x_i}, \frac{1}{f}\cdot\frac{\partial f}{\partial x_i}\right)$$
$$= {}^t\!\left(\frac{r-1}{f}\cdot\frac{\partial f}{\partial x_1}, \ldots, \frac{r-1}{f}\cdot\frac{\partial f}{\partial x_n}, r\right)$$

and hence we have

$$\det(d\varphi)_x = \det\left(\frac{1}{f}\cdot\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{1}{f^2}\,\frac{\partial f}{\partial x_i}\,\frac{\partial f}{\partial x_j}\right)$$
$$= \det\left[\begin{array}{c|c} \left(\frac{1}{f}\cdot\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{1}{f^2}\cdot\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\right) & \begin{matrix} \frac{1}{f}\,\frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{1}{f}\cdot\frac{\partial f}{\partial x_n} \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$
$$= \det\left[\begin{array}{c|c} \left(\frac{1}{f}\cdot\frac{\partial^2 f}{\partial x_i \partial x_j}\right) & \begin{matrix} \frac{1}{f}\,\frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{1}{f}\cdot\frac{\partial f}{\partial x_n} \end{matrix} \\ \hline \frac{1}{f}\cdot\frac{\partial f}{\partial x_1}, \ldots, \frac{1}{f}\,\frac{\partial f}{\partial x_n} & 1 \end{array}\right]$$
$$= \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)\cdot\frac{1}{f(x)^n}\cdot\left(1 - \frac{r}{r-1}\right),$$

i.e.,

$$\det(d\varphi)_x = \frac{-1}{r-1}\cdot\frac{H_f(x)}{f(x)^n}(x \in V - S).$$

Hence we obtain our assertion. Q.E.D.

*Remark* 11. If $\deg f = 1$ and $\det\left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\right) \neq 0$, then $H_{f^2}(x) \neq 0$ since $\frac{\partial^2 f^2}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(2f\cdot\frac{\partial f}{\partial x_j}\right) = 2\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}$ $\left(\text{note that } \frac{\partial^2 f}{\partial x_i \partial x_j} = 0 \text{ since } \deg f = 1\right)$. Hence $(G, \rho, V)$ is regular if and only if there exists a relative invariant $f(x)$ such that $\operatorname{grad}\log f$ is generically surjective. In general, a prehomogeneous vector space $(G, \rho, V)$ is called *quasi-regular* if there exist $\omega \in \mathfrak{g}^*$ (= the dual space of $\mathfrak{g}$) and a regular rational map $\varphi\colon V - S \to V^*$ satisfying the following conditions:

(1) $\varphi(\rho(g)x) = \rho^*(g)\varphi(x)$ $(g \in G,\ x \in V - S)$,

(2) $\langle d\rho(A), \varphi(x)\rangle = \omega(A)$ $(A \in \mathfrak{g},\ x \in V - S)$,

(3) $\varphi$ is generically surjective, i.e., $\overline{\varphi(V - S)} = V^*$.

In this case it can be proved that a triplet $(G, \rho^*, V^*)$ is also a P. V. and $\varphi$ is a biregular rational map of $V - S$ onto $V^* - S^*$. Moreover, the number of irreducible hypersurfaces in $S$ is the same as in $S^*$. Note that in general the dual triplet $(G, \rho^*, V^*)$ is not a P. V. even if $(G, \rho, V)$ is a P. V. For example, $G = \left\{\begin{pmatrix} 1 & b \\ & a \end{pmatrix} \middle| a, b \in \boldsymbol{C},\ a \neq 0\right\}$, $V = \boldsymbol{C}^2$, $\rho\left(\begin{pmatrix} 1 & b \\ & a \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + by \\ ay \end{pmatrix}$. By Propositions 9 and 10, a regular P. V. is always quasi-regular. But the converse is not true. For example, put $\tilde{G} = \left\{\begin{pmatrix} a & b \\ & a \end{pmatrix} \middle| a, b \in \boldsymbol{C},\ a \neq 0\right\}$, $\tilde{V} = \boldsymbol{C}^2$, $\tilde{\rho}\left(\begin{pmatrix} a & b \\ & a \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ ay \end{pmatrix}$. Then this triplet $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is a quasi-regular P. V. but not regular. However, if $G$ is reductive, then the regularity and the quasi-regularity are equivalent. Hence we omit the detail of quasi-regularity.

Now we shall return to the irreducible case.

PROPOSITION 12. *Let $(G, \rho, V)$ be an irreducible P. V. Then there is, up to a constant multiple, at most one irreducible relative invariant polynomial $f(x)$, and hence any relative invariant is of the form $c\cdot f(x)^m$ $(c \in \boldsymbol{C}^\times,\ m \in \boldsymbol{Z})$.*

*Proof*. Assume that there exist two such polynomials $f_1(x)$, $f_2(x)$. Let $r_i$ be the degree of $f_i(x)$ $(1 \leq i \leq 2)$. Then the quotient $f(x) = f_1(x)^{r_2}/f_2(x)^{r_1}$ is not constant by Propositions 5. Moreover, it is an absolute invariant by Theorem 1 in §1. This is a contradiction in view of Proposition 3 in § 2. Q.E.D.

DEFINITION 13. Let $(G, \rho, V)$ be an irreducible P. V. By *the* relative invariant of $(G, \rho, V)$, we mean the irreducible relative invariant polynomial which is, if it exists, uniquely determined up to a constant multiple by Proposition 12. Note that its degree is uniquely determined by $(G, \rho, V)$.

PROPOSITION 14. *Let $(G, \rho, V)$ be an irreducible regular P. V. and let $f(x)$ be the relative invariant and $\chi$ its character. Put $r = \deg f(x)$ and $n = \dim V$. Then we have $r \mid 2n$ and $\chi(g)^{2n/r} = \det \rho(g)^2$.*

*Proof.* By Proposition 8, there exists a relative invariant $F(x)$ with the character $\det \rho(g)^2$. Then by Proposition 12, we have $F(x) = c \cdot f(x)^m$ for some $c \in \boldsymbol{C}^\times$, $m \in \boldsymbol{Z}$. Hence we have $\chi(g)^m = \det \rho(g)^2$. If we take $g$ such as $\rho(g) = tI_n$ $(t \in \boldsymbol{C}^\times)$, we have $t^{mr} = t^{2n}$ $(t \in \boldsymbol{C}^\times)$ and hence $m = \dfrac{2n}{r}$. Q.E.D.

PROPOSITION 15 (The degree formula). *Let $(G, \rho, V)$ be an irreducible regular P. V. and let $f(x)$ be the relative invariant. Assume that there exists an orbit $\rho(G)x_0$ of codimension one, i.e., $\overline{\rho(G)x_0} = \{x \in V \mid f(x) = 0\}$. Then*

$$(4.1) \qquad \deg f(x) = \frac{\operatorname{tr} d\rho(A) + \operatorname{tr} \operatorname{ad}_{\mathfrak{g}_{x_0}} A}{\operatorname{tr} d\rho(A)} \cdot \dim V \qquad (A \in \mathfrak{g}_{x_0},\ \operatorname{tr} d\rho(A) \neq 0)$$

*where $\operatorname{tr} d\rho(A)$ is the trace of $d\rho(A)$ in $V$ and $\operatorname{tr} \operatorname{ad}_{\mathfrak{g}_{x_0}} A$ is the a trace of the adjoint representation of the isotropy subalgebra $\mathfrak{g}_{x_0}$ at $x_0$.*

*Proof.* By differentiating the equality $\chi(g)^{2n/r} = \det \rho(g)^2$ in Proposition 14, we have $\dfrac{n}{r} \delta\chi(A) = \operatorname{tr} d\rho(A)$ $(A \in \mathfrak{g} =$ the Lie algebra of $G)$, i.e., $\deg f = \dfrac{\delta\chi(A)}{\operatorname{tr} d\rho(A)} \cdot \dim V$ $(A \in \mathfrak{g})$. Let $V_{x_0} = V / d\rho(\mathfrak{g}) \cdot x_0$ be the normal vector space at $x_0$. Then the isotropy subgroup $G_{x_0}$ at $x_0$ acts on $V_{x_0}$ since $\rho(G_{x_0}) \cdot d\rho(\mathfrak{g}) \cdot x_0 \subset d\rho(\mathfrak{g}) \cdot x_0$. We shall show that $\chi(g) = \det_{V_{x_0}} g$ for $g \in G_{x_0}$. Since $f(x_0) = 0$ and $df(x_0) \neq 0$, we have $f(x) = \langle x - x_0, df(x_0) \rangle$ + higher term of $(x - x_0)$. Since $f(\rho(g)x) = \chi(g) f(x)$ and $\rho(g)x - x_0 = \rho(g)(x - x_0)$ for $g \in G_{x_0}$, we have $\langle \rho(g)x - x_0, df(x_0) \rangle = \chi(g) \langle x - x_0, df(x_0) \rangle$. On the other hand, Euler's identity says that $\langle x_0, df(x_0) \rangle = \deg f \cdot f(x_0)$

$= 0$, and hence we have $\langle \rho(g)x, df(x_0)\rangle = \chi(g)\langle x, df(x_0)\rangle$, i.e., $\langle \rho(g)x - \chi(g)x, df(x_0)\rangle = 0$ $(g \in G_{x_0})$. Since $\langle d\rho(A)x_0, df(x_0)\rangle = \delta\chi(A)f(x_0) = 0$ for any $A \in \mathfrak{g}$ (see Proposition 9, (2); note that $\langle d\rho(A)x \ \mathrm{grad}\, f(x)\rangle = \langle d\rho(A)x, df(x)\rangle$ by definition), and $\mathrm{codim}_V\, d\rho(\mathfrak{g})x_0 = 1$, we have $\rho(g)x \equiv \chi(g)x \bmod d\rho(\mathfrak{g})\cdot x_0$. In fact if $\rho(g)x - \chi(g)x \notin d\rho(\mathfrak{g})x_0$ we have $\langle V\ df(x_0)\rangle = 0$, i.e., $df(x_0) = 0$. This is a contradiction. Hence we have $\det_{V_{x_0}} g = \chi(g)$ (note that $\dim V_{x_0} = 1$). By differentiation of this equality, we have $\delta\chi(A) = \mathrm{tr}_{V_{x_0}} A$ $(A \in \mathfrak{g}_{x_0})$. Since $\mathfrak{g}$ is reductive, we have $\mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}} A = 0$ and hence $\mathrm{tr}_{V_{x_0}} A = \mathrm{tr}\, d\rho(A) - \mathrm{tr}_{d\rho(\mathfrak{g})\cdot x_0} A = \mathrm{tr}\, d\rho(A) + \mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}_{x_0}} A$, i.e., $\delta\chi(A) = \mathrm{tr}\, d\rho(A) + \mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}_{x_0}} A$ $(A \in \mathfrak{g}_{x_0})$. Together with $\deg f = \dfrac{\delta\chi(A)}{\mathrm{tr}\, d\rho(A)}\cdot \dim V$, we obtain our assertion. Q.E.D.

PROPOSITION 16. *Let $(G, \rho, V)$ be an irreducible P. V. satisfying $\dim G = \dim V$ $(= n)$. Then it is regular and there is a relative invariant polynomial $f(x)$ of degree $n$. Moreover, if there exists an orbit of codimension one, then $f(x)$ is irreducible.*

*Proof*. Since $G$ is reductive and the generic isotropy subgroup is finite, it is regular by the following Proposition 25. By choosing a basis we may assume that $V \simeq \boldsymbol{C}^n$ and $G \subset GL(n, \boldsymbol{C})$. Let $\mathfrak{g}(\subset \mathfrak{gl}(n, \boldsymbol{C}))$ be the Lie algebra of $G$, and let $A_1, \cdots, A_n$ be a basis of $\mathfrak{g}$ over $\boldsymbol{C}$. Then $A_1, \cdots, A_n$ are $n \times n$ matrices. Define a polynomial $f(x)$ by $f(x) = \det(A_1x, \cdots, A_nx)$ for $x \in \boldsymbol{C}^n$. We shall show that $f(x)$ is a relative invariant. Let $(c_{ij}(g))$ be an $n \times n$ matrix of the adjoint representation of $G$ w. r. t. a basis $A_1, \cdots, A_n$, i.e., $(g^{-1}A_1g, \cdots, g^{-1}A_ng) = (A_1, \cdots, A_n)(c_{ij}(g))$. Then we have

$$\begin{aligned} f(gx) &= \det g\cdot\det(g^{-1}A_1gx, \cdots, g^{-1}A_ngx) \\ &= \det g\cdot\Big(\sum_i c_{i1}(g)A_ix, \cdots, \sum c_{in}(g)A_ix\Big) \\ &= \det g\cdot\det(c_{ij}(g))\cdot f(x)\ . \end{aligned}$$

Assume that there exists an orbit $Gx_0$ of codimension one. Then $\dim \mathfrak{g}_{x_0} = \dim\mathfrak{g} - (n-1) = 1$ and hence $\mathfrak{g}_{x_0}$ is abelian. This implies that $\mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}_{x_0}} A = 0$ $(A \in \mathfrak{g}_{x_0})$, and hence the degree of the relative invariant is $n$ by Proposition 15. Therefore $f(x)$ is the relative invariant of $(G, \rho, V)$, i.e., irreducible. Q.E.D.

*Remark* 17. Consider a triplet $(G, \rho, V)$ where $G = GL(2, \boldsymbol{C})$, $V =$

$M(2, \boldsymbol{C})$, $\rho(A)X = AX$ $(A \in G, X \in V)$. Although it is not irreducible, it is regular and its singular set $S$ is an irreducible hypersurface: $S = \{x \in V \mid \det x = 0\}$. Moreover, we have

$$\dim G = \dim V, \deg f = \frac{\delta\chi(A)}{\operatorname{tr}_V A} \dim V$$

where $\delta\chi$ is the differential character of $f(x) = \det x$. However, there is no orbit of codimension 1, and $\deg f = \frac{1}{2} \dim V$.

Now we shall consider relative invariants and regularity of the castling transform.

Let $(G \times SL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ be a triplet with $m > n \geq 1$. Identify $V(m) \otimes V(n)$ with all $m \times n$ matrices $M(m, n, \boldsymbol{C})$. We may assume that $G \subset GL(m)$. Then we have $\rho \otimes \Lambda_1(g_1, g_2)X = g_1 X {}^t g_2$ for $X \in M(m, n, \boldsymbol{C})$, $g_1 \in G$, $g_2 \in SL(n)$. Let $f(x_{11}, x_{12}, \cdots, x_{mn})$ $(X = (x_{ij}) \in M(m, n, \boldsymbol{C}))$ be a relative invariant of this triplet. Put

$$X_{i_1, \cdots, i_n} = \det \begin{pmatrix} x_{i_1 1}, & \cdots, & x_{i_1 n} \\ \vdots & & \\ x_{i_n 1}, & \cdots, & x_{i_n n} \end{pmatrix} \qquad (1 \leq i_1, \cdots, i_n \leq m) .$$

Since $f$ is invariant under the action of $SL(n)$, the first main theorem for $SL(n)$ (see p. 45 in [1]) says that there exists a polynomial $F$ of $\binom{m}{n}$-variables satisfying $f((x_{ij})) = F((X_{i_1, \cdots, i_n}))$ $(1 \leq i, i_1, \cdots, i_n \leq m, 1 \leq j \leq n)$. Now we shall consider the castling transform $(G \times SL(m-n), \rho^* \otimes \Lambda_1, V(m) \otimes V(m-n))$ of this triplet. Similarly, we may assume that $V(m) \otimes V(m-n) = M(m, m-n, \boldsymbol{C})$ and $G \subset GL(m)$. Then we have

$$(\rho^* \otimes \Lambda_1)(g_1, g_2)X = {}^t g_1^{-1} X {}^t g_2 \qquad \text{for } X = (x_{ij})_{\substack{1 \leq i \leq m \\ n+1 \leq j \leq m}} \in M(m, m-n, \boldsymbol{C}) .$$

Put

$$X_{i_{n+1}, \cdots, i_m} = \det \begin{pmatrix} x_{i_{n+1}, n+1}, & \cdots, & x_{i_{n+1}, m} \\ \vdots & & \\ x_{i_m, n+1}, & \cdots, & x_{i_m, m} \end{pmatrix},$$

and to each $X_{i_1, \cdots, i_n}$, correspond $X_{i_{n+1}, \cdots, i_m}$ under the condition that $\{i_1, \cdots, i_n, i_{n+1}, \cdots, i_m\} = \{1, \cdots, m\}$ and $\operatorname{sgn} \begin{pmatrix} 1, & \cdots, & m \\ i_1, & \cdots, & i_m \end{pmatrix} = 1$. This gives a one-to-one correspondence between $X_{i_1, \cdots, i_n}$ and $X_{i_{n+1}, \cdots, i_m}$.

Hence in this case, we denote $X_{i_{n+1},\cdots,i_m}$ by $X'_{i_1,\cdots,i_n}$. We shall show that $\tilde{f}(x_{ij}) = F((X'_{i_1,\cdots,i_n}))$ $(1 \leq i \leq m,\ n+1 \leq j \leq m)$ is a relative invariant of $(G \times SL(m-n),\ \rho^* \otimes \Lambda_1,\ V(m) \otimes V(m-n))$. Since each $X'_{i_1,\cdots,i_n}$ is invariant, $\tilde{f}$ is also invariant under the action of $SL(m-n)$. To see $\tilde{f}$ invariant under the action $\rho^*$ of $G$, we may assume that $G \subset SL(m)$. Then an element of the Lie algebra $\mathfrak{g}$ of $G$ is an $m \times m$ matrix of trace 0. Since $\tilde{f}$ is absolutely invariant under the action $\rho^*$ of $G$ if and only if it becomes zero under the action $d\rho^*$ of $\mathfrak{g}$, we may consider infinitesimally.

It is sufficient to show that the action $d\rho$ of $\mathfrak{g}$ to $X_{i_1,\cdots,i_n}$ is the same as the contragredient action $d\rho^*$ of $\mathfrak{g}$ to $X'_{i_1,\cdots,i_n}$ since in that case $f = F((X_{i_1,\cdots,i_n}))$ is invariant under the action $\rho$ of $G$ if and only if $\tilde{f} = F((X'_{i_1,\cdots,i_n}))$ is invariant under the contragredient action $\rho^*$ of $G$. To show this, we may assume that $X_{i_1,\cdots,i_n} = X_{1,\cdots,n}$. Then $d\rho$ induces

$$X_{1,\cdots,n} \mapsto \sum_{k=1}^{n}\sum_{\ell=1}^{m} a_{k\ell} X_{1,\cdots,\overset{k}{\underset{\ell}{\vee}},\cdots,n} = (a_{11} + \cdots + a_{nn})X_{1,\cdots,n}$$
$$+ \sum_{k=1}^{n}\sum_{\ell=n+1}^{m} a_{k\ell} X_{1,\cdots,\overset{k}{\underset{\ell}{\vee}},\cdots,n} \qquad \text{for } A = (a_{ij}) \in \mathfrak{g}\ .$$

Note that $\sum_{i=1}^{m} a_{ii} = 0$. On the other hand, the contragredient action $d\rho^*$ induces

$$X'_{1,\cdots,n} = X_{n+1,\cdots,m} \mapsto \sum_{r=n+1}^{m}\sum_{s=1}^{m} (-a_{sr}) X_{n+1,\cdots,\overset{r}{\underset{s}{\vee}},\cdots,m}$$
$$= -(a_{n+1,n+1} + \cdots + a_{mm})X_{n+1,\cdots,m} + \sum_{r=n+1}^{m}\sum_{s=1}^{n} (-a_{sr}) X_{n+1,\cdots,\overset{r}{\underset{s}{\vee}},\cdots,m}$$
$$= (a_{11} + \cdots + a_{nn})X'_{1,\cdots,n} + \sum_{s=1}^{n}\sum_{r=n+1}^{m} a_{sr}(X_{1,\cdots,\overset{s}{\underset{r}{\vee}},\cdots,n})$$
$$\text{for } A = (a_{ij}) \in \mathfrak{g}\ .$$

Hence we obtain our assertion that $\tilde{f}$ is a relative invariant of $(G \times SL(m-n),\ \rho^* \otimes \Lambda_1,\ V(m) \otimes V(m-n))$. Since this correspondence is one-to-one, if $f$ is irreducible, then $\tilde{f}$ is also irreducible. Therefore, we obtain the following proposition.

PROPOSITION 18. *There is a one-to-one correspondence between relative invariants $f(x)$ of $(G \times SL(n),\ \rho \otimes \Lambda_1,\ V(m) \otimes V(n))$ $(m > n \geq 1)$ and relative invariants $\tilde{f}(\tilde{x})$ of its castling transform $(G \times SL(m-n),\ \rho^* \otimes \Lambda_1,\ V(m) \otimes V(m-n))$. Moreover, there exists a positive integer $d$ for each $f(x)$ such that* $\deg f(x) = nd$ *and* $\deg \tilde{f}(\tilde{x}) = (m-n)d$. *If $f$ is irreducible,* *then $\tilde{f}$ is also irreducible.*

Next we shall consider some conditions that a given $P.V.$ has no relative invariant.

PROPOSITION 19. *Let $(G, \rho, V)$ be a $P.V.$ where $\rho$ is not necessariyl irreducible. Let $\chi$ $(\neq 1)$ be a rational character of $G$. Then there exists a relative invariant $f(x)$ corresponding to $\chi$ if and only if the restriction $\chi|_{G_{x_0}}$ of $\chi$ to a generic isotropy subgroup $G_{x_0}$ $(x_0 \in V - S)$ is identity. Note that the property $\chi|G_{x_0} = 1$ does not depend on a generic point $x_0$ since $G_{\rho(g)x_0} = gG_{x_0}g^{-1}$.*

*Proof.* If there exists a relative invariant $f(x)$ satisfying $f(\rho(g)x) = \chi(g)f(x)$, we have $f(x_0) = \chi(g)f(x_0)$ and $f(x_0) \neq 0, \infty$ for $g \in G_{x_0}$, $x_0 \in V - S$. This implies that $\chi(g) = 1$ for any $g \in G_{x_0}$. Conversely, assume that $\chi | G_{x_0} = 1$ for $x_0 \in V - S$. Then $\chi$ can be regarded as a rational regular function on $G/G_{x_0} \approx V - S$, and hence there exists a rational function $f(x)$ on $V$ satisfying $f(\rho(g)x_0) = \chi(g)$ for any $g \in G$. Clearly, $f(x)$ is a relative invariant corresponding to the character $\chi$. Q.E.D.

PROPOSITION 20. *Let $(G, \rho, V)$ be an irreducible $P.V.$ and let $\mathfrak{g}$ be the Lie algebra of $G$. Denote by $\mathfrak{sl}(V)$ the Lie algebra of all endomorphisms of $V$ of trace 0. Then*

(1) *If $d\rho(\mathfrak{g}) \subset \mathfrak{sl}(V)$, there exists no (non-constant) relative invariant and hence $(G, \rho, V)$ is not regular.*

(2) *Let $\mathfrak{g}_{x_0}$ $(x_0 \in V - S)$ be a generic isotropy subalgebra. If $d\rho(\mathfrak{g}_{x_0}) \not\subset \mathfrak{sl}(V)$, there exists no (non-constant) relative invariant and hence $(G, \rho, V)$ is not regular.*

(3) *If $d\rho(\mathfrak{g}) \not\subset \mathfrak{sl}(V)$ and $d\rho(\mathfrak{g}_{x_0}) \subset \mathfrak{sl}(V)$, then there exists a (non-constant) relative invariant of $(G, \rho, V)$.*

*Proof.* (1) By Theorem 1 in § 1, $d\rho(\mathfrak{g}) \subset \mathfrak{sl}(V)$ implies that $d\rho(\mathfrak{g})$ is a semi-simple Lie algebra, and hence we may assume that $\mathfrak{g}$ is semi-simple. If there exists a non-constant relative invariant $f(x)$ satisfying $f(\rho(g)x) = \chi(g)f(x)$, then the infinitesimal character $\delta\chi$ of $\chi$ is not identically zero since $G$ is connected. Hence $\delta x : \mathfrak{g} \to \mathfrak{gl}(1)$ is an onto map and its kernel $\mathrm{Ker}\,\delta\chi$ is an ideal of $\mathfrak{g}$ of codimension one. Since $\mathfrak{g}$ is semi-simple, this is a contradiction.

(2) Assume that there exists a relative invariant $f(x)$ corresponding to $\chi$. Then by Theorem 1 in § 1, $\delta\chi = c\,\mathrm{tr}_V$ for some $c \in \boldsymbol{C}$ where $\mathrm{tr}_V$ is

the trace of $d\rho$ in $V$, and hence $\delta\chi|_{\mathfrak{g}_{x_0}} \neq 0$. In particular, we have $\chi|_{G_{x_0}} \neq 1$. This is a contradiction in view of Proposition 19.

(3) In this case $\mathrm{tr}_V$ is not identically zero and $\det\rho(G_{x_0})$ is a finite group. Hence some power $(\det\rho)^\ell$ satisfies the condition $(\det\rho)^\ell \neq 1$, and $(\det\rho)^\ell|_{G_{x_0}} = 1$. By Proposition 19, there exists a relative invariant corresponding to $(\det\rho)^\ell$. Q.E.D.

*Remark* 21. Let $(G, \rho, V)$ be an irreducible P. V. and let $G_{x_0}$ be a generic isotropy subgroup. Let $SL^{\pm}(V)$ (resp. $SL(V)$) be the group of all non-singular transformations of $V$ of determinant $\pm 1$ (resp. 1). If $(G, \rho, V)$ is regular, then we have $\rho(G_{x_0}) \subset SL^{\pm}(V)$ by Propositions 8 and 19. In general, if $\rho(G) \not\subset SL(V)$, then $\rho(G_{x_0}) \subset SL^{\pm}(V)$ (resp. $\rho(G_{x_0}) \subset SL(V)$) if and only if there exists a relative invariant of degree $2n$ (resp. $n$) corresponding to the character $\det\rho(g)^2$ (resp. $\det\rho(g)$) where $n = \dim V$. (Note that since $G$ is connected, $\rho(G) \not\subset SL(V)$ implies that $d\rho(\mathfrak{g}) \not\subset \mathfrak{sl}(V)$). In particular, if $\dim G = \dim V$, then $\rho(G_{x_0})$ is a finite group contained in $SL(V)$ by Proposition 16.

PROPOSITION 22 (D. Luna). *Let $G$ be a reductive algebraic group which acts on a smooth affine variety $X$. For $x \in X$, let $G_x$ be the isotropy subgroup of $G$ at $x$. Assume that, for any point $x$ in $X$, there exists a non-degenerate symmetric $G_x$-invariant bilinear form on the tangent space at $x$. Then there exists an open dense subset of $X$ which consists of closed $G$-orbits in $X$. In pvrticular, an open orbit in $X$ is closed.*

*Proof.* See ([21]).

COROLLARY 23 (J. Dorfmeister). *Let $(G, \rho, V)$ be a regular P.V. with a reductive algebraic group $G$ where $\rho$ is not necessarily irreducible. Then its generic isotropy subgroup is also reductive.*

*Proof.* Let $f(x)$ be a relative invariant satisfying $H_f(x) \neq 0$. We may assume that $f(x)$ is a polynomial, and hence its Hessian $H_f(x)$ is also a polynomial. Put $X = \{x \in V \,|\, H_f(x) \neq 0\}$. Then $X$ is a smooth affine variety and, for each $x \in X$, $B_x(u, u) = \sum_{i,j} \dfrac{\partial^2 f}{\partial x_i \partial x_j}(x) u_i v_j$ is a non-degenerate symmetric $G_x$-invariant bilinear form on the tangent space of $X$ at $x$ (see the proof of Proposition 8). Since $H_f(x)$ is also a relative invariant, the open $G$-orbit in $V$ is contained in $X$. It is dense in

$X$, and closed by Proposition 22; hence it coincides with $X$. Since $G$ is reductive and the open orbit is affine, the generic isotropy subgroup is reductive. Q.E.D.

Finally, we shall prove the converse of Corollary 23.

Let $(G, \rho, V)$ be a P. V. with a reductive algebraic group $G$, where $\rho$ is not necessarily irreducible. Let $(G, \rho^*, V^*)$ be the dual P. V. of $(G, \rho, V)$, i.e., $\rho^*$ is the contragredient representation of $\rho$ on the dual vector space $V^*$ of $V$. Choosing a basis of $V$, we may identify $V$ with $\boldsymbol{C}^n$ ($n = \dim V$) and $V^*$ with $\boldsymbol{C}^n$ by the inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. In this case we may assume that $G \subset GL(n)$. It can be proved that a reductive algebraic group $G$ is a Zariski closure of a maximal compact subgroup $K$. Moreover, by choosing a suitable basis of $V$, we have $K \subset U(n)$ where $U(n)$ is the group of all $n \times n$ unitary matrices, i.e., $U(n) = \{g \in GL(n) \mid {}^t g \bar{g} = I_n\}$. Then we have $\rho^*(g)y = {}^t g^{-1} y = \bar{g} y$ for $g \in K$, $y \in \boldsymbol{C}^n \simeq V^*$. Let $f(x)$ be any relative invariant polynomial of $(G, \rho, V)$ corresponding to a character $\chi$. Put $f^*(y) = \overline{f(\bar{y})}$ where $\bar{\ }$ is the complex conjugate ($y \in \boldsymbol{C}^n \simeq V^*$). Then for each $g \in K$, we have $f^*(\rho^*(g)y) = f^*(\bar{g}y) = \overline{f(g\bar{y})} = \overline{\chi(g) f(\bar{y})} = \chi^{-1}(g) f^*(y)$ ($y \in \boldsymbol{C}^n \simeq V^*$). Note that $|\chi(g)| = 1$ for $g \in K \subset U(n)$. Since $G$ is the Zariski closure of $K$, we have $f^*(\rho^*(g)y) = \chi^{-1}(g) f^*(y)$ for any $g \in G$, $y \in V^*$, i.e., $f^*$ is a relative invariant polynomial of $(G, \rho^*, V^*)$ corresponding to $\chi^{-1}$. Put

$$f(x)^m = \sum_{i_1 + \cdots + i_n = rm} a^{(m)}_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

where $r = \deg f = \deg f^*$. Then we have

$$f^*(y)^m = \sum_{i_1 + \cdots + i_n = rm} \overline{a^{(m)}_{i_1, \cdots, i_n}} y_1^{i_1} \cdots y_n^{i_n}$$

and

$$f^* (\mathrm{grad}_x)^m f(x)^m = \sum_{i_1 + \cdots + i_n = rm} |a^{(m)}_{i_1, \cdots, i_n}|^2 \cdot (i_1 !) \cdots (i_n !)$$

where

$$f^* (\mathrm{grad}_x)^m = \sum_{i_1 + \cdots + i_n = rm} \overline{a^{(m)}_{i_1, \cdots, i_n}} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}$$

and $m$ is any non-negative integer. By choosing a suitable basis of $V$, we may assume that $f((1, 0, \cdots, 0)) \neq 0$, i.e., $a = a_{r0, \dots, 0} \neq 0$. Then we haev $f((1, 0, \cdots, 0))^m = a_{rm, 0, \cdots, 0} = a^m$. Since $(f^* (\mathrm{grad}_x) f^m)(gx) = f^* (\mathrm{grad}_{gx}) \cdot$

$f^m(gx) = \chi(g)^{m-1}\cdot f^*(\mathrm{grad}_x) f(x)^m$, $f^*(\mathrm{grad}_x) f(x)^m$ is a relative invariant corresponding to $\chi^{m-1}$, and hence there exists a constant $b(m)$ satisfying $f^*(\mathrm{grad}_x) f(x)^m = b(m) f(x)^{m-1}$. Clearly $b(m)$ is a polynomial in $m$ with degree $\leq r$. Put $r' = \deg b(m)$. Then there exists a constant $C$ satisfying $|b(m)| \leq Cm^{r'}$ for any $m \geq 1$. Hence we have

$$\begin{aligned} C^m\cdot(m!)^{r'} &\geq |b(m)b(m-1)\cdots b(1)| = |f^*(\mathrm{grad}_x)^m f(x)^m| \\ &= \sum_{i_1+\cdots+i_n=rm} |a^{(m)}_{i_1,\dots,i_n}|^2\cdot(i_1!)\cdots(i_n!) \geq |a|^{2m}\cdot(rm)! \end{aligned}$$

for any $m \geq 1$. Since there exists a positive number $m_0$ satisfying $m! > C^m/|a|^{2m}$ for any $m \geq m_0$, if $r' < r$, we have $(m!)^r \geq (m!)^{r'+1} \geq (rm)!$ for any $m \geq m_0$. This is a contradiction for $r \geq 2$. If $r = 1$ and $r' = 0$, we have $C^m \geq |a|^{2m}\cdot m!$ for any $m \geq 0$ and again a contradiction. Hence we have $\deg b(m) = r$ $(= \deg f = \deg f^*)$. Since

$$\begin{aligned} f^*(\mathrm{grad}_x) f(x)^m &= m^r\cdot f^*(\mathrm{grad}\, f(x))\cdot f(x)^{m-r} + (\text{lower term in } m) \\ &= b(m) f(x)^{m-1} = m^r b_0 f(x)^{m-1} + (\text{lower term in } m)\,, \end{aligned}$$

we have $f^*(\mathrm{grad}\, f(x))\cdot f(x)^{m-r} = b_0 f(x)^{m-1}$, i.e., $f^*(\mathrm{grad}\log f(x)) = b_0/f(x)$ where $b_0 \neq 0$ since $\deg b(m) = r$. Therefore, we obtain the following proposition.

PROPOSITION 24. *Let $(G, \rho, V)$ be a P. V. with a reductive algebraic group $G$ where $\rho$ is not necessarily irreducible. Assume that there exists a relative invariant polynomial corresponding to a character $\chi$. Then there exists a relative invariant polynomial $f^*(y)$ of $(G, \rho^*, V^*)$ corresponding to $\chi^{-1}$, where $\rho^*$ is the contragredient representation of $\rho$ on the dual vector space $V^*$ of $V$. Moreover, we have $f^*(\mathrm{grad}\log f(x)) \neq 0$ for $x \in V - S$.*

Now we are ready to prove the following proposition.

PROPOSITION 25. *Let $(G, \rho, V)$ be a P. V. with a reductive algebraic group $G$ where $\rho$ is not necessarily irreducible. Assume that a generic isotropy subgroup $G_{x_0}$ $(x_0 \in V - S)$ is also reductive. Then it is a regular P. V.*

*Proof.* It is well known that the quotient $G/G_{x_0} \approx V - S$ of reductive algebraic group $G$ is an affine variety if and only if $G_{x_0}$ is reductive (see [8]). Since $V - S$ is affine if and only if $S$ is a hypersurface, in our case the singular set $S$ is a hypersurface. Let $f(x)$ be a relative invariant polynomial satisfying $S = \{x \in V \mid f(x) = 0\}$. Then by Proposition 24, there exists a relative invariant polynomial $f^*$ of $(G, \rho^*, V^*)$ corresponding to $f(x)$. Moreover, the singular set $S^*$ of $(G, \rho^*, V^*)$ is also a hypersurface defined by $f^*$, i.e., $S^* = \{y \in V^* \mid f^*(y) = 0\}$. Since $f^*(\operatorname{grad}\log f(x)) \neq 0$ for $x \in V - S$ by Proposition 24, we have $\operatorname{grad}\log f(V - S) = V^* - S^*$, i.e., $\operatorname{grad}\log f$ is generically surjective. This shows that $(G, \rho, V)$ is regular by Remark 11. Q.E.D.

*Remark* 26. Let $(G, \rho, V)$ be a $P.V.$ with a reductive algebraic group $G$. Then it is regular if and only if its generic isotropy subgroup is reductive, and hence the regularity is invariant under the castling transform (see Proposition 9 in § 2). Since its generic isotropy subgroup is reductive if and only if the singular set $S$ is a hypersurface, a reductive $P.V.$ is regular if and only if its singular set is a hypersurface. However, if $G$ is not reductive, it is false. For example, put $G = \left\{\begin{pmatrix} 1 & b \\ & a \end{pmatrix} \middle| a, b \in \boldsymbol{C}, a \neq 0\right\}$, $V = \boldsymbol{C}^2$, $\rho\begin{pmatrix} 1 & b \\ & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + by \\ ay \end{pmatrix}$. Then a triplet $(G, \rho, V)$ is a $P.V.$ and its singular set is a hypersurface defined by $y = 0$. But it is not regular. Conversely, it is an open problem that the singular set of any regular $P.V.$ is a hypersurface or not.

## § 5. The prehomogeneity and relative invariants of reduced triplets obtained in § 3

In this section we shall investigate the reduced triplets obtained in Theorem 24 in § 3, especially we shall determine their prehomogeneity.

(1) $$(G \times GL(m),\ \rho \otimes \Lambda_1,\ V(n) \otimes V(m))$$

where $\rho$ is an $n$-dimensional irreducible representation of a connected semi-simple algebraic group $G\,(\neq SL(n))$ with $m \geqq n \geqq 2$. This is a trivial P.V. (See Definition 5 in § 3). We may identify $V(n) \otimes V(m)$ with the totality of $n \times m$ matrices $M(n, m)$. We may assume that $G \subset SL(n)$. Then $(\rho \otimes \Lambda_1)(g_1, g_2)X = g_1 X {}^t g_2$ for $g_1 \in G$, $g_2 \in GL(m)$, and $X \in M(n, m)$. If $m = n$, then the isotropy subgroup at $I_n$ is $\{(g_1, {}^t g_1^{-1}) \mid g_1 \in G\} \cong G$ and hence it is a regular P.V. by Proposition 25 in § 4. The relative invariant is $\det X$. If $m > n$, the generic isotropy subgroup $G_{x_0}$ at $x_0 = [I_n O_{n,m-n}]$ where $O_{n,m-n}$ is the $n \times (m - n)$ zero matrix, is the totality of $(g_1, g_2)$ of the form

$$g_2 = \left(\begin{array}{c|c} {}^t g_1^{-1} & 0 \\ \hline * & * \end{array}\right),$$

and hence $G_{x_0}$ is isomorphic to the semi-direct product of $(G \times GL(m-n))$ and the $n(m-n)$-dimensional vector group $(G_a)^{n(m-n)}$ i.e. $G_{x_0} \cong (G \times GL((m-n)) \cdot (G_a)^{n(m-n)}$ where $(G_a)$ denotes the one-dimensional additive group: $G_a \cong \boldsymbol{C}$. By (2) in Proposition 20 in § 4, it is not regular. There are no relative invariants.

(2) $$(SL(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$$
$$(m/2 \geqq n \geqq 1 \text{ or } n = m \geqq 1)$$

This is also a trivial P.V. Similarly it is regular if $n = m$ and not regular if $n < m$. (1) and (2) are essentially of the same type. The reason why we separated (2) from (1) is only to avoid the triplets $(SL(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ $(m > n > m/2)$ which is not reduced.

PROPOSITION 1. *A trivial P.V.*, $(G \times GL(m), \rho \otimes \Lambda_1, V(n) \otimes V(m))$ $(n \leq m)$ *is regular if and only if* $m = n$.

(3) $$(GL(1) \times G,\ \square \otimes \text{adjoint rep.},\ V(1) \otimes V(n))$$

where $G$ is an almost simple algebraic group of dimension $n \geqq 3$.

Let $\mathfrak{g}$ be the Lie algebra of $G$. We may assume that $G \subset GL(n)$ and $\mathfrak{g} \subset \mathfrak{gl}(n)$. Then the adjoint representation $Ad$ is given by $Ad(g)X = gXg^{-1}$ for any $g \in G$, $X \in \mathfrak{g}$. Put $\mathfrak{g}(X) = \{Y \in \mathfrak{g} \mid \mathrm{ad}(X)^n Y = 0 \text{ for some } n > 0\}$ for $X \in \mathfrak{g}$. An element $X$ in $\mathfrak{g}$ is called a *regular element* if $\dim \mathfrak{g}(X)$ is minimum. If $X_0$ is a regular element, then $\mathfrak{h} = \mathfrak{g}(X_0)$ is a Cartan subalgebra. Note that $\dim \mathfrak{g}(X)$ is the multiplicity of the eigenvalue 0 of $\mathrm{ad}(X)$. Let $A_1, \cdots, A_n$ be a basis of $\mathfrak{g}$ and put $\det(tI - \mathrm{ad}(X)) = t^n + \varphi_1(x_1, \cdots, x_n)t^{n-1} + \cdots + \varphi_{n-\ell}(x_1, \cdots, x_n)t^\ell$ $(\ell \geqq 1)$ for $X = \sum_{i=1}^n x_i A_i \in \mathfrak{g}$, where $\ell = \mathrm{rank}\,\mathfrak{g} = \dim \mathfrak{h}$. Then the totality of non-regular elements is a hypersurface $S$ given by $S = \{X = \sum_{i=1}^n x_i A_1 \in \mathfrak{g} \mid \varphi_{n-\ell}(x_1, \cdots, x_n) = 0\}$. Since $\mathfrak{g}(gXg^{-1}) = g\mathfrak{g}(X)g^{-1}$, this hypersurface is $G$-invariant, and hence if this triplet is a P.V., then a generic point must be a regular element. Let $X_0$ be a regular element. Let $\tilde{\mathfrak{g}}_{X_0}$ be the isotropy subalgebra $\tilde{\mathfrak{g}}_{X_0} = \{(c, X) \in \mathfrak{gl}(1) \oplus \mathfrak{g} \mid cX_0 + [X, X_0] = 0\}$ at $X_0$. Then it is a P.V. if and only if $\dim \tilde{\mathfrak{g}}_{X_0} = \dim(GL(1) \times G) - \dim \mathfrak{g} = 1$. Since $\mathfrak{h} = \mathfrak{g}(X_0)$ is a Cartan subalgebra and hence commutative, we have $\mathfrak{h} = \{(0, X) \in \mathfrak{gl}(1) \oplus \mathfrak{g} \mid [X, X_0] = 0\} \subset \tilde{\mathfrak{g}}_{X_0}$. This shows that $\dim \tilde{\mathfrak{g}}_{X_0} \geqq \dim \mathfrak{h} = \mathrm{rank}\,\mathfrak{g}$ and hence it is not a P.V.

if $\operatorname{rank}\mathfrak{g} > 1$. Assume that $\operatorname{rank}\mathfrak{g} = 1$. Then $\mathfrak{g} = \mathfrak{sl}(2)$ $(= \mathfrak{sp}(1) = \mathfrak{o}(3))$ (See Example 45 in § 1), and it is equivalent to $(GL(2), 2\Lambda_1, V(3))$. The representation space can be considered as the space of all binary quadratic forms $F_x(u, v) = x_1u^2 + x_2uv + x_3v^2$ with $x = (x_1, x_2, x_3) \in C^3$. The action $\rho = 2\Lambda_1$ of $GL(2)$ is given by $(\rho(g)F_x)(u, v) = F_x((u, v)g)$ for $g \in GL(2)$. Since the isotropy subgroup $G_{X_0}$ at $X_0 = uv$ is

$$O(2) = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \middle| \alpha \neq 0\right\},$$

it is a regular P.V. by Proposition 25 in § 4. The relative invariant $f(x)$ is a discriminant of $F_x$ i.e. $f(x) = x_2^2 - 4x_1x_3$ for $x = F_x$.

PROPOSITION 2. *The composition of scalar multiplications and the adjoint representation of an almost simple algebraic group $G$ is prehomogeneous if and only if* $\operatorname{rank} G = 1$. *If* $\operatorname{rank} G = 1$, *it is a regular P.V. equivalent to* $(GL(2), 2\Lambda_1, V(3))$.

(4) $$(GL(n),\ 2\Lambda_1,\ V(\tfrac{1}{2}n(n+1))) \qquad (n \geqq 3)$$

Put $V = \{X \in M(n, C) \mid {}^tX = X\}$. Then the action $\rho = 2\Lambda_1$ of $GL(n)$ is given by $\rho(A)X = AX^tA$ for $A \in GL(n)$, $X \in V$. The isotropy subgroup at the unit matrix $I_n$ is by definition the orthogonal group $O(n) = \{A \in GL(n) \mid A^tA = 1\}$ and since $\dim O(n) = \frac{1}{2}n(n-1) = \dim GL(n) - \dim V(\frac{1}{2}n(n+1))$, it is a P.V. By Proposition 25 in § 4, it is regular and the relative invariant is given by $\det X$ for $X \in V$.

PROPOSITION 3. *A triplet* $(GL(n),\ 2\Lambda_1,\ V(\frac{1}{2}n(n+1)))$ *is a regular P.V. and its generic isotropy subgroup is the orthogonal group* $O(n)$.

(5) $$(GL(n),\ \Lambda_2,\ V(\tfrac{1}{2}n(n-1))) \qquad (n \geqq 5)$$

Put $V = \{X \in M(n, C) \mid {}^tX = -X\}$. Then the action $\rho = \Lambda_2$ of $GL(n)$ is given by $\rho(A)X = AX^tA$ for $A \in GL(n)$, $X \in V$. Assume that $n$ is even, i.e., $n = 2m$. Then the isotropy subgroup at

$$J = \left(\begin{array}{c|c} 0 & I_m \\ \hline -I_m & 0 \end{array}\right)$$

is by definition the symplectic group $Sp(m) = \{A \in GL(2m) \mid {}^tAJA = J\}$ since ${}^tAJA = J$ if and only if $AJ^tA = AJ({}^tAJA)A^{-1}J^{-1} = AJ(J)A^{-1}J^{-1} = -J^{-1} = J$. Since $\dim Sp(m) = m(2m+1) = \dim GL(2m) - \dim V(m(2m-1))$, it is a P.V. By Proposition 25 in § 4, it is regular and the relative invariant is the Pfaffian of a skew-symmetric matrix $X$ in $V$. Next assume that $n$ is odd i.e. $n = 2m + 1$. Put

$$x_0 = \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}\right).$$

Since

$$(5.1)\qquad \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}\right){}^t\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} AJ^tA & AJ^tC \\ \hline CJ^tA & CJ^tC \end{array}\right) = \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}\right)$$
$$\text{for } \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in G_{x_0},$$

we have

$$G_{x_0} = \left\{\left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right) \middle| AJ^tA = J\right\} \cong (Sp(m) \times GL(1))\cdot(G_a)^{2m}.$$

Since

$$\begin{aligned} &\dim (Sp(m) \times GL(1))\cdot(G_a)^{2m} \\ &\quad = (m+1)(2m+1) \\ &\quad = \dim GL(2m+1) - \dim V(m(2m+1)), \end{aligned}$$

it is a $P.V.$ By (2) in Proposition 20 in § 4, there is no relative invariant and hence it is not regular. Thus we have the following proposition.

PROPOSITION 4. *A triplet $(GL(2m), \Lambda_2, V(m(2m-1)))$ is a regular $P.V.$ and its generic isotropy subgroup is the symplectic group $Sp(m)$. The relative invariant is the Pfaffian and hence of degree $m$.*

PROPOSITION 5. *A triplet $(GL(2m+1), \Lambda_2, V(m(2m+1)))$ is a $P.V.$ There is no relative invariant and hence not regular. The generic isotropy subgroup is isomorphic to the semi-direct product $(Sp(m) \times GL(1))\cdot(G_a)^{2m}$.*

(6) $(GL(2), 3\Lambda_1, V(4))$

The representation space can be identified with the space of all binary cubic forms $F_x(u, v) = x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3$ with $x = (x_1, x_2, x_3, x_4) \in C^4$. Then the action $\rho = 3\Lambda_1$ of $GL(2)$ is given by $(\rho(g)F_x)(u, v) = F_x((u, v)g)$ (See (6) in Example 24 in § 1). For each binary cubic form $F_x = F_x(u, v) = x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3$, let $F'_x$ be the binary quadratic form defined by

$$F'_x = \frac{1}{4}\det\begin{bmatrix} \dfrac{\partial^2 F}{\partial u \partial u} & \dfrac{\partial^2 F}{\partial u \partial v} \\ \dfrac{\partial^2 F}{\partial v \partial u} & \dfrac{\partial^2 F}{\partial v \partial v} \end{bmatrix} = \det\begin{pmatrix} 3x_1u + x_2v, & x_2u + x_3v \\ x_2u + x_3v, & x_3u + 3x_4v \end{pmatrix}$$
$$= (3x_1x_3 - x_2^2)u^2 + (9x_1x_4 - x_2x_3)uv + (3x_2x_4 - x_3^2)v^2 .$$

Then the correspondence $F_x \mapsto F'_x$ gives a generically surjective $GL(2)$-equivariant morphism $\varphi$ of $V$ into $V'$ where $V$ (resp. $V'$) denotes the space of all binary cubic (resp. binary quadratic) forms. As we have seen in (3) just before Proposition 2, a triplet $(GL(2), 2\Lambda_1, V')$ is a regular P.V. and the generic isotropy subgroup $G_{X'_0}$ at $X'_0 = 9uv$ is given by $G_{X'_0} = O(2)$ $= \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \middle| \alpha \neq 0\right\}$. The fiber $\varphi^{-1}(X'_0)$ of $\varphi$ at $X'_0$ is by definition $\varphi^{-1}(X'_0)$ $= \{F_x(u, v) \in V \mid 3x_1x_3 - x_2^2 = 3x_2x_4 - x_3^2 = 0,\ 9x_1x_4 - x_2x_3 = 9\} = \{F_x(u, v) =$ $x_1u^3 + x_1^{-1}v^3 \mid x_1 \neq 0\}$. Since an element $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ of $G_{X'_0}$ acts on the fiber $\varphi^{-1}(X'_0)$ such as $x_1u^3 + x_1^{-1}v^3 \mapsto (\alpha x_1)u^3 + (\alpha x_1)^{-1}v^3$, $\varphi^{-1}(X'_0)$ is $G_{X'_0}$-homogeneous, and hence by Lemma 5 in §2, a triplet $(GL(2), 3\Lambda_1, V(4))$ is a P.V. We shall determine the isotropy subgroup $G_{X_0}$ at $X_0 = u^3 + v^3$ in $\varphi^{-1}(X'_0)$. If we identify $V$ with $\boldsymbol{C}^4$ by an isomorphism $F_x(u, v) \mapsto x = {}^t(x_1, x_2, x_3, x_4) \in \boldsymbol{C}^4$, we have

$$\rho(g)x = \begin{bmatrix} \alpha^3 & \alpha^2\beta & \alpha\beta^2 & \beta^3 \\ 3\alpha^2\gamma & \alpha^2\delta + 2\alpha\beta\gamma & 2\alpha\beta\delta + \gamma\beta^2 & 3\beta^2\delta \\ 3\alpha\gamma^2 & 2\alpha\gamma\delta + \gamma^2\beta & \alpha\delta^2 + 2\beta\gamma\delta & 3\beta\delta^2 \\ \gamma^3 & \gamma^2\delta & \gamma\delta^2 & \delta^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} \tag{5.2}$$

$$\text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2)$$

and hence

$$G_{X_0} = \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2) \mid \alpha^3 + \beta^3 = \gamma^3 + \delta^3 = 1,\ \alpha^2\gamma + \beta^2\delta = \alpha\gamma^2 + \beta\delta^2 = 0\right\}$$
$$= \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| \alpha^3 = \delta^3 = 1\right\} \cup \left\{\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \middle| \beta^3 = \gamma^3 = 1\right\} .$$

From (5.2), the kernel of $\rho$ is the central cyclic group $\mathscr{Z}$ of order 3:

$$\mathscr{Z} = \left\{\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \middle| \omega^3 = 1\right\} .$$

Thus the isotropy subgroup $G_{X_0}$ is a finite group of order 18 and its image $\rho(G_{X_0}) \cong G_{X_0}/\mathscr{X}$ is a finite group of order 6. In particular, it is a regular P.V. by Proposition 25 in § 4. Since $\varphi$ is equivariant, the discriminant of $F'_x$ i.e.

$$\begin{aligned}(9x_1x_4 - x_2x_3)^2 &- 4(3x_1x_3 - x_2^2)(3x_2x_4 - x_3^2) \\ &= -3(x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_1x_3^3 - 4x_2^3x_4 - 27x_1^2x_4^2)\end{aligned}$$

is a relative invariant, and its character is $(\det g)^6$ for $g \in GL(2)$. Note that $P(x) = (x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_1x_3^3 - 4x_2^3x_4 - 27x_1^2x_4^2)$ is the discriminant of the binary cubic form $F_x(u, v)$.

Let $\chi$ be any rational character of $GL(2)$ i.e. $\chi(g) = (\det g)^m$ for some integer $m$.

Since the restriction $\chi|_{G_{X_0}}$ of $\chi$ to $G_{X_0}$ is identity if and only if $(\alpha\delta)^m = (-\beta\gamma)^m = 1$ for any $\alpha, \beta, \gamma, \delta$ satisfying $\alpha^3 = \beta^3 = \gamma^3 = \delta^3 = 1$ i.e., $m = 6n$ for some integer $n$. Hence any relative invariant is of the form $cP(x)^n$ ($c \in \boldsymbol{C}$, $n \in \boldsymbol{Z}$) by Proposition 19 in § 4. In particular $P(x)$ must be irreducible. The existence of the equivariant polynomial map $\varphi$ of degree 2 is based on the fact that the symmetric product $S^2(V)$ of the representation space $V$ of $3\Lambda_1$ decomposes into the dirəct sum $S^2(V) = V_1 \oplus V_2$ as a representation space of $GL(n)$ where $V_1$ (resp. $V_2$) is corresponding $6\Lambda_1$ (resp. $2\Lambda_1 + 2\Lambda_2$) i.e.

$$S^2(3\Lambda_1) = 6\Lambda_1 \oplus (2\Lambda_1 + 2\Lambda_2)$$

or

$$S^2(\ydiagram{3}) = (\ydiagram{6}) \oplus \ydiagram{4,2}$$

by the Young diagram (See § 1). For $n = 2$, we have $(SL(2), 2\Lambda_1 + 2\Lambda_2, V_2) \cong (SL(2), 2\Lambda_1, V)$ and hence we know that $\varphi$ is a map obtained by the composition $V \to S^2(V) = V_1 \oplus V_2 \to V_2 \cong V'$. Since $\dim GL(2) = \dim V$, we can also use Proposition 16 in § 4 to construct a relative invariant. By (6) in Example 30 in § 1, we have

$$d\rho(A)x = \begin{pmatrix} 3a & b & 0 & 0 \\ 3c & 2a + d & 2b & 0 \\ 0 & 2c & a + 2d & 3b \\ 0 & 0 & c & 3d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{5.3}$$

$$\text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}(2) .$$

Note that (5.3) is also obtained from (5.2) by putting

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in (5.2) and taking out the coefficients of the linear term $t$ since

$$\rho\left(\exp t \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \exp t d\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for any $t \in \boldsymbol{C}$. The isotropy subalgebra at $X_0 = u^3 + v^3$ (i.e. $X_0 = {}^t(1, 0, 0, 1)$) is 0 from (5.3) and hence it is a regular P.V. by Proposition 16 in § 4. The isotropy subalgebra $\mathfrak{g}_{X'_0}$ at $X'_0 = u^2 v$ (i.e. $X'_0 = {}^t(0, 1, 0, 0)$) is

$$\mathfrak{g}_{X'_0} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in \mathfrak{gl}(2) \right\}$$

and hence the orbit of $X'_0$ is of codimension one. By Proposition 16 in § 4, there exists an irreducible relative invariant polynomial $f(x)$ of degree $4\,(= \dim V)$. We shall construct it according to the proof of Proposition 16 in § 4. Let $A_1$ (resp. $A_2, A_3, A_4$) be the $4 \times 4$ matrix with $a = 1$ (resp. $b, c, d = 1$), all remaining entries zero in (5.3). Then $\{A_1, \cdots, A_4\}$ is a basis of $d\rho(\mathfrak{gl}(2))$ and for $x = {}^t(x_1, x_2, x_3, x_4)$, we have

$$(5.4) \qquad \begin{aligned} f(x) = \det(A_1 x, \cdots, A_4 x) &= \det \begin{bmatrix} 3x_1 & x_2 & 0 & 0 \\ 2x_2 & 2x_3 & 3x_1 & x_2 \\ x_3 & 3x_4 & 2x_2 & 2x_3 \\ 0 & 0 & x_3 & 3x_4 \end{bmatrix} \\ &= 3(x_2^2 x_3^2 + 18 x_1 x_2 x_3 x_4 - 4 x_1 x_3^3 - 4 x_2^3 x_4 - 27 x_1^2 x_4^2)\,. \end{aligned}$$

PROPOSITION 6. *A triplet* $(GL(2), 3\Lambda_1, V(4))$ *is a regular P.V. and its generic isotropy subgroup is a finite group of order* 18. *The representation space can be identified with the space of binary cubic forms* $F_x(u, v) = x_1 u^3 + x_2 u^2 v + x_3 u v^2 + x_4 v^3$ $((x_1, x_2, x_3, x_4) \in \boldsymbol{C}^4)$ *and then the action* $\rho = 3\Lambda_1$ *of* $GL(2)$ *is given by* $((\rho(g)F_x)(u, v) = F_x((u, v)g)$ *for* $g \in GL(2)$. *The relative invariant* $f(x)$ *is the discriminant of* $F_x$ *i.e.* $f(x) = x_2^2 x_3^2 + 18 x_1 x_2 x_3 x_4 - 4 x_1 x_3^3 - 4 x_2^3 x_4 - 27 x_1^2 x_4^2$.

(7) $(GL(6),\ \Lambda_3,\ V(20))$

Let $V_1 = \sum_{i=1}^{6} \boldsymbol{C} u_i$ be a vector space with a basis $\{u_1, \cdots, u_6\}$. Then $GL(6)$ acts on $V_1$ as $(u_1, \cdots, u_6) \mapsto (u_1, \cdots, u_6)g$ for $g \in GL(6)$. In general, let $V_\nu = \sum_{i_1<\cdots<i_\nu} \boldsymbol{C}u_{i_1}\wedge\cdots\wedge u_{i_\nu}$ be the skew-symmetric tensors of rank $\nu$ $(1\le\nu\le 6)$. Then $\dim V_\nu = \binom{6}{\nu}$, $V_6=\boldsymbol{C}\tau$ where $\tau = u_1\wedge\cdots\wedge u_6$, and the representation space of $(GL(6), \Lambda_3, V(20))$ can be identified with $V_3$ under the induced action of $GL(6)$. Denote by $A$ the Grassmann algebra generated by $V_1$ i.e. $A=\boldsymbol{C}+V_1+\cdots+V_6$, and define the polarization $D_\nu\colon V_\nu\to V_\nu\otimes V_1$ as a part of the derivation map $D$ from $A$ to a left $A$-module $A\otimes V_1$ determined by $D_1\colon V_1\to\boldsymbol{C}\otimes V_1$ which is given by $D_1(\omega) = 1\otimes\omega$. Namely $D_\nu$ is defined by $D_\nu(u_{i_1}\wedge\cdots\wedge u_{i_\nu}) = \sum_{\mu=1}^{\nu}(-1)^{\nu-\mu}u_{i_1}\wedge\cdots\wedge u_{i_{\mu-1}}\wedge u_{i_{\mu+1}}\wedge\cdots\wedge u_{i_\nu}\otimes u_{i_\mu}$. For each $\eta\in V_4$ and $\theta\in V_3$, we have $(\eta\otimes 1)\wedge D_3(\theta)\in V_6\otimes V_1 = \boldsymbol{C}\tau\otimes V_1 = \tau\otimes V_1$, and hence there exists a bilinear map $L\colon V_4\times V_3\to V_1$ satisfying $(\eta\otimes 1)\wedge D_3(\theta) = \tau\otimes L(\eta,\theta)$ for any $\eta\in V_4$, $\theta\in V_3$. Now for each $\theta\in V_3$, define a linear endomorphism $S_\theta$ of $V_1$ by $S_\theta(\omega) = L(\omega\wedge\theta,\theta)$ for $\omega\in V_1$. Note that matrix elements of $S_\theta$ are quadratic forms of coefficients $x_{ijk}$ of $\theta = \sum x_{ijk}u_i\wedge u_j\wedge u_k\in V_3$. For $\theta_0 = u_1\wedge u_2\wedge u_3 + u_4\wedge u_5\wedge u_6$, from $\tau\otimes S_\theta(\omega) = ((\omega\wedge\theta)\otimes 1)\wedge D_3(\theta)$, we have $S_{\theta_0}(u_i) = u_i (1\le i\le 3)$ and $S_\theta(u_i) = -u_i(4\le i\le 6)$. Hence we can divide $V_1$ into eigenspaces $V_{\theta_0}^{\pm}$ of $S_{\theta_0}$.

$$V_1 = V_{\theta_0}^+ + V_{\theta_0}^- \qquad \text{where } V_{\theta_0}^+ = \sum_{i=1}^{3}\boldsymbol{C}u_i,\ V_{\theta_0}^- = \sum_{i=4}^{6}\boldsymbol{C}u_i\,. \tag{5.5}$$

On the other hand, for $\omega\in V_1$, $T\in GL(6)$, we have $\tau\otimes S_{T(\theta)}\circ T(\omega) = \tau\otimes L(T(\omega)\wedge T(\theta), T(\theta)) = (T(\omega)\wedge T(\theta)\otimes 1)\wedge D_3T(\theta) = T((\omega\wedge\theta)\otimes 1\wedge D_3(\theta)) = T(\tau\otimes S_\theta(\omega)) = (\det T)\tau\otimes T\circ S_\theta(\omega)$ and hence $S_{T(\theta)} = (\det T)T\circ S_\theta\circ T^{-1}$. Let $G_{\theta_0}$ be the isotropy subgroup $\{T\in GL(6)\,|\,T(\theta_0)=\theta_0\}$ of $GL(6)$. Then we have $S_{\theta_0}T(u_1) = (\det T)T(u_1)$ for $T\in G_{\theta_0}$. This implies $(\det T) = \pm 1$ since eigenvalues of $S_{\theta_0}$ is $\pm 1$. If $\det T=1$, then we have $S_{\theta_0}T = TS_{\theta_0}$ and hence $T(V_{\theta_0}^+)\subset V_{\theta_0}^+$ and $T(V_{\theta_0}^-)\subset V_{\theta_0}^-$. Therefore $T(\theta_0) = \det(T|_{V_{\theta_0}^+})u_1\wedge u_2\wedge u_3 + \det(T|_{V_{\theta_0}^-})u_4\wedge u_5\wedge u_6 = \theta_0$ and hence $T$ must be in $SL(V_{\theta_0}^+)\times SL(V_{\theta_0}^-)$. Conversely it is clear that $SL(V_{\theta_0}^+)\times SL(V_{\theta_0}^-)\subset G_{\theta_0}$. Put

$$T_0 = \left(\begin{array}{c|c} & I_3 \\ \hline I_3 & \end{array}\right).$$

Then $T_0\in G_{\theta_0}$ and $\det T_0 = -1$, moreover for any $T$ in $G_{\theta_0}$ with $\det T = -1$, $TT_0^{-1}$ is in $SL(V_{\theta_0}^+)\times SL(V_{\theta_0}^-)$. Hence the isotropy subgroup $G_{\theta_0}$ is given by

$$G_{\theta_0} = \left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B\end{array}\right)\,\middle|\, A, B\in SL(3)\right\}\times\left\{\left(\begin{array}{c|c} 0 & I_3 \\ \hline I_3 & 0\end{array}\right),\ I_6\right\}$$

$$\cong (SL(3) \times SL(3)) \times \{\pm 1\}.$$

Since $\dim G_{\theta_0} = 16 = \dim GL(6) - \dim V_3$, a triplet $(GL(6), \Lambda_3, V_3)$ is a P.V. and by Proposition 25 in § 4, it is regular. Hence we obtain the first assertion of the following proposition.

PROPOSITION 7. (1) *A triplet* $(GL(6), \Lambda_3, V(20))$ *is a regular P.V. and its generic isotropy subgroup is isomorphic to* $(SL(3) \times SL(3)) \times \{\pm 1\}$. (2) *trace* $S_\theta = 0$ *for any* $\theta \in V_3$. (3) *There exists an irreducible relative invariant polynomial P of degree* 4 *satisfying* $S_\theta^2 = P(\theta)I_6$ *for any* $\theta \in V_3$. (4) *Let K be a subfield of* $\boldsymbol{C}$, *and put* $V_k = \sum_{i=1}^{6} Ku_i$. *Then* $SL(V_k)$ *operates transitively on K-rational points of the hypersurface* $W = \{\theta \in V_3 \mid P(\theta) = 1\}$.

*Proof.* (2) Since eigenvalues of $S_{\theta_0}$ is $1, 1, 1, -1, -1, -1$, we have $\operatorname{trace} S_{\theta_0} = 1 + 1 + 1 - 1 - 1 - 1 = 0$. For $\theta = T\theta_0$ $(\det T \neq 0)$,

$$\operatorname{trace} S_\theta = \operatorname{trace} (\det T)TS_{\theta_0}T^{-1} = 0 .$$

Since the Zariski closure of $\{\theta \in V_3 \mid \theta = T(\theta_0),\ \det T \neq 0\}$ is $V_3$, we have our assertion. (3) Since $S_{\theta_0}^2(u_i) = u_i$ $(1 \leq i \leq 6)$, we have $S_{\theta_0}^2 = I_6$. For $\theta = T\theta_0(\det T \neq 0)$, we have $S_\theta^2 = ((\det T)TS_{\theta_0}T^{-1})^2 = (\det T)^2 I_6$. Since the Zariski closure of $\{\theta \in V_3 \mid \theta = T(\theta_0),\ \det T \neq 0\}$ is $V_3$, there exists a polynomial $P$ on $V_3$ satisfying $S_\theta^2 = P(\theta)I_6$. Clearly $P(\theta)$ is a relative invariant corresponding to $(\det T)^2$.

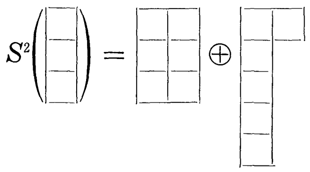
Since $(\det T)^m|_{G_0} = 1$ if and only if $m$ is even i.e. $m = 2n$ $(n \in \boldsymbol{Z})$, $P(\theta)$ is irreducible and any relative invariant is of the form $cP(\theta)^n$ $(c \in \boldsymbol{C}, n \in \boldsymbol{Z})$ by Proposition 19 in § 4. For $T = tI_6$ $(t \in \boldsymbol{C}^\times)$, since $(\det T)^2 = t^{12}$ and $P(T\theta) = t^{3 \deg P}$. $P(\theta)$, we have $t^{12} = t^{3 \cdot \deg P}$ $(t \in \boldsymbol{C}^\times)$ and hence $\deg P = 4$. (4) Let $\theta$ be a $K$-rational point in $W$. Then $S_\theta$ is defined over $K$ and $V_{\theta,K}^{\pm} = \frac{1}{2}(I_6 \pm S_\theta)V_K$ is an eigenspace of $S_\theta$ corresponding to $\pm 1$ since $S_\theta^2 = I_6$. Note that $V_k = V_{\theta,K}^+ + V_{\theta,K}^-$.

Since $P(\theta) \neq 0$, there exists $T \in GL(6)$ satisfying $\theta = T(\theta_0)$. Put $\tilde{v}_i = Tu_i$ $(1 \leq i \leq 6)$. Then we have $\theta = \tilde{v}_1 \wedge \tilde{v}_2 \wedge \tilde{v}_3 + \tilde{v}_4 \wedge \tilde{v}_5 \wedge \tilde{v}_6$ and $S_\theta(\tilde{v}_i) = (\det T)TS_{\theta_0}T^{-1}(T\theta_i) = (\det T)\tilde{v}_i$ $(1 \leq i \leq 3)$, $S_\theta(\tilde{v}_i) = -(\det T)\tilde{v}_i$ $(4 \leq i \leq 6)$. Since $V_3 = V_{\theta,C}^+ + V_{\theta,C}^-$ where $V_{\theta,C}^{\pm} = \frac{1}{2}(1 \pm S_\theta)V_3$, an eigenvalue of $S_\theta$ is $\pm 1$, and hence $(\det T) = \pm 1$. Therefore by changing indices if necessary, we may assume that $\theta = \tilde{v}_1 \wedge \tilde{v}_2 \wedge \tilde{v}_3 + \tilde{v}_4 \wedge \tilde{v}_5 \wedge \tilde{v}_6$, $V_{\theta,C}^+ = \sum_{i=1}^{3} \boldsymbol{C}\tilde{v}_i$ and $V_{\theta,C}^- = \sum_{i=4}^{6} \boldsymbol{C}\tilde{v}_i$. There exist $v_i'$ $(1 \leq i \leq 6)$ in $V_K$ and $c_i \in \boldsymbol{C}$ $(i = 1, 2)$ such that $\theta = c_1 v_1' \wedge v_2' \wedge v_3' + c_2 v_4' \wedge v_5' \wedge u_6'$, $V_{\theta,K}^+ = \sum_{i=1}^{3} Kv_i'$ and $V_{\theta,K}^- = \sum_{i=4}^{6} Kv_i'$.

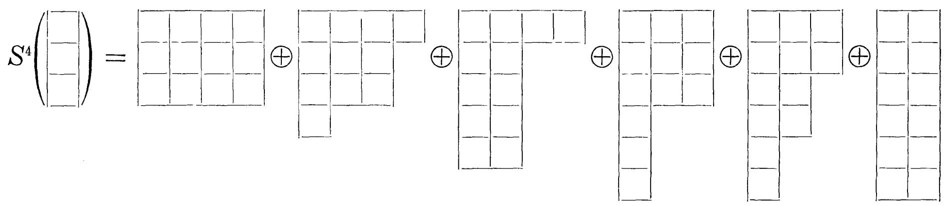
If $c_1 = 0$ or $c_2 = 0$, we have $P(\theta) = 0$ by simple calculation. This is a contradiction since $\theta \in W$. Therefore $c_1 \neq 0$, $c_2 \neq 0$ and moreover $c_1$ and $c_2$ are in $K$ since $\theta$ is $K$-rational.

Put $v_1 = c_1 v_1'$, $v_4 = c_2 v_4'$, $v_i' = v_i'$ $(i \neq 1, 4)$. Then we have $\theta = v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6$, $V_{\theta,K}^{+} = \sum_{i=1}^{3} K v_i$ and $V_{\theta,K}^{-} = \sum_{i=4}^{6} K v_i$. Hence there exists $T'$ in $GL(V_K)$ satisfying $T'(u_i) = v_i$ $(1 \leq i \leq 6)$. Since $S_\theta = (\det T')T' \circ S_{\theta_0} \circ T'^{-1}$, we have $S_\theta(v_i) = (\det T')v_i = v_i$ $(1 \leq i \leq 3)$ and $S_\theta(v_i) = (-\det T')v_i = -v_i$ $(4 \leq i \leq 6)$, and hence $\det T' = 1$ i.e. $T' \in SL(V_K)$. Q.E.D.

The existence of the equivariant polynomial map $\theta \mapsto S_\theta$ of degree 2 from $V_3$ to $\mathfrak{sl}(6, \boldsymbol{C})$ is based on the fact that $S^2(\Lambda_3) = 2\Lambda_3 \otimes (\Lambda_1 + \Lambda_5)$ i.e.



by the Young diagram. The representation $(\Lambda_1 + \Lambda_5)$ of $GL(6)$ is the composition of the adjoint representation of $SL(6)$ and scalar multiplications $(\det g)$ $(g \in GL(6))$. We can also say that the existence of a relative invariant of degree 4 is based on the fact that



for $GL(6)$. The representation space $2\Lambda_6$ is one-dimensional and the action of $GL(6)$ is given by scalar multiplications $(\det g)^2$ for $g \in GL(6)$.

We shall determine the explicit form of $P(x)$ for $x = \sum x_{ijk} u_i \wedge u_j \wedge u_k \in V(20)$ where $(x_{ijk})$ forms an alternating tensor of rank three with coefficients in $\boldsymbol{C}$.

For example $x_{123} = x_{231} = x_{312} = -x_{132} = -x_{321} = -x_{213}$. Since $\deg P(x) = 4$, $P(x)$ is a linear combination of $x_{i_1 i_2 i_3} x_{i_4 i_5 i_6} x_{i_7 i_8 i_9} x_{i_{10} i_{11} i_{12}}$ where $1 \leq i_1, \cdots, i_{12} \leq 6$. Under the action of $u_i \mapsto a u_i$, $u_j \mapsto u_j$ $(j \neq i)$ for $1 \leq i \leq 6$, each term $x_{i_1 i_2 i_3} \cdots x_{i_{10} i_{11} i_{12}}$ is multiplied by $a^2$ and hence we have $\{i_1, \cdots, i_{12}\} = \{1, 1, 2, 2, \cdots, 6, 6\}$ as a set. Moreover $P(x)$ is invariant under the action of $\mathfrak{S}_6 \subset GL(6)$ where $\mathfrak{S}_6$ denotes the permutation group of $\{1, \cdots, 6\}$

and hence $P(x)$ is of the form

$$\begin{aligned} P(x) = a \sum_{\sigma \in \mathfrak{S}_6}{}' x^2_{\sigma(1)\sigma(2)\sigma(3)} x^2_{\sigma(4)\sigma(5)\sigma(6)} &+ b \sum_{\sigma \in \mathfrak{S}_6}{}' x_{\sigma(1)\sigma(2)\sigma(3)} x_{\sigma(1)\sigma(2)\sigma(4)} x_{\sigma(3)\sigma(5)\sigma(6)} x_{\sigma(4)\sigma(5)\sigma(6)} \\ &+ c \sum_{\sigma \in \mathfrak{S}_6}{}' x_{\sigma(1)\sigma(2)\sigma(3)} x_{\sigma(1)\sigma(4)\sigma(5)} x_{\sigma(2)\sigma(4)\sigma(6)} x_{\sigma(3)\sigma(5)\sigma(6)} \end{aligned}$$

where $\sum'$ denotes the sum of distinct terms.

Since $P(\theta_0) = 1$ for $\theta_0 = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$, we have $a = 1$. For $\theta = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_2 \wedge u_4 + u_3 \wedge u_5 \wedge u_6 + u_4 \wedge u_5 \wedge u_6$ we have $S_\theta(u_i) = 0$ $(1 \leq i \leq 6)$ from $\tau \otimes S_\theta(u_i) = ((u_i \wedge \theta) \otimes 1) \wedge D_3(\theta)$. Hence we have $P(\theta) = 2a + b = 2 + b = 0$ i.e. $b = -2$. For $\theta = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5 + u_2 \wedge u_4 \wedge u_6 + u_3 \wedge u_5 \wedge u_6$, we have $S_\theta(u_1) = -2u_6$, $S_\theta(u_6) = -2u_1$ and hence $P(\theta)u_1 = S^2_\theta(u_1) = S_\theta(-2u_6) = 4u_1$ i.e. $P(\theta) = c = 4$. Hence we have $a = 1$, $b = -2$, and $c = 4$.

We can also express $P(x)$ as follows.

Put

$$X = \begin{pmatrix} x_{423} & x_{143} & x_{124} \\ x_{523} & x_{153} & x_{125} \\ x_{623} & x_{163} & x_{126} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_{156} & x_{416} & x_{451} \\ x_{256} & x_{426} & x_{452} \\ x_{356} & x_{436} & x_{453} \end{pmatrix}.$$

We denote by $X_{ij}$ (resp. $Y_{ij}$) the matrix obtained from $X$ (resp. $Y$) by crossing out its $i$-th line and $j$-th column. Put $x_0 = x_{123}$ and $y_0 = x_{456}$. Then we have

$$\begin{aligned} P(x) = (x_0 y_0 - \operatorname{tr} XY)^2 &+ 4x_0 \det Y + 4y_0 \det X \\ &- 4 \sum_{i,j} \det(X_{ij}) \cdot \det(Y_{ji}) . \end{aligned}$$

One can also use Proposition 15 in §4 to check that the degree of the relative invaiant is four since the orbit of $X_0' = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5 + u_2 \wedge u_4 \wedge u_6$ is of codimension one.

(8) $(GL(7), \Lambda_3, V(35))$

Let $V_1$ be a vector space spanned by $u_1, \cdots, u_7$. Then $GL(7)$ acts on $V_1$ as $\rho_1(g)(u_1, \cdots, u_7) = (u_1, \cdots, u_7)g$ for $g \in GL(7)$. Let $V$ be a vector space spanned by skew tensors $u_i \wedge u_j \wedge u_k$ $(1 \leq i < j < k \leq 7)$. Then $GL(7)$ acts on $V$ as $\rho(g)(u_i \wedge u_j \wedge u_k) = \rho_1(g)u_i \wedge \rho_1(g)u_j \wedge \rho_1(g)u_k$ for $g \in GL(7)$, and $\rho = \Lambda_3$, $V(35) = V$. We shall prove the prehomogeneity by seeking a generic point $X_0$ in $V$. Let $H$ be a subgroup of $GL(7)$ defined by $H = \{g = (g_{ij}) \in GL(7) \mid g_{k1} = 0 \text{ for } 2 \leq k \leq 7\}$. Then $H$ is isomorphic to

$(GL(1) \times GL(6)) \cdot (G_a)^6$ and $GL(6)$ acts prehomogeneously on the subspace of $V$ spanned by $u_i \wedge u_j \wedge u_k$ $(2 \leq i < j < k \leq 7)$. As we have seen in (7), $u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7$ is a generic point and a generic isotropy subgroup is isomorphic to $(SL(3) \times SL(3)) \times \{\pm 1\}$. Therefore we may assume that $X_0$ is of the form

$$u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge \left( \sum_{2 \leq i < j \leq 7} a_{ij} u_i \wedge u_j \right),$$

and $(GL(1) \times SL(3) \times SL(3)) \cdot (G_a)^6$ leaves this form invariant. By the action of $(G_a)^6$, we may assume that $a_{ij} = 0$ $(2 \leq i, j \leq 4$ or $5 \leq i, j \leq 6)$. In fact for $g = (g_{ij})$ where $g_{ii} = 1$ $(1 \leq i \leq 7)$, $g_{12} = -a_{34}$, $g_{13} = a_{24}$, $g_{14} = -a_{23}$, $g_{15} = -a_{67}$, $g_{16} = a_{57}$, $g_{17} = -a_{56}$, all remaining entries 0, we have

$$\begin{aligned} \rho(g) X_0 = {} & u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 \\ & + u_1 \wedge \left( \sum_{i=2}^{4} \sum_{j=5}^{7} a_{ij} u_i \wedge u_j \right), \end{aligned}$$

and $GL(1) \times SL(3) \times SL(3) \cong SL(3) \times GL(3)$ leaves this form invariant. The action of $SL(3) \times GL(3)$ on the $3 \times 3$ matrices $(a_{ij})_{\substack{2 \leq i \leq 4 \\ 5 \leq j \leq 7}}$ is isomorphic to a triplet $(SL(3) \times GL(3),\ \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(3))$. As we have seen in (2), it is a regular trivial P.V. and $(a_{ij}) = I_3$ is a generic point with the isotropy subgroup $SL(3)$. Hence we have $X_0 = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$ and the (connected component) of the isotropy subgroup $H_{X_0}$ of $H$ is isomorphic to $SL(3)$. Since $\dim H_{X_0} = 8 = \dim H - \dim V(35)$, a triplet $(H, \Lambda_3|_H, V_{(35)}$ is a P.V. and hence a triplet $(GL(7), \Lambda_3, V(35))$ is a P.V. with a generic point $X_0 = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$. Let $\mathfrak{g}_{X_0}$ be the isotropy subalgebra of $\mathfrak{g} = \mathfrak{gl}(7, \boldsymbol{C})$ at $X_0$, and let $\mathfrak{h}$ be the Lie algebra of $H$, i.e., $\mathfrak{h} = \{A = (a_{ij}) \in \mathfrak{gl}(7) \mid a_{k1} = 0 \text{ for } 2 \leq k \leq 7\}$. Then we have $\dim \mathfrak{g}_{X_0} = = \dim GL(7) - \dim V(35) = 14$ and

$$\mathfrak{g}_{X_0} \cap \mathfrak{h} = \left\{ \left( \begin{array}{c|c|c} & & \\ \hline & X & \\ \hline & & -{}^tX \end{array} \right) \middle| \; X \in \mathfrak{sl}(3, \boldsymbol{C}) \right\}.$$

Therefore to determine $\mathfrak{g}_{X_0}$, it is sufficient to show that $d\rho(A_i) X_0 = 0$ $(1 \leq i \leq 6)$ where $A_1$ (resp. $A_2, \cdots, A_6$) is the matrix of the form (1.8) in Example 30 in § 1 with $a = 1$ (resp. $b = 1, \cdots, f = 1$), all remaining entries zero.

For example we shall show that $d\rho(A_1) X_0 = 0$. Since $A_1$ acts on $V_1$

as $u_1 \mapsto u_2$, $u_3 \mapsto u_7$, $u_4 \mapsto -u_6$, $u_5 \mapsto 2u_1$, $u_j \mapsto 0$ $(j = 2, 6, 7)$, we have

$$\begin{aligned} &d\rho(A_1)u_2 \wedge u_3 \wedge u_4 = u_2 \wedge u_7 \wedge u_4 - u_2 \wedge u_3 \wedge u_6 \,, \\ &d\rho(A_1)u_5 \wedge u_6 \wedge u_7 = 2u_1 \wedge u_6 \wedge u_7 \,, \\ &d\rho(A_1)u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7) \\ &\qquad = u_2 \wedge (u_3 \wedge u_6 + u_4 \wedge u_7) + u_1 \wedge (u_7 \wedge u_6 - u_6 \wedge u_7) \,, \end{aligned}$$

and hence $d\rho(A_1)X_0 = 0$.

This shows that the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0$ is a simple Lie algebra of type $(G_2)$ (See Example 30 in § 1) and hence a triplet $(GL(7), \Lambda_3, V(35))$ is a regular P.V. by Proposition 25 in § 4.

Now we shall determine the isotropy subgroup $G_{X_0}$, not infinitesimally but globally. First we shall show that $G_{X_0} \subset GO(7)$. There exists, up to constant, uniquely a quadratic form $q(x) = {}^t xAx$ $(x \in V(7), A \in M(7))$ which is invariant under the action of $\mathfrak{g}_{X_0}$ i.e. ${}^tBA + AB = 0$ for any $B \in \mathfrak{g}_{X_0}$ (See the case of $d = 1$ in (32)). In general the quadratic form $q(g^{-1}x) = {}^t x({}^t g^{-1}Ag^{-1})x$ is invariant under the action of the isotropy subalgebra $\mathfrak{g}_{gX_0} = g\mathfrak{g}_{X_0}g^{-1}$ at $gX_0$ where $g \in GL(7)$. In particular, if $g \in G_{X_0}$ i.e. $gX_0 = X_0$, $q(g^{-1}x)$ is invariant under the action of $\mathfrak{g}_{X_0}$ and hence it must coincide with $q(x)$ up to a constant multiple. This implies that $G_{X_0} \subset GO(7)$. Next we shall show that an element $T$ of $GO(7)$ can be written uniquely as $T = cI_7 \cdot T_0$ where $c \in \boldsymbol{C}^\times$ and $T \in SO(7)$.

Assume that $T = cT_0 = c'T_0'$ where $T_0, T_0' \in SO(7)$. Put $a = c'c^{-1}$. Then we have $aI_7 = T_0T_0'^{-1} \in SO(7)$ and hence $\det(aI_7) = a^7 = 1$, $a^2A = A$, i.e., $a^2 = 1$ where $q(x) = {}^t xAx$. Therefore we have $a = 1$, i.e., $c = c'$ and hence $T_0 = T_0'$. Finally we shall show that $G_{X_0}$ can be written as the direct product of the connected component $G_{X_0}^0$ and the finite group $H$ which is contained in the centralizer of $G_{X_0}^0$ in $GL(7)$. Note that by the Schur's lemma, an element of $H$ is of the form $cI_7$ where $c \in \boldsymbol{C}^\times$.

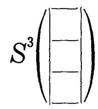
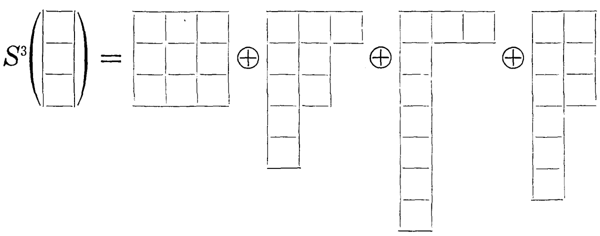
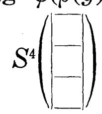
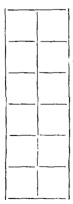
In general let $G$ be a connected and simply connected semi-simple algebraic group with the Lie algebra $\mathfrak{g}$. Then we have the exact sequence $1 \to Z(G) \to G \to \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathrm{Dynkin}) \to 1$ where $Z(G)$ is the center of $G$ and Aut (Dynkin) is the automorphism of the Dynkin diagram of $\mathfrak{g}$. Since the Dynkin diagram of $\mathfrak{g}_{X_0} = (\mathfrak{g}_2)$ is $0 \Rrightarrow 0$, we have $\mathrm{Aut}(\mathrm{Dynkin}) = 1$ and hence $G \to \mathrm{Aut}(\mathfrak{g}_{X_0}) \to 1$ (exact). Since $G$ is connected, $\mathrm{Aut}(\mathfrak{g}_{X_0})$ is also connected. On the other hand, we have the exact sequence

$$1 \longrightarrow H \longrightarrow G_{X_0} \xrightarrow{\varphi} \mathrm{Aut}(\mathfrak{g}_{X_0}) \longrightarrow 1$$

where $\varphi$ is defined by $\varphi(g)X = gXg^{-1}$ for $g \in G_{X_0}$ and $X \in \mathfrak{g}_{X_0}$. As we have $g\mathfrak{g}_{X_0}g^{-1} = \mathfrak{g}_{\rho(g)X_0} = \mathfrak{g}_{X_0}$ for $g \in G_{X_0}$, $\varphi(g)$ is well-defined for $g \in G_{X_0}$. Since the Lie algebra of $G^0_{X_0}$ and that of $\mathrm{Aut}(\mathfrak{g}_{X_0})$ are $\mathfrak{g}_{X_0}$, $\rho(G^0_{X_0})$ is an open subgroup (and hence closed subgroup) of $\mathrm{Aut}(\mathfrak{g}_{X_0})$, and hence we have $\rho(G^0_{X_0}) = \mathrm{Aut}(\mathfrak{g}_{X_0})$. This implies that an element $T$ of $G_{X_0}$ can be written as $T = cI \cdot T_0$ where $cI \in H$ and $T_0 \in G^0_{X_0} \subset SO(7)$. Since this expression is unique, we have $G_{X_0} = G^0_{X_0} \times H$.

If $cI_7 \in H$, then $\rho(cI_7)X_0 = c^3X_0 = X_0$, and hence we have $G_{X_0} = (G_2) \times \{\omega I_7 \mid \omega^3 = 1\}$. Any rational character $\chi$ of $GL(7)$ can be written as $\chi(g) = (\det g)^m$ for some integer $m$ $(g \in GL(7))$. Since $\chi|_{G_{x_0}} = 1$ if and only if $m \equiv 0 \bmod 3$, by Proposition 19 in § 4, any relative invariant is of the form $cf(x)^\ell$ $(c \in \boldsymbol{C}^\times,\ \ell \in \boldsymbol{Z})$ where $f(x)$ is the relative invariant with the character $(\det g)^3$. For $g = tI_7 \in GL(7)$, we have $(\det g)^3 = t^{21} = t^{3\deg f(x)}$, i.e., $\deg f(x) = 7$.

PROPOSITION 8. *A triplet* $(GL(7), \Lambda_3, V(35))$ *is a regular P.V. and its generic isotropy subgroup is* $(G_2) \times \{\omega I_7 \mid \omega^3 = 1\}$. *The relative invariant is of degree* 7.

*Remark* 9. Let $W$ be the totality of $7 \times 7$ symmetric matrices. By the inner product $\langle X, Y\rangle = \mathrm{tr}\, XY$ $(X, Y \in W)$, we may identify the dual vector space $W^*$ of $W$ with $W$. Since the symmetric product  decomposes as  there, exists a polynomial map $\varphi$ of degree 3 from $V(35)$ to $W$ satisfying $\varphi(\rho(g)x) = (\det g)g \cdot \varphi(x)^t g$ for $g \in GL(7)$, $x \in V(35)$. On the other hand,  contains . Note that this fact corresponds to the existence of the relative invariant of degree 4 of $(GL(6), \Lambda_3, V(20))$. Therefore there exists a polynomial map $\varphi^*$ of degree 4 from $V(35)$ to $W^*$ satisfying $\varphi^*(\rho(g)x) = (\det g)^2 \cdot {}^t g^{-1} \varphi^*(x) g^{-1}$. Hence $f(x) = \langle \varphi(x), \varphi^*(x) \rangle = \operatorname{tr} \varphi(x) \cdot \varphi^*(x)$ is a relative invariant of degree 7 satisfying $f((\rho(g)x) = (\det g)^3 f(x)$ for $g \in GL(7)$ and $x \in V(35)$. However, it is necessary to prove that $\langle \varphi(x), \varphi^*(x) \rangle$ is not identically zero.

One can also use Proposition 15 in § 4 to check that $\deg f = 7$ since the orbit of $X_0' = u_2 \wedge u_3 \wedge u_5 + u_3 \wedge u_4 \wedge u_6 + u_1 \wedge u_2 \wedge u_7 - u_1 \wedge u_4 \wedge u_5$ is of codimension one.

(9) $(GL(8), \Lambda_3, V(56))$

Let $V_1$ be a vector space spanned by $u_1, \cdots, u_8$ over $\boldsymbol{C}$. Then $GL(8)$ acts on $V_1$ as $\rho_1(g)(u_1, \cdots, u_8) = (u_1, \cdots, u_8)g$ for $g \in GL(8)$. Let $V$ be a vector space spanned by skew-tensors $u_i \wedge u_j \wedge u_k$ $(1 \leq i < j < k \leq 8)$ of rank 3 over $\boldsymbol{C}$. We identify $V(56)$ with $V$. Then the action $\rho = \Lambda_3$ of $GL(8)$ is given by $\rho(g)(u_i \wedge u_j \wedge u_k) = \rho_1(g)u_i \wedge \rho_1(g)u_j \wedge \rho_1(g)u_k$ $(1 \leq i < j < k \leq 8)$. We shall determine the prehomogeneity by seeking a generic point $X_0$. By the action of $GL(7)$, we have $X_0 = \omega_0 + u_8 \wedge \eta$ where $\omega_0 = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$ and $\eta$ is a 2-form of $u_1, \cdots, u_7$ (see (8)). The isotropy subgroup $(G_2)$ of $GL(7)$ at $\omega_0$ acts on the space $V(21)$ of all 2-forms of $u_1, \cdots, u_7$. Let $\mathfrak{h}$ be the Cartan subalgebra of $(\mathfrak{g}_2)$ defined in Example 30 in § 1. Then the weights of this action of $(\mathfrak{g}_2)$ on $V(21)$ w. r. t. $\mathfrak{h}$ is given by $\{0, \pm\lambda_1, \pm\lambda_2, \pm(\lambda_1 + \lambda_2)\} \cup \{0, 0, \pm\lambda_1, \pm\lambda_2, \pm\lambda_1 \pm \lambda_2, \pm(\lambda_1 + 2\lambda_2), \pm(2\lambda_1 + \lambda_2)\}$ and hence this action of $(G_2)$ is $\Lambda_1 \oplus \Lambda_2$, i.e., $V(21)$ decomposes into the direct sum of the 7-dimensional representation space $V(7)$ and the adjoint representation space $V(14) \cong (\mathfrak{g}_2)$: $V(21) = V(7) \oplus V(14)$ (see Definition 4 and Example 30 in § 1).

Let $\eta_i = \partial\omega_0/\partial u_i$ $(1 \leqq i \leqq 7)$ be the polarizations of $\omega_0$ i.e. $\eta_1 = u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7$, $\eta_2 = u_3 \wedge u_4 - u_1 \wedge u_5$, $\eta_3 = -u_2 \wedge u_4 - u_1 \wedge u_6$, $\eta_4 = u_2 \wedge u_3 - u_1 \wedge u_7$, $\eta_5 = u_6 \wedge u_7 + u_1 \wedge u_2$, $\eta_6 = -u_5 \wedge u_7 + u_1 \wedge u_3$, $\eta_7 = u_5 \wedge u_6 + u_1 \wedge u_4$.

Since

$$\begin{aligned} \frac{\partial\omega_0}{\partial u_i}(u_1, \cdots, u_7) &= \frac{\partial}{\partial u_i}\omega_0((u_1, \cdots, u_7)g) \\ &= \sum_k \frac{\partial\omega_0}{\partial u_k}((u_1, \cdots, u_7)g)\frac{\partial \sum_j u_j g_{jk}}{\partial u_i} \end{aligned}$$

$$= \sum_k \frac{\partial\omega_0}{\partial u_k}((u_1, \cdots, u_7)g)g_{ik}$$

for $g = (g_{ij}) \in (G_2)$, we have

$$\left(\frac{\partial\omega_0}{\partial u_1}((u_1, \cdots, u_7)g), \cdots, \frac{\partial\omega_0}{\partial u_7}((u_1, \cdots, u_7)g)\right) = \left(\frac{\partial\omega_0}{\partial u_1}, \cdots, \frac{\partial\omega_0}{\partial u_7}\right){}^t g^{-1}$$

for $g \in (G_2)$, and hence $\{\eta_1, \cdots, \eta_7\}$ is a basis of $V(7)$. Put $X_0 = \omega_0 + u_8 \wedge \eta' + u_8 \wedge \eta_0$ where $\eta_0 = a_1\eta_1 + \cdots + a_7\eta_7$ and $\eta' \in V(14)$. Then by the action of $u_\nu \mapsto u_\nu - a_\nu u_8$ $(1 \leq \nu \leq 7)$ and $u_8 \mapsto u_8$, we have $X_0 = \omega_0 + u_8 \wedge \eta'$ where $\eta' \in V(14) \cong (\mathfrak{g}_2)$. Moreover we may assume that $\eta'$ is a regular element of $(\mathfrak{g}_2)$ (see (3)). A regular element $\eta'$ can be transferred to an element of the Cartan subalgebra $\mathfrak{h}$ by the action of $(G_2)$. We shall determine the subspace of $V(14)$ corresponding to $\mathfrak{h}$. Let $\varphi\colon (\mathfrak{g}_2) \to V(14)$ be the $(G_2)$-equivariant isomorphism. Since $\mathfrak{h}$ is abelian, we have $A\varphi(X) = \varphi([A, X]) = \varphi(0) = 0$ for $A, X \in \mathfrak{h}$, and hence $\varphi(\mathfrak{h}) = \{x \in V(14) \mid Ax = 0$ for any $A \in \mathfrak{h}\} = \{c_1u_2 \wedge u_5 + c_2u_3 \wedge u_6 + c_3u_4 \wedge u_7 \mid c_1 + c_2 + c_3 = 0\}$.

Therefore we may assume that $X_0 = \omega_0 + u_8 \wedge (c_1u_2 \wedge u_5 + c_2u_3 \wedge u_6 + c_3u_4 \wedge u_7)$ with $c_1 + c_2 + c_3 = 0$. By changing indices and generalizing this form, we shall consider the 6-dimensional subvariety $V'$ of $V$ consisting of the forms $u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 + u_7 \wedge (a_1u_1 \wedge u_4 + a_2u_2 \wedge u_5 + a_3u_3 \wedge u_6) + u_8 \wedge (b_1u_1 \wedge u_4 + b_2u_2 \wedge u_5 + b_3u_3 \wedge u_6)$. Let $H$ be the subgroup of $GL(8)$ defined as follows

$$(5.6)\qquad H = \left\{\begin{pmatrix} \Lambda & & 0 \\ & \Lambda' & \\ 0 & & A \end{pmatrix} \middle| \Lambda, \Lambda' \in SL(3),\ A \in GL(2),\ \Lambda, \Lambda' = \text{diagonal}\right\}.$$

Then the subgroup $H$ acts on the subvariety $V'$ as

$$X = {}^t\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \mapsto \Lambda\Lambda' X A\ .$$

Obviously this action is prehomogeneous, and we may take $X = {}^t\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ as a generic point. Hence the triplet $(GL(8), \Lambda_3, V(56))$ is a P. V. and the corresponding point $X_0 = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 + u_7 \wedge (u_1 \wedge u_4 - u_2 \wedge u_5) + u_8 \wedge (u_1 \wedge u_4 - u_3 \wedge u_6)$ is a generic point. We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0$. The infinitesimal action $d\rho$ of $\rho$ is given by $d\rho(A)(u_i \wedge u_j \wedge u_k) = d\rho_1(A)u_i \wedge u_j \wedge u_k + u_i$

$\wedge\, d\rho_1(A)u_j \wedge u_k + u_i \wedge u_j \wedge d\rho_1(A)u_k$. For example $d\rho(A)(u_1 \wedge u_2 \wedge u_3) = (a_{11}u_1 + a_{41}u_4 + a_{51}u_5 + a_{61}u_6 + a_{71}u_7 + a_{81}u_8) \wedge u_2 \wedge u_3 + u_1 \wedge (a_{22}u_2 + a_{42}u_4 + a_{52}u_5 + a_{62}u_6 + a_{72}u_7 + a_{82}u_8) \wedge u_3 + u_1 \wedge u_2 \wedge (a_{33}u_3 + a_{43}u_4 + a_{53}u_5 + a_{63}u_6 + a_{73}u_7 + a_{83}u_8)$ where $A = (a_{ij}) \in \mathfrak{gl}(8)$. Such a calculation shows that $d\rho(A)X_0 = \sum a_{ijk}u_i \wedge u_j \wedge u_k$ where $a_{ijk}$ are given by the following table.

| $ijk$ | $a_{ijk}$ | $ijk$ | $a_{ijk}$ | $ijk$ | $a_{ijk}$ | $ijk$ | $a_{ijk}$ |
|---|---|---|---|---|---|---|---|
| 123 | $a_{11}+a_{22}+a_{33}$ | 148 | $a_{11}+a_{44}+a_{88}+a_{87}$ | 247 | $a_{21}-a_{45}$ | 358 | $-a_{56}$ |
| 124 | $a_{43}-a_{27}-a_{28}$ | 156 | $a_{14}$ | 248 | $a_{21}$ | 367 | $-a_{78}$ |
| 125 | $a_{53}-a_{17}$ | 157 | $a_{54}-a_{12}$ | 256 | $a_{24}-a_{67}$ | 368 | $-a_{33}-a_{66}-a_{88}$ |
| 126 | $a_{63}$ | 158 | $a_{54}$ | 257 | $-a_{22}-a_{55}-a_{77}$ | 378 | $-a_{76}$ |
| 127 | $a_{15}+a_{24}+a_{73}$ | 167 | $a_{64}$ | 258 | $-a_{87}$ | 456 | $a_{44}+a_{55}+a_{66}$ |
| 128 | $a_{24}+a_{83}$ | 168 | $a_{64}-a_{13}$ | 267 | $-a_{65}$ | 457 | $a_{76}-a_{51}-a_{42}$ |
| 134 | $-a_{37}-a_{38}-a_{42}$ | 178 | $a_{74}-a_{84}$ | 268 | $-a_{23}$ | 458 | $a_{86}-a_{51}$ |
| 135 | $-a_{52}$ | 234 | $a_{41}$ | 278 | $a_{85}$ | 467 | $-a_{61}-a_{75}$ |
| 136 | $-a_{18}-a_{62}$ | 235 | $a_{37}+a_{51}$ | 345 | $a_{36}$ | 468 | $-a_{43}-a_{61}-a_{85}$ |
| 137 | $a_{34}-a_{72}$ | 236 | $a_{61}-a_{28}$ | 346 | $a_{48}-a_{35}$ | 478 | $a_{81}-a_{71}$ |
| 138 | $a_{16}+a_{34}-a_{82}$ | 237 | $a_{71}-a_{35}$ | 347 | $a_{31}$ | 567 | $a_{62}+a_{74}$ |
| 145 | $a_{16}+a_{57}+a_{58}$ | 238 | $a_{26}+a_{81}$ | 348 | $a_{31}-a_{46}$ | 568 | $a_{84}-a_{53}$ |
| 146 | $a_{67}+a_{68}-a_{15}$ | 245 | $a_{26}+a_{47}$ | 356 | $a_{34}+a_{58}$ | 578 | $-a_{82}$ |
| 147 | $a_{11}+a_{44}+a_{77}+a_{78}$ | 246 | $-a_{25}$ | 357 | $-a_{32}$ | 678 | $a_{73}$ |

Hence by simple calculation we have $d\rho(A)X_0 = 0$, i.e. $a_{ijk} = 0$ for $1 \leq i < j < k \leq 8$, if and only if $A$ is of the following form (5.7).

$$
(5.7) \qquad A = \left|\begin{array}{c|c|c|c|c|c|c|c|}
\alpha_1 & 0 & 0 & 0 & \gamma_3 & \gamma_2 & \beta_1 & \beta_1 \\
0 & \alpha_2 & 0 & -\gamma_3 & 0 & -\gamma_1 & -2\beta_2 & \beta_2 \\
0 & 0 & \alpha_3 & -\gamma_2 & \gamma_1 & 0 & -\beta_3 & 2\beta_3 \\
\hline
0 & -\beta_3 & -\beta_2 & -\alpha_1 & 0 & 0 & \gamma_1 & \gamma_1 \\
\beta_3 & 0 & \beta_1 & 0 & -\alpha_2 & 0 & -2\gamma_2 & \gamma_2 \\
\beta_2 & -\beta_1 & 0 & 0 & 0 & -\alpha_3 & -\gamma_3 & 2\gamma_3 \\
\hline
\gamma_1 & -\gamma_2 & 0 & \beta_1 & -\beta_2 & 0 & \multicolumn{2}{c|}{0} \\
\gamma_1 & 0 & \gamma_3 & \beta_1 & 0 & \beta_3 & \multicolumn{2}{c|}{} 
\end{array}\right.
$$

with $\alpha_1 + \alpha_2 + \alpha_3 = 0$ .

Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}_{X_0} = \{A \in \mathfrak{gl}(7) \mid d\rho(A)X_0 = 0\}$ consisting of the diagonal matrices $H(\alpha_1, \alpha_2, \alpha_3)$ in (5.7).

For $1 \leq i \leq 3$, let $E_i$ (resp. $F_i$) be the element of the form (5.7) with $\beta_i = 1$ (resp. $\gamma_i = 1$), all remaining entries zero. Then one can easily check that for each $i = 1, 2, 3$, $\mathrm{ad}\,(H)E_i = \alpha_i E_i$ and $\mathrm{ad}\,(H)F_i =$

$-\alpha_i F_i$ where $H = H(\alpha_1, \alpha_2, \alpha_3) \in \mathfrak{h}$. This shows that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_{X_0}$ and the root system of $\mathfrak{g}_{X_0}$ w. r. t. $\mathfrak{h}$ is given by $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 0\}$ i.e. $\mathfrak{g}_{X_0} \cong \mathfrak{sl}(3)$ and $A$ in (5.7) is the adjoint representation $\Lambda_1 + \Lambda_2$ of $\mathfrak{sl}(3)$. Hence this triplet is regular by Proposition 25 in § 4.

Put $X_0' = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_5 \wedge u_6 + u_2 \wedge u_4 \wedge u_6 + u_7 \wedge (u_1 \wedge u_4 - u_2 \wedge u_5) + u_8 \wedge (u_1 \wedge u_4 - u_3 \wedge u_6)$. Then by the same calculation as the case of $X_0$, the isotropy subalgebra $\mathfrak{g}_{X_0'}$ is given by

$$\mathfrak{g}_{X_0'} = \left\{ \left( \begin{array}{cc|ccc|ccc} \alpha & \beta_1 & \gamma_2 & \gamma_3 & \gamma_4 & -\beta_3 & 2\beta_2 & 2\beta_2 \\ \beta_1 & \alpha & \gamma_1 & -\gamma_4 & \gamma_5 & -\beta_2 & -4\beta_3 & 2\beta_3 \\ \hline & & -2\alpha & \beta_3 & \beta_2 & & & \\ & 0 & -\beta_3 & -2\alpha & \beta_1 & & 0 & \\ & & \beta_2 & \beta_1 & -2\alpha & & & \\ \hline \beta_3 & -\beta_2 & \gamma_3 - \gamma_5 & \gamma_2 & \gamma_1 & \alpha & 2\beta_1 & -4\beta_1 \\ \beta_2 & \beta_3 & 0 & -\gamma_1 & \gamma_2 & 0 & \alpha & 0 \\ \beta_2 & 0 & \gamma_4 & -\gamma_1 & 0 & -\beta_1 & 0 & \alpha \end{array} \right) \right\}$$

$$\text{(5.8)} \qquad \cong \left\{ (\alpha) \oplus \begin{bmatrix} \dfrac{\beta_1}{2} & \dfrac{\beta_2 - \beta_3}{2} \\ \dfrac{\beta_2 + \beta_3}{2} & -\dfrac{\beta_1}{2} \end{bmatrix} \right\} \oplus (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus V(5)$$

where the second $\oplus$ denotes the semi-direct sum and $V(5)$ denotes the Lie algebra of the 5-dimensional vector group. Hence the isotropy subgroup $G_{X_0'}$ at $X_0'$ is locally isomorphic to $(GL(1) \times SL(2)) \cdot (G_a)^5$.

Since $\dim GL(8) - \dim G_{X_0'} = 55$, the orbit of $X_0'$ is of codimension one. For an element $A$ of $\mathfrak{g}_{X_0'}$, we have $\mathrm{tr}_V A = -21\alpha$ and $\mathrm{tr\,ad}_{\mathfrak{g}_{X_0'}} A = 15\alpha$, and hence $\deg f = \dfrac{-21\alpha + 15\alpha}{-21\alpha} \times 56 = 16$ by Proposition 15 in § 4.

The orbital decomposition of this space was completed by I. Ozeki (see [20]).

PROPOSITION 10. *A triplet $(GL(8), \Lambda_3, V(56))$ is a regular P.V. and its generic isotropy subgroup is locally isomorphic to $SL(3)$. The relative invariant is of degree* 16.

(10) $\qquad (SL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$ .

We identify $V(6) \otimes V(2)$ with $V = \{X = (X_1, X_2) \mid X_1, X_2 \in M(3),\ {}^tX_1 =$

$X_1, {}^tX_2 = X_2\}$. Then the action $\rho = 2\Lambda_1 \otimes \Lambda_1$ of $SL(3) \times GL(2)$ is given by $\rho(A, B)X = (A(aX_1 + bX_2){}^tA, A(cX_1 + dX_2){}^tA)$ where $X = (X_1, X_2) \in V$, $(A, B) \in SL(3) \times GL(2)$, and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For each $X = (X_1, X_2)$ in $V$, we can obtain the binary cubic form $F_X(u, v) = \det(uX_1 + vX_2)$ which is invariant under the action of $SL(3)$. Let $G_{X_0}$ be the isotropy subgroup of $G = SL(3) \times GL(2)$ at $X_0 = \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}\right)$ where $\omega^3 = 1$, $\omega \neq 1$, and let $(A, B)$ be an element of $G_{X_0}$ where $A \in SL(3)$, $B \in GL(2)$. Then $B$ must be in the isotropy subgroup

$$\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a^3 = d^3 = 1\right\} \cup \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| b^3 = c^3 = 1\right\}$$

of $GL(2)$ at $F_{X_0}(u, v) = u^3 + v^3$ (see (6)). In the case of $B = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, we have $A{}^tA = I_3$, $A\begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}{}^tA = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$ and hence

$$A\begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} A\,.$$

This implies that $A$ is diagonal. Since $A{}^tA = I_3$, we have $A = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}$, i.e., $\{A\}$ is an abelian group of order 8 of type $(2, 2, 2)$. Similarly we have

$$A = \begin{pmatrix} \pm\omega & & \\ & \pm\omega & \\ & & \pm\omega \end{pmatrix}\left(\text{resp. } \begin{pmatrix} & & \pm\omega \\ \pm\omega & & \\ & \pm\omega & \end{pmatrix}, \begin{pmatrix} \pm 1 & & \\ & & \pm\omega \\ & \pm\omega^2 & \end{pmatrix}\right)$$

when $B = \begin{pmatrix} \omega & \\ & \omega \end{pmatrix}\left(\text{resp. } \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right)$. This implies that $G_{X_0}$ is a finite group of order $8 \times 18 = 144$. Hence it is a regular P.V. by Proposition 16 in § 4. The kernel of $\rho$ is a finite group $\left\{\left(\pm\begin{pmatrix} \omega & & \\ & \omega & \\ & & \omega \end{pmatrix}, \begin{pmatrix} \omega & \\ & \omega \end{pmatrix}\right) \middle| \omega^3 = 1\right\}$ of order 6, and hence the image $\rho(G_{X_0})$ is a finite group of order 24. Since $X \mapsto F_X(u, v)$ is equivariant, the discriminant $f(X)$ of $F_X(u, v)$ is a relative invariant of $SL(3) \times GL(2)$ (see (6)). Any rational character $\chi$ of $SL(3) \times GL(2)$ is of the form $\chi((A, B)) = (\det B)^m$ for some $m \in \mathbf{Z}$ where $(A, B) \in SL(3) \times GL(2)$. Since $\deg f(X) = 12$, if $(\det B)^m$ is the character of $f(X)$, we have $(\det B)^m = t^{2m} = t^{12}$ for $B = \begin{pmatrix} t & \\ & t \end{pmatrix}$ and hence

$m=6$. Since $\chi|G_{X_0}=1$ if and only if $m=6n$ for some $n\in \boldsymbol{Z}$, by Proposition 19 in §4, any relative invariant is of the form $cf(X)^n$ and hence $f(X)$ is irreducible.

PROPOSITION 11. *A triplet* $(SL(3)\times GL(2),\ 2\Lambda_1\otimes\Lambda_1,\ V(6)\otimes V(2))$ *is a regular P.V. and its generic isotropy subgroup is a finite group of order* 144. *The relative invariant is of degree* 12.

$$\text{(11)}\qquad \left(SL(n)\times GL(2),\ \Lambda_2\otimes\Lambda_1,\ V\left(\frac{n(n-1)}{2}\right)\otimes V(2)\right)\qquad (n\geqq 5)$$

I) *The case of* $n=2m$.

We identify $V=V(m(2m-1))\otimes V(2)$ with $\{X=(X_1,X_2)\in M(2m)\oplus M(2m)\,|\,{}^tX_1=-X_1,\ {}^tX_2=-X_2\}$. Then the action $\rho=\Lambda_2\otimes\Lambda_1$ on $V$ is given by $\rho(g)X=(A(\alpha X_1+\beta X_2){}^tA,\ A(\gamma X_1+\delta X_2){}^tA)$ for

$$g=\left(A,\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)\in SL(m)\times GL(2)\ .$$

For each $X=(X_1,X_2)$ in $V$, we can obtain the binary $m$-form $F_X(u,v)=Pff(uX_1+vX_2)$ which is invariant under the action of $SL(m)$ where $Pff$ denotes the Pfaffian. This map $\varphi: X\to F_X(u,v)$ is clearly equivariant. Moreover it is generically surjective since $\varphi(X)=\pm(u-\lambda_1 v)\cdots(u-\lambda_m v)$ for

$$X=\left(\left(\begin{array}{c|c}0 & I_m\\ \hline -I_m & 0\end{array}\right),\left(\begin{array}{c|c}0 & -\Lambda\\ \hline \Lambda & 0\end{array}\right)\right)\qquad \text{where } \Lambda=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_m\end{pmatrix}.$$

For $m\geqq 4$, a triplet $(GL(2), m\Lambda_1, V(m+1))$ is not a P.V. and hence the triplet $(SL(2m)\times GL(2),\ \Lambda_2\otimes\Lambda_1,\ V(2m^2-m)\otimes V(2))$ is not a P.V. for $m\geqq 4$ by Lemma 5 in §2. Assume that $m=3$. Put

$$X_0=\left\{\left(\begin{array}{c|c}0 & I_3\\ \hline -I_3 & 0\end{array}\right),\left(\begin{array}{c|c}0 & \Lambda\\ \hline -\Lambda & 0\end{array}\right)\right\}\qquad \text{where } \Lambda=\begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}.$$

In general the infinitesimal action $d\rho$ of $\rho$ is given by

$$d\rho\left(A,\ \begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)X=(AX_1+X_1{}^tA+aX_1+bX_2,\ AX_2+X_2{}^tA+cX_1+dX_2)$$

where $X=(X_1,X_2)$ in $V$, $A\in\mathfrak{sl}(2m,\boldsymbol{C})$ and $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathfrak{gl}(2,\boldsymbol{C})$. Hence we have $d\rho\left(A,\ \begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)X_0=(X_0^1,X_0^2)$ where $A=(a_{ij})\in\mathfrak{sl}(2m)$, and $X_0^1=(c_{ij})$,

$X_0^2 = (c'_{ij})$ are $6 \times 6$ skew-symmetric matrices given as follows.

(5.9)

| | | | | | |
|---|---|---|---|---|---|
| $c_{12}$ | $a_{24} - a_{15}$ | $c_{35}$ | $a_{32} + a_{56}$ | $c'_{23}$ | $a_{26}$ |
| $c_{13}$ | $a_{34} - a_{16}$ | $c_{36}$ | $a - b + a_{33} + a_{66}$ | $c'_{24}$ | $a_{21}$ |
| $c_{14}$ | $a + b + a_{11} + a_{44}$ | $c_{45}$ | $a_{42} - a_{51}$ | $c'_{25}$ | $c$ |
| $c_{15}$ | $a_{12} + a_{54}$ | $c_{46}$ | $a_{43} - a_{61}$ | $c'_{26}$ | $-a_{23}$ |
| $c_{16}$ | $a_{13} + a_{64}$ | $c_{56}$ | $a_{53} - a_{62}$ | $c'_{34}$ | $a_{31} - a_{46}$ |
| $c_{23}$ | $a_{35} - a_{26}$ | $c'_{12}$ | $a_{24}$ | $c'_{35}$ | $-a_{56}$ |
| $c_{24}$ | $a_{21} + a_{45}$ | $c'_{13}$ | $a_{16} + a_{34}$ | $c'_{36}$ | $c - d - a_{33} - a_{66}$ |
| $c_{25}$ | $a + a_{22} + a_{55}$ | $c'_{14}$ | $c + d + a_{11} + a_{44}$ | $c'_{45}$ | $-a_{51}$ |
| $c_{26}$ | $a_{23} + a_{65}$ | $c'_{15}$ | $a_{54}$ | $c'_{46}$ | $-a_{43} - a_{61}$ |
| $c_{34}$ | $a_{31} + a_{46}$ | $c'_{16}$ | $a_{64} - a_{13}$ | $c'_{56}$ | $-a_{53}$ |

Therefore the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0$ is given by

$$\begin{aligned} \mathfrak{g}_{X_0} &= \left\{\left(A, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathfrak{sl}(6) \oplus \mathfrak{gl}(2) \,\middle|\, c_{ij} = c'_{ij} = 0 \text{ in } (5.9)\right\} \\ &= \left\{\left(\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & -\alpha \end{array}\right) \oplus (0) \,\middle|\, \alpha, \beta, \gamma \text{ are diagonal } 3 \times 3 \text{ matrices}\right\} \qquad (5.10) \\ &\cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \,. \end{aligned}$$

Since $\dim \mathfrak{g}_{X_0} = 9 = \dim SL(6) \times GL(2) - \dim V(15) \otimes V(2)$, it is a P.V., and by Proposition 25 in § 4, it is regular.

The discriminant $f(X)$ of the binary cubic form $F_X(u, v)$ $(= Pff(uX_1 + vX_2))$ is a relative invariant of degree 12 (see Proposition 6). We shall show that $f(X)$ is irreducible.

Put

$$J = \begin{pmatrix} J_1 & & 0 \\ & J_1 & \\ 0 & & J_1 \end{pmatrix} \qquad \text{where } J_1 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} .$$

Let $X$ be a $3 \times 3$ quaternion matrix. In general a quaternion can be represented as a $2 \times 2$ matrix $\begin{pmatrix} x & -y \\ \bar{y} & \bar{x} \end{pmatrix}$ and hence we can consider $X$ as a $6 \times 6$ matrix. In this case, if $X$ is a quaternion hermitian i.e. ${}^t\overline{X} = X$, then ${}^t(XJ) = -(XJ)$. This gives a one-to-one correspondence from $3 \times 3$ quaternion hermitian matrices to $6 \times 6$ skew-symmetric matrices. Moreover we can define the determinant of quaternion hermitian matrices which corresponds to the Pfaffian of skew-symmetric matrices. Therefore $f(X)$ can be considered as a discriminant of the binary cubic form $\det(uX_1 + vX_2)$ where $X_1, X_2$ are $3 \times 3$ quaternion hermitian matrices.

We have seen in (10) that the restriction of $f(X)$ to $X = (X_1, X_2)$ where $X_1, X_2$ are $3 \times 3$ symmetric matrices over $\boldsymbol{C}$, is irreducible and hence $f(X)$ itself must be irreducible.

PROPOSITION 12. *A triplet* $(SL(2m) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(m(2m-1)) \otimes V(2))$ *is not a P.V. if* $m \geqq 4$. *A triplet* $(SL(6) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $SL(2) \times SL(2) \times SL(2)$. *The relative invariant is of degree* 12.

II) *The case of* $n = 2m + 1$.

We identify $V = V(m(2m+1)) \otimes V(2)$ with $\{(X_1, X_2) \in M(2m+1) \oplus M(2m+1) \mid {}^tX_1 = -X_1, {}^tX_2 = -X_2\}$. The subgroups $GL(2m+1) = SL(2m+1) \times GL(1)$ of $SL(2m+1) \times GL(2)$ acts on $V$ as $X = (X_1, X_2) \mapsto (AX_1{}^tA, AX_2{}^tA)$ for $A \in GL(2m+1)$. Let $X_0$ be a point

$$X_0 = \left\{ \begin{pmatrix} & & I_n \\ & 0 & \\ -I_n & & \end{pmatrix}, \left(\begin{array}{c|cc} & \multicolumn{2}{c}{0} \\ \hline 0 & & -I_n \\ & I_n & \end{array}\right) \right\}.$$

Then the isotropy subalgebra $\mathfrak{g}_{X_0}$ of $\mathfrak{gl}(2m+1)$ at $X_0$ is

$$(5.11) \qquad \mathfrak{g}_{X_0} = \left\{ \begin{array}{l} m+1\left\{ \vphantom{\begin{array}{c}a\\a\\ \cdot\\a\end{array}} \right. \\ m\left\{ \vphantom{\begin{array}{c}a\\a\\ \vdots\\a\end{array}} \right. \end{array} \left(\begin{array}{cccc|ccc} a_0 & & & & & & \\ & a_0 & & & & 0 & \\ & & \ddots & & & & \\ & & & a_0 & & & \\ \hline a_1 & a_2 & \cdots & a_{m+1} & -a_0 & & \\ a_2 & a_3 & \cdots & a_{m+2} & & \ddots & \\ \vdots & & & & & & \\ a_m & & & a_{2m} & & & -a_0 \end{array}\right) \in \mathfrak{gl}(2m+1) \right\}.$$

Since $\dim \mathfrak{g}_{X_0} = 2m + 1 = \dim \mathfrak{gl}(2m+1) - \dim V$, it is a P.V. and there is no relative invariant by (2) in Proposition 20 in §4. Hence $(SL(2m+1) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(m(2m+1)) \otimes V(2))$ is a P.V. and there is no relative invariant.

PROPOSITION 13. *A triplet* $(SL(2m+1) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(m(2m+1)) \otimes V(2))$ *is a P.V. There is no relative invariant and hence it is not regular.*

$$(12) \qquad (SL(5) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(3)).$$

Let $V_1$ be a vector space spanned by $u_1, \cdots, u_5$. Then $SL(5)$ acts on $V_1$ as $\rho_1(g)(u_1, \cdots, u_5) = (u_1, \cdots, u_5)g$ for $g \in SL(5)$. Let $V_2$ be a vector space spanned by 2-forms $u_i \wedge u_j$ $(1 \leq i < j \leq 5)$. Then $SL(5)$ acts on $V_2$ as $\rho_2(g)(u_i \wedge u_j) = \rho_1(g)u_i \wedge \rho_1(g)u_j$ for $g \in SL(5)$. The infinitesimal action $d\rho_2$ of $\rho_2$ is given by $d\rho_2(A)(u_i \wedge u_j) = d\rho_1(A)u_i \wedge u_j + u_k \wedge d\rho_1(A)u_j$ for $A \in \mathfrak{sl}(5)$. Put $\omega_1 = u_1 \wedge u_2$, $\omega_2 = u_2 \wedge u_3$, $\omega_3 = u_1 \wedge u_3$, $\omega_4 = u_2 \wedge u_4$, $\omega_5 = u_1 \wedge u_4$, $u_6 = u_3 \wedge u_4$, $\omega_7 = u_4 \wedge u_5$, $\omega_8 = u_2 \wedge u_5$, $\omega_9 = u_3 \wedge u_5$, $\omega_{10} = u_1 \wedge u_5$. Then $\{\omega_1, \cdots, \omega_{10}\}$ is a basis of $V_2$, and for $A = (a_{ij}) \in \mathfrak{sl}(5)$, by simple calculation we have

$$d\rho_2(A)(\omega_1, \cdots, \omega_{10}) = (\omega_1, \cdots, \omega_{10})(A_1 \,|\, A_2)$$

$$A_1 = \left|\begin{array}{c|c|c|c|c}
a_1 + a_2 & -a_{13} & a_{23} & -a_{14} & a_{24} \\
-a_{31} & a_2 + a_3 & a_{21} & a_{34} & 0 \\
a_{32} & a_{12} & a_1 + a_3 & 0 & a_{34} \\
-a_{41} & a_{43} & 0 & a_2 + a_4 & a_{21} \\
a_{42} & 0 & a_{43} & a_{12} & a_1 + a_4 \\
\hline
0 & -a_{42} & -a_{41} & a_{32} & a_{31} \\
0 & 0 & 0 & -a_{52} & -a_{51} \\
-a_{51} & a_{53} & 0 & a_{54} & 0 \\
0 & -a_{52} & -a_{51} & 0 & 0 \\
a_{52} & 0 & a_{53} & 0 & a_{54}
\end{array}\right| \tag{5.12}$$

$$A_2 = \left|\begin{array}{c|c|c|c|c}
0 & 0 & -a_{15} & 0 & a_{25} \\
-a_{24} & 0 & a_{35} & -a_{25} & 0 \\
-a_{14} & 0 & 0 & -a_{15} & a_{35} \\
a_{23} & -a_{25} & a_{45} & 0 & 0 \\
a_{13} & -a_{15} & 0 & 0 & a_{45} \\
\hline
a_3 + a_4 & -a_{35} & 0 & a_{45} & 0 \\
-a_{53} & a_4 + a_5 & a_{42} & a_{43} & a_{41} \\
0 & a_{24} & a_2 + a_5 & a_{23} & a_{21} \\
a_{54} & a_{34} & a_{32} & a_3 + a_5 & a_{31} \\
0 & a_{14} & a_{12} & a_{13} & a_1 + a_5
\end{array}\right|$$

where $a_{ii} = a_i$ and $\sum a_i = 0$.

We denote this matrix also by $d\rho_2(A)$, i.e., $d\rho_2(A) \in M(10)$. Identify $V = V(10) \otimes V(3)$ with $10 \times 3$ matrices $M(10, 3)$. Then the action $\rho = \Lambda_2 \otimes \Lambda_1$ on $V$ is given by $\rho(g)X = \rho_2(g_1)X^t g_2$ for $g = (g_1, g_2) \in SL(5) \times GL(3)$, $X \in M(10, 3)$. The infinitesimal action $d\rho$ of $\rho$ is given by $d\rho(\tilde{A})X = d\rho_2(A)X + X^t B$ for $\tilde{A} = (A, B) \in \mathfrak{g} = \mathfrak{sl}(5) \oplus \mathfrak{gl}(3)$. We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0 = {}^t(I_3\, 0 \,|\, I_3\, 0) \in M(10, 3)$. Then for $\tilde{A} = (A, B)$ with $A = (a_{ij}) \in \mathfrak{sl}(5)$, $B = (b_{ij}) \in \mathfrak{gl}(3)$, we have

$$d\rho(\tilde{A})X_0 = d\rho_2(A)X_0 + X_0{}^tB = \left|\begin{array}{c|c|c} a_1 + a_2 + b_1 & b_{21} - a_{13} & a_{23} - a_{15} + b_{31} \\ b_{12} - a_{31} - a_{24} & a_2 + a_3 + b_2 & a_{21} + a_{35} + b_{32} \\ a_{32} - a_{14} + b_{13} & a_{12} + b_{23} & a_1 + a_3 + b_3 \\ a_{23} - a_{41} & a_{43} - a_{25} & a_{45} \\ a_{42} + a_{13} & -a_{15} & a_{43} \\ a_3 + a_4 + b_1 & b_{21} - a_{42} - a_{35} & b_{31} - a_{41} \\ b_{12} - a_{53} & a_4 + a_5 + b_2 & a_{42} + b_{32} \\ b_{13} - a_{51} & a_{24} + a_{53} + b_{23} & a_2 + a_5 + b_3 \\ a_{54} & a_{34} - a_{52} & a_{32} - a_{51} \\ a_{52} & a_{14} & a_{12} + a_{53} \end{array}\right| .$$

where $d\rho_2(A)$ is given by (5.12).

Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{\tilde{A} \mid d\rho(\tilde{A})X_0 = 0\}$ is given as follows.

$$\mathfrak{g}_{X_0} = \left\{ \left[\begin{array}{c|c|c|c|c} 0 & -\beta & \gamma & & \\ -3\gamma & 2\alpha & & -2\beta & \\ 3\beta & & -2\alpha & & 2\gamma \\ & -\gamma & & 4\alpha & \\ & & \beta & & -4\alpha \end{array}\right] \oplus \left[\begin{array}{c|c|c} -2\alpha & \beta & \\ \gamma & & \beta \\ & \gamma & 2\alpha \end{array}\right] \right\} \cong \mathfrak{sl}(2) .$$

This is $4\Lambda_1 \oplus 2\Lambda_1$ of $\mathfrak{sl}(2)$. Since $\dim \mathfrak{g}_{X_0} = 3 = \dim SL(5) \times GL(3) - \dim V(10) \otimes V(3)$, it is a P.V. and moreover it is regular by Proposition 25 in § 4.

Next we shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0'}$ at

$$X_0' = {}^t\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} .$$

Similarly, we have

$$d\rho(\tilde{A})X_0' = \left|\begin{array}{c|c|c} a_1 + a_2 + b_1 & a_{25} + b_{21} & b_{31} - a_{13} \\ b_{13} - a_{31} & b_{23} - a_{24} & a_2 + a_3 + b_3 \\ a_{32} & a_{35} - a_{14} & a_{12} \\ -a_{41} & a_{23} & a_{43} - a_{25} \\ a_{42} & a_{13} + a_{45} & -a_{15} \\ b_{12} & a_3 + a_4 + b_2 & b_{32} - a_{42} - a_{35} \\ b_{13} & a_{41} - a_{53} + b_{23} & a_4 + a_5 + b_3 \\ -a_{51} & a_{21} & a_{53} + a_{24} \\ 0 & a_{31} + a_{54} & a_{34} - a_{52} \\ a_{52} + b_{12} & a_1 + a_5 + b_2 & a_{14} + b_{32} \end{array}\right| .$$

Hence the isotropy subalgebra $\mathfrak{g}_{X_0'} = \{\tilde{A} \mid d\rho(\tilde{A})X_0' = 0\}$ at $X_0'$ is given as follows.

$$
(5.13) \qquad \mathfrak{g}_{X_0'} = \left\{ \tilde{A} = \left[ \begin{array}{c|c|c|c|c} 2\alpha + 4\beta & & \delta & & \\ & -2\alpha - 2\beta & & & \gamma \\ & & \alpha & & \\ & & \gamma & \beta & -\delta \\ & & & & -\alpha - 3\beta \end{array} \right] \oplus \left( \begin{array}{c|c|c} -2\beta & & \\ -\gamma & -\alpha - \beta & \\ \delta & & \alpha + 2\beta \end{array} \right) \right\} .
$$

Since $\dim \mathfrak{g} - \dim \mathfrak{g}_{X_0'} = 33 - 4 = 29$, the orbit of $X_0'$ is of codimension one. For $\tilde{A} \in \mathfrak{g}_{X_0'}$, the trace $\mathrm{tr}_V \tilde{A}$ on $V = V(10) \otimes V(3)$ is $10(-2\beta - (\alpha + \beta) + (\alpha + 2\beta)) = -10\beta$ and the trace $\mathrm{tr}\,\mathrm{ad}\,\mathfrak{g}_{X_0'}\tilde{A}$ of the adjoint representation on $\mathfrak{g}_{X_0'}$ is $5\beta$. By the degree formula, i.e., Proposition 15 in § 4, the degree of the relative invariant $f(x)$ is given by

$$
\deg f(x) = \frac{\mathrm{tr}_V \tilde{A} + \mathrm{tr}\,\mathrm{ad}\,\mathfrak{g}_{X_0'}\tilde{A}}{\mathrm{tr}_V \tilde{A}} \cdot \dim V = \frac{-10\beta + 5\beta}{-10\beta} \times 30 = 15 .
$$

PROPOSITION 14. *A triplet* $(SL(5) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(3))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $SL(2)$. *The relative invariant is of degree* 15.

$$
(13) \qquad (SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4)) .
$$

We identify $V = V(10) \otimes V(4)$ with $10 \times 4$ matrices $M(10, 4)$. Then the action of $\rho_2 = \Lambda_2 \otimes \Lambda_1$ is given by $\rho(g)X = \rho_2(g_1)X{}^tg_2$ for $g = (g_1, g_2) \in G = SL(5) \times GL(4)$, $X \in V$ where $\rho_2$ is defined as in (12). The infinitesimal action $d\rho$ of $\rho$ is given by $d\rho(\tilde{A})X = d\rho_2(A)X + X{}^tB$ for $\tilde{A} = (A, B) \in \mathfrak{g} = \mathfrak{sl}(5) \oplus \mathfrak{gl}(4)$ where $d\rho_2(A)$ is the $10 \times 10$ matrix of (5.12). We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0 = {}^t(I_4\, 0 \mid I_4\, 0) \in M(10, 4)$. Similarly as in (12), we have

$$
d\rho(\tilde{A})X_0 = d\rho_2(A)X_0 + X_0{}^tB
$$

$$
= \left|\begin{array}{c|c|c|c}
a_1 + a_2 + b_1 & b_{21} - a_{13} & a_{23} - a_{15} + b_{31} & b_{41} - a_{14} \\
b_{12} - a_{31} - a_{24} & a_2 + a_3 + b_2 & a_{21} + a_{35} + b_{32} & a_{34} - a_{25} + b_{42} \\
a_{32} - a_{14} + b_{13} & a_{12} + b_{23} & a_1 + a_3 + b_3 & b_{43} - a_{15} \\
a_{23} - a_{41} + b_{14} & a_{43} - a_{25} + b_{24} & a_{45} + b_{34} & a_2 + a_4 + b_4 \\
a_{42} + a_{13} & -a_{15} & a_{43} & a_{12} \\
a_3 + a_4 + b_1 & b_{21} - a_{42} - a_{35} & b_{31} - a_{41} & a_{32} + a_{45} + b_{41} \\
b_{12} - a_{53} & a_4 + a_5 + b_2 & a_{42} + b_{32} & a_{43} - a_{52} + b_{42} \\
b_{13} - a_{51} & a_{24} + a_{53} + b_{23} & a_2 + a_5 + b_3 & a_{54} + a_{23} + b_{43} \\
b_{14} + a_{54} & a_{34} - a_{52} + b_{24} & a_{32} - a_{51} + b_{34} & a_3 + a_5 + b_4 \\
a_{52} & a_{14} & a_{12} + a_{53} & a_{13}
\end{array}\right|
$$

Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{\tilde{A} \in \mathfrak{g} \mid d\rho(\tilde{A})X_0 = 0\}$ at $X_0$ is zero. Since $\dim \mathfrak{g}_{X_0} = 0 = \dim SL(5) \times GL(4) - \dim V(10) \otimes V(4)$, it is a regular P.V.

Since $\dim G = \dim V$, there is a relative invariant polynomial of degree 40 by Proposition 16 in § 4.

Similarly one can check that the isotropy subalgebra $\mathfrak{g}_{X_0'}$ at

$$
X_0' = (\omega_7, \omega_3 + \omega_8, \omega_2 + \omega_5, \omega_4 + \omega_{10}) = {}^t\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

is of one-dimension, i.e., the orbit of $X_0'$ is of codimension one. Hence by Proposition 16 in § 4, the relative invariant of degree 40 is irreducible. This point $X_0'$ was found by I. Ozeki ([20]).

PROPOSITION 15. *A triplet* $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$ *is a regular P.V. and its generic isotropy subgroup is a finite group. The relative invariant is of degree* 40.

(14) $\quad (SL(n) \times SL(n) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(n) \otimes V(2)) \quad (n \geqq 3)$ .

We identify $V(n) \otimes V(n) \otimes V(2)$ with $V = M(n) \oplus M(n)$. Then the action $\rho = \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$ is given by $\rho(g)X = (A(\alpha X_1 + \beta X_2)^t B, A(\gamma X_1 + \delta X_2)^t B)$ where $g = \left(A, B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) \in SL(n) \times SL(n) \times GL(2)$ and $X = (X_1, X_2) \in V$. For each $X = (X_1, X_2)$ in $V$, we can obtain a binary $n$-form $F_X(u, v) = \det(uX_1 + vX_2)$ which is invariant under the action of $SL(n) \times SL(n)$. Clearly the map $\varphi : X \mapsto F_X(u, v)$ is equivariant. Moreover, it is generically surjective since $\varphi(X) = (u - \lambda_1 v) \cdots (u - \lambda_n v)$ for

$$X = \left(I_n, \begin{pmatrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_n \end{pmatrix}\right)$$

in $V$. Therefore by Lemma 5 in §2, if $n \geqq 4$, then it is not a P.V. since $(GL(2), n\Lambda_1, V(n+1))$ is not a P.V. for $n \geqq 4$. Assume that $n = 3$. The infinitesimal representation $d\rho$ of $\rho$ is given by

$$d\rho\left(A, B, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)X$$
$$= (AX_1 + X_1{}^tB + aX_1 + bX_2,\ AX_2 + X_2{}^tB + cX_1 + dX_2)$$

for $\left(A, B, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathfrak{sl}(n) \oplus \mathfrak{sl}(n) \oplus \mathfrak{gl}(2)$. Hence for

$$X_0 = \left\{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\right\},$$

we have

$$d\rho\left(A, B, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)X_0 = \left(\left(\begin{array}{c|c|c} a + b + a_{11} + b_{11} & a_{12} + b_{21} & a_{13} + b_{31} \\ a_{21} + b_{12} & a + a_{22} + b_{22} & a_{23} + b_{32} \\ a_{31} + b_{13} & a_{32} + b_{23} & a - b + a_{33} + b_{33} \end{array}\right)\right.$$
$$\left.\left(\begin{array}{c|c|c} c + d + a_{11} + b_{11} & b_{21} & -a_{13} + b_{31} \\ a_{21} & c & -a_{23} \\ a_{31} - b_{13} & -b_{23} & c - d - a_{33} - b_{33} \end{array}\right)\right)$$

where $A = (a_{ij}),\ B = (b_{ij}) \in \mathfrak{sl}(3), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}(2)$.

Therefore the isotropy subalgebra

$$\mathfrak{g}_{X_0} = \left\{\left(A, B, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathfrak{sl}(3) \oplus \mathfrak{sl}(3) \oplus \mathfrak{gl}(2) \,\middle|\, d\rho\left(A, B, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)X_0 = 0\right\}$$
$$= \left\{\left(\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}\begin{pmatrix} -\alpha & & \\ & -\beta & \\ & & -\gamma \end{pmatrix}(0)\right) \,\middle|\, \alpha + \beta + \gamma = 0\right\} \cong \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)\,.$$

Since $\dim \mathfrak{g}_{X_0} = 2 = \dim SL(3) \times SL(3) \times GL(2) - \dim V$, it is a regular P.V. by Proposition 25 in §4. The discriminant $f(X)$ of the binary cubic form $F_X(u, v) = \det(uX_1 + vX_2)$ is a relative invariant of degree 12. This is irreducible since we have seen in (10) that the restriction of $f(X)$ to $\{X = (X_1, X_2) \in M(3) \oplus M(3) \mid {}^tX_1 = X_1, {}^tX_2 = X_2\}$ is irreducible.

PROPOSITION 16. *A triplet* $(SL(n) \times SL(n) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(n) \otimes V(2))$ *is not a P.V. if* $n \geqq 4$. *A triplet* $(SL(3) \times SL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $GL(1) \times GL(1)$. *The relative invariant is of degree* 12.

(15) $\quad (Sp(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(m)) \qquad (n \geqq m \geqq 1)$.

We identify $V = V(2n) \otimes V(m)$ with $2n \times m$ matrices $M(2n, m)$. Then the action $\rho = \Lambda_1 \otimes \Lambda_1$ is given by $\rho(g)X = g_1 X {}^t g_2$ where $X \in M(2n, m)$ and $g = (g_1, g_2) \in Sp(n) \times GL(m)$. Let $\mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{gl}(m)$ be the Lie algebra of $Sp(n) \times GL(m)$. Then the infinitesimal action $d\rho$ of $\rho$ is given by $d\rho(A)X = A_1 X + X {}^t A_2$ where $A = (A_1, A_2) \in \mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{gl}(m)$.

I) The case when $m$ is even, i.e., $m = 2\ell$ $(n \geqq 2\ell \geqq 2)$. We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ at

$$X_0 = {}^t\left(\begin{array}{cc|cc} I_\ell & 0 & 0 & \\ \hline 0 & & I_\ell & 0 \end{array}\right) \in M(2n, 2\ell)$$

where $I_\ell$ denotes the identity matrix of size $\ell$. An element $A$ of $\mathfrak{g}$ can be written as follows.

$$A = \left[\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & {}^tB_2 & B_4 \\ \hline C_1 & C_2 & -{}^tA_1 & -{}^tA_3 \\ {}^tC_2 & C_4 & -{}^tA_2 & -{}^tA_4 \end{array}\right] \oplus \left(\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right) \tag{5.14}$$

where $A_1, B_1, C_1, D_j \in M(\ell)$ for $1 \leq j \leq 4$; $A_2, B_2, C_2 \in M(\ell, n-\ell)$; $A_3 \in M(n-\ell, \ell)$; $A_4, B_4, C_4 \in M(n-\ell)$ and ${}^tB_1 = B_1$, ${}^tB_4 = B_4$, ${}^tC_1 = C_1$, ${}^tC_4 = C_4$. Then we have

$$\begin{aligned} d\rho(A)X_0 &= \left[\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & {}^tB_2 & B_4 \\ C_1 & C_2 & -{}^tA_1 & -{}^tA_3 \\ {}^tC_2 & C_4 & -{}^tA_2 & -{}^tA_4 \end{array}\right] \left[\begin{array}{c|c} I_\ell & 0 \\ 0 & 0 \\ 0 & I_\ell \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} I_\ell & 0 \\ 0 & 0 \\ 0 & I_\ell \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} {}^tD_1 & {}^tD_3 \\ {}^tD_2 & {}^tD_4 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_1 + {}^tD_1 & B_1 + {}^tD_3 \\ A_3 & {}^tB_2 \\ C_1 + {}^tD_2 & -{}^tA_1 + {}^tD_4 \\ {}^tC_2 & -{}^tA_2 \end{array}\right]. \end{aligned} \tag{5.15}$$

Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{A \in \mathfrak{g} \mid d\rho(A)X_0 = 0\}$ is given as follows.

$$\mathfrak{g}_{X_0} = \left\{ \left( \begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_4 & 0 & B_4 \\ \hline C_1 & 0 & -{}^tA_1 & 0 \\ 0 & C_4 & 0 & -{}^tA_4 \end{array} \right) \oplus \left( -{}^t\begin{pmatrix} A_1 & B_1 \\ C_1 & -{}^tA_1 \end{pmatrix} \right) \right\} \cong \mathfrak{sp}(\ell) \oplus \mathfrak{sp}(n-\ell) \, .$$

Since $\dim \mathfrak{g}_{X_0} = \ell(2\ell+1) + (n-\ell)(2n-2\ell+1) = \{n(2n+1) + 4\ell^2\} - 4n\ell = \dim Sp(n) \times GL(2\ell) - \dim V(2n) \otimes V(2\ell)$, this triplet is a regular P.V. by Proposition 25 in §4.

The relative invariant $f(X)$ is given by $Pff({}^tXJX)$ where

$$J = \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) ,$$

and $Pff$ denotes the Pfaffian of the $2\ell \times 2\ell$ skew-symmetric matrix ${}^tXJX$. Since $Sp(n) = \{g_1 \in GL(2n) \,|\, {}^tg_1Jg_1 = J\}$, we have $f(\rho(g)X) = f(g_1X{}^tg_2) = Pff(g_2{}^tX{}^tg_1Jg_1X{}^tg_2) = Pff(g_2{}^tXJX{}^tg_2) = \det g_2 \cdot Pff({}^tXJX) = \det g_2 \cdot ff(X)$ for $g = (g_1, g_2) \in Sp(n) \times GL(2\ell)$. By Proposition 18 in §4, the degree of any relative invariant is multiple of $2\ell$. Since $\deg f(X) = 2\ell$, $f(X)$ is irreducible.

PROPOSITION 17. *A triplet $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ $(n \geqq 2m \geqq 2)$ is a regular P.V. and its generic isotropy subgroup is locally isomorphic to $Sp(m) \times Sp(n-m)$. The relative invariant is of degree $2m$.*

Note that this proposition holds even if $2n \geqq 2m > n$, but in this case, it is not reduced.

II) *The case when $m$ is odd, i.e., $m = 2\ell + 1$.*

We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ at

$$X_0 = {}^t\left( \begin{array}{cc|cc} I_\ell & 0 & 0 & \\ \hline & 0 & I_{\ell+1} & 0 \end{array} \right) \in M(2n, 2\ell+1) \, .$$

An element $A$ of $\mathfrak{g}$ can be written as follows.

$$(5.16) \qquad A = \begin{array}{r} \\ \ell\{ \\ 1\{ \\ n-\ell-1\{ \\ \ell\{ \\ 1\{ \\ n-\ell-1\{ \end{array} \!\! \begin{array}{c} \begin{array}{cccccc} \overbrace{\quad}^{\ell} & \overbrace{\quad}^{1} & \overbrace{\quad}^{n-\ell-1} & \overbrace{\quad}^{\ell} & \overbrace{\quad}^{1} & \overbrace{\quad}^{n-\ell-1} \end{array} \\ \left( \begin{array}{ccc|ccc} A_1 & A_{12} & A_{13} & B_1 & B_1' & B_2' \\ A_{21} & A_2 & A_{23} & {}^tB_1' & B_2 & B_3' \\ A_{31} & A_{32} & A_3 & {}^tB_2' & {}^tB_3' & B_3 \\ \hline C_1 & C_1' & C_2' & -{}^tA_1 & -{}^tA_{21} & -{}^tA_{31} \\ {}^tC_1' & C_2 & C_3' & -{}^tA_{12} & -{}^tA_2 & -{}^tA_{32} \\ {}^tC_2' & {}^tC_3' & C_3 & -{}^tA_{13} & -{}^tA_{23} & -{}^tA_3 \end{array} \right) \end{array} \oplus \begin{array}{c} \begin{array}{ccc} \overbrace{\quad}^{\ell} & \overbrace{\quad}^{\ell} & \overbrace{\quad}^{1} \end{array} \\ \left( \begin{array}{ccc} D_1 & D_{12} & D_{13} \\ D_{21} & D_2 & D_{23} \\ D_{31} & D_{32} & D_3 \end{array} \right) \end{array} \!\! \begin{array}{l} \\ \}\ell \\ \}\ell \\ \}1 \end{array}$$

Then we have

(5.17) $d\rho(A)X_0 =$

$$
\left[\begin{array}{ccc|ccc}
A_1 & A_{12} & A_{13} & B_1 & B'_1 & B'_2 \\
A_{21} & A_2 & A_{23} & {}^tB'_1 & B_2 & B'_3 \\
A_{31} & A_{32} & A_3 & {}^tB'_2 & {}^tB'_3 & B_3 \\
\hline
C_1 & C'_1 & C'_2 & -{}^tA_1 & -{}^tA_{21} & -{}^tA_{31} \\
{}^tC'_1 & C_2 & C'_3 & -{}^tA_{12} & -{}^tA_2 & -{}^tA_{32} \\
{}^tC'_2 & {}^tC'_3 & C_3 & -{}^tA_{13} & -{}^tA_{23} & -{}^tA_3
\end{array}\right]
\left[\begin{array}{c|c|c}
I_\ell & & \\
 & & \\
 & & \\
 & I_\ell & \\
 & & 1 \\
 & & 
\end{array}\right]
+
\left[\begin{array}{c|c|c}
I_\ell & & \\
 & & \\
 & & \\
 & I_\ell & \\
 & & 1 \\
 & & 
\end{array}\right]
\left[\begin{array}{c|c|c}
{}^tD_1 & {}^tD_{21} & {}^tD_{31} \\
{}^tD_{12} & {}^tD_2 & {}^tD_{32} \\
{}^tD_{13} & {}^tD_{23} & {}^tD_3
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c|c}
A_1 + {}^tD_1 & B_1 + {}^tD_{21} & B'_1 + {}^tD_{31} \\
A_{21} & {}^tB'_1 & B_2 \\
A_{31} & {}^tB'_2 & {}^tB'_3 \\
C_1 + {}^tD_{12} & -{}^tA_1 + {}^tD_2 & -{}^tA_{21} + {}^tD_{32} \\
{}^tC'_1 + {}^tD_{13} & -{}^tA_{12} + {}^tD_{23} & -{}^tA_2 + {}^tD_3 \\
{}^tC'_2 & -{}^tA_{13} & -{}^tA_{23}
\end{array}\right]
$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{A \in \mathfrak{g} \,|\, d\rho(A)X_0 = 0\}$ is given as follows.

$$
\mathfrak{g}_{X_0} = \left\{ \left[\begin{array}{ccc|ccc}
A_1 & A_{12} & 0 & B_1 & 0 & 0 \\
0 & A_2 & 0 & 0 & 0 & 0 \\
0 & A_{32} & A_3 & 0 & 0 & B_3 \\
\hline
C_1 & C'_1 & 0 & -{}^tA_1 & 0 & 0 \\
{}^tC'_1 & C_2 & C'_3 & -{}^tA_{12} & -{}^tA_2 & -{}^tA_{32} \\
0 & {}^tC'_3 & C_3 & 0 & 0 & -{}^tA_3
\end{array}\right] \oplus \left( - {}^t\!\left(\begin{array}{cc|c}
A_1 & B_1 & \\
C_1 & -{}^tA_1 & 0 \\
\hline
C'_1 & -{}^tA_{12} & -{}^tA_2
\end{array}\right)\right) \right\}
$$

$$
\cong \left\{ \left[\begin{array}{c|c|c|c}
-{}^tA_2 & {}^tC'_1 - {}^tA_{12} & C'_3 - {}^tA_{32} & C_2 \\
\hline
0 & \begin{matrix} A_1 & B_1 \\ C_1 & -{}^tA_1 \end{matrix} & 0 & \begin{matrix} A_{12} \\ C'_1 \end{matrix} \\
\hline
0 & 0 & \begin{matrix} A_3 & B_3 \\ C_3 & -{}^tA_3 \end{matrix} & \begin{matrix} A_{32} \\ {}^tC'_3 \end{matrix} \\
\hline
0 & 0 & 0 & A_2
\end{array}\right] \oplus \left( - {}^t\!\left(\begin{array}{cc|c}
A_1 & B_1 & \\
C_1 & -{}^tA_1 & 0 \\
\hline
C'_1 & -{}^tA_{12} & -{}^tA_2
\end{array}\right)\right) \right\}
$$

$$
\cong (\mathfrak{gl}(1) \oplus \mathfrak{sp}(\ell) \oplus \mathfrak{sp}(n - \ell - 1)) \oplus \mathfrak{u}(2n - 1)
$$

where $\mathfrak{u}(2n - 1)$ is the nilpotent Lie algebra of dimension $(2n - 1)$. The first isomorphism is obtained by changing rows and columns from $\{1, \cdots, 6\}$ to $\{5, 1, 4, 3, 6, 2\}$. Since

$$
\begin{aligned}
\dim \mathfrak{g}_{X_0} &= 1 + \ell(2\ell + 1) + (n - \ell - 1)(2n - 2\ell - 1) + (2n - 1) \\
&= n(2n + 1) + (2\ell + 1)^2 - 2n(2\ell + 1)
\end{aligned}
$$

$$= \dim Sp(n) \times GL(2\ell + 1) - \dim V ,$$

it is a P.V. Since $\mathfrak{g}_{X_0} \not\subset \mathfrak{sl}(V)$, by (2) in Proposition 20 in § 4, there is no relative invariant. In particular, it is not regular.

PROPOSITION 18. *A triplet* $(Sp(n) \times GL(2m+1), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m+1))$ $(n \geqq 2m+1 \geqq 1)$ *is a P.V. and its generic isotropy subgroup is locally isomorphic to* $(GL(1) \times Sp(m) \times Sp(n-m)) \cdot U(2n-1)$ *where* $U(2n-1)$ *is a unipotent group of dimension* $(2n-1)$. *There is no relative invariant and hence it is not regular.*

(16) $\quad (GL(1) \times Sp(n) \times SO(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(m))$
$(2n \geqq m \geqq 3)$

By Proposition 13 in § 2, this triplet is a P.V. if and only if $(GL(1) \times SO(m), \square \otimes \Lambda_2, V(1) \otimes V(\frac{1}{2}m(m-1)))$ is a P.V. Since $\Lambda_2$ is the adjoint representation of $SO(m)$, it is a P.V. only when rank $SO(m) = 1$, i.e., $m = 3$, by Proposition 2. Assume that $m = 3$. We identify $V = V(1) \otimes V(2n) \otimes V(3)$ with $2n \times 3$ matrices $M(2n, 3)$. Then the action $\rho = \square \otimes \Lambda_1 \otimes \Lambda_1$ is given by $\rho(g)X = cAX{}^tB$ where $g = (c, A, B) \in GL(1) \times Sp(n) \times SO(m)$ and $X \in M(2n, 3)$. Let $\mathfrak{g}$ be the Lie algebra of $GL(1) \times Sp(n) \times SO(m)$ i.e.

$$\mathfrak{g} = \left\{ (d) \oplus \left[\begin{array}{c|c} A & B \\ \hline C & -{}^tA \end{array}\right] \oplus \begin{bmatrix} 0 & a & b \\ -b & c & 0 \\ -a & 0 & -c \end{bmatrix} \middle| \begin{array}{l} A, B, C \in M(n) \\ {}^tB = B,\ {}^tC = C \end{array} \right\} .$$

The infinitesimal action $d\rho$ of $\rho$ is given by $d\rho(d, A, B)X = dX + AX + X{}^tB$ for $(d, A, B) \in \mathfrak{g}$. Put

$$X_0 = {}^t\!\left(\begin{array}{ccc|cc} 1 & 0 & & & \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & & 01 & \end{array}\right) \in V = M(2n, 3) .$$

Then we have

(5.18) $\quad d\rho(d, A, B)X_0$

$$
= \begin{pmatrix} d & 0 & \\ 0 & d & \\ & & \\ \hline 0 & 0 & 0 \\ & & d \\ & & \end{pmatrix} + \left(\begin{array}{cccc|cccc} a_1 & a_{12} & \cdots & a_{1n} & b_1 & b_{12} & \cdots & b_{1n} \\ a_{21} & a_2 & & \vdots & b_{12} & b_2 & & \vdots \\ \vdots & & \ddots & & \vdots & & \ddots & \\ a_{n1} & & \cdots & a_n & b_{1n} & & \cdots & b_n \\ \hline c_1 & c_{12} & \cdots & c_{1n} & -a_1 & -a_{21} & & -a_{n1} \\ c_{12} & c_2 & & \vdots & -a_{12} & -a_2 & & \vdots \\ \vdots & & \ddots & & \vdots & & \ddots & \\ c_{1n} & & \cdots & c_n & -a_{1n} & & \cdots & -a_n \end{array}\right) \begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ & & \\ \hline & & 0 \\ & & 1 \\ & & \end{pmatrix}
$$

$$
+ \begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ & & \\ \hline & & 0 \\ & & 1 \\ & & \end{pmatrix} \begin{pmatrix} 0 & -b & -a \\ a & c & 0 \\ b & 0 & -c \end{pmatrix}
$$

$$
= {}^t\begin{pmatrix} a_1 + d & a_{21} + a & a_{31} \cdots a_{n1} & c_1 & c_{12} + b & c_{13} \cdots c_{1n} \\ a_{12} - b & a_2 + d + c & a_{32} \cdots a_{n2} & c_{12} & c_2 & c_{23} \cdots c_{2n} \\ b_{12} - a & b_2 & b_{23} \cdots b_{2n} & -a_{21} & -a_2 + d - c & -a_{23} \cdots -a_{2n} \end{pmatrix}
$$

Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(d, A, B) \mid d\rho(d, A, B)X_0 = 0\}$ at $X_0$ is given by

$$
\mathfrak{g}_{X_0} = \left\{ (0) \oplus \left(\begin{array}{cccccc|cccccc} 0 & 0 & a_{13} & \cdots & a_{1n} & & b_1 & 0 & b_{13} & \cdots & b_{1n} & \\ 0 & -c & 0 & \cdots & 0 & & 0 & 0 & 0 & \cdots & 0 & \\ 0 & 0 & a_3 & \cdots & a_{3n} & & b_{13} & 0 & b_3 & \cdots & b_{3n} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & a_{n3} & \cdots & a_n & & b_{1n} & 0 & b_{3n} & \cdots & b_n & \\ \hline 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & 0 & \cdots & 0 & \\ 0 & 0 & 0 & \cdots & 0 & & 0 & c & 0 & \cdots & 0 & \\ 0 & 0 & c_3 & \cdots & c_{3n} & & -a_{13} & 0 & -a_3 & \cdots & -a_{n3} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & c_{3n} & \cdots & c_n & & -a_{1n} & 0 & -a_{3n} & \cdots & -a_n & \end{array}\right) \oplus \begin{pmatrix} 0 & & \\ & c & \\ & & -c \end{pmatrix} \right\}
$$

Since $\dim \mathfrak{g}_{X_0} = 2n^2 - 5n + 4 = \dim GL(1) \times Sp(n) \times SO(3) - \dim M(2n, 3)$, this triplet is a P.V.

However, its generic isotropy subgroup is not reductive since it is locally isomorphic to the semi-direct product of $Sp(n-2) \times SO(2)$ with a unipotent group $U(2n-3)$ of dimension $(2n-3)$.

By Corollary 23 in § 4, this P.V. is not regular. Since $\mathfrak{g}_{X_0} \subset sl(V)$, there exists a relative invariant $f(x)$ by (3) in Proposition 20 in § 4. In fact, a polynomial $f(x) = \mathrm{tr}\,({}^tXJXK)^2$ is a relative invariant where $Sp(n) = \{A \in GL(2n) \mid {}^tAJA = J\}$ and $SO(m) = \{B \in SL(m) \mid {}^tBKB = K\}$ since $f(AX{}^tB) = \mathrm{tr}\,(B{}^tX{}^tAJAX{}^tBK)^2 = \mathrm{tr}\,(B{}^tXJXKB^{-1})^2 = \mathrm{tr}\,({}^tXJXK)^2 = f(X)$ for $A \in Sp(n)$, $B \in SO(m)$. Since $f(X_0) = 2$, $f(X)$ is not identically zero. Note that $\mathrm{tr}\,{}^tXJXK$ is identically zero.

Since this P.V. is not regular, the Hessian of $f(X) = \mathrm{tr}\,({}^tXJXK)^2$ must be identically zero. We can check this directly as follows.

The infinitesimal character $\delta\chi$ of $f(X)$ is given by $\delta\chi(\tilde{A}) = 4d$ for $\tilde{A} = (d, A, B) \in \mathfrak{g}$. By Proposition 10 in § 4, Hessian $H_f(x)$ of $f(x)$ is not identically zero if and only if $\mathrm{grad}\log f : V - S \to V^*$ is generically surjective. In view of (1) in Proposition 9 in § 1, the map $\mathrm{grad}\log f$ is generically surjective if and only if $\mathrm{grad}\log f(X_0)$ is a generic point of the dual P.V. By the inner product $\langle X, Y\rangle = \mathrm{tr}\,{}^tXY$ for $X, Y \in M(2n, 3)$, we may identify this P.V. with its dual. By (2) in Proposition 9 in § 1, we have $\langle d\rho(d, A, B)X_0, \mathrm{grad}\log f(X_0)\rangle = \delta\chi(d, A, B) = 4d$ for any $(d, A, B) \in \mathfrak{g}$. This condition completely characterizes $\mathrm{grad}\log f(X_0)$ since $\{d\rho(d, A, B)X_0 \mid (d, A, B) \in \mathfrak{g}\} = M(2n, 3)$.

From (5.18), we have

$$\mathrm{grad}\log f(X_0) = {}^t\left(\begin{array}{ccc|ccc} 0 & 0 & & & & \\ 0 & 2 & 0 & & & 0 \\ 0 & 0 & & 0 & 2 & \end{array}\right). \tag{5.19}$$

Since the rank of this $2n \times 3$ matrix is 2, it cannot be a generic point and hence the Hessian of $f(X)$ is identically zero.

PROPOSITION 19. *A triplet $(GL(1) \times Sp(n) \times SO(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(m))$ is not a P.V. for $2n \geqq m \geqq 4$. A triplet $(GL(1) \times Sp(n) \times SO(3), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(3))$ is a P.V. and there exists a relative invariant of degree 4. However it is not a regular P.V.*

(17) $(GL(1) \times Sp(n), \square \otimes \Lambda_2, V(1) \otimes V((n-1)(2n+1)))$ $(n \geqq 3)$

Let $V' = \{X \in M(2n) \mid {}^tX = -X\}$ and $\rho'(g)X = sAX{}^tA$ for $g = (s, A) \in GL(1) \times Sp(n)$. Then

$$J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) \in V'$$

and its complement $V$ can be identified with $V((n-1)(2n+1))$ (see Example 25 in § 1). Then $\rho = \square \otimes \Lambda_2$ is the restriction of $\rho'$ to $V$. Since ${}^tAJA = J$ for $A \in Sp(n)$, we have $(\rho(g)X)J = sAX{}^tAJ = sAXJA^{-1}$ and hence eigen-values of $XJ$ are invariant under the action of $g = (1, A) \in GL(1) \times Sp(n)$. Therefore if $n \geqq 3$, it cannot be a P.V. Note that the trace, i.e., the sum of eigen-values, of skew-symmetric matrices is zero.

PROPOSITION 20. *A triplet*

$$(GL(1) \times Sp(n), \square \otimes \Lambda_2, V(1) \otimes V((n-1)(2n+1)))$$

*is not a P.V. for* $n \geqq 3$.

(18) $(GL(1) \times Sp(n) \times Sp(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(2m))$

$(n \geqq m \geqq 2)$

Let $V$ be the totality of $2m \times 2m$ skew-symmetric matrices, i.e., $V = \{X \in M(2m) \mid {}^tX = -X\}$. Define the action $\rho_{\boxminus}$ of $G = GL(1) \times Sp(m)$ on $V$ by $\rho_{\boxminus}(g)X = tAX{}^tA$ where $X \in V$ and $g = (t, A) \in GL(1) \times Sp(m)$. Note that $\rho_{\boxminus}$ is not irreducible. By Proposition 13 in § 2, our triplet is a if and only if a triplet $(G, \rho_{\boxminus}, V)$ is a P.V. As a representation space of $G$, $V$ decomposes to the direct sum $V = V(1) \oplus V((m-1)(2m+1))$ where $V(1)$ is a one-dimensional vector space spanned by

$$J = \left(\begin{array}{c|c} 0 & I_m \\ \hline -I_m & 0 \end{array}\right),$$

and the action of $\rho$ on $V((m-1)(2m+1))$ is equivalent to a triplet $(GL(1) \times Sp(m), \square \otimes \Lambda_2, V(1) \otimes V((m-1)(2m+1)))$ (See (17)). The projection of $V$ into $V((m-1)(2m+1))$ is clearly surjective and $G$-equivariant. By Proposition 20, a triplet $(GL(1) \times Sp(m), \square \otimes \Lambda_2, V(1) \otimes V((m-1)(2m+1)))$ is not a P.V. for $m \geqq 3$, by Lemma 5 in § 2. Assume that a triplet $(G, \rho_{\boxminus}, V)$ is a P.V. for $m = 2$ and let $X_0 = (x_1, x_2) \in V = V(1) \oplus V(5)$ be its generic point. Then $x_2$ is a generic point of a triplet $(GL(1) \times Sp(2), \square \otimes \Lambda_2, V(1) \otimes V(5)) \cong (GL(1) \times SO(5), \square \otimes \Lambda_1, V(1) \otimes V(5))$ and the isotropy subgroup at $x_2$ is isomorphic to $SO(4) \times \{\pm 1\}$ by Proposition 23. Since the action of $SO(4) \times \{\pm 1\}$ on $x_1$ is given by $x_1 \mapsto \pm x_1$, $X_0 = (x_1, x_2)$ can not be generic point. This is a contradiction and hence a triplet $(G, \rho_{\boxminus}, V)$ is not a P.V. for $m \geqq 2$, i.e., our triplet is not a P.V.

PROPOSITION 21. *A triplet* $(GL(1) \times Sp(n) \times Sp(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(2m))$ $(n \geqq m \geqq 2)$ *is not a P.V.*

(19) $$(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$$

First we consider $(GL(6), \Lambda_3, V(20))$ (See (7)). If we restrict $\Lambda_3$ to the subgroup $GL(1) \times Sp(3)$ of $GL(6)$, $V(20)$ decomposes into a direct sum $V(20) = V(6) \oplus V(14)$ since the restriction of weights $\{\lambda_i + \lambda_j + \lambda_k \mid 1 \leq i < j < k \leq 6\}$ of $GL(6)$ decomposes into $\{\pm\lambda_i \mid i = 1, 2, 3\} \cup \{\pm\lambda_1 \pm \lambda_2 \pm \lambda_3, \pm\lambda_1, \pm\lambda_2, \pm\lambda_3\}$. The action $\rho$ of $GL(1) \times Sp(3)$ on $V(14)$ is $\square \otimes \Lambda_3$. Note that the same notation $\Lambda_3$ is used for $GL(6)$ and $Sp(3)$. Every element of $V(20)$ can be written uniquely in the form $x = \sum_{i<j<k} x_{ijk} u_i \wedge u_j \wedge u_k$ with $(x_{ijk})$ forming an alternating tensor of rank three with coefficients in $\boldsymbol{C}$. Moreover the element $x$ is contained in $V(14)$ if and only if $x_{i14} + x_{i25} + x_{i36} = 0$ for $1 \leq i \leq 6$. Let $\mathfrak{g}$ be the Lie algebra of $GL(1) \times Sp(3)$.

$$(5.20) \quad \mathfrak{g} = \left\{ A = \left[\begin{array}{ccc|ccc} d + a_1 & a_{12} & a_{13} & b_1 & b_{12} & b_{13} \\ a_{21} & d + a_2 & a_{23} & b_{12} & b_2 & b_{23} \\ a_{31} & a_{32} & d + a_3 & b_{13} & b_{23} & b_3 \\ \hline c_1 & c_{12} & c_{13} & d - a_1 & -a_{21} & -a_{31} \\ c_{12} & c_2 & c_{23} & -a_{12} & d - a_2 & -a_{32} \\ c_{13} & c_{23} & c_3 & -a_{13} & -a_{23} & d - a_3 \end{array}\right] \in \mathfrak{gl}(6) \right\}$$

We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0 = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$. Since $d\rho(A)X_0 = (3d + a_1 + a_2 + a_3)u_1 \wedge u_2 \wedge u_3 + (3d - a_1 - a_2 - a_3)\, u_4 \wedge u_5 \wedge u_6 + b_1 u_1 \wedge u_5 \wedge u_6 + b_2 u_4 \wedge u_2 \wedge u_6 + b_3 u_4 \wedge u_5 \wedge u_3 + c_1 u_4 \wedge u_2 \wedge u_3 + c_2 u_1 \wedge u_5 \wedge u_3 \wedge c_3 u_1 \wedge u_2 \wedge u_6 + (c_{12}u_3 - b_{12}u_6) \wedge (u_1 \wedge u_4 - u_2 \wedge u_5) + (c_{23}u_1 - b_{23}u_4) \wedge (u_2 \wedge u_5 - u_3 \wedge u_6) + (c_{13}u_2 - b_{13}u_5) \wedge (u_3 \wedge u_6 - u_1 \wedge u_4)$, we have

$$\begin{aligned} \mathfrak{g}_{X_0} &= \{A \in \mathfrak{g} \mid d\rho(A)X_0 = 0\} \\ &= \{A \in \mathfrak{g} \mid d = a_1 + a_2 + a_3 = 0, (b_{ij}) = (c_{ij}) = 0\} \\ &= \left\{ \left( \begin{array}{c|c} A & 0 \\ \hline 0 & -{}^tA \end{array} \right) \middle| A \in \mathfrak{sl}(3) \right\} \cong \mathfrak{sl}(3) \,. \end{aligned}$$

Since $\dim \mathfrak{g}_{X_0} = 8 = \dim GL(1) \times Sp(3) - \dim V(14)$, it is a regular P.V. by Proposition 25 in § 4. Similarly, for $X_0' = u_1 \wedge u_2 \wedge u_6 + u_3 \wedge (u_1 \wedge$

$u_4 - u_2 \wedge u_5)$, we have $d\rho(A)X_0' = \{(3d + a_3)u_3 + c_3u_6\} \wedge (u_1 \wedge u_4 - u_2 \wedge u_5) - \{(a_{13} + a_{32})u_1 + c_{13}u_4\} \wedge (u_2 \wedge u_5 - u_3 \wedge u_6) - \{(a_{23} + a_{31})u_2 + c_{23}u_5\} \wedge (u_3 \wedge u_6 - u_1 \wedge u_4) + (c_2u_6 - 2a_{12}u_3) \wedge u_1 \wedge u_5 - (c_1u_6 + 2a_{21}u_3) \wedge u_2 \wedge u_4 - 2c_{12}u_3 \wedge u_4 \wedge u_5 + (b_3 + 2b_{12})u_1 \wedge u_2 \wedge u_3 + (3d + a_1 + a_2 - a_3)u_1 \wedge u_2 \wedge u_6$ and hence $\mathfrak{g}_{X_0'} = \{A \in \mathfrak{g} \mid d\rho(A)X_0' = 0\}$ is given by

$$\mathfrak{g}_{X_0'} = \left\{ A = \left(\begin{array}{ccc|ccc} -2d+\alpha & 0 & a_{13} & b_1 & b_{12} & b_{13} \\ 0 & -2d-\alpha & a_{23} & b_{12} & b_2 & b_{23} \\ -a_{23} & -a_{13} & -2d & b_{13} & b_{23} & -2b_{12} \\ \hline & & & 4d-\alpha & 0 & a_{23} \\ & 0 & & 0 & 4d+\alpha & a_{13} \\ & & & -a_{13} & -a_{23} & 4d \end{array}\right) \right\} \tag{5.21}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{so}(3)) \oplus \mathfrak{u}(5)$$

where $\mathfrak{u}(5)$ is the Lie algebra of a 5-dimensional vector group. Since $\dim \{d\rho(A)X_0' \mid A \in \mathfrak{g}\} = 13$, the orbit of $X_0'$ is of codimension 1. For $A \in \mathfrak{g}_{X_0'}$, the trace $\mathrm{tr}_V A$ in $V = V(14)$ is $14 \times 3d = 42d$ and the trace $\mathrm{tr}\,\mathrm{ad}\,\mathfrak{g}_{X_0'}A$ of the adjoint representation is $-30d$, and hence we have

$$\deg f = \frac{\mathrm{tr}_V A + \mathrm{tr}\,\mathrm{ad}\,A}{\mathrm{tr}_V A} \cdot \dim V = \frac{42d - 30d}{42d} \times 14 = 4$$

by Proposition 15 in §4, where $f(x)$ is an irreducible relative invariant polynomial. This shows that the restriction of the relative invariant of degree 4 of $(GL(6), \Lambda_3, V(20))$ is still irreducible, and it is the relative invariant of $(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$.

Put

$$X = \begin{pmatrix} x_{423} & x_{143} & x_{124} \\ x_{523} & x_{153} & x_{125} \\ x_{623} & x_{163} & x_{126} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} x_{156} & x_{416} & x_{451} \\ x_{256} & x_{426} & x_{452} \\ x_{356} & x_{436} & x_{453} \end{pmatrix}.$$

Since $x_{i14} + x_{i25} + x_{i36} = 0$, $X$ and $Y$ are symmetric matrices. We denote by $X_{ij}$ the matrix obtained from $X$ by crossing out its $i$-th line and $j$-th column. Put $x_0 = x_{123}$ and $y_0 = x_{456}$. Then the relative invariant $f(x)$ for $x = \sum x_{ijk}u_i \wedge u_j \wedge u_k \in V(14)$ is given by $f(x) = (x_0y_0 - \mathrm{tr}\,XY)^2 + 4x_0 \det Y + 4y_0 \det X - 4\sum_{i,j} \det(X_{ij}) \cdot \det(Y_{ji})$ (See (7)). This space was investigated in detail by J. Igusa (See [2]).

PROPOSITION 22. *A triplet* $(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to*

$SL(3)$. *The relative invariant is of degree* 4.

(20) $(SO(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m)) \qquad \left(n \geqq 3, \frac{n}{2} \geqq m \geqq 1\right)$.

The special orthogonal group $SO(n)$ is defined by

$$SO(n) = \{g \in SL(n) \mid {}^t gKg = K\}$$

for some symmetric $n \times n$ non-singular matrix $K$. Since we consider them over the algebraically closed field $\boldsymbol{C}$, we may assume that $K = I_n$, i.e., the identity matrix of size $n$. We identify $V = V(n) \otimes V(m)$ with $n \times m$ matrices $M(n, m)$. Then the action $\rho = \Lambda_1 \otimes \Lambda_1$ is given by $\rho(g)X = g_1 X {}^t g_2$ for $g = (g_1, g_2) \in SO(n) \times GL(m)$. We shall calculate the isotropy subgroup $G_{X_0}$ at $X_0 = {}^t(I_m O) \in M(n, m)$. An element of $SO(n)$ can be written as follows.

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \text{where} \quad A_1 \in M(m), A_2, {}^tA_3 \in M(m, n-m)\,, \tag{5.22}$$

$A_4 \in M(n-m)$, and ${}^tAA = 1_n$, $\det A = 1$. Then for $g = (A, B) \in SO(n) \times GL(m)$, we have

$$\rho(g)X_0 = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} I_m \\ 0 \end{pmatrix} {}^tB = \begin{pmatrix} A_1 {}^tB \\ A_3 {}^tB \end{pmatrix} \tag{5.23}$$

and hence the isotropy subgroup $G_{X_0}$ is given by $G_{X_0} = \{(A, B) \in SO(n) \times GL(m) \mid A_1 {}^tB = I_n, A_3 {}^tB = 0\}$. The equation $A_1 {}^tB = I_n$, $A_3 {}^tB = 0$ implies that $A_3 = (A_3 {}^tB) {}^tB^{-1} = 0$ and $B = {}^tA_1^{-1}$. Since $\begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$ is in $SO(n)$ if and only if $\det A_1 \cdot \det A_4 = 1$, ${}^tA_1 A_1 = I_m$, ${}^tA_1 A_2 = 0$ (i.e. $A_2 = 0$) and ${}^tA_4 A_4 = I_{n-m}$, we have

$$\begin{aligned} G_{X_0} &= \left\{ \left( \left( \begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_4 \end{array} \right), ({}^tA_1^{-1}) \right) \middle| A_1 \in O(m), A_4 \in O(n-m), \det A_1 \cdot \det A_4 = 1 \right\} \\ &\cong SO(m) \times SO(n-m) \times \{\pm 1\}\,. \end{aligned}$$

Since $\dim G_{x_0} = \frac{1}{2}m(m+1) + \frac{1}{2}(n-m)(n-m+1) = \frac{1}{2}n(n+1) + m^2 - nm = \dim SO(n) \times GL(m) - \dim V$, it is a regular P.V. by Proposition 25 in §4. The relative invariant $f(X)$ is given by $f(X) = \det({}^tXX)$. In general, $f(X) = \det({}^tXKX)$ if $SO(n)$ is given by $\{g \in SL(n) \mid {}^tgKg = K\}$ since $f(AX{}^tB) = \det(B {}^tX {}^tAKAX {}^tB) = \det(B {}^tXKX {}^tB) = (\det B)^2 \cdot f(X)$ for

$(A, B) \in SO(n) \times GL(m)$. Any rational character $\chi$ of $SO(n) \times GL(m)$ is of the form $\chi(g) = (\det B)^{\ell}$ for some $\ell \in \boldsymbol{Z}$ where $g = (A, B) \in SO(n) \times GL(m)$. Since $\chi|_{G_{X_0}} \equiv 1$ if and only if $\ell$ is even, any relative invariant is of the form $c \cdot f(X)^r$ $(r \in \boldsymbol{Z}, c \in \boldsymbol{C}^{\times})$ by Proposition 19 in § 4 and hence $f(X) = \det({}^tXKX)$ is irreducible.

PROPOSITION 23. *A triplet* $(SO(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ *with* $n \geqq 3$, $\frac{n}{2} \geqq m \geqq 1$, *is a regular P.V. and its generic isotropy subgroup is isomorphic to* $SO(m) \times SO(n-m) \times \{\pm 1\}$. *The relative invariant is of degree* $2m$.

Note that Proposition 23 holds even if $n \geqq m > \frac{n}{2}$ although in this case, it is not reduced.

(21) $\quad (GL(1) \times SO(n) \times Sp(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(n) \otimes V(2m))$ $\quad (n > 2m \geqq 4)$

By Proposition 14 in § 2, it is a P.V. if and only if a triplet $(GL(1) \times Sp(m), \square \otimes 2\Lambda_1, V(1) \otimes V(m(2m+1)))$ is a P.V. Since $2\Lambda_1$ is the adjoint representation of $Sp(m)$ and rank $Sp(m) \geqq 2$ $(m \geqq 2)$, it is not a P.V. by Proposition 2.

PROPOSITION 24. *A triplet* $(GL(1) \times SO(n) \times Sp(m), \square \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(n) \otimes V(2m))$ *is not a P.V.*

A short outline of the theory of the spin representation necessary for the following exposition will be presented below.

Let $V$ be a vector space over the complex number field $\boldsymbol{C}$ of even dimension $n = 2m$. Let $Q$ be a non-degenerate quadratic form on $V$, and let $B(x, y)$ be the associated bilinear form, i.e., $B(x, y) = Q(x+y) - Q(x) - Q(y)$ for $x, y \in V$.

Then there exists a basis $\{e_1, \cdots, e_m, f_1, \cdots, f_m\}$ of $V$ satisfying $B(e_i, e_j) = B(f_i, f_j) = 0$, $B(e_i, f_j) = \delta_{ij}$ and we have

$$Q\left(\sum_{i=1}^{m} x_i e_i + \sum_{i=1}^{m} y_i f_i\right) = \sum_{i=1}^{m} x_i y_i$$

for any $x_i, y_i \in \boldsymbol{C}$. Let $B_0$ be the bilinear form on $V \times V$ defined by

$$B_0\left(\sum x_i e_i + \sum y_i f_i, \sum x'_i e_i + \sum y'_i f_i\right) = \sum x'_i y_i \,. \tag{5.24}$$

Note that $B_0(x, x) = Q(x)$ for $x \in V$.

Let $T(V) = \sum_{k=0}^{\infty} \overbrace{V \otimes \cdots \otimes V}^{k}$ be the tensor algebra over $V$, and let $I_Q$ be the two-sided ideal of $T(V)$ generated by the subset $\{x \otimes x - Q(x) \cdot 1 \mid x \in V\}$. Then the quotient algebra $C(Q) = T(V)/I_Q$ is called the *Clifford algebra*. Put $T^+(V) = \sum_{k=0}^{\infty} \overbrace{V \otimes \cdots \otimes V}^{2k}$ and let $\varphi: T(V) \to C(Q)$ be the canonical map from $T(V)$ onto $C(Q)$. The image $\varphi(T^+(V))$ of $T^+(V)$ is called the *even Clifford algebra* and denoted by $C^+(Q)$. It is known that $C(Q) \cong M(2^m, \boldsymbol{C})$ and $C^+(Q) \cong M(2^{m-1}, \boldsymbol{C}) \oplus M(2^{m-1}, \boldsymbol{C})$. The orthogonal group $O(Q, V)$w, r, t, $Q$ and the special orthogonal group $SO(Q, V)$w, r, t, $Q$ are defined as follows.

$$(5.25) \qquad \begin{aligned} O(Q, V) &= \{g \in GL(V) \mid Q(gx) = Q(x) \text{ for any } x \in V\} \\ SO(Q, V) &= O(Q, V) \cap SL(V) \end{aligned}$$

We shall also define the *Clifford group* $\Gamma(Q)$ and the *even Clifford group* $\Gamma^+(Q)$ as follows.

$$(5.26) \qquad \begin{aligned} \Gamma(Q) &= \{s \in C(Q) \mid \exists s^{-1}, sVs^{-1} \subset V\} \\ \Gamma^+(Q) &= \Gamma(Q) \cap C^+(Q) \end{aligned}$$

Let $\chi$ be the representation of $\Gamma(Q)$ on $V$ defined by $\chi(s)v = svs^{-1}$ for $s \in \Gamma(Q)$ and $x \in V$. Since $Q(\chi(s)v) = (svs^{-1})^2 = sv^2s^{-1} = sQ(v)s^{-1} = Q(v)$, we have $\chi(s) \in O(Q, V)$. This $\chi$ is called the *vector representation* of $\Gamma(Q)$.

Let $\alpha$ be the anti-automorphism of $T(V)$ defined by $\alpha(v_1 \otimes \cdots \otimes v_k) = v_k \otimes \cdots \otimes v_1$. As $\alpha(I_Q) \subset I_Q$, $\alpha$ induces the anti-automorphism on $C(Q)$, which is also denoted by $\alpha$. Note that $\alpha$ fixes an element of $V$ and hence $svs^{-1} = \alpha(svs^{-1}) = \alpha(s)^{-1}\alpha(v)\alpha(s) = \alpha(s)^{-1}v\alpha(s)$ for $s \in \Gamma(Q)$, $v \in V$. This implies that $\alpha(s)sv = v\alpha(s)s$ and hence $\alpha(s)s$ is an element of the center $\boldsymbol{C}$ of $C(Q)$. Since $s$ is invertible, we have $\alpha(s)s \in \boldsymbol{C}^\times$. We shall define the spin group Spin$(Q)$ as Spin$(Q) = \{s \in \Gamma^+(Q) \mid \alpha(s)s = 1\}$. It is connected, simply connected and semisimple. Moreover if $n \neq 4$, then it is simple. It is well-known that the following exact sequence (5.27) holds.

$$(5.27) \qquad 1 \longrightarrow \{\pm 1\} \longrightarrow Spin(Q) \xrightarrow{\chi} SO(Q, V) \longrightarrow 1 \qquad \text{(exact)}.$$

Now we shall construct the half-spin representation of the spin group $Spin(Q)$. Let $\Lambda(V) = \sum_{k=0}^{n} \Lambda^k(V)$ be the exterior algebra of $V$. For each $x \in V$, let $\rho(x)$ be an element of $\mathrm{End}_{\boldsymbol{C}}(\Lambda(V))$ defined by $\rho(x)\lambda = (L_x +$

$\delta_x)\lambda(\lambda \in \Lambda(V))$ where $L_x\lambda = x \wedge \lambda$ and

$$\delta_x(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} B_0(x, v_i) v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_k .$$

One can easily check that $\rho(x)^2 = Q(x)\cdot 1$, and hence the representation of $V$ on $\Lambda(V)$ can be extended to that of the Clifford algebra $C(Q)$ on $\Lambda(V)$. Put $f = f_1 \wedge \cdots \wedge f_m$ and $M = \Lambda(V) \wedge f$. Then one can check that $M$ is a $\rho$-invariant subspace of $\Lambda(V)$.

Let $E$ be the subspace of $V$ generated by $\{e_1, \cdots, e_m\}$. Then the map $\varphi: \Lambda(E) \to M$ defined by $\varphi(\mu) = \mu \wedge f$ for $\mu \in \Lambda(E)$, is clearly a linear isomorphism. We identify $\Lambda(E)$ with $M$ by this map, and hence we obtain the representation $\rho$ of $C(Q)$ on $\Lambda(E)$. The subspaces $\Lambda^+(E) = \sum_{k:\text{ even}} \Lambda^k(E)$ and $\Lambda^-(E) = \sum_{k:\text{ odd}} \Lambda^k(E)$ of $\Lambda(E)$ are the irreducible representation spaces with respect to the restriction of $\rho$ to the spin group Spin $(Q)$. This representation of Spin $(Q)$ on $\Lambda^+(E)$ (resp. $\Lambda^-(E)$) is called the even (resp. odd) half-spin representation.

These two half-spin representations are inequivalent, however, they are transformed to each other by the outer automorphism of Spin $(Q)$. Therefore we shall consider only the even half-spin representation of Spin $(Q)$.

Now we shall calculate the infinitesimal representation $d\rho$ of the half-spin representation.

Let $E_{ij}$ be the matrix unit of degree $m$ $(1 \leq i, j \leq m)$ and put $E'_{ij} = E_{ij} - E_{ji}$. Then an element $A$ of the Lie algebra $\mathfrak{g} = \mathfrak{o}(2m, \boldsymbol{C})$ of the spin group Spin $(Q)$ can be written as follows (See § 1).

$$\begin{aligned} A = \sum_{i \neq j} a_{ij} \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} + \sum_{i<j} b_{ij} \begin{pmatrix} 0 & E'_{ij} \\ 0 & 0 \end{pmatrix} \\ + \sum_{i<j} c_{ij} \begin{pmatrix} 0 & 0 \\ E'_{ij} & 0 \end{pmatrix} + \sum_{i=1}^{m} a_i \begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix} . \end{aligned} \tag{5.28}$$

By the definition the product in $C(Q)$ is given by $e_i f_i + f_i e_i = 1, e_i^2 = f_i^2 = 0$, $e_i f_j = -f_j e_i$, $e_i e_j = -e_j e_i$ and $f_i f_j = -f_j f_i$ $(j \neq i, i, j = 1, \cdots, m)$. We shall consider an element $s = 1 + t e_i f_j$ $(t \in \boldsymbol{C},\ i \neq j)$ of $C(Q)$. Then $\alpha(s) = 1 + t f_j e_i$ and hence $\alpha(s)s = 1$ i.e. $s^{-1} = \alpha(s)$. Since $\chi(s)e_k = s e_k s^{-1} = (1 + t e_i f_j) e_k (1 + t f_j e_i)$, we have $\chi(s)e_k = e_k$ for any $k \neq j$ and $\chi(s)e_j = e_j + t e_i$. Also we have $\chi(s)f_k = f_k$ for any $k \neq i$ and $\chi(s)f_i = f_i - t f_j$. This implies that $s = 1 + t e_i f_j \in Spin(Q)$ and

$$\chi(1+te_if_j)=\exp t\begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}\in SO(Q,V)\ .$$

Similarly we have the following relations.

1) $\chi(1+te_if_j)=\exp t\begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}\in SO(Q,V)\ \ (i\neq j).$

2) $\chi(1+te_ie_j)=\exp t\begin{pmatrix}0 & E'_{ij}\\ 0 & 0\end{pmatrix}\in SO(Q,V)\ \ (i<j).$

3) $\chi(1+tf_if_j)=\exp t\begin{pmatrix}0 & 0\\ E'_{ij} & 0\end{pmatrix}\in SO(Q,V)\ \ (i<j).$

4) $$\chi\left(\sqrt{t}\,e_kf_k+\frac{1}{\sqrt{t}}f_ke_k\right)=\begin{array}{c}\begin{array}{ccccc} & k & & m+k & \\ & \vee & & \vee & \end{array}\\ \begin{pmatrix}1 & & & & \\ & \ddots & & & \\ & & t & & \\ & & & \ddots & \\ & & & & \frac{1}{t} \\ & & & & & \ddots \\ & & & & & & 1\end{pmatrix}\end{array}\in SO(Q,V).$$

Since $\chi$ is an isomorphism in a neighborhood of the identity, we have

$$d\rho\begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}\lambda=\lim_{t\to 0}\frac{1}{t}\left(\rho\left(\exp t\begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}\right)-\rho(1)\right)\lambda$$
$$=\lim_{t\to 0}\frac{1}{t}(\rho(1+te_if_j)-\rho(1))\lambda=\rho(e_i)\rho(f_j)\lambda=e_i\delta_{f_j}\lambda$$
$$\text{for } i\neq j,\ \lambda\in\Lambda^+(E)\ .$$

Similarly we have

1′) $d\rho\begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}\lambda=\rho(e_i)\rho(f_j)\lambda=e_i\delta_{f_j}\lambda$

2′) $d\rho\begin{pmatrix}0 & E'_{ij}\\ 0 & 0\end{pmatrix}\lambda=\rho(e_i)\rho(e_j)\lambda=e_i\wedge e_j\wedge\lambda$

3′) $d\rho\begin{pmatrix}0 & 0\\ E'_{ij} & 0\end{pmatrix}\lambda=\rho(f_i)\rho(f_j)\lambda=\delta_{f_i}\delta_{f_j}\lambda$

4′) $d\rho\begin{array}{c}\begin{array}{cc} k & m+k\\ \vee & \vee\end{array}\\ \begin{pmatrix}1 & \\ & -1\end{pmatrix}\end{array}\lambda=\rho(e_k)\rho(f_k)\lambda-\frac{1}{2}\lambda$

and hence for an element $A$ in (5.28) of $\mathfrak{so}(2m)$, we have the following (5.29).

$$(5.29)\quad d\rho(A)\lambda = \sum_{i\neq j} a_{ij}e_i\delta_{f_j}\lambda + \sum_{i=1}^{m} a_i(e_i\delta_{f_i} - \tfrac{1}{2})\lambda + \sum_{i<j} b_{ij}e_ie_j\lambda + \sum_{i<j} c_{ij}\delta_{f_i}\delta_{f_j}\lambda\,.$$

Next we shall consider the case when the dimension $n$ of $V$ is odd, i.e., $n = 2m + 1$. Let $Q$ be a nondegenerate quadratic form on $V$, and let $B(x, y)$ be the associated bilinear form. Let $V_0$ denote the subspace of $V$ generated by $e_1, \cdots, e_m, f_1, \cdots, f_m$ satisfying $B(e_i, e_j) = B(f_i, f_j) = 0$ and $B(e_i, f_j) = \delta_{ij}$. Then the subspace of $V$ orthogonal to $V_0$ is of the form $\boldsymbol{C}v_0$ for some $v_0$ in $V$ satisfying $Q(v_0) \neq 0$. We may assume that $Q(v_0) = 1$. Now consider a vector space $V_1$ of dimension $n + 1$ and represent it as $V_1 = V + \boldsymbol{C}v_1$. Let $Q_1$ be the quadratic form on $V_1$ defined by $Q_1(V + \lambda v_1) = Q(v) - \lambda^2$ for $v \in V$, $\lambda \in \boldsymbol{C}$, and let $B_1(x, y)$ be the associated bilinear form of $Q_1$. Then, if we put $e_{m+1} = \frac{1}{2}(v_0 + v_1)$ and $f_{m+1} = \frac{1}{2}(v_0 - v_1)$, we have $B_1(e_i, e_j) = B_1(f_i, f_j) = 0$ and $B_1(e_i, f_j) = \delta_{ij}$ for $1 \leq i, j \leq m + 1$. Let $\mathrm{Spin}(Q_1)$ be the spin group of $(V_1, Q_1)$, and let $\chi_1$ be its vector representation. Then the spin group $Spin(Q)$ is defined by $Spin(Q) = \{s \in Spin(Q_1) \mid \chi_1(s)v_1 = v_1\}$. By restricting the half-spin representation of $Spin(Q_1)$ to $Spin(Q)$, we obtain the spin representation of $Spin(Q)$ (See [2], [3]).

In the following, we denote the spin group by $Spin(n)$ instead of $Spin(Q)$, and denote the element $e_{i_1} \wedge \cdots \wedge e_{i_k}$ of $\Lambda^+(E)$ by $e_{i_1} \cdots e_{i_k}$.

(22) $(Spin(7) \times GL(d),\ spin\ rep. \otimes \Lambda_1,\ V(8) \otimes V(d))$ $\quad (1 \leq d \leq 4)$

First of all, we shall calculate the half-spin representation $d\rho_1$ of $\mathfrak{o}(8, \boldsymbol{C})(=D_4)$. The representation space $V(8)$ is spanned by $1$, $e_ie_j$, $e_1e_2e_3e_4$ $(1 \leq i < j \leq 4)$. We may assume that an element $A$ of $\mathfrak{o}(8, \boldsymbol{C})$ is of the form (5.28). Then by (5.29) we can calculate $d\rho_1$. For example $d\rho_1(A)e_1e_2 = \sum_{i\neq 1} a_{i1}e_ie_2 - \sum_{i\neq 2} a_{i2}e_ie_1 + \frac{1}{2}(a_1 + a_2 - a_3 - a_4)e_1e_2 + b_{34}e_3e_4e_1e_2 + c_{12}\delta_{f_1}\delta_{f_2}e_1e_2 = -a_{31}e_2e_3 - a_{41}e_2e_4 + a_{32}e_1e_3 + a_{42}e_1e_4 + \frac{1}{2}(a_1 + a_2 - a_3 - a_4)e_1e_2 + b_{34}e_1e_2e_3e_4 - c_{12}$. Hence we have

$$(5.30)\quad d\rho_1(A)x = \left(\begin{array}{cccc|cccc}
A_1 & -c_{12} & -c_{13} & -c_{14} & 0 & c_{34} & -c_{24} & c_{23} \\
b_{12} & A_2 & a_{23} & a_{24} & -c_{34} & 0 & -a_{14} & a_{13} \\
b_{13} & a_{32} & A_3 & a_{34} & c_{24} & a_{14} & 0 & -a_{12} \\
b_{14} & a_{42} & a_{43} & A_4 & -c_{23} & -a_{13} & a_{12} & 0 \\
\hline
0 & b_{34} & -b_{24} & b_{23} & -A_1 & -b_{12} & -b_{13} & -b_{14} \\
-b_{34} & 0 & a_{41} & -a_{31} & c_{12} & -A_2 & -a_{32} & -a_{42} \\
b_{24} & -a_{41} & 0 & a_{21} & c_{13} & -a_{23} & -A_3 & -a_{43} \\
-b_{23} & a_{31} & -a_{21} & 0 & c_{14} & -a_{24} & -a_{34} & -A_4
\end{array}\right)
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$

where $x = x_1 + x_2e_1e_2 + x_3e_1e_3 + x_4e_1e_4 + x_5e_1e_2e_3e_4 - x_6e_3e_4 + x_7e_2e_4 - x_8e_2e_3$,

$$A_1 = -\frac{a_1 + a_2 + a_3 + a_4}{2}, \quad A_2 = \frac{a_1 + a_2 - a_3 - a_4}{2},$$

$$A_3 = \frac{a_1 - a_2 + a_3 - a_4}{2}, \qquad A_4 = \frac{a_1 - a_2 - a_3 + a_4}{2}.$$

Since $d\rho_1(A)$ in (5.30) is the same form as in (5.28), $d\rho_1(A) \in \mathfrak{o}(8, \boldsymbol{C})$ and it leaves the quadratic form $q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8$ invariant (See Example 28 in § 1).

Put $v_1 = {}^t(0001000-1) \in \boldsymbol{C}^8$. Then the Lie algebra $\mathfrak{o}(7, \boldsymbol{C})$ of $Spin(7)$ is given by $\mathfrak{o}(7, \boldsymbol{C}) = \{A \in \mathfrak{o}(8, \boldsymbol{C}) \mid Av_1 = 0\}$ and hence an element $A$ of $\mathfrak{o}(8, \boldsymbol{C})$ of the form (5.28) is in $\mathfrak{o}(7, \boldsymbol{C})$ if and only if

$$Av_1 = {}^t(a_{14} - b_{14}, a_{24} - b_{24}, a_{34} - b_{34}, a_4, c_{14} + a_{41}, c_{24} + a_{42}, c_{34} + a_{43}, a_4) = 0 .$$

Thus the spin representation $d\rho_1$ of $\mathfrak{o}(7, \boldsymbol{C})$ is given as follows.

$$(5.31) \qquad d\rho_1(A)x = \left(\begin{array}{c|c|c|c|c|c|c|c} A_1' & -c_{12} & -c_{13} & -c_{14} & 0 & c_{34} & -c_{24} & c_{23} \\ b_{12} & A_2' & a_{23} & b_{24} & -c_{34} & 0 & -b_{14} & a_{13} \\ b_{13} & a_{32} & A_3' & b_{34} & c_{24} & b_{14} & 0 & -a_{12} \\ b_{14} & -c_{24} & -c_{34} & A_4' & -c_{23} & -a_{13} & a_{12} & 0 \\ \hline 0 & b_{34} & -b_{24} & b_{23} & -A_1' & -b_{12} & -b_{13} & -b_{14} \\ -b_{34} & 0 & -c_{14} & -a_{31} & c_{12} & -A_2' & -a_{32} & c_{24} \\ b_{24} & c_{14} & 0 & a_{21} & c_{13} & -a_{23} & -A_3' & c_{34} \\ -b_{23} & a_{31} & -a_{21} & 0 & c_{14} & -b_{24} & -b_{34} & -A_4' \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$

where $x = x_1 + x_2e_1e_2 + x_3e_1e_3 + x_4e_1e_4 + x_5e_1e_2e_3e_4 - x_6e_3e_4 + x_7e_2e_4 - x_8e_2e_3$, $A_1' = -\dfrac{a_1 + a_2 + a_3}{2}$, $A_2' = \dfrac{a_1 + a_2 - a_3}{2}$, $A_3' = \dfrac{a_1 - a_2 + a_3}{2}$ and $A_4' = \dfrac{a_1 - a_2 - a_3}{2}$.

I) The case of $d = 1$.

Put $X_0 = 1 + e_1e_2e_3e_4 = {}^t(10001000) \in V(8)$. We shall calculate the iso- isotropy subalgebra $\mathfrak{g}_{X_0}$ of $\mathfrak{g} = \mathfrak{gl}(1) \oplus \mathfrak{o}(7, \boldsymbol{C})$ at $X_0$. Since

$$dX_0 + d\rho_1(A)X_0 = {}^t\Big(d - \frac{a_1 + a_2 + a_3}{2}, \ b_{12} - c_{34}, \ b_{13} + c_{24}, \ b_{14} - c_{23},$$

$$d + \frac{a_1 + a_2 + a_3}{2}, \ c_{12} - b_{34}, \ c_{13} + b_{24}, \ c_{14} - b_{23}\Big)$$

for $(d, A) \in \mathfrak{g}$, the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0$ is given as follows.

$$
\mathfrak{g}_{X_0} = \left\{ A = \left|\begin{array}{cccc|cccc}
a_1 & a_{12} & a_{13} & b_{14} & 0 & b_{12} & b_{13} & b_{14} \\
a_{21} & a_2 & a_{23} & b_{24} & -b_{12} & 0 & b_{23} & b_{24} \\
a_{31} & a_{32} & a_3 & b_{34} & -b_{13} & -b_{23} & 0 & b_{34} \\
-b_{23} & b_{13} & -b_{12} & 0 & -b_{14} & -b_{24} & -b_{34} & 0 \\
\hline
0 & b_{34} & -b_{24} & b_{23} & -a_1 & -a_{21} & -a_{31} & b_{23} \\
-b_{34} & 0 & b_{14} & -b_{13} & -a_{12} & -a_2 & -a_{32} & -b_{13} \\
b_{24} & b_{14} & 0 & b_{12} & -a_{13} & -a_{23} & -a_3 & b_{12} \\
-b_{23} & b_{13} & -b_{12} & 0 & -b_{14} & -b_{24} & -b_{34} & 0
\end{array}\right| \; \middle| \; a_1 + a_2 + a_3 = 0 \right\} \tag{5.32}
$$

Since $\dim \mathfrak{g}_{X_0} = 14 = \dim GL(1) \times \mathrm{Spin}\,(7) - \dim V(8)$, it is a P.V. Let $S$ be the element of $GL(8)$ defined by

$$
S^{-1} = \left|\begin{array}{c|c|c|c}
0 & \begin{matrix}\frac{1}{2}\\ 1\end{matrix} & 0 & \begin{matrix}-\frac{1}{2}\\ 1\end{matrix} \\
\hline
I_3 & 0 & 0 & 0 \\
\hline
0 & 0 & -I_3 & 0
\end{array}\right| , \qquad
S = \left|\begin{array}{c|c|c|c}
0 & 0 & I_3 & 0 \\
\hline
1 & \frac{1}{2} & 0 & 0 \\
\hline
0 & 0 & 0 & -I_3 \\
\hline
-1 & \frac{1}{2} & 0 & 0
\end{array}\right| \in GL(8) \,. \tag{5.33}
$$

Then we have

$$
S^{-1}AS = \left(\begin{array}{cc|ccc|ccc}
0 & 0 & & 0 & & & 0 & \\
\hline
0 & 0 & -2b_{23} & 2b_{13} & -2b_{12} & 2b_{14} & 2b_{24} & 2b_{34} \\
\hline
 & b_{14} & a_1 & a_{12} & a_{13} & 0 & -b_{12} & -b_{13} \\
0 & b_{24} & a_{21} & a_2 & a_{23} & b_{12} & 0 & -b_{23} \\
 & b_{34} & a_{31} & a_{32} & a_3 & b_{13} & b_{23} & 0 \\
\hline
 & -b_{23} & 0 & -b_{34} & b_{24} & -a_1 & -a_{21} & -a_{31} \\
0 & b_{13} & b_{34} & 0 & -b_{14} & -a_{12} & -a_2 & -a_{32} \\
 & -b_{12} & -b_{24} & b_{14} & 0 & -a_{13} & -a_{23} & -a_3
\end{array}\right) \tag{5.34}
$$

with $a_1 + a_2 + a_3 = 0$ .

By (1.8) in Example 30 in § 1, this is an element of $(\mathfrak{g}_2)$ i.e. $\mathfrak{g}_{X_0} \cong (\mathfrak{g}_2)$, and hence it is a regular P.V. by Proposition 25 in § 4. The relative invariant is the quadratic form $q(x)$. J. Igusa completed the orbital decomposition of this triplet (See [2]).

PROPOSITION 25. *A triplet* $(GL(1) \times Spin\,(7), \;\square \otimes spin\ rep.,\; V(1) \otimes V(8))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $(G_2)$. *The relative invariant is a quadratic form.*

II) *The case of* $d = 2$.

We identify $V = V(8) \otimes V(2)$ with $8 \times 2$ matrices $M(8,2)$. Put $X_0 = {}^t\begin{pmatrix}10000000\\00001000\end{pmatrix} \in V$. We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ of

$\mathfrak{g} = \mathfrak{o}(7, \boldsymbol{C}) \oplus \mathfrak{gl}(2)$ at $X_0$. From (5.31), we have

$$d\rho_1(A)X_0 + X_0{}^tD = {}^t\left(\begin{array}{c|ccc|c|ccc} d_{11} - \dfrac{a_1 + a_2 + a_3}{2} & b_{12} & b_{13} & b_{14} & d_{12} & -b_{34} & b_{24} & -b_{23} \\ d_{21} & -c_{34} & c_{24} & -c_{23} & d_{22} + \dfrac{a_1 + a_2 + a_3}{2} & c_{12} & c_{13} & c_{14} \end{array}\right) \in V\,,$$

where $A \in \mathfrak{o}(7, \boldsymbol{C})$ and $D = (d_{ij}) \in \mathfrak{gl}(2)$. Hence the isotropy subalgebra $\mathfrak{g}_{X_0}$ is given by

$$(5.35)\quad \mathfrak{g}_{X_0} = \left\{\left(\begin{array}{cc|cc} 2\alpha I_3 + A_0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -2\alpha I_3 - {}^tA_0 & 0 \\ & & 0 & 0 \end{array}\right) \oplus \begin{pmatrix} 3\alpha & 0 \\ 0 & -3\alpha \end{pmatrix} \,\middle|\, A_0 \in \mathfrak{sl}(3)\right\}$$

$$\cong \mathfrak{sl}(3) \oplus \mathfrak{o}(2)\,.$$

Since $\dim \mathfrak{g}_{X_0} = 9 = \dim \mathrm{Spin}\,(14) \times GL(2) - \dim V(8) \otimes V(2)$, it is a regular P.V. by Proposition 25 in § 4. Since $\mathrm{Spin}\,(7) \hookrightarrow SO(8)$ by the spin representation, there exists an irreducible relative invariant of degree 4 by Proposition 23.

PROPOSITION 26. *A triplet* ($\mathit{Spin}\,(7) \times GL(2)$, *spin rep.* $\otimes \Lambda_1$, $V(8) \otimes V(2)$) *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $SL(3) \times O(2)$. *The relative invariant is of degree* 4.

III) *The case of* $d = 3$.

We identify $V = V(8) \otimes V(3)$ with $8 \times 3$ matrices $M(8, 3)$, and put $X_0 = {}^t\begin{pmatrix} 10000000 \\ 00001000 \\ 01000100 \end{pmatrix} \in V$. Then from (5.31), we have

$$(5.36)\quad d\rho_1(A)X_0 + X_0{}^tD = {}^t\left(\begin{array}{c|c|c|c|c|c|c|c} A'_1 + d_{11} & b_{12} + d_{13} & b_{13} & b_{14} & d_{12} & d_{13} - b_{34} & b_{24} & -b_{23} \\ d_{21} & -c_{34} + d_{23} & c_{24} & -c_{23} & -A'_1 + d_{22} & c_{12} + d_{23} & c_{13} & c_{14} \\ c_{34} - c_{12} + d_{31} & A'_2 + d_{33} & a_{32} + b_{14} & -c_{24} - a_{13} & b_{34} - b_{12} + d_{32} & d_{33} - A'_2 & c_{14} - a_{23} & a_{31} - b_{24} \end{array}\right) \in V\,.$$

Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(A, D) \mid d\rho_1(A)X_0 + X_0{}^tD = 0\}$ of

$\mathfrak{g} = \mathfrak{o}(7, \boldsymbol{C}) \oplus \mathfrak{gl}(3)$ at $X_0$ is given as follows.

$$(5.37)\qquad \mathfrak{g}_{X_0} = \left\{ \left(\begin{array}{cc|cc|cc|cc} a_1 & a_{12} & 0 & 0 & 0 & b_{12} & 0 & 0 \\ a_{21} & a_2 & 0 & 0 & -b_{12} & 0 & 0 & 0 \\ \hline 0 & 0 & a_3 & -b_{12} & 0 & 0 & 0 & -b_{12} \\ 0 & 0 & c_{12} & 0 & 0 & 0 & b_{12} & 0 \\ \hline 0 & c_{12} & 0 & 0 & -a_1 & -a_{21} & 0 & 0 \\ -c_{12} & 0 & 0 & 0 & -a_{12} & -a_2 & 0 & 0 \\ \hline 0 & 0 & 0 & -c_{12} & 0 & 0 & -a_3 & -c_{12} \\ 0 & 0 & c_{12} & 0 & 0 & 0 & b_{12} & 0 \end{array}\right) \oplus \left(\begin{array}{c|c|c} a_3 & 0 & -b_{12} \\ 0 & -a_3 & -c_{12} \\ 2c_{12} & 2b_{12} & 0 \end{array}\right) \right\}$$

$$\cong \mathfrak{sl}(2) \oplus \mathfrak{o}(3)\,. \qquad\qquad \text{with } a_1 + a_2 = a_3$$

Since $\dim \mathfrak{g}_{X_0} = 6 = \dim \mathrm{Spin}\,(7) \times GL(3) - \dim V(8) \otimes V(3)$, it is a regular P.V. by Proposition 25 in § 4. Since $\mathrm{Spin}\,(7) \hookrightarrow SO(8)$ by the spin representation, there exists an irreducible relative invariant polynomial of degree 6 by Proposition 23.

PROPOSITION 27. *A triplet* ($Spin\,(7) \times GL(3)$, *spin rep.* $\otimes \Lambda_1$, $V(8) \otimes V(3)$) *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $SL(2) \times O(3)$. *The relative invariant is of degree* 6.

IV) *The case of* $d = 4$.

Assume that this triplet is a P.V. and let $H$ be a generic isotropy subgroup. Then we have $\dim H = \dim Spin\,(7) \times GL(4) - \dim V(8) \otimes V(4) = 5$. Since $Spin\,(7) \hookrightarrow SO(8)$ by the spin representation, we may consider that this P.V. is contained in a regular P.V. ($SO(8) \times GL(4)$, $\Lambda_1 \otimes \Lambda_1$, $V(8) \otimes V(4)$) and hence by Proposition 23 we have $H = Spin\,(7) \times GL(4) \cap SO(4) \times SO(4) \times \{\pm 1\} \supset SO(4)$ (See (20) in § 5). This implies that $\dim H \geqq \dim SO(4) = 6$, i.e., a contradiction, and hence we obtain the following proposition.

PROPOSITION 28. *A triplet* ($Spin\,(7) \times GL(4)$, *spin rep.* $\otimes \Lambda_1$, $V(8) \otimes V(4)$) *is not a P.V.*

(23) ($Spin\,(7) \times GL(2)$, *spin rep.* $\otimes 2\Lambda_1$, $V(8) \otimes V(3)$) .

Assume that this triplet is a P.V., and let $H$ be its generic isotropy subgroup. Then we have $\dim H = \dim Spin\,(7) \times GL(2) - \dim V(8) \otimes V(3) = 1$. Since $(SL(2), 2\Lambda_1, V(3)) \cong (SO(3), \Lambda_1, V(3))$, we may consider that this triplet is contained in ($Spin\,(7) \times GL(3)$, spin rep. $\otimes \Lambda_1$, $V(8) \otimes V(3)$) and hence by Proposition 27, $H^0 = SL(2) \times O(3) \cap Spin\,(7) \times SO(3)$

$\supset SO(3)$ where $H^0$ denotes the connected component of $H$. This implies that $\dim H \geqq \dim SO(3) = 3$ i.e. a contradiction, and hence we obtain the following proposition.

PROPOSITION 29. *A triplet* $(Spin\,(7) \times GL(2),\ spin\ rep. \otimes 2\Lambda_1,\ V(8) \otimes V(3))$ *is not a P.V.*

(24) $(GL(1) \times Spin\,(7) \times Sp\,(2),\ \square \otimes spin\ rep. \otimes \Lambda_1,\ V(1) \otimes V(8) \otimes V(4))$

We may consider that this triplet is contained in a triplet $(Spin\,(7) \times GL(4)$, spin rep. $\otimes \Lambda_1$, $V(8) \otimes V(4))$ which is not a P.V. by Proposition 28, and hence it is not a P.V.

PROPOSITION 30. *A triplet* $(GL(1) \times Spin\,(7) \times Sp\,(2),\ \square \otimes spin\ rep. \otimes \Lambda_1,\ V(1) \otimes V(8) \otimes V(4))$ *is not a P.V.*

(25) $(Spin\,(10) \times GL(d),\ half\text{-}spin\ rep. \otimes \Lambda_1,\ V(16) \otimes V(d))$ $(1 \leq d \leq 3)$

First of all, we shall calculate the half-spin representation $d\rho_1$ of $\mathfrak{o}(10, C)$ by (5.29). The representation space $V(16)$ is spanned by 1, $e_ie_j$ $(1 \leq i < j \leq 5)$, $e_ie_je_ke_\ell$ $(1 \leq i < j < k < \ell \leq 5)$. We may assume that an element $A$ in $\mathfrak{o}(10, C)$ is of the form (5.28). Then by (5.29) we have

1) $d\rho_1(A)\cdot 1 = -\dfrac{a_1 + a_2 + a_3 + a_4 + a_5}{2} + \sum_{i<j} b_{ij}e_ie_j$

2) $d\rho_1(A)e_ke_\ell = \sum_{i \neq k} a_{ik}e_ie_\ell - \sum_{i \neq \ell} a_{i\ell}e_ie_k + \dfrac{a_k + a_\ell}{2}e_ke_\ell - \sum_{s \neq k,\ell} \dfrac{a_s}{2}e_ke_\ell$

$+ \sum_{i<j} b_{ij}e_ie_je_ke_\ell - c_{k\ell}$ $(k < \ell)$

3) $d\rho_1(A)e_ke_\ell e_me_n = a_{sk}e_se_\ell e_me_n - a_{s\ell}e_se_ke_me_n + a_{sm}e_se_ke_\ell e_n - a_{sn}e_se_ke_\ell e_m$

$+ \dfrac{a_k + a_\ell + a_m + a_n - a_s}{2}e_ke_\ell e_me_n - c_{k\ell}e_me_n + c_{km}e_\ell e_n - c_{kn}e_\ell e_m$

$- c_{\ell m}e_ke_n + c_{\ell n}e_ke_m - c_{mn}e_ke_\ell$ where $1 \leq k < \ell < m < n \leq 5$ and $\{s, k, \ell, m, n\} = \{1, \cdots, 5\}$.

Hence we have

$$d\rho_1(A)x = \left(\begin{array}{c|c} \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \alpha_4 \end{array}\right)\begin{pmatrix} x_1 \\ \vdots \\ x_{16} \end{pmatrix}$$

where $\alpha_1, \cdots, \alpha_4$ are given as follows.

$$
\alpha_1 = \begin{array}{|c|c|c|c|c|c|c|c|}
A_1 & -c_{12} & -c_{13} & -c_{14} & -c_{15} & -c_{23} & -c_{24} & -c_{25} \\
b_{12} & A_2 & a_{23} & a_{24} & a_{25} & -a_{13} & -a_{14} & -a_{15} \\
b_{13} & a_{32} & A_3 & a_{34} & a_{35} & a_{12} & & \\
b_{14} & a_{42} & a_{43} & A_4 & a_{45} & & a_{12} & \\
b_{15} & a_{52} & a_{53} & a_{54} & A_5 & & & a_{12} \\
b_{23} & -a_{31} & a_{21} & & & A_6 & a_{34} & a_{35} \\
b_{24} & -a_{41} & & a_{21} & & a_{43} & A_7 & a_{45} \\
b_{25} & -a_{51} & & & a_{21} & a_{53} & a_{54} & A_8
\end{array} ,
$$

$$
\alpha_2 = \begin{array}{|c|c|c|c|c|c|c|c|}
-c_{34} & -c_{35} & -c_{45} & & & & & \\
 & & & -c_{34} & -c_{35} & -c_{45} & & \\
-a_{14} & -a_{15} & & c_{24} & c_{25} & & -c_{45} & \\
a_{13} & & -a_{15} & -c_{23} & & c_{25} & c_{35} & \\
 & a_{13} & a_{14} & & -c_{23} & -c_{24} & -c_{34} & \\
-a_{24} & -a_{25} & & -c_{14} & -c_{15} & & & -c_{45} \\
a_{23} & & -a_{25} & c_{13} & & -c_{15} & & c_{35} \\
 & a_{23} & a_{24} & & c_{13} & c_{14} & & -c_{34}
\end{array} ,
$$

(5.38)

$$
\alpha_3 = \begin{array}{|c|c|c|c|c|c|c|c|}
b_{34} & & -a_{41} & a_{31} & & -a_{42} & a_{32} & \\
b_{35} & & -a_{51} & & a_{31} & -a_{52} & & a_{32} \\
b_{45} & & & -a_{51} & a_{41} & & -a_{52} & a_{42} \\
 & b_{34} & -b_{24} & b_{23} & & b_{14} & -b_{13} & \\
 & b_{35} & -b_{25} & & b_{23} & b_{15} & & -b_{13} \\
 & b_{45} & & -b_{25} & b_{24} & & b_{15} & -b_{14} \\
 & & b_{45} & -b_{35} & b_{34} & & & \\
 & & & & & b_{45} & -b_{35} & b_{34}
\end{array} ,
$$

$$
\alpha_4 = \begin{array}{|c|c|c|c|c|c|c|c|}
A_9 & a_{45} & -a_{35} & -c_{12} & & & -c_{15} & -c_{25} \\
a_{54} & A_{10} & a_{34} & & -c_{12} & & c_{14} & c_{24} \\
-a_{53} & a_{43} & A_{11} & & & -c_{12} & -c_{13} & -c_{23} \\
b_{12} & & & A_{12} & a_{45} & -a_{35} & a_{25} & -a_{15} \\
 & b_{12} & & a_{54} & A_{13} & a_{34} & -a_{24} & a_{14} \\
 & & b_{12} & -a_{53} & a_{43} & A_{14} & a_{23} & -a_{13} \\
b_{15} & -b_{14} & b_{13} & a_{52} & -a_{42} & a_{32} & A_{15} & a_{12} \\
b_{25} & -b_{24} & b_{23} & -a_{51} & a_{41} & -a_{31} & a_{21} & A_{16}
\end{array}
$$

where

$$2A_1 = -a_1 - a_2 - a_3 - a_4 - a_5 , \qquad 2A_2 = a_1 + a_2 - a_3 - a_4 - a_5 ,$$

$$2A_3 = a_1 - a_2 + a_3 - a_4 - a_5 , \qquad 2A_4 = a_1 - a_2 - a_3 + a_4 - a_5 ,$$

$$2A_5 = a_1 - a_2 - a_3 - a_4 + a_5 , \qquad 2A_6 = -a_1 + a_2 + a_3 - a_4 - a_5 ,$$

$$2A_7 = -a_1 + a_2 - a_3 + a_4 - a_5 , \qquad 2A_8 = -a_1 + a_2 - a_3 - a_4 + a_5 ,$$

$$2A_9 = -a_1 - a_2 + a_3 + a_4 - a_5 , \qquad 2A_{10} = -a_1 - a_2 + a_3 - a_4 + a_5 ,$$

$$2A_{11} = -a_1 - a_2 - a_3 + a_4 + a_5\,, \qquad 2A_{12} = a_1 + a_2 + a_3 + a_4 - a_5\,,$$
$$2A_{13} = a_1 + a_2 + a_3 - a_4 + a_5\,, \qquad 2A_{14} = a_1 + a_2 - a_3 + a_4 + a_5\,,$$
$$2A_{15} = a_1 - a_2 + a_3 + a_4 + a_5\,, \qquad 2A_{16} = -a_1 + a_2 + a_3 + a_4 + a_5\,,$$

and

$$\begin{aligned} x = x_1 &+ x_2e_1e_2 + x_3e_1e_3 + x_4e_1e_4 + x_5e_1e_5 + x_6e_2e_3 + x_7e_2e_4 + x_8e_2e_5 \\ &+ x_9e_3e_4 + x_{10}e_3e_5 + x_{11}e_4e_5 + x_{12}e_1e_2e_3e_4 + x_{13}e_1e_2e_3e_5 \\ &+ x_{14}e_1e_2e_4e_5 + x_{15}e_1e_3e_4e_5 + x_{16}e_2e_3e_4e_5\,. \end{aligned}$$

We identify $V(16)$ with $\boldsymbol{C}^{16}$ by an isomorphisms $x \mapsto {}^t(x_1, \cdots, x_{16}) \in \boldsymbol{C}^{16}$.

I) *The case of* $d = 1$.

Put $X_0 = 1 + e_1e_2e_3e_4 = {}^t(10 \cdots 010000) \in V(16)$. We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ of $\mathfrak{g} = \mathfrak{gl}(1) \oplus \mathfrak{o}(10, \boldsymbol{C})$ at $X_0$. From (5.38), we have $aX_0 + d\rho_1(A)X_0 = {}^t(a + A_1, b_{12} - c_{34}, b_{13} + c_{24}, b_{14} - c_{23}, b_{15}, b_{23} - c_{14}, b_{24} + c_{13}, b_{25}, b_{34} - c_{12}, b_{35}, b_{45}, a + A_{12}, a_{54}, -a_{53}, a_{52}, -a_{51}) \in V(16)$ where $(a, A) \in \mathfrak{gl}(1) \oplus \mathfrak{o}(10)$. Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(a, A) \mid aX_0 + d\rho_1(A)X_0 = 0\}$ is given as follows.

$$(5.39) \qquad \mathfrak{g}_{X_0} = \{(a) \oplus \left|\begin{array}{ccccc|ccccc} a_1 & a_{12} & a_{13} & a_{14} & a_{15} & 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ a_{21} & a_2 & a_{23} & a_{24} & a_{25} & -\alpha_1 & 0 & \beta_1 & \beta_2 & 0 \\ a_{31} & a_{32} & a_3 & a_{34} & a_{35} & -\alpha_2 & -\beta_1 & 0 & \beta_3 & 0 \\ a_{41} & a_{42} & a_{43} & a_4 & a_{45} & -\alpha_3 & -\beta_2 & -\beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \beta_3 & -\beta_2 & \beta_1 & c_{15} & -a_1 & -a_{21} & -a_{31} & -a_{41} & 0 \\ -\beta_3 & 0 & \alpha_3 & -\alpha_2 & c_{25} & -a_{12} & -a_2 & -a_{32} & -a_{42} & 0 \\ \beta_2 & -\alpha_3 & 0 & \alpha_1 & c_{35} & -a_{13} & -a_{23} & -a_3 & -a_{43} & 0 \\ -\beta_1 & \alpha_2 & -\alpha_1 & 0 & c_{45} & -a_{14} & -a_{24} & -a_{34} & -a_4 & 0 \\ -c_{15} & -c_{25} & -c_{35} & -c_{45} & 0 & -a_{15} & -a_{25} & -a_{35} & -a_{45} & -2a \end{array}\right| \; \Bigg|\; a_1 + a_2 + a_3 + a_4 = 0\}$$

Since $\dim \mathfrak{g}_{X_0} = 30 = \dim (GL(1) \times \mathit{Spin}\,(10)) - \dim V(16)$, this triplet is a P.V. Since $\mathfrak{g}_{X_0} \not\subset \mathfrak{sl}(V)$, there is no relative invariant by Proposition 20 in § 4. From (5.39) we have $\mathfrak{g}_{X_0} \cong (\mathfrak{gl}(1) \oplus \mathfrak{o}(7)) \oplus V(8)$ where $V(8)$ is the Lie algebra of the vector group of dimension eight. This space was investigated by J. Igusa (See [2]).

PROPOSITION 31. *A triplet* $(GL(1) \times \mathit{Spin}\,(10),\ \square \otimes$ *half-spin rep.*, $V(1) \otimes V(16))$ *is a P.V. and its generic isotropy subgroup is locally isomorphic to* $(GL(1) \times \mathit{Spin}\,(7)) \cdot (G_a)^8$. *There is no relative invariant and* *hence it is not regular.*

II) *The case of* $d = 2$.

We identify $V = V(16) \otimes V(2)$ with $16 \times 2$ matrices $M(16, 2)$. Then the action $d\rho$ of $\mathfrak{g} = \mathfrak{o}(10) \oplus \mathfrak{gl}(2)$ is given by $d\rho(A, D)X = d\rho_1(A)X + X{}^tD$ for $X \in V$, $(A, D) \in \mathfrak{g}$. Put

$$X_0 = (1 + e_1e_2e_3e_4,\ e_1e_5 + e_2e_3e_4e_5) = {}^t\begin{pmatrix} 10 \cdots 010000 \\ 000010 \cdots 01 \end{pmatrix} \in V\ .$$

Then from (5.38), we have

$$d\rho(A, D)X_0 = {}^t\left(\begin{array}{c|c|c|c|c|c} d_{11} + A_1 & b_{12} - c_{34} & b_{13} + c_{24} & b_{14} - c_{23} & d_{21} + b_{15} & b_{23} - c_{14} \\ d_{12} - c_{15} & a_{25} & a_{35} & a_{45} & d_{22} + A_5 & -c_{45} \end{array}\right.$$
$$\left|\begin{array}{c|c|c|c|c} b_{24} + c_{13} & b_{25} & b_{34} - c_{12} & b_{35} & b_{45} \\ c_{35} & a_{21} - c_{34} & -c_{25} & a_{31} + c_{24} & a_{41} - c_{23} \end{array}\right.$$
$$\left|\left.\begin{array}{c|c|c|c|c} d_{11} + A_{12} & a_{54} & -a_{53} & a_{52} & d_{21} - a_{51} \\ d_{12} - a_{15} & b_{23} + a_{14} & b_{24} - a_{13} & b_{34} + a_{12} & d_{22} + A_{16} \end{array}\right)\right.$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(A, D) \mid d\rho(A, D)X_0 = 0\}$ is given as follows.

(5.40)
$$\mathfrak{g}_{X_0} = \left\{ \left|\begin{array}{ccccc|ccccc} 0 & a_{12} & a_{13} & a_{14} & d_{12} & 0 & a_{21} & a_{31} & a_{41} & -d_{21} \\ a_{21} & a_2 & a_{23} & a_{24} & 0 & -a_{21} & 0 & -a_{14} & a_{13} & 0 \\ a_{31} & a_{32} & a_3 & a_{34} & 0 & -a_{31} & a_{14} & 0 & -a_{12} & 0 \\ a_{41} & a_{42} & a_{43} & a_4 & 0 & -a_{41} & -a_{13} & a_{12} & 0 & 0 \\ d_{21} & 0 & 0 & 0 & 2d_{11} & d_{21} & 0 & 0 & 0 & 0 \\ \hline 0 & -a_{12} & -a_{13} & -a_{14} & d_{12} & 0 & -a_{21} & -a_{31} & -a_{41} & -d_{21} \\ a_{12} & 0 & a_{41} & -a_{31} & 0 & -a_{12} & -a_2 & -a_{32} & -a_{42} & 0 \\ a_{13} & -a_{41} & 0 & a_{21} & 0 & -a_{13} & -a_{23} & -a_3 & -a_{43} & 0 \\ a_{14} & a_{31} & -a_{21} & 0 & 0 & -a_{14} & -a_{24} & -a_{34} & -a_4 & 0 \\ -d_{12} & 0 & 0 & 0 & 0 & -d_{12} & 0 & 0 & 0 & -2d_{11} \end{array}\right| \right.$$
$$\left. \oplus \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & -d_{11} \end{pmatrix} \right\} \quad \text{with } a_2 + a_3 + a_4 = 0\ .$$

Since $\dim \mathfrak{g}_{X_0} = 17 = \dim Spin\,(10) \times GL(2) - \dim V(16) \otimes V(2)$, it is a P.V. Let $A$ be the $10 \times 10$ matrix in (5.40) and put $S$ in $GL(10)$ as follows.

$$(5.41)\quad S^{-1} = \left(\begin{array}{c|c|c|c|c|c} & & 1 & & & \\ \hline & & & & & 1 \\ \hline \frac{1}{2} & & & \frac{1}{2} & & \\ \hline -1 & & & 1 & & \\ \hline & I_3 & & & & \\ \hline & & & & I_3 & \end{array}\right), \qquad S = \left(\begin{array}{c|c|c|c|c|c} & & 1 & -\frac{1}{2} & & \\ \hline & & & & I_3 & \\ \hline 1 & & & & & \\ \hline & & 1 & \frac{1}{2} & & \\ \hline & 1 & & & & I_3 \end{array}\right)$$

where $I_3$ is the identity matrix of size three. Then by simple calculation, we have

$$(5.42)\quad S^{-1}AS = \left(\begin{array}{ccc|ccccccc} 2d_{11} & 0 & 2d_{21} & & & & & & & \\ 0 & -2d_{11} & -2d_{12} & & & & 0 & & & \\ d_{12} & -d_{21} & 0 & & & & & & & \\ \hline & & & 0 & -2a_{12} & -2a_{13} & -2a_{14} & -2a_{21} & -2a_{31} & -2a_{41} \\ & & & -a_{21} & a_2 & a_{23} & a_{24} & 0 & -a_{14} & a_{13} \\ & & & -a_{31} & a_{32} & a_3 & a_{34} & a_{14} & 0 & -a_{12} \\ & 0 & & -a_{41} & a_{42} & a_{43} & a_4 & -a_{13} & a_{12} & 0 \\ & & & -a_{12} & 0 & a_{41} & -a_{31} & -a_2 & -a_{32} & -a_{42} \\ & & & -a_{13} & -a_{41} & 0 & a_{21} & -a_{23} & -a_3 & -a_{43} \\ & & & -a_{14} & a_{31} & -a_{21} & 0 & -a_{24} & -a_{34} & -a_4 \end{array}\right)$$

with $a_2 + a_3 + a_4 = 0$.

Therefore we have $\mathfrak{g}_{X_0} \cong (\mathfrak{g}_2) \oplus \mathfrak{sl}(2)$ (See (1.8) in § 1), and hence this triplet is regular by Proposition 25 in § 4. Put

$$X_0' = (1 + e_1e_2e_3e_4, e_1e_2 + e_2e_3e_4e_5) = {}^t\begin{pmatrix} 10 \cdots 010000 \\ 010 \cdots\cdots 01 \end{pmatrix} \in V\,.$$

Then from (5.38), we have

$$d\rho_1(A)X_0' + X_0'{}^tD = {}^t\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c} A_1 + d_{11} & b_{12} - c_{34} + d_{12} & b_{13} + c_{24} & b_{14} - c_{23} & b_{15} & b_{23} - c_{14} & b_{24} + c_{13} & b_{25} & b_{34} - c_{12} & b_{35} & b_{45} & A_{12} + d_{11} & a_{54} & -a_{53} & a_{52} & d_{12} - a_{51} \\ d_{21} - c_{12} & A_2 + d_{22} & a_{32} & a_{42} & a_{52} & -a_{31} - c_{45} & c_{35} - a_{41} & -a_{51} - c_{34} & -c_{25} & c_{24} & -c_{23} & b_{34} - a_{15} + d_{21} & b_{35} + a_{14} & b_{45} - a_{13} & a_{12} & d_{22} + A_{16} \end{array}\right)$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_0'}$ at $X_0'$ is given as follows.

$$
(5.43)\qquad \mathfrak{g}_{X_0'} = \left\{ \left(\begin{array}{c|c|c|c|c|c|c|c|c|c}
d_{11}+d_{22} & 0 & 0 & 0 & 2c_{12} & 0 & 2c_{34} & 0 & 0 & 0 \\
a_{21} & -2d_{22} & a_{23} & a_{24} & a_{25} & -2c_{34} & 0 & c_{14} & -c_{13} & 0 \\
-c_{45} & 0 & a_3 & a_{34} & a_{35} & 0 & -c_{14} & 0 & c_{12} & 0 \\
c_{35} & 0 & a_{43} & a_4 & a_{45} & 0 & c_{13} & -c_{12} & 0 & 0 \\
-c_{34} & 0 & 0 & 0 & 2d_{11} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & c_{12} & c_{13} & c_{14} & c_{15} & -d_{11}-d_{22} & -a_{21} & c_{45} & -c_{35} & c_{34} \\
-c_{12} & 0 & 0 & 0 & 0 & 0 & 2d_{22} & 0 & 0 & 0 \\
-c_{13} & 0 & 0 & c_{34} & c_{35} & 0 & -a_{23} & -a_3 & -a_{43} & 0 \\
-c_{14} & 0 & -c_{34} & 0 & c_{45} & 0 & -a_{24} & -a_{34} & -a_4 & 0 \\
-c_{15} & 0 & -c_{35} & -c_{45} & 0 & -2c_{12} & -a_{25} & -a_{35} & -a_{45} & -2d_{11}
\end{array}\right) \oplus \begin{pmatrix} d_{11} & -c_{34} \\ c_{12} & d_{22} \end{pmatrix} \right\} \quad \text{with } a_3 + a_4 = d_{22} - d_{11}
$$

Since $\dim \mathfrak{g} - \dim \mathfrak{g}_{X_0'} = 49 - 18 = 31$, the orbit of $X_0'$ is of codimension one. For $(A, D) \in \mathfrak{g}_{X_0'}$, the trace $\mathrm{tr}_V(A, D)$ in $V$ is $16(d_{11} + d_{22})$ and the trace $\mathrm{tr\,ad}\,\mathfrak{g}_{X_0'} A$ of the adjoint representation is by simple calculation $-14(d_{11} + d_{22})$ and hence

$$
\deg f = \frac{16(d_{11} + d_{22}) - 14(d_{11} + d_{22})}{16(d_{11} + d_{22})} \times 32 = 4
$$

by Proposition 15 in § 4. The explicit form of this irreducible relative invariant $f(x)$ of degree 4 is given by Kawahara (See [13]).

PROPOSITION 32. *A triplet* $(Spin(10) \times GL(2)$, *half-spin rep.* $\otimes \Lambda_1$, $V(16) \otimes V(2))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $(G_2) \times SL(2)$. *The relative invariant is a quartic form.*

III) *The case of* $d = 3$.

We identify $V = V(16) \otimes V(3)$ with $16 \times 3$ matrices $M(16, 3)$. Then the action $d\rho$ of $\mathfrak{g} = \mathfrak{o}(10) \oplus \mathfrak{gl}(3)$ is given by $d\rho(A, D)X = d\rho_1(A)X + X{}^tD$ for $(A, D) \in \mathfrak{g}$, $X \in M(16, 3)$.

Put

$$
\begin{aligned}
X_0 &= (1 + e_1e_2e_3e_4,\ e_1e_5 + e_2e_3e_4e_5,\ e_1e_2 + e_1e_3e_4e_5) \\
&= {}^t\begin{pmatrix} 10 \cdots 010000 \\ 000010 \cdots 01 \\ 010 \cdots\cdots 010 \end{pmatrix} \in M(16, 3)\,.
\end{aligned}
$$

We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0$. By (5.38) we have

$$
\begin{aligned}
d\rho(A, D)X_0 &= d\rho_1(A)X_0 + X_0{}^tD \\
&= {}^t\left(\begin{array}{c|c|c|c|c}
A_1 + d_{11} & b_{12} - c_{34} + d_{13} & b_{13} + c_{24} & b_{14} - c_{23} & b_{15} + d_{12} \\
d_{21} - c_{15} & a_{25} + d_{23} & a_{35} & a_{45} & A_5 + d_{22} \\
d_{31} - c_{12} & A_2 + d_{33} & a_{32} - c_{45} & a_{42} + c_{35} & a_{52} - c_{34} + d_{32}
\end{array}\right. \\
&\qquad \left|\begin{array}{c|c|c|c|c|c}
b_{23} - c_{14} & b_{24} + c_{13} & b_{25} & b_{34} - c_{12} & b_{35} & b_{45} \\
-c_{45} & c_{35} & a_{21} - c_{34} & -c_{25} & a_{31} + c_{24} & a_{41} - c_{23} \\
-a_{31} & -a_{41} & -a_{51} & -c_{15} & c_{14} & -c_{13}
\end{array}\right. \\
&\qquad \left|\left.\begin{array}{c|c|c|c|c}
A_{12} + d_{11} & a_{54} & -a_{53} & d_{13} + a_{52} & d_{12} - a_{51} \\
d_{21} - a_{15} & b_{23} + a_{14} & b_{24} - a_{13} & b_{34} + a_{12} + d_{23} & A_{16} + d_{22} \\
b_{34} + a_{25} + d_{31} & b_{35} - a_{24} & b_{45} + a_{23} & A_{15} + d_{33} & d_{32} + a_{21}
\end{array}\right)\right.
\end{aligned}
$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(A, D) \mid d\rho(A, D)X_0 = 0\}$ is given as follows.

$$
\tag{5.44}
\mathfrak{g}_{X_0} = \left\{
\left|\begin{array}{c|c|c|c|c|c|c|c|c|c|}
 & -3c_{12} & & & & & 3c_{34} & & & \\
c_{34} & d_{11} & & & -2c_{12} & -3c_{34} & & & & \\
 & & a_3 & a_{34} & & & & & c_{12} & \\
 & & a_{43} & -a_3 & & & & -c_{12} & & \\
 & 2c_{34} & & & 2d_{11} & & & & & \\
\hline
 & c_{12} & & & & & -c_{34} & & & \\
-c_{12} & & & & & 3c_{12} & -d_{11} & & & \\
 & & & c_{34} & & & & -a_3 & -a_{43} & \\
 & & -c_{34} & & & & & -a_{34} & a_3 & \\
 & & & & & & & & & -2d_{11}
\end{array}\right.
\oplus
\left[\begin{array}{c|c|c}
d_{11} & & -2c_{34} \\
 & -d_{11} & 2c_{12} \\
c_{12} & -c_{34} &
\end{array}\right]
\right\}
$$

$$
\cong \left\{\begin{pmatrix} a_3 + d_{11}/2 & a_{34} \\ a_{43} & -a_3 - d_{11}/2 \end{pmatrix} \oplus \begin{bmatrix} d_{11} & & -2c_{34} \\ & -d_{11} & 2c_{12} \\ c_{12} & -c_{34} & 0 \end{bmatrix}\right\} = \mathfrak{sl}(2) \oplus \mathfrak{o}(3) .
$$

Since $\dim \mathfrak{g}_{X_0} = 6 = \dim Spin(10) \times GL(3) - \dim V(16) \otimes V(3)$, it is a P.V. and by Proposition 25 in §4, it is regular. Put

$$
\begin{aligned}
X_0' &= (1 + e_1e_2e_3e_4,\ e_1e_5 + e_2e_3e_4e_5,\ e_1e_2 + e_3e_5 + e_1e_2e_4e_5) \\
&= {}^t\begin{pmatrix} 10 \cdots\cdots 010000 \\ 000010 \cdots\cdots 01 \\ 010 \cdots 01000100 \end{pmatrix} \in M(16, 3) .
\end{aligned}
$$

By (5.38), we have

$$
\begin{aligned}
d\rho(A, D)X_0' &= d\rho_1(A)X_0' + X_0'{}^tD \\
&= {}^t\left(\begin{array}{l}
\begin{array}{c|c|c|c}
A_1 + d_{11} & b_{12} - c_{34} + d_{13} & b_{13} + c_{24} & b_{14} - c_{23} \\
d_{21} - c_{15} & a_{25} + d_{23} & a_{35} & a_{45} \\
d_{31} - c_{12} - c_{35} & A_2 - c_{45} + d_{33} & a_{32} - a_{15} & a_{42} + c_{25}
\end{array} \\[1ex]
\begin{array}{|c|c|c|c}
b_{15} + d_{12} & b_{23} - c_{14} & b_{24} + c_{13} & b_{25} \\
A_5 + d_{22} & -c_{45} & c_{35} & a_{21} - c_{34} \\
a_{52} + a_{13} - c_{24} + d_{32} & -a_{31} - a_{15} & -a_{41} - c_{15} & a_{23} + c_{14} - a_{51}
\end{array} \\[1ex]
\begin{array}{|c|c|c|c}
b_{34} - c_{12} & b_{35} + d_{13} & b_{45} & A_{12} + d_{11} \\
-c_{25} & a_{31} + c_{24} + d_{23} & a_{41} - c_{23} & d_{21} - a_{15} \\
a_{45} & A_{10} + d_{33} & a_{43} - c_{12} & b_{34} + d_{31} - a_{35}
\end{array} \\[1ex]
\begin{array}{|c|c|c|c}
a_{54} & d_{13} - a_{53} & a_{52} & d_{12} - a_{51} \\
b_{23} + a_{14} & b_{24} - a_{13} + d_{23} & a_{12} + b_{34} & A_{16} + d_{22} \\
b_{35} + b_{12} + a_{34} & b_{45} + A_{14} + d_{33} & a_{32} - b_{14} & d_{32} - b_{24} - a_{31}
\end{array}
\end{array}\right)
\end{aligned}
$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_0'} = \{(A, D) \mid d\rho(A, D)X_0' = 0\}$ is given as follows.

$$
(5.45) \qquad \mathfrak{g}_{X_0'} = \left\{ \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
0 & 0 & -\alpha_3 & -\alpha_2 & 0 & 0 & \beta_2 & -\alpha_3 & 0 & -\alpha_1 \\
\beta_1 + \beta_2 & 2d & \alpha_1 - \alpha_2 & \beta_3 & -\alpha_3 & -\beta_2 & 0 & \alpha_2 & -2\alpha_3 & 0 \\
\alpha_3 & 0 & 0 & \beta_1 - \beta_2 & 0 & \alpha_3 & -\alpha_2 & 0 & 0 & -\beta_1 \\
0 & 0 & 0 & -2d & 0 & 0 & 2\alpha_3 & 0 & 0 & 0 \\
\alpha_1 & 0 & \beta_1 & 0 & 2d & \alpha_1 & 0 & \beta_1 & 0 & 0 \\
\hline
0 & 0 & 2\alpha_3 & \alpha_2 & 0 & 0 & -\beta_1 - \beta_2 & -\alpha_3 & 0 & -\alpha_1 \\
0 & 0 & 0 & -2\alpha_3 & 0 & 0 & -2d & 0 & 0 & 0 \\
-2\alpha_3 & 0 & 0 & \beta_1 + \beta_2 & 0 & \alpha_3 & \alpha_2 - \alpha_1 & 0 & 0 & -\beta_1 \\
-\alpha_2 & 2\alpha_3 & -\beta_1 - \beta_2 & 0 & 0 & \alpha_2 & -\beta_3 & \beta_2 - \beta_1 & 2d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & -2d
\end{array} \right.
$$

$$
\left. \oplus \begin{bmatrix} d & \alpha_1 & \beta_1 \\ 0 & -d & \alpha_3 \\ 0 & -\alpha_3 & -d \end{bmatrix} \right\}
$$

Since $\dim Spin(10) \times GL(3) - \dim \mathfrak{g}_{X_0'} = 54 - 7 = 47$, the orbit of $X_0'$ is of codimension one, and the trace $\mathrm{tr}_V(A, D)$ on $V$ is $16(d - d - d) = -16d$ and the trace $\mathrm{ad}\,\mathfrak{g}_{X_0'}(A, D)$ of the adjoint representation is $12d$. By Proposition 15 in § 4, the degree of the irreducible relative invariant polynomial $f(x)$ is given by $\deg f = ((-16d + 12d)/-16d) \times 48 = 12$. The orbital decomposition of this space is completed by Kawahara (See [13]).

PROPOSITION 33. *A triplet* $(Spin(10) \times GL(3)$, *half-spin rep.* $\otimes \Lambda_1$, $V(16) \otimes V(3))$ *is a regular P.V. and its generic isotropy subgroup is locally* *isomorphic to* $SL(2) \times O(3)$. *The relative invariant is of degree* 12.

(26) $(Spin(10) \times GL(2),\ \text{half-spin rep.} \otimes 2\Lambda_1,\ V(16) \otimes V(3))$ .

Assume that this triplet is a P.V. and let $H$ be its generic isotropy subgroup. Then we have $\dim H = \dim Spin(10) \times GL(2) - \dim V(16) \otimes V(3) = 1$. Since we may consider that this triplet is contained in $(Spin(10) \times GL(3),\ \text{half-spin rep.} \otimes \Lambda_1,\ V(16) \otimes V(3))$, we have $H \supset Spin(10) \times GL(2) \cap SL(2) \times SO(3) \supset SL(2)$ by (5.44) and hence $\dim H \geqq \dim SL(2) = 3$, i.e., a contradiction. Thus we obtain the following proposition.

PROPOSITION 34. *A triplet* $(Spin(10) \times GL(2),$ *half-spin rep.* $\otimes 2\Lambda_1,$ $V(16) \otimes V(3))$ *is not a P.V.*

(27) $(Spin(9) \times GL(d),$ *spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(d))$ $\quad (1 \leq d \leq 2)$

Put $v_1 = e_5 - f_5 = {}^t(000010000\text{–}1)$. Then $\mathfrak{o}(9, \boldsymbol{C}) = \{A \in \mathfrak{o}(10, \boldsymbol{C}) \mid Av_1 = 0\}$. We may assume that an element $A$ of $\mathfrak{o}(10, \boldsymbol{C})$ is of the form (5.28). Then $A$ is in $\mathfrak{o}(9, \boldsymbol{C})$ if and only if $Av_1 = {}^t(a_{15} + b_{15},\ a_{25} + b_{25},\ a_{35} + b_{35},\ a_{45} + b_{45},\ a_5,\ c_{15} - a_{51},\ c_{25} - a_{52},\ c_{35} - a_{53},\ c_{45} - a_{54},\ -a_5) = 0$ i.e. $a_{i5} = -b_{i5}$, $a_{5i} = c_{i5}$, $a_5 = 0$ $(1 \leq i \leq 4)$.

I) *The case of* $d = 1$.

Put $X_0 = 1 + e_1e_2e_3e_4 = (10 \cdots 010000) \in V(16)$. Then by (5.38) we have $aX_0 + d\rho_1(A)X_0 = {}^t(a - (a_1 + a_2 + a_3 + a_4)/2,\ b_{12} - c_{34},\ b_{13} + c_{24},\ b_{14} - c_{23},\ b_{15},\ b_{23} - c_{14},\ b_{24} + c_{13},\ b_{25},\ b_{34} - c_{12},\ b_{35},\ b_{45},\ a + (a_1 + a_2 + a_3 + a_4)/2,\ a_{54},\ -a_{53},\ a_{52},\ -a_{51})$ with $a_5 = 0$, $a_{i5} = -b_{i5}$, $a_{5i} = c_{i5}$ $(1 \leq i \leq 4)$ and $(a, A) \in \mathfrak{gl}(1) \oplus \mathfrak{o}(9)$. Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(a, A) \mid aX_0 + d\rho_1(A)X_0 = 0\}$ is given as follows.

(5.46)
$$\mathfrak{g}_{X_0} = \{(0) \oplus \left|\begin{array}{cccc|c|cccc|c} a_1 & a_{12} & a_{13} & a_{14} & 0 & 0 & b_{12} & b_{13} & b_{14} & 0 \\ a_{21} & a_2 & a_{23} & a_{24} & 0 & -b_{12} & 0 & b_{23} & b_{24} & 0 \\ a_{31} & a_{32} & a_3 & a_{34} & 0 & -b_{13} & -b_{23} & 0 & b_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_4 & 0 & -b_{14} & -b_{24} & -b_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & b_{34} & -b_{24} & b_{23} & 0 & -a_1 & -a_{21} & -a_{31} & -a_{41} & 0 \\ -b_{34} & 0 & b_{14} & -b_{13} & 0 & -a_{12} & -a_2 & -a_{32} & -a_{42} & 0 \\ b_{24} & -b_{14} & 0 & b_{12} & 0 & -a_{13} & -a_{23} & -a_3 & -a_{43} & 0 \\ -b_{23} & b_{13} & -b_{12} & 0 & 0 & -a_{14} & -a_{24} & -a_{34} & -a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

with $a_1 + a_2 + a_3 + a_4 = 0\} \cong \mathfrak{o}(7, \boldsymbol{C})$ .

Since $\dim \mathfrak{g}_{X_0} = 21 = \dim GL(1) \times Spin(9) - \dim V(1) \otimes V(16)$, it is a P.V. and by Proposition 25 in §4, it is regular. This space was investigated by J. Igusa (See [2]).

The relative invariant is a quadratic form $q(x)$, and the explicit form which is due to J. Igusa, is given as follows.

$$q(x) = x_0 y_0 + Pff(x_{ij}) + \sum_i x_i y_i \tag{5.47}$$

where

$$x = x_0 + \sum_{i<j\leq 4} x_{ij} e_i e_j + y_0 e_1 e_2 e_3 e_4 + \left( \sum_i x_i e_i + \sum_i y_i e_i^* \right) e_5$$

and

$$e_i e_i^* = e_1 e_2 e_3 e_4 e_5 \qquad (1 \leq i \leq 4)\ .$$

PROPOSITION 35. *A triplet* $(GL(1) \times Spin(9),\ \square \otimes$ *spin rep.*, $V(1) \otimes V(16))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $Spin(7)$. *The relative invariant is a quadratic form.*

II) *The case of* $d = 2$.

Assume that this triplet is a P.V. and let $H$ be its generic isotropy subgroup. Then we have $\dim H = \dim Spin(9) \times GL(2) - \dim V(16) \otimes V(2) = 8$. This implies that for a suitable $x_0$ in $V(10)$, the isotropy subalgebra $\mathfrak{g}_{x_0} = \{A \in \mathfrak{g} \subset \mathfrak{o}(10) \mid Ax_0 = 0\}$ of $\mathfrak{g} \cong (\mathfrak{g}_2) \oplus \mathfrak{sl}(2)$ which consists of $10 \times 10$ matrices $A$ in (5.40), is of dimension eight. Therefore there exists a 9-dimensional orbit of $\mathfrak{g}'$ which consists of $10 \times 10$ matrices in (5.42), since $\dim \mathfrak{g}' - 8 = 9$. However, it is impossible since there exist algebraically independent two quadratic forms $f_1(x)$ and $f_2(x)$ where $f_1(x) = f_1(x_1, x_2, x_3)$ is absolutely invariant under the action of $\mathfrak{sl}(2)$ and $f_2(x) = f_2(x_4, \cdots, x_{10})$ is absolutely invariant under the action of $(\mathfrak{g}_2)$ (See Proposition 25 or (32)). Hence it is not a P.V.

PROPOSITION 36. *A triplet* $(Spin(9) \times GL(2),$ *spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(2))$ *is not a P.V.*

(28) $(Spin(12) \times GL(d),$ *half-spin rep.* $\otimes \Lambda_1,\ V(32) \otimes V(d))$ $(1 \leq d \leq 2)$

I) *The case of* $d = 1$.

The representation space $V = V(1) \otimes V(32)$ is spanned by $1$, $e_i e_j$, $e_k e_\ell e_m e_n$, $e_1 e_2 e_3 e_4 e_5 e_6$ $(1 \leq i < j \leq 6,\ 1 \leq k < \ell < m < n \leq 6)$. Put $X_0 = 1$

$+ e_1e_2e_3e_4e_5e_6$. We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ of $\mathfrak{g} = \mathfrak{gl}(1) \oplus \mathfrak{o}(12)$ at $X_0$. We may assume that an element $A$ of $\mathfrak{o}(12)$ is of the form (5.28)..

Then by (5.29), we have $d\rho_1(A)\cdot 1 = -\frac{1}{2}(a_1 + \cdots + a_6) + \sum_{i<j} b_{ij}e_ie_j$ and $d\rho_1(A)\, e_1e_2e_3e_4e_5e_6 = \frac{1}{2}(a_1 + \cdots + a_6)\, e_1e_2e_3e_4e_5e_6 + \sum_{i<j} c_{ij}\delta_{fi}\delta_{fj}e_1e_2e_3e_4e_5e_6$ and hence $aX_0 + d\rho_1(A)X_0 = (a - \frac{1}{2}(a_1 + \cdots + a_6)) + \sum_{i<j} b_{ij}e_ie_j + \sum_{i<j} c_{ij}\delta_{fi}\delta_{fj}e_1e_2e_3e_4e_5e_6 + (a + \frac{1}{2}(a_1 + \cdots + a_6))\, e_1e_2e_3e_4e_5e_6$ for $(a, A) \in \mathfrak{gl}(1) \oplus \mathfrak{o}(12)$. Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(a, A) \mid aX_0 + d\rho_1(A)X_0 = 0\}$ is given by

$$\mathfrak{g}_{X_0} = \left\{(0) \oplus \left[\begin{array}{c|c} A & 0 \\ \hline 0 & -{}^tA \end{array}\right] \middle| A \in \mathfrak{sl}(6)\right\} \cong \mathfrak{sl}(6)\ .$$

Since $\dim \mathfrak{g}_{X_0} = 35 = \dim GL(1) \times Spin(12) - \dim V(1) \otimes V(32)$, it is a P.V. and by Proposition 25 in § 4, it is regular. This space was investigated by J. Igusa and the explicit form of the relative invariant quartic form $f(x)$ is given as follows (See [2]).

$$f(x) = x_0 Pff((y_{ij})) + y_0 Pff((x_{ij})) + \sum_{i<j} Pff(X_{ij})Pff(Y_{ij}) \tag{5.48}$$
$$-\frac{1}{4}\Big(x_0y_0 - \sum_{i<j} x_{ij}y_{ij}\Big)^2$$

for $x = x_0 + \sum_{i<j} x_{ij}e_ie_j + \sum_{i<j} y_{ij}e^*_{ij} + y_0e_1e_2e_3e_4e_5e_6$ in which, e.g., $(x_{ij})$ is the alternating matrix determined by $x_{ij}$ and $X_{ij}$ the alternating matrix obtained from $(x_{ij})$ by crossing out its $i$-th and $j$-th lines and columns, and $e^*_{ij} = (-1)^{i+j-1}e_1 \cdots e_{i-1}e_{i+1} \cdots e_{j-1}e_{j+1} \cdots e_6$.

PROPOSITION 37. *A triplet $(GL(1) \times Spin(12)$, $\square \otimes$ half-spin rep., $V(1) \otimes V(32))$ is a regular P.V. and its generic isotropy subgroup is locally isomorphic to $SL(6)$. The relative invariant is a quartic form.*

II) *The case of $d = 2$.*

We identify $V = V(32) \otimes V(2)$ with $V(32) \oplus V(32)$. Assume that this triplet is a P.V. and let $X_0 = (x_0', x_0'')$ is a generic point. Then the isotropy subgroup $H$ at $X_0$ is of dimension $6\,(= \dim Spin(12) \times GL(2) - \dim V(32) \otimes V(2))$. From ths case of $d = 1$, we may assume that $x_0' = 1 + e_1e_2e_3e_4e_5e_6$. As a representation space of the isotropy subgroup $SL(6)$ at $x_0'$, $V(32)$ decomposes into $V(32) = V_1(1) \oplus V_2(1) \oplus V_3(15) \oplus V_4(15)$ i.e. $1 \oplus 1 \oplus \Lambda_2 \oplus \Lambda_4$ since the weights $\frac{1}{2}(\pm a_1 \pm a_2 \cdots \pm a_6)$ of $\mathfrak{sl}(6)$ where $a_1 + \cdots + a_6 = 0$ and the number of the signature $+$ is even, decomposes to $\{0\} \cup \{0\} \cup \{a_i + a_j \mid 1 \le i < j \le 6\} \cup \{-a_i - a_j \mid 1 \le i < j \le 6\}$. Note that $\Lambda_4$

is the contragredient representation of $\Lambda_2$. We may assume that $x_0 \in V_1(1)$ and $x'_0 = x'_2 + x'_3 + x'_4$ with $x'_2 \in V_2(1)$, $x'_3 \in V_3(15)$, $x'_4 \in V_4(15)$. Then together with scalar multiplications, $SL(2)$ acts on $V_3(15)$ prehomogeneously and we may assume that $x'_3$ is its generic point. Then the isotropy subgroup at $x'_3$ is $Sp(3)$ by Proposition 4. As a representation space of $Sp(3)$, $V_4(15)$ decomposes to $V_4(15) = V_5(1) \oplus V_6(14)$ i.e. $1 \oplus \Lambda_2$, since $\{-a_i - a_j \mid 1 \leq i < j \leq 6\}$ with $a_4 = -a_1$, $a_5 = -a_2$, $a_6 = -a_3$ decomposes to $\{0\} \cup \{0, 0, \pm a_i \pm a_j,\ i < j = 1, 2, 3\}$. An element of $V_6(14)$ is generically transferred to the form $\lambda_1 u_1 \wedge u_4 + \lambda_2 u_2 \wedge u_5 + \lambda_3 u_3 \wedge u_6$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, under the action of $Sp(3)$, and the isotropy subgroup at this point contains $SL(2) \times SL(2) \times SL(2)$. This implies that the generic isotropy subgroup $H$ at $X_0 = (x'_0, x''_0)$ contains $SL(2) \times SL(2) \times SL(2)$ and hence $6 = \dim H \geqq \dim SL(2) \times SL(2) \times SL(2) = 9$, i.e., a contradiction. Therefore it is not a P.V.

PROPOSITION 38. *A triplet* $(Spin(12) \times GL(2)$, *half-spin rep.* $\otimes \Lambda_1$, $V(32) \otimes V(2))$ *is not a P.V.*

(29) $(GL(1) \times Spin(11),\ \square \otimes \textit{spin rep.},\ V(1) \otimes V(32))$ .

Put $v_1 = e_6 - f_6 = {}^t(00000100000 - 1) \in V(12)$. Then we have $\mathfrak{o}(11, \boldsymbol{C}) = \{A \in \mathfrak{o}(12, \boldsymbol{C}) \mid Av_1 = 0\}$ and hence an element $A$ of the form (5.28) is in $\mathfrak{o}(11, \boldsymbol{C})$ if and only if $a_6 = 0$, $a_{i6} = b_{i6}$ and $a_{6i} = -c_{i6}$ for $1 \leq i \leq 5$. Therefore the isotropy subalgebra $\mathfrak{g}_{X_0}$ of $\mathfrak{g} = \mathfrak{gl}(1) \oplus \mathfrak{o}(11)$ at $X_0 = 1 + e_1e_2e_3e_4e_5e_6$ is given by

$$\mathfrak{g}_{X_0} = \mathfrak{o}(11) \cap \left\{ \left( \begin{array}{c|c} A' & \\ \hline & -{}^tA' \end{array} \right) \middle| A' \in \mathfrak{sl}(6) \right\} = \left\{ \left( \begin{array}{c|c} A' & \\ \hline & -{}^tA' \end{array} \right) \middle| A' \in \mathfrak{sl}(5) \right\} \cong \mathfrak{sl}(5)$$

by Proposition 37. Since $\dim \mathfrak{g}_{X_0} = 24 = \dim GL(1) \times Spin(11) - \dim V(1) \otimes V(32)$, this triplet is a P.V. and it is regular by Proposition 25 in § 4. The relative invariant is the quartic form in (5.48). This space was investigated by J. Igusa (See [2]).

PROPOSITION 39. *A triplet* $(GL(1) \times Spin(11)$, $\square \otimes$ *spin rep.*, $V(1) \otimes V(32))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $SL(5)$. *The relative invariant is the same quartic form as in Proposition* 37.

(30) $(GL(1) \times Spin(14),\ \square \otimes \textit{half-spin rep.},\ V(1) \otimes V(64))$

The representation space $V = V(1) \otimes V(64)$ is spanned by 1, $e_ie_j$ $(1 \leq$

$i < j \leq 7)$, $e_ie_je_ke_\ell$ $(1 \leq i < j < k < \ell \leq 7)$ and $e_ie_je_ke_\ell e_me_n$ $(1 \leq i < j < k < \ell < m < n \leq 7)$. Put $X_0 = 1 + e_1e_2e_3e_7 + e_4e_5e_6e_7 + e_1e_2e_3e_4e_5e_6$. We shall calculate the isotropy subalgebra $\mathfrak{g}_{X_0}$ of $\mathfrak{g} = \mathfrak{gl}(1) \oplus \mathfrak{o}(14)$ at $X_0$. We may assume that an element $A$ of $\mathfrak{o}(14, \boldsymbol{C})$ is of the form (5.28). Let $d\rho_1$ be the half-spin representation of $\mathfrak{o}(14, \boldsymbol{C})$. Then by (5.29) we have

$$\begin{aligned} d\rho_1(A)\cdot 1 &= -\tfrac{1}{2}(a_1 + \cdots + a_7) + \sum_{i<j} b_{ij}e_ie_j \,, \\ d\rho_1(A)e_1e_2e_3e_7 &= \sum_{i=4}^{6} a_{i1}e_2e_3e_ie_7 - \sum_{i=4}^{6} a_{i2}e_1e_3e_ie_7 + \sum_{i=4}^{6} a_{i3}e_1e_2e_ie_7 + \sum_{i=4}^{6} a_{i7}e_1e_2e_3e_i \\ &\quad + \tfrac{1}{2}(a_1 + a_2 + a_3 - a_4 - a_5 - a_6 + a_7)e_1e_2e_3e_7 \\ &\quad + \sum_{4 \leq i < j \leq 6} e_1e_2e_3e_ie_je_7 - c_{17}e_2e_3 + c_{27}e_1e_3 - c_{37}e_1e_2 \\ &\quad - (c_{23}e_1 - c_{13}e_2 + c_{12}e_3)e_7 \,, \end{aligned}$$

and

$$\begin{aligned} d\rho_1(A)e_1e_2 \cdots e_6 &= \sum_{j=1}^{6} (-1)^{j-1}a_{7j}e_1 \cdots e_{j-1}e_{j+1} \cdots e_7 \\ &\quad + \tfrac{1}{2}(a_1 + \cdots + a_6 - a_7)e_1e_2 \cdots e_6 \\ &\quad + \sum_{i<j\leq 6} (-1)^{i+j}e_1 \cdots e_{i-1}e_{i+1} \cdots e_{j-1}e_{j+1} \cdots e_6 \,. \end{aligned}$$

By changing indices from $(1, \cdots, 7)$ to $(4, 5, 6, 1, 2, 3, 7)$, we obtain $d\rho_1(A)$ $e_4e_5e_6e_7$ from $d\rho_1(A)e_1e_2e_3e_7$. Hence the isotropy subalgebra $\mathfrak{g}_{X_0} = \{(a, A) \in \mathfrak{gl}(1) \oplus \mathfrak{o}(14) \mid aX_0 + d\rho_1(A)X_0 = 0\}$ is given by

$$\begin{aligned} \mathfrak{g}_{X_0} &= \left\{(0) \oplus \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & -{}^tA_1 \end{array}\right) \in \mathfrak{gl}(1) \oplus \mathfrak{o}(14) \mid A_1, A_2, A_3 \text{ are of the form (5.49)}\right\} \\ &\cong \{(A_4) \oplus (A_5) \mid A_4, A_5 \text{ are of the form (5.49)}\} \cong (\mathfrak{g}_2) \oplus (\mathfrak{g}_2) \end{aligned}$$

(See (1.8) in § 1).

$$A_1 = \left(\begin{array}{ccc|ccc|c} & & & & & & a \\ & X & & & 0 & & b \\ & & & & & & c \\ \hline & & & & & & d \\ & 0 & & & Y & & e \\ & & & & & & f \\ \hline u & v & w & \lambda & \mu & \nu & 0 \end{array}\right) \quad \text{with } X, Y \in \mathfrak{sl}(3) \,.$$

$$A_2 = \left[\begin{array}{ccc|ccc|c} 0 & w & -v & & & & -a \\ -w & 0 & u & & 0 & & -b \\ v & -u & 0 & & & & -c \\ \hline & & & 0 & -\nu & \mu & d \\ & 0 & & \nu & 0 & -\lambda & e \\ & & & -\mu & \lambda & 0 & f \\ \hline a & b & c & -d & -e & -f & 0 \end{array}\right],$$

(5.49)
$$A_3 = \left[\begin{array}{ccc|ccc|c} 0 & -c & b & & & & u \\ c & 0 & -a & & 0 & & v \\ -b & a & 0 & & & & w \\ \hline & & & 0 & f & -e & -\lambda \\ & 0 & & -f & 0 & d & -\mu \\ & & & e & -d & 0 & -\nu \\ \hline -u & -v & -w & \lambda & \mu & \nu & 0 \end{array}\right],$$

$$A_4 = \left[\begin{array}{c|ccc|ccc} 0 & 2u & 2v & 2w & 2a & 2b & 2c \\ \hline a & & & & 0 & w & -v \\ b & & X & & -w & 0 & u \\ c & & & & v & -u & 0 \\ \hline u & 0 & -c & b & & & \\ v & c & 0 & -a & & -{}^tX & \\ w & -b & a & 0 & & & \end{array}\right],$$

$$A_5 = \left[\begin{array}{c|ccc|ccc} 0 & 2\lambda & 2\mu & 2\nu & 2d & 2e & 2f \\ \hline d & & & & 0 & \nu & -\mu \\ e & & Y & & -\nu & 0 & \lambda \\ f & & & & \mu & -\lambda & 0 \\ \hline \lambda & 0 & -f & e & & & \\ \mu & f & 0 & -d & & -{}^tY & \\ \nu & -e & d & 0 & & & \end{array}\right].$$

Since $\dim \mathfrak{g}_{X_0} = 28 = \dim GL(1) \times Spin((14) - \dim V(1) \otimes V(64)$, this triplet is a P.V. and it is regular by Proposition 25 in § 4. The prehomogeneity of this space was proved by T. Shintani (1970) and the orbital decomposition was completed by I. Ozeki and the second author (1973) (See [12]). Put $X_0' = 1 + e_1e_2e_3e_7 + e_1e_4e_5e_7 + e_2e_4e_6e_7 + e_1e_2e_3e_4e_5e_6$. Then by the similar calculation, the isotropy subalgebra $\mathfrak{g}_{X_0'}$ at $X_0'$ is given by

$$\mathfrak{g}_{X_0'} = \left\{ \left(\frac{a_7}{2}\right) \oplus \left[\begin{array}{c|c} A_1' & A_2' \\ \hline A_3' & -{}^tA_1' \end{array}\right] \middle| A_1', A_2', A_3' \text{ are given in (5.50)} \right\}$$

$$A_1' = \begin{pmatrix} a_1 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & c_{35} \\ a_{21} & a_2 & a_{23} & a_{24} & a_{25} & a_{26} & c_{36} \\ 0 & 0 & a_3 & 0 & a_{35} & a_{36} & -c_{15} \\ a_{36} & -a_{35} & a_{43} & a_4 & a_{45} & a_{46} & c_{56} \\ 0 & 0 & -a_{24} & 0 & a_5 & -a_{21} & c_{13} \\ 0 & 0 & a_{14} & 0 & -a_{12} & a_6 & c_{23} \\ 0 & 0 & -2c_{47} & 0 & c_{27} & -2c_{17} & a_7 \end{pmatrix} \quad \begin{aligned} &\text{with} \\ &a_7 = -(a_1 + a_2 + a_3) \\ &\quad = a_4 + a_5 + a_6 \\ &a_1 + a_3 = a_4 + a_6 \\ &a_2 + a_3 = a_4 + a_5 \\ &a_{43} = a_{25} - a_{16} \end{aligned}$$

$$(5.50) \quad A_2' = \begin{pmatrix} 0 & c_{37} & -c_{27} & c_{57} & -c_{47} & 0 & 2c_{23} \\ -c_{37} & 0 & c_{17} & c_{67} & 0 & -c_{47} & -2c_{13} \\ c_{27} & -c_{17} & 0 & 0 & 0 & 0 & 0 \\ -c_{57} & -c_{67} & 0 & 0 & c_{17} & c_{27} & -2c_{15} \\ c_{47} & 0 & 0 & -c_{17} & 0 & 0 & 0 \\ 0 & c_{47} & 0 & -c_{27} & 0 & 0 & 0 \\ -2c_{23} & 2c_{13} & 0 & 2c_{15} & 0 & 0 & 0 \end{pmatrix},$$

$$A_3' = \begin{pmatrix} 0 & 0 & c_{13} & 0 & c_{15} & 0 & c_{17} \\ 0 & 0 & c_{23} & 0 & 0 & c_{15} & c_{27} \\ -c_{13} & -c_{23} & 0 & 0 & c_{35} & c_{36} & c_{37} \\ 0 & 0 & 0 & 0 & c_{23} & -c_{13} & c_{47} \\ -c_{15} & 0 & -c_{35} & -c_{23} & 0 & c_{56} & c_{57} \\ 0 & -c_{15} & -c_{36} & c_{13} & -c_{56} & 0 & c_{67} \\ -c_{17} & -c_{27} & -c_{37} & -c_{47} & -c_{57} & -c_{67} & 0 \end{pmatrix}.$$

Since $\dim \mathfrak{g}_{X_0'} = 29$, the orbit of $X_0'$ is of codimension one. Since the trace $\mathrm{tr}_V A$ of an element $A$ of $\mathfrak{g}_{X_0'}$ on $V$ is $64 \times (a_7/2) = 32a_7$ and the adjoint representation on $\mathfrak{g}_{X_0'}$ is $-28a_7$, the degree of the irreducible relative invariant polynomial $f(x)$ is given by $\deg f = (32a_7 - 28a_7)/32a_7 \times 64 = 8$.

PROPOSITION 40. *A triplet* $(GL(1) \times Spin(14), \square \otimes$ *half-spin rep.*, $V(1) \otimes V(64))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $(G_2) \times (G_2)$. *The relative invariant is of degree* 8.

(31) $(GL(1) \times Spin(13), \square \otimes$ *spin rep.*, $V(1) \otimes V(64))$.

This space was investigated by T. Shintani (1971). We shall prove the non-prehomogeneity of this triplet after T. Shintani.

Assume that this triplet is a P.V. Let $\mathfrak{h}$ be the generic isotropy subalgebra of the regular P.V. in Proposition 40. Then there exists a 14-dimensional vector $x_0$ such that $\mathfrak{h} \cap (\mathfrak{gl}(1) \oplus \{A \in \mathfrak{o}(14) \mid Ax_0 = 0\}) = \{A \in \mathfrak{h} \mid Ax_0 = 0\}$ is of dimension 15 ($= \dim GL(1) \times Spin(13) - \dim V(1) \otimes V(64)$). By Proposition 40, $\mathfrak{h} \cong (\mathfrak{g}_2) \oplus (\mathfrak{g}_2)$ and $\mathfrak{h}$ is contained in $\mathfrak{o}(14)$ as the direct sum of 7-dimensional representations, this implies that the (non-irreducible) triplet $(GL(1)\times(G_2)\times(G_2),\ \square\otimes(\Lambda_2\otimes 1+1\otimes\Lambda_2),\ V(1)\otimes(V_1(7)\oplus V_2(7))$ is a P.V. By Proposition 25, there exists a quadratic invariant $q_i(x)$ on $V(1)\otimes V_i(7)$ $(i=1,2)$ and the quotient $q(x)=q_1(x)/q_2(x)$ is a non-constant absolute invariant of this triplet. By Proposition 3 in §2, it is not a P.V. and hence our triplet is not a P.V.

PROPOSITION 41. *A triplet* $(GL(1)\times Spin(13),\ \square\otimes$ *spin rep.*, $V(1)\otimes V(64))$ *is not a P.V.*

(32) $\qquad ((G_2)\times GL(d),\ \Lambda_2\otimes\Lambda_1,\ V(7)\otimes V(d)) \qquad (1\le d\le 3)$

I) *The case of* $d=1$.

Let $\mathfrak{g}$ be the Lie algebra of $GL(1)\times(G_2)$. Then by Example 30 in §1, we have

(5.51)

$$\mathfrak{g}=\left\{A=\left(\begin{array}{c|ccc|ccc} \xi & 2d & 2e & 2f & 2a & 2b & 2c \\ \hline a & \xi+\lambda_1 & \lambda_{12} & \lambda_{13} & 0 & f & -e \\ b & \lambda_{21} & \xi+\lambda_2 & \lambda_{23} & -f & 0 & d \\ c & \lambda_{31} & \lambda_{32} & \xi+\lambda_3 & e & -d & 0 \\ \hline d & 0 & -c & b & \xi-\lambda_1 & -\lambda_{21} & -\lambda_{31} \\ e & c & 0 & -a & -\lambda_{12} & \xi-\lambda_2 & -\lambda_{32} \\ f & -b & a & 0 & -\lambda_{13} & -\lambda_{23} & \xi-\lambda_3 \end{array}\right) \middle|\ \lambda_1+\lambda_2+\lambda_3=0\right\}$$

Put $X_0={}^t(1,0,0,0,0,0,0)$. Then $AX_0={}^t(\xi,a,b,c,d,e,f)$. Therefore the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0$ is given by

$$\begin{aligned}\mathfrak{g}_{X_0}&=\{A\in\mathfrak{g}\,|\,AX_0=0\}=\{A\in\mathfrak{g}\,|\,\xi=a=b=c=d=e=f=0\}\\ &=\left\{\left(\begin{array}{c|c|c} 0 & & \\ \hline & \Lambda & \\ \hline & & -{}^t\Lambda\end{array}\right)\middle|\ \Lambda\in\mathfrak{sl}(3,C)\right\}\cong\mathfrak{sl}(3,C)\,.\end{aligned}$$

Since $\dim\mathfrak{g}_{X_0}=8=\dim\mathfrak{g}-\dim V$ and $\mathfrak{g}_{X_0}$ is reductive, it is a regular P.V. by Proposition 25 in §4. By Proposition 25 (in §5), there exists a relative invariant quadratic form, i.e., $(G_2)\subset SO(7)$.

We shall determine the relative invariant $f(x)=\sum_{1\le i\le j\le 7}c_{ij}x_ix_j$. Let $A$ be a diagonal matrix in (5.51). Then

$$\begin{aligned}\langle Ax,\operatorname{grad}f(x)\rangle&=\sum_{j=1}^{7}\xi x_1\frac{\partial f}{\partial x_j}+\sum_{j=1}^{7}(\xi+\lambda_1)x_2\frac{\partial f}{\partial x_j}+\cdots\\ &\quad+\sum_{j=1}^{7}(\xi-\lambda_3)x_7\frac{\partial f}{\partial x_j}=2\xi f(x)\end{aligned}$$

and hence $f(x) = c_{11}x_1^2 + c_{25}x_2x_5 + c_{36}x_3x_6 + c_{47}x_4x_7$. Let $A_1$ (resp. $A_2$) be the matrix with $a = 1$ (resp. $b = 1$), all remaining entries zero in (5.51). Then $\langle A_1x, \operatorname{grad} f(x)\rangle = (4c_{11} + c_{25})x_1x_5 + (c_{47} - c_{36})x_3x_4 = 0$ and hence $f(x) = c_{11}(x_1^2 - 4x_2x_5) + c_{36}(x_3x_6 + x_4x_7)$. Since $\langle A_2x, \operatorname{grad} f(x)\rangle = c_{11}(4x_1x_6 - 4x_2x_4) + c_{36}(x_1x_6 - x_2x_4) = 0$, we have $f(x) = c_{11}(x_1^2 - 4x_2x_5 - 4x_3x_6 - 4x_4x_7)$.

PROPOSITION 42. *A triplet $(GL(1) \times (G_2), \square \otimes \Lambda_2, V(1) \otimes V(7))$ is a regular P.V. and its generic isotropy subgroup is locally isomorphic to $SL(3)$. The relative invariant is a quadratic form.*

II) *The case of $d = 2$.*

If we identify $V(7) \otimes V(2)$ with $7 \times 2$ matrices $M(7,2)$, the action $d\rho$ of the Lie algebra $\mathfrak{g} = (\mathfrak{g}_2) \oplus \mathfrak{gl}(2)$ of $(G_2) \times GL(2)$ is given by $d\rho(A)X = BX + X^tC$ where $X \in M(7,2)$, $A = (B, C) \in \mathfrak{g} = (\mathfrak{g}_2) \oplus \mathfrak{gl}(2)$. Put

$$X_0 = {}^t\begin{pmatrix} 0100000 \\ 0000100 \end{pmatrix}.$$

Then

$$(5.52)\qquad d\rho(A)X_0 = \left(\begin{array}{c|ccc|ccc} 0 & 2d & 2e & 2f & 2a & 2b & 2c \\ \hline a & \lambda_1 & \lambda_{12} & \lambda_{13} & 0 & f & -e \\ b & \lambda_{21} & \lambda_2 & \lambda_{23} & -f & 0 & d \\ c & \lambda_{31} & \lambda_{32} & \lambda_3 & e & -d & 0 \\ \hline d & 0 & -c & b & -\lambda_1 & -\lambda_{21} & -\lambda_{31} \\ e & c & 0 & -a & -\lambda_{12} & -\lambda_2 & -\lambda_{32} \\ f & -b & a & 0 & -\lambda_{13} & -\lambda_{23} & -\lambda_3 \end{array}\right)\left(\begin{array}{c|c} & \\ \hline 1 & \\ & \\ & \\ \hline & 1 \\ & \\ & \end{array}\right) + \left(\begin{array}{c|c} & \\ \hline 1 & \\ & \\ & \\ \hline & 1 \\ & \\ & \end{array}\right)\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

$$= \left(\begin{array}{c|c} 2d & 2a \\ \hline \lambda_1 + \alpha & \gamma \\ \lambda_{21} & -f \\ \lambda_{31} & e \\ \hline \beta & -\lambda_1 + \delta \\ c & -\lambda_{12} \\ -b & -\lambda_{13} \end{array}\right).$$

Hence the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0$ is by definition $\mathfrak{g}_{X_0} = \{A \in \mathfrak{g} \mid d\rho(A)X_0 = 0\}$ and hence we have

(5.53)

$$\mathfrak{g}_{X_0} = \left\{ \left( \begin{array}{c|ccc|ccc} 0 & & 0 & & & 0 & \\ \hline & \lambda_1 & 0 & 0 & & & \\ 0 & 0 & \lambda_2 & \lambda_{23} & & 0 & \\ & 0 & \lambda_{32} & \lambda_3 & & & \\ \hline & & & & -\lambda_1 & 0 & 0 \\ 0 & & 0 & & 0 & -\lambda_2 & -\lambda_{32} \\ & & & & 0 & -\lambda_{23} & -\lambda_3 \end{array} \right) \oplus \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \middle| \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\} \cong \mathfrak{gl}(2) .$$

Since $\dim \mathfrak{g}_{X_0} = 4 = \dim \mathfrak{g} - \dim V$, it is a regular P.V. by Proposition 25 in § 4. Since $(G_2) \times GL(2) \subset SO(7) \times GL(2)$, there exists an irreducible relative invariant polynomial of degree 4 (See (20)).

PROPOSITION 43. *A triplet* $((G_2) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(7) \otimes V(2))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $GL(2)$. *The relative invariant is a quartic form.*

III) *The case of* $d = 3$.

Identify $V(7) \otimes V(3)$ with $7 \times 3$ matrices $M(7, 3)$. Since $(G_2) \times GL(3) \subset SO(7) \times GL(3)$, there exists an irreducible relative invariant polynomial $f_1(x)$ of degree 6 (See (20)). On the other hand, $(G_2)$ is the isotropy subgroup at $X_0 = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$ (See (8)), and hence the group $(G_2) \times SL(3)$ leaves the following polynomial $f_2(x)$ invariant.

(5.54)

$$f_2(X) = \det \begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix} + \det \begin{pmatrix} x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \\ x_7 & y_7 & z_7 \end{pmatrix} + \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_5 & y_5 & z_5 \end{pmatrix} + \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_6 & y_6 & z_6 \end{pmatrix}$$
$$+ \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ x_7 & y_7 & z_7 \end{pmatrix} \qquad \text{where } X = {}^t\begin{pmatrix} x_1 & \cdots & x_7 \\ y_1 & \cdots & y_7 \\ z_1 & \cdots & z_7 \end{pmatrix} \in V(7) \otimes V(3) .$$

Since the quotient $f_1(x)/f_2(x)^2$ is a non-constant absolute invariant, it is not a P.V. by Proposition 3 in § 2 (or Proposition 12 in § 4).

PROPOSITION 44. *A triplet* $((G_2) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(7) \otimes V(3))$ *is not a P.V. There exist two irreducible relative invariant of degree* 3 *and degree* 6.

(33) $\quad (F_4 \times GL(d), \Lambda_4 \otimes \Lambda_1, V(26) \otimes V(d)) \qquad (1 \leq d \leq 2)$

I) *The case of* $d=1$.

By Proposition 39 in §1, $f_1(X) = \operatorname{Tr} X \circ X$ and $f_2(X) = \operatorname{Tr} X \circ X \circ X$ $(x \in \mathscr{J}_0)$ is a relative invariant of $GL(1) \times F_4$, and hence $f(x) = f_2(x)^2 \cdot f_1(x)^{-3}$ is an absolute invariant. Since

$$f\left(\begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}\right) = 0 \quad \text{and} \quad f\left(\begin{pmatrix}1 & & \\ & -2 & \\ & & 1\end{pmatrix}\right) \neq 0\,,$$

$f(x)$ is not constant, and hence it is not a P.V. by Proposition 3 in §2.

PROPOSITION 45. *A triplet* $(GL(1) \times F_4,\ \square \otimes \Lambda_4,\ V(1) \otimes V(26))$ *is not a P.V. There exist two irreducible relative invariants of degree* 2 *and degree* 3.

II) *The case of* $d=2$.

Identify $V(26) \otimes V(2)$ with $\mathscr{J}_0 \oplus \mathscr{J}_0$ and put $f_1(X) = \operatorname{Tr} X \circ X$, $f_2(X) = \operatorname{Tr} X \circ X \circ X$ for $X \in \mathscr{J}_0$. For $X = (X_1, X_2) \in \mathscr{J}_0 \oplus \mathscr{J}_0$, the polynomial $f_1(uX_1 + vX_2)$ is a binary quadratic form which is invariant under the action of $F_4$. Therefore, its discriminant $g_1(X)$ is a relative invariant of $F_4 \times GL(2)$ (See (3)). Similarly the discriminant $g_2(X)$ of the binary cubic form $f_2(uX_1 + vX_2)$ is also a relative invariant of $F_4 \times GL(2)$ (See (6)). As $\deg g_1 = 4$ and $\deg g_2 = 12$, $f(x) = g_2(x) g_1(x)^{-3}$ for $x \in \mathscr{J}_0 \oplus \mathscr{J}_0$ is an absolute invariant of $F_4 \times GL(2)$. Since

$$f\left(\begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}\begin{pmatrix}0 & 1 & \\ 1 & 0 & \\ & & 0\end{pmatrix}\right) = 0 \quad \text{and} \quad f\left(\begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}\begin{pmatrix}0 & & \\ & 1 & \\ & & -1\end{pmatrix}\right) \neq 0\,,$$

$f(x)$ is not constant, and hence it is not a P.V. by Proposition 3 in §2.

PROPOSITION 46. *A triplet* $(F_4 \times GL(2),\ \Lambda_4 \otimes \Lambda_1,\ V(26) \otimes V(2))$ *is not a P.V.*

(34) $\quad (E_6 \times GL(d),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(d)) \qquad (1 \leq d \leq 3)$

I) *The case of* $d=1$.

Identify $V(1) \otimes V(27)$ with the exceptional simple Jordan algebra $\mathscr{J}$. The Lie algebra $\mathfrak{g}$ of $E_6 \times GL(1)$ is $\mathfrak{g} = \{A = D + R_Y + \alpha \mid D \in \mathscr{D},\ Y \in \mathscr{J}_0,\ \alpha \in \boldsymbol{C}\}$ (See Example 39 in §1). Hence an element $A = D + R_Y + \alpha$ of $\mathfrak{g}$ is contained in the isotropy subalgebra $\mathfrak{g}_{X_0}$ at $X_0 = \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix} \in \mathscr{J}$ if and only if $AX_0 = Y + \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha \end{pmatrix} = 0$. Since $\operatorname{Tr} Y = 0$, this implies that $Y = 0$ and $\alpha = 0$. Therefore the isotropy subalgebra $\mathfrak{g}_{X_0}$ is the exceptional simple algebra $\mathscr{D} \cong F_4$. Since $\dim E_6 \times GL(1) - \dim F_4 = 27 = \dim \mathscr{J}$, it is a regular P.V. by Proposition 25 in §4. The relative invariant is the determinant $N(X) = \det X$ $(X \in \mathscr{J})$ (See Example 39 in §1).

PROPOSITION 47. *A triplet* $(GL(1) \times E_6, \square \otimes \Lambda_1, V(1) \otimes V(27))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $F_4$. *The relative invariant is the determinant of the exceptional simple Jordan algebra* $\mathscr{J} (\cong V(1) \otimes V(27))$.

II) *The case of* $d = 2$.

Identify $V(27) \otimes V(2)$ with $\mathscr{J} \oplus \mathscr{J}$. Let $\mathfrak{g}_{X_0}$ be the isotropy subalgebra of $E_6 \oplus \mathfrak{gl}(2)$ at $X_0 = \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\right)$. Then it is obvious that $\mathscr{D}_0 \subseteq \mathfrak{g}_{X_0}$ (See Definition 34 in §1). We shall show that $\mathscr{D}_0 \supseteq \mathfrak{g}_{X_0}$. Let $z = X + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathfrak{g}_{X_0}$, where $X = D + (\alpha)'_1 + (\beta)'_2 + (\gamma)'_3 + R_Y$ $(D \in \mathscr{D}_0$, $\operatorname{Tr} Y = 0$, See Definition 36 in §1). Then

$$z \cdot X_0 = \left( Y + \begin{pmatrix} a+b & & \\ & a & \\ & & a-b \end{pmatrix}, X\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} + \begin{pmatrix} c+d & & \\ & c & \\ & & c-d \end{pmatrix} \right) = 0$$

implies that $Y + \begin{pmatrix} a+b & & \\ & a & \\ & & a-b \end{pmatrix} = 0$. Since $\operatorname{Tr} Y = 0$, we obtain that $Y = \begin{pmatrix} b & & \\ & 0 & \\ & & -b \end{pmatrix}$ and $a = 0$. Since $D\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} = 0$ for $D \in \mathscr{D}_0$, we obtain that

$$X\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} + \begin{pmatrix} c+d & & \\ & c & \\ & & c-d \end{pmatrix}$$
$$+ \begin{bmatrix} b+c+d & -\dfrac{\gamma}{4} & -\dfrac{\beta}{2} \\ -\dfrac{\bar{\gamma}}{4} & c & -\dfrac{\alpha}{4} \\ -\dfrac{\bar{\beta}}{2} & -\dfrac{\bar{\alpha}}{4} & b+c-d \end{bmatrix} = 0\,.$$

Together with $Y = \begin{pmatrix} -b & & \\ & 0 & \\ & & b \end{pmatrix}$ and $a = 0$, we have $z = D \in \mathscr{D}_0$. Therefore we obtain that $\mathfrak{g}_{X_0} \cong \mathscr{D}_0 \cong \mathfrak{o}(8, \boldsymbol{C})$ (See Proposition 35 in §1). Since $\dim \mathfrak{g}_{X_0} = 28 = \dim E_6 \times GL(1) - \dim \mathscr{J} \oplus \mathscr{J}$, it is a regular P.V. by Proposition 25 in §4. For $X = (X_1, X_2) \in \mathscr{J} \oplus \mathscr{J}$, the discriminant $f(X)$ of the binary cubic form $N(uX_1 + vX_2)$ is a relative invariant of degree 12. This is irreducible since the restriction of $f(X)$ to $\{(X_1, X_2) \in \mathscr{J} \oplus \mathscr{J} \mid X_1, X_2 \in M(3, \boldsymbol{C}), {}^tX_1 = X_1, {}^tX_2 = X_2\}$ is irreducible (See (10)).

PROPOSITION 48. *A triplet* $(E_6 \times GL(2), \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(2))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to the orthogonal group* $O(8, \boldsymbol{C})$. *The relative invariant is of degree* 12.

*Remark* 49. The following four P.V.'s correspond to $\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}, \mathfrak{C}$ respectively.

1) $(SL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$;
2) $(SL(3) \times SL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$;
3) $(SL(6) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2))$;
4) $(E_6 \times GL(2), \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(2))$.

Their representation spaces can be regarded as the pair of $3 \times 3$ hermitian matrices over some algebra $A$, where $A = \boldsymbol{C} (= \boldsymbol{R} \otimes_{\boldsymbol{R}} \boldsymbol{C})$, $\boldsymbol{C} \oplus \boldsymbol{C}$ $(= \boldsymbol{C} \otimes_{\boldsymbol{R}} \boldsymbol{C})$, $M_2(\boldsymbol{C})$ $(= \boldsymbol{H} \otimes_{\boldsymbol{R}} \boldsymbol{C})$ and $\mathfrak{C}$ respectively. The rank $m$ of $A$ is $m = 2^k$ $(k = 0, 1, 2, 3)$, and the dimension of a generic isotropy group is given by $(k + 1)(m - 1)$ $(k = 0, 1, 2, 3)$. The degree of the irreducible relative invariant is 12.

III) *The case of* $d = 3$.

We may identify $V = V(27) \otimes V(3)$ with $\mathscr{J} \oplus \mathscr{J} \oplus \mathscr{J}$ where $\mathscr{J}$ is the exceptional simple Jordan algebra.

Let $N(X_1)$ be the determinant of $X_1$ in $\mathscr{J}$ (See Example 39 in §1). For each $X = (X_1, X_2, X_3) \in V$, we can define the ternary cubic form $\varphi(X) = N(uX_1 + vX_2 + wX_3)$ which is invariant under the action of $E_6$. This $\varphi$ is an equivariant map from $\mathscr{J} \oplus \mathscr{J} \oplus \mathscr{J}$ to the space of ternary cubic forms. Since a triplet $(E_6 \times GL(3), 1 \otimes 3\Lambda_1, V(10)) \cong (GL(3), 3\Lambda_1, V(10))$ is clearly not a P.V., if $\varphi$ is generically surjective, a triplet $(E_6 \times GL(3), \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(3))$ is not a P.V. by Lemma 5 in §2. For $t \in \boldsymbol{C}$, put

$$X(t) = \left( \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} -1 & -t & -t \\ -t & 0 & -t \\ -t & -t & 0 \end{pmatrix} \right) \in V .$$

Then $\varphi(X(t)) = uv(u + v - w) - \gamma w^3$ where $\gamma = 2t^3 - t^2$. For $A = (a_{ij})$

$\in \mathfrak{gl}(3)$, $A\varphi(X(t)) = a_{12}u^3 + a_{21}v^3 - 3\gamma a_{33}w^3 + (2a_{11} + 2a_{12} + a_{22} - a_{13})u^2v + (a_{32} - a_{12})u^2w + (2a_{21} + 2a_{22} + a_{11} - a_{23})v^2u + (a_{31} - a_{21})v^2w - (a_{32} + 3\gamma a_{13})w^2u - (a_{31} + 3a_{23})w^2v + (2a_{31} + 2a_{32} - a_{11} - a_{22} - a_{33})uvw$ and hence the isotropy subalgebra at $X_0$ is $\{A \in \mathfrak{gl}(3) \mid A\varphi(X(t)) = 0\} = 0$ if $\gamma \neq 0$. This implies that if $\gamma \neq 0$, the $GL(3)$-orbit of $\varphi(X(t))$ is of dimension 9. On the other hand, $\varphi(X(t)) = 0$ is an elliptic curve for $\gamma \neq 0, -\frac{1}{27}$, and its modular invariant is given by $J = \dfrac{-27\gamma(1 + 8\gamma)^3}{\gamma^3 + 27\gamma^4}$. Since $J$ is invariant under the action of $GL(3)$, the dimension of the union of $GL(3)$-orbits of $\varphi(X(t))$ is 10. This implies that $\varphi$ is generically surjective, and hence a triplet $(E_6 \times GL(3), \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(3))$ is not a P.V.

PROPOSITION 50. *A triplet* $(E_6 \times GL(3), \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(3))$ *is not a P.V.*

(35) $$(E_6 \times GL(2), \Lambda_1 \otimes 2\Lambda_1, V(27) \otimes V(3)) .$$

Since $(E_6 \times GL(2), \Lambda_1 \otimes 2\Lambda_1) \subset (E_6 \times GL(3), \Lambda_1 \otimes \Lambda_1)$, it is clearly not a P.V. by Proposition 50.

PROPOSITION 51. *A triplet* $(E_6 \times GL(3), \Lambda_1 \otimes 2\Lambda_1, V(27) \otimes V(3))$ *is not a P.V.*

(36) $$(E_7 \times GL(d), \Lambda_6 \otimes \Lambda_1, V(56) \otimes V(d)) \qquad (1 \leq d \leq 2) .$$

I) *The case of* $d = 1$.

We may identify $V(56) \otimes V(1)$ with $\mathfrak{M} = \boldsymbol{C} \oplus \boldsymbol{C} \oplus \mathscr{J} \oplus \mathscr{J}$ (See Example 40 in § 1). Let $\mathfrak{g}_{X_0}$ be the isotropy subalgebra of $E_7 \oplus \mathfrak{gl}(1)$ at $X_0 = (1, 1, 0, 0)$ in $\mathfrak{M}$. For $A = a \oplus \bar{b} \oplus 2mR_I \oplus L \oplus k \in E_7 \oplus \mathfrak{gl}(1)$ where $a, b \in \mathscr{J}$, $m, k \in \boldsymbol{C}$, and $L \in E_6$, by (1.14) we have $[a, X_0] = (0, 0, a, 0)$, $[\bar{b}, X_0] = (0, 0, 0, -b)$, $[2mR_I, X_0] = (3m, -3m, 0, 0)$, $[k, X_0] = (k, k, 0, 0)$ and $[L, X_0] = (0, 0, 0, 0)$, and hence $[A, X_0] = (3m + k, -3m + k, a, -b)$. Therefore we have $\mathfrak{g}_{X_0} = \{A \in E_7 \oplus \mathfrak{gl}(1) \mid [A, X_0] = 0\} = \{L \in E_6\} \cong E_6$. Since $\dim \mathfrak{g}_{X_0} = 78 = \dim E_7 \times GL(1)$ $(= 134) - \dim V$ $(= 56)$, it is a regular P.V. by Proposition 25 in § 4. The relative invariant is a quartic form $q(X)$ in (1.16).

The orbital decomposition is completed by S. J. Harris (See [7]).

PROPOSITION 52. *A triplet* $(GL(1) \times E_7, \square \otimes \Lambda_6, V(1) \otimes V(56))$ *is a regular P.V. and its generic isotropy subgroup is locally isomorphic to* $E_6$. *The relative invariant is a quartic form* $q(X)$ *given in* (1.16).

II) *The case of* $d=2$.

We may identify $V=V(56)\otimes V(2)$ with $\mathfrak{M}\oplus\mathfrak{M}$. Put $f_1(X)=\{X_1, X_2\}=\xi_1\eta_2-\xi_2\eta_1+T(x_1y_2)-T(x_2y_1)$ for $X=(X_1, X_2)=((\xi_1, \eta_1, x_1, y_1), (\xi_2, \eta_2, x_2, y_2))\in V$. Then this is a relative invariant of $E_7\times GL(2)$ i.e. $E_7\subset Sp(28)$ (See Example 40 in § 1). Let $q(X)$ be the invariant quartic form of $E_7$, i.e., $q(X)=T(x^{\#}, y^{\#})-\xi N(x)-\eta N(y)-\frac{1}{4}(T(x, y)-\xi\eta)^2$ for $X=(\xi, \eta, x, y)\in\mathfrak{M}$, where $a^{\#}=a^2-\mathrm{Tr}\,(a)\cdot a+\frac{1}{2}\{(\mathrm{Tr}\,a)^2-\mathrm{Tr}\,(a^2)\}\cdot 1$ (See Example 40 in § 1). For each $X=(X_1, X_2)\in\mathfrak{M}\oplus\mathfrak{M}$, we have a binary quartic form $\varphi(X)=q(uX_1+vX_2)$. In general for a binary quartic form $\tilde{x}=\sum_{i=0}^4 x_iu^{4-i}v^i$, one can easily check that the polynomial $h(\tilde{x})=x_2^2-3x_1x_3+12x_0x_4$ is relatively invariant under the action of $GL(2)$. Since $\varphi$ is an equivariant map, the polynomial $f_2(X)=h(\varphi(X))$ is a relative invariant of $E_7\times GL(2)$. Since $\deg f_1(X)=2$ and $\deg f_2(X)=8$, the quotient $f(X)=f_1(X)^4\cdot f_2(X)^{-1}$ is an absolute invariant. For $X=((1,0,0,0), (0,1,0,0))\in\mathfrak{M}\oplus\mathfrak{M}$, $f_1(X)=1$ and $q((u, v, 0, 0))=-\frac{1}{4}u^2v^2$, i.e., $f_2(X)=\frac{1}{16}$, and hence $f(X)\neq 0$. On the other hand, for

$$X=\left(\left(0,0,\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix},0\right),\left(0,0,0,\begin{pmatrix}1&&\\&-1&\\&&0\end{pmatrix}\right)\right)\in\mathfrak{M}\oplus\mathfrak{M}\,,$$

we have $f_1(X)=0$,

$$q\left(\left(0,0,\begin{pmatrix}u&&\\&u&\\&&0\end{pmatrix},\begin{pmatrix}v&&\\&-v&\\&&0\end{pmatrix}\right)\right)=T\left(\begin{pmatrix}u&&\\&u&\\&&0\end{pmatrix}^{\#},\begin{pmatrix}v&&\\&-v&\\&&0\end{pmatrix}^{\#}\right)=-u^2v^2$$

i.e. $f_2(X)=1$, and hence $f(X)=0$. This implies that $f(X)$ is a non-constant absolute invariant, and hence this triplet is not a P.V. by Proposition 3 in § 2.

PROPOSITION 53. *A triplet* $(E_7\times GL(2), \Lambda_6\otimes\Lambda_1, V(56)\otimes V(2))$ *is not a P.V.*

THEOREM 54. *Let* $(\tilde{G}, \tilde{\rho}, \tilde{V})$ *be a reduced P.V. and let* $\tilde{\mathfrak{g}}$ *be the Lie algebra of* $\tilde{\rho}(\tilde{G})$. *Assume that the center of* $\tilde{\mathfrak{g}}$ *is one-dimensional. Then it is equivalent to one of the following P.V.'s.*

I) *A regular P.V.*

(1) $(G\times GL(m), \rho\otimes\Lambda_1, V(m)\otimes V(m))$ *where* $\rho: G\to GL(V(m))$ *is an* $m$*-dimensional irreducible representation of a semi-simple algebraic*

*group* $G$.

(2) $(GL(n),\ 2\Lambda_1,\ V(\frac{1}{2}n(n+1)))\qquad (n \geqq 2)$

(3) $(GL(2m),\ \Lambda_2,\ V(m(2m-1)))\qquad (m \geqq 3)$

(4) $(GL(2),\ 3\Lambda_1,\ V(4))$

(5) $(GL(6),\ \Lambda_3,\ V(20))$

(6) $(GL(7),\ \Lambda_3,\ V(35))$

(7) $(GL(8),\ \Lambda_3,\ V(56))$

(8) $(SL(3) \times GL(2),\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(2))$

(9) $(SL(6) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(15) \otimes V(2))$

(10) $(SL(5) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$

(11) $(SL(5) \times GL(4),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(4))$

(12) $(SL(3) \times SL(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(3) \otimes V(2))$

(13) $(Sp(n) \times GL(2m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m))\qquad (n \geqq 2m \geqq 2)$

(14) $(GL(1) \times Sp(3),\ \square \otimes \Lambda_3,\ V(1) \otimes V(14))$

(15) $(SO(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))\qquad (n \geqq 3,\ n/2 \geqq m \geqq 1)$

(16) $(GL(1) \times \mathrm{Spin}\,(7),\ \square \otimes$ *spin rep.*, $V(1) \otimes V(8))$

(17) $(\mathrm{Spin}\,(7) \times GL(2),$ *spin rep.* $\otimes \Lambda_1,\ V(8) \otimes V(2))$

(18) $(\mathrm{Spin}\,(7) \times GL(3),$ *spin rep.* $\otimes \Lambda_1,\ V(8) \otimes V(3))$

(19) $(GL(1) \times \mathrm{Spin}\,(9),\ \square \otimes$ *spin rep.*, $V(1) \otimes V(16))$

(20) $(\mathrm{Spin}\,(10) \times GL(2),$ *half-spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(2))$

(21) $(\mathrm{Spin}\,(10) \times GL(3),$ *half-spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(3))$

(22) $(GL(1) \times \mathrm{Spin}\,(11),\ \square \otimes$ *spin rep.*, $V(1) \otimes V(32))$

(23) $(GL(1) \times \mathrm{Spin}\,(12),\ \square \otimes$ *half-spin rep.*, $V(1) \otimes V(32))$

(24) $(GL(1) \times \mathrm{Spin}\,(14),\ \square \otimes$ *half-spin rep.*, $V(1) \otimes V(64))$

(25) $(GL(1) \times (G_2),\ \square \otimes \Lambda_2,\ V(1) \otimes V(7))$

(26) $((G_2) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(2))$

(27) $(GL(1) \times E_6,\ \square \otimes \Lambda_1,\ V(1) \otimes V(27))$

(28) $(E_6 \times GL(2),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(2))$

(29) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$

II) *A non-regular P.V. with relative invariants.*

$$(Sp(n) \times GL(2),\ \Lambda_1 \otimes 2\Lambda_1,\ V(2n) \otimes V(3))$$
$$\cong (GL(1) \times Sp(n) \times SO(3),\ \square \otimes \Lambda_1 \otimes \Lambda_1,\ V(1) \otimes V(2n) \otimes V(3))$$

III) *A non-regular P.V. without relative invariants.*

(1) $(G \times GL(m),\ \rho \otimes \Lambda_1,\ V(n) \otimes V(m))$ *where* $\rho : G \to GL(V(n))$ *is an* $n$*-dimensional irreducible representation of a semi-simple algebraic group* $G(\neq SL(n))$ *with* $m > n \geqq 3$.

(2) $(SL(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(m/2 \geqq n \geqq 1)$

(3) $(GL(2m+1),\ \Lambda_2,\ V(m(2m+1)))$ $(m \geqq 2)$

(4) $(SL(2m+1) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$ $(m \geqq 2)$

(5) $(Sp(n) \times GL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$ $(n > 2m+1 \geqq 1)$

(6) $(GL(1) \times \text{Spin}\,(10),\ \square \otimes$ *half-spin rep.*, $\square \otimes V(16))$

*Proof.* By Theorem 24 in §3 and Proposition 1 ~ Proposition 53, we obtain our assertion. Q.E.D.

## §6. Semi-simple case

Let $(G, \rho, V)$ be an irreducible P.V. Then the Lie algebra $\mathfrak{g}$ of $\rho(G)$ is reductive with center at most one-dimensional by Theorem 1 in §1. In the previous sections, we have considered the case when the center of $\mathfrak{g}$ is of one dimension. In this section, we shall consider the case when the center of $\mathfrak{g}$ is of zero dimension. Then $\mathfrak{g}$ is semi-simple and hence $\mathfrak{g} \subset \mathfrak{sl}(V)$. Therefore, there is no relative invariant by Proposition 20 in §4. Then a triplet $(GL(1) \times G,\ \square \otimes \rho,\ V(1) \otimes V)$ is a P.V. without relative invariant and hence by Theorem 54 in §5, it belongs to the same castling class as one of the following reduced P.V.'s.

(1) $(G \times GL(m),\ \rho \otimes \Lambda_1,\ V(n) \otimes V(m))$ where $\rho$ is any $n$-dimensional irreducible representation of a connected semi-simple algebraic group $G$ $(\neq SL(n))$ with $m > n \geq 3$.

(2) $(SL(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ with $m/2 \geq n \geq 1$.

(3) $(GL(2m+1),\ \Lambda_2,\ V(m(2m+1)))$.

(4) $(SL(2m+1) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$.

(5) $Sp(n) \times GL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$ with $n \geq 2m+1 \geq 1$.

(6) $(GL(1) \times \text{Spin}\,(10),\ \square \otimes \text{half-spin rep.},\ V(1) \otimes V(16))$.

As we have seen in §5, the generic isotropy subalgebras $\mathfrak{h}$ of these P.V.'s are not contained in $\mathfrak{sl}(V)$, and hence these are P.V. even if without the action of scalar multiplications. Therefore, we obtain the following proposition.

PROPOSITION 1. *Let $(\tilde{G}, \tilde{\rho}, \tilde{V})$ be a reduced P.V. where $\tilde{G}$ is a connected semi-simple algebraic group. Then it is equivalent to one of the following reduced P.V.'s.*

(1) $(G \times SL(m),\ \rho \otimes \Lambda_1,\ V(n) \otimes V(m))$ *where $\rho$ is any $n$-dimensional*

*irreducible representation of a connected semi-simple algebraic group* $G$ $(\neq SL(n))$ *with* $m > n \geqq 3$.

(2) $(SL(n) \times SL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ *with* $m/2 \geq n \geq 1$.

(3) $(SL(2m+1),\ \Lambda_2,\ V(m(2m+1)))$.

(4) $(SL(2m+1) \times SL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$.

(5) $(Sp(n) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$ *with* $n \geq 2m + 1 \geq 1$.

(6) $(Spin\,(10),$ *half-spin rep.*, $V(16))$.

## § 7. Table of irreducible reduced prehomogeneous vector spaces

In general, we denote by $H$ a generic isotropy subgroup of a reduced P.V. If two groups $H_1$ and $H_2$ are isomorphic (resp. locally isomorphic) to each other, we denote this relation by $H_1 \simeq H_2$ (resp. $H_1 \sim H_2$). The degree of the irreducible relative invariant polynomial $f(x)$ will be denoted by $\deg f$. Then from § 1 ~ § 6, we obtain the following table.

I) A Regular Prehomogeneous Vector Space.

(1) $(G \times GL(m),\ \rho \otimes \Lambda_1,\ V(m) \otimes V(m))$ where $\rho: G \to GL(V(m))$ is an $m$-dimensional irreducible representation of a connected semi-simple algebraic group $G$ (or $G = \{1\}$ and $m = 1$).
(i) $H \simeq G$, (ii) $\deg f = m$, (iii) $f(x) = \det x$ for $x \in M(m) \simeq V(m) \otimes V(m)$ (see Proposition 1 in § 5).

(2) $(GL(n),\ 2\Lambda_1,\ V(\frac{1}{2}n(n+1)))$ $(n \geq 2)$.
(i) $H \simeq O(n)$, (ii) $\deg f = n$, (iii) $f(x) = \det x$ for $x \in \{x \in M(n) \mid {}^t x = x\} \simeq V(\frac{1}{2}n(n+1))$ (see Proposition 3 in § 5).

(3) $(GL(2m),\ \Lambda_2,\ V(m(2m-1))$ $(m \geq 3)$.
(i) $H \simeq Sp(m)$, (ii) $\deg f = m$, (iii) $f(x) = Pff(x)$ for $x \in \{x \in M(2m) \mid {}^t x = -x\} \simeq V(m(2m-1))$ (see Proposition 4 in § 5).

(4) $(GL(2),\ 3\Lambda_1,\ V(4))$.
(i) $H \sim \{1\}$, $\sharp H = 18$, (ii) $\deg f = 4$, (iii) $f(x) = x_2^2 x_3^2 + 18 x_1 x_2 x_3 x_4 - 4 x_1 x_3^3 - 4 x_2^3 x_4 - 27 x_1^2 x_4^2$ for $x = x_1 u^3 + x_2 u^2 v + x_3 u v^2 + x_4 v^3$, i.e., the discriminant of a binary cubic form $x$ (see Proposition 6 in § 5).

(5) $(GL(6),\ \Lambda_3,\ V(20))$.
(i) $H \simeq (SL(3) \times SL(3)) \cdot \{\pm 1\}$, (ii) $\deg f = 4$, (iii) $f(x) = (x_0 y_0 - \operatorname{tr} XY)^2 + 4x_0 \det Y + 4y_0 \det X - 4 \sum_{i,j} \det(X_{ij}) \cdot \det(Y_{ji})$ (see after Proposition 7 in § 5).

(6) $(GL(7),\ \Lambda_3,\ V(35))$.

(i) $H \simeq (G_2) \times \{\omega I_7 \mid \omega^3 = 1\}$, (ii) $\deg f = 7$ (see Proposition 8 in § 5).

(7) $(GL(8),\ \Lambda_3,\ V(56))$.
(i) $H \sim SL(3)$, (ii) $\deg f = 16$ (see Proposition 10 in § 5).

(8) $(SL(3) \times GL(2),\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(2))$.
(i) $H \sim \{1\}$, $\sharp H = 144$, (ii) $\deg f = 12$, (iii) $f(x)$ is given by the discriminant of a binary cubic form $\det(uX + vY)$ for $x = (X, Y) \in \{(X, Y) \mid X, Y \in M(3),\ {}^tX = X,\ {}^tY = Y\} \simeq V(6) \otimes V(2)$ (see Proposition 11 in § 5).

(9) $(SL(6) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(15) \otimes V(2))$.
(i) $H \sim SL(2) \times SL(2) \times SL(2)$, (ii) $\deg f = 12$, (iii) $f(x)$ is given by the discriminant of a binary cubic form $Pff(uX + vY)$ for $x = (X, Y) \in \{(X, Y) \mid X, Y \in M(6),\ {}^tX = -X,\ {}^tY = -Y\} \simeq V(15) \otimes V(2)$ (see Proposition 12 in § 5).

(10) $(SL(5) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$.
(i) $H \sim SL(2)$, (ii) $\deg f = 15$ (see Proposition 14 in § 5).

(11) $(SL(5) \times GL(4),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(4))$.
(i) $H \sim \{1\}$, (ii) $\deg f = 40$, (iii) $f(x)$ is given by the proof of Proposition 16 in § 4 (see Proposition 15 in § 5).

(12) $(SL(3) \times SL(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(3) \otimes V(2))$.
(i) $H \sim GL(1) \times GL(1)$, (ii) $\deg f = 12$, (iii) $f(x)$ is given by the discriminant of a binary cubic form $\det(uX + vY)$ for $x = (X, Y) \in M(3) \oplus M(3) \simeq V(3) \otimes V(3) \otimes V(2)$.

(13) $(Sp(n) \times GL(2m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m)$ $\quad (n \geq 2m \geq 2)$.
(i) $H \sim Sp(m) \times Sp(n - m)$, (ii) $\deg f = 2m$, (iii) $f(x) = Pff({}^tXJX)$ for $X \in M(2n, 2m)$ (see Proposition 17 in § 5).

(14) $(GL(1) \times Sp(3),\ \square \otimes \Lambda_3,\ V(1) \otimes V(14))$.
(i) $H \sim SL(3)$, (ii) $\deg f = 4$, (iii) $f(x)$ is given by the restriction of the relative invariant of $(GL(6),\ \Lambda_3,\ V(20))$ (see Proposition 22 in § 5).

(15) $(SO(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $\quad (n \geq 3,\ n/2 \geq m \geq 1)$.
(i) $H \simeq (SO(m) \times SO(n - m)) \cdot \{\pm 1\}$, (ii) $\deg f = 2m$, (iii) $f(X) = \det {}^tXKX$ for $X \in M(n, m) \simeq V(n) \otimes V(m)$ (see Proposition 23 in § 5).

(16) $(GL(1) \times \mathrm{Spin}(7),\ \square \otimes \text{spin rep.},\ V(1) \otimes V(8))$.
(i) $H \sim (G_2)$, (ii) $\deg f = 2$, (iii) By the spin representation, we have $\mathrm{Spin}(7) \hookrightarrow SO(8)$ and hence a relative invariant is that of $(GL(1) \times SO(8),\ \square \otimes \Lambda_1,\ V(1) \otimes V(8))$ (see (15), or Proposition 25 in § 5).

(17) (Spin (7) $\times$ $GL(2)$, *spin rep.* $\otimes \Lambda_1$, $V(8) \otimes V(2)$).
(i) $H \sim SL(3) \times O(2)$, (ii) $\deg f = 4$, (iii) By the spin representation, we have Spin (7) $\hookrightarrow SO(8)$ and hence a relative invariant is that of $(SO(8) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(8) \otimes V(2))$ (see (15), or Proposition 26 in § 5).

(18) (Spin (7) $\times$ $GL(3)$, *spin rep.* $\otimes \Lambda_1$, $V(8) \otimes V(3)$).
(i) $H \sim SL(2) \times O(3)$, (ii) $\deg f = 6$, (iii) By the spin representation, we have Spin (7) $\hookrightarrow SO(8)$ and hence a relative invariant is that of $(SO(8) \times GL(3), \Lambda_1 \otimes \Lambda_1, V(8) \otimes V(3))$ (see (15), or Proposition 27 in § 5).

(19) ($GL(1)$ $\times$ Spin (9), $\square \otimes$ *spin rep.*, $V(1) \otimes V(16)$).
(i) $H \sim$ Spin (7), (ii) $\deg f = 2$, (iii) $f(x) = x_0 y_0 + Pff(x_{ij}) + \sum_i x_i y_i$ for $x = x_0 + \sum_{i<j\leq 4} x_{ij} e_i e_j + y_0 e_1 e_2 e_3 e_4 + (\sum_i x_i e_i + \sum_i y_i e_i^*) e_5$ where $e_i^* e_i = e_1 e_2 e_3 e_4 e_5$ for $1 \leq i \leq 4$ (see Proposition 35 in § 5).

(20) (Spin (10) $\times$ $GL(2)$, *half-spin rep.* $\otimes \Lambda_1$, $V(16) \otimes V(2)$).
(i) $H \sim (G_2) \times SL(2)$, (ii) $\deg f = 4$ (see Proposition 32).

(21) (Spin (10) $\times$ $GL(3)$, *half-spin rep.* $\otimes \Lambda_1$, $V(16) \otimes V(3)$).
(i) $H \sim SL(2) \times O(3)$, (ii) $\deg f = 12$ (see Proposition 33 in § 5).

(22) ($GL(1)$ $\times$ Spin (11), $\square \otimes$ *spin rep.*, $V(1) \otimes V(32)$).
(i) $H \sim SL(5)$, (ii) $\deg f = 4$, (iii) $f(x) = x_0 Pff((y_{ij})) + y_0 Pff((x_{ij})) + \sum_{i<j} Pff(X_{ij}) \cdot Pff(Y_{ij}) - \frac{1}{4}(x_0 y_0 - \sum_{i<j} x_{ij} y_{ij})^2$ for $x = x_0 + \sum_{i<j} x_{ij} e_i e_j + \sum_{i<j} y_{ij} e_{ij}^* + y_0 e_1 e_2 e_3 e_4 e_5 e_6$ where $e_{ij}^* = (-1)^{i+j-1} e_1 \cdots e_{i-1} e_{i+1} \cdots e_{j-1} e_{j+1} \cdots e_6$ (see Proposition 39 in § 5).

(23) ($GL(1)$ $\times$ Spin (12), $\square \otimes$ *half-spin rep.*, $V(1) \otimes V(32)$).
(i) $H \sim SL(6)$, (ii) $\deg f = 4$, (iii) $f(x)$ is the same as (22) (see Proposition 37 in § 5).

(24) ($GL(1)$ $\times$ Spin (14), $\square \otimes$ *half-spin rep.*, $V(1) \otimes V(64)$).
(i) $H \sim (G_2) \times (G_2)$, (ii) $\deg f = 8$ (see Proposition 40 in § 5).

(25) $((GL(1) \times (G_2), \square \otimes \Lambda_2, V(1) \otimes V(7))$.
(i) $H \sim SL(3)$, (ii) $\deg f = 2$, (iii) By $\Lambda_2$, we have $(G_2) \hookrightarrow SO(7)$ and hence a relative invariant is that of $(GL(1) \times SO(7), \square \otimes \Lambda_1, V(1) \otimes V(7))$ (see Proposition 42 in § 5).

(26) $((G_2) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(7) \otimes V(2))$.
(i) $H \sim GL(2)$, (ii) $\deg f = 4$, (iii) By $\Lambda_2$, we have $(G_2) \hookrightarrow SO(7)$ and hence a relative invariant is that of $(SO(7) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(7) \otimes V(2))$ (see (15), or Proposition 43 in § 5).

(27) $(GL(1) \times E_6,\ \square \otimes \Lambda_1,\ V(1) \otimes V(27))$.
(i) $H \sim F_4$, (ii) $\deg f = 3$, (iii) $f(x) = \det x = \xi_1\xi_2\xi_3 + \operatorname{tr} x_1x_2x_3 - \xi_1 x_1\bar{x}_1 - \xi_2 x_2 \bar{x}_2 - \xi_3 x_3 \bar{x}_3$ for

$$x = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \in \mathscr{J} \simeq V(1) \otimes V(27)$$

(see Proposition 47 in § 5).

(28) $(E_6 \times GL(2),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(2))$.
(i) $H \sim O(8)$, (ii) $\deg f = 12$, (iii) $f(x)$ is given by the discriminant of a binary cubic form $\det(uX + vY)$ for $x = (X, Y) \in \mathscr{J} \oplus \mathscr{J} \simeq V(27) \otimes V(2)$ (see Proposition 48 in § 5).

(29) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$.
(i) $H \sim E_6$, (ii) $\deg f = 4$, (iii) $f(X) = T(x^\#, y^\#) - \xi N(x) - \eta N(y) - \frac{1}{4}(T(x, y) - \xi\eta)^2$ (see (1.16), or Proposition 52 in § 5).

II) A Non-regular Prehomogeneous Vector Space, with relative invariants.
$(GL(1) \times Sp(n) \times SO(3),\ \square \otimes \Lambda_1 \otimes \Lambda_1,\ V(1) \otimes V(2n) \otimes V(3))$.
(i) $H \sim (Sp(n-2) \times SO(2)) \cdot U(2n-3)$, (ii) $\deg f = 4$, (iii) $f(X) = \operatorname{tr}({}^t XJXK)^2$ for $X \in M(2n, 3) \simeq V(1) \otimes V(2n) \otimes V(3)$ (see Proposition 19 in § 5).

III) A Non-regular Prehomogeneous Vector Space, without relative invariant.

(1) $(G \times GL(m),\ \rho \otimes \Lambda_1,\ V(n) \otimes V(m))$.
(1′) $(G \times SL(m),\ \rho \otimes \Lambda_1,\ V(n) \otimes V(m))$,
where $\rho: G \to GL(V(n))$ is an $n$-dimensional irreducible representation of a semi-simple algebraic group $G$ ($\neq SL(n)$) with $m > n \geq 3$.
(2) $(SL(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(m/2 \geq n \geq 1)$.
(2′) $(SL(n) \times SL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$.
(3) $(GL(2m+1),\ \Lambda_2,\ V(m(2m+1)))$ $(m \geq 2)$.
(3′) $(SL(2m+1),\ \Lambda_2,\ V(m(2m+1)))$.
(4) $(SL(2m+1) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$ $(m \geq 2)$.
(4′) $(SL(2m+1) \times SL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$.
(5) $(Sp(n) \times GL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$
$(n > 2m+1 \geq 1)$.

(5′) $(Sp(n) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$.

(6) $(GL(1) \times Spin\,(10),\ \square \otimes$ *half-spin rep.*, $V(1) \otimes V(16))$.

(6′) $(Spin\,(10)$, *half-spin rep.*, $V(16))$.

## § 8. Prehomogeneous vector spaces with finitely many orbits

In this section, we shall study some necessary conditions that a number of orbits of a given irreducible triplet $(G, \rho, V)$ is finite. Clearly, such a triplet must be a P.V. from the dimension reasons. By Theorem 1 in § 1, the Lie algebra $\mathfrak{g}$ of $\rho(G)$ is reductive with at most one-dimensional center. First we shall consider the case when the center of $\mathfrak{g}$ is of one dimension. Then as we have seen in § 3, we may assume that $G = GL(1) \times G_1 \times \cdots \times G_k$, $\rho = \square \otimes \rho_1 \otimes \cdots \otimes \rho_k$, $V = V(1) \otimes V(d_1) \otimes \cdots \otimes V(d_k)$ with $d_1 \geq d_2 \geq \cdots \geq d_k \geq 2$, where each $G_i$ is a connected almost simple algebraic group, $\rho_i$ is an irreducible representation of $G_i$ on the $d_i$-dimensional vector space $V(d_i)$ $(1 \leq i \leq k)$, and $\square$ is the standard representation of $GL(1)$ on the one-dimensional vector space $V(1)$.

PROPOSITION 1. *Let $\rho: G' \to GL(V')$ be an irreducible representation of a semi-simple algebraic group $G'$ on $V'$. Assume that a number of orbits of a triplet $(G' \times GL(n),\ \rho \otimes \Lambda_1,\ V' \otimes V(n))$ is finite. Then for any $k \leq n$, a number of orbits of a triplet $(G' \times GL(k),\ \rho \otimes \Lambda_1,\ V' \otimes V(k))$ is also finite.*

*Proof.* Identify $V = V' \otimes V(n)$ with $V' \oplus \cdots \oplus V'$ ($n$-copies) and for any point $v = (v_1, \cdots, v_n)$ of $V$, let $\varphi(v)$ be a subspace of $V'$ generated by $v_1, \cdots, v_n$. Then $\varphi$ is a surjective map from $V$ to a set $T = \mathrm{Grass}_\ell(V') \cup \cdots \cup \mathrm{Grass}_0(V')$ where $\ell = \min(\dim V', n)$ and $\mathrm{Grass}_r(V')$ denotes the Grassmann variety consisting of $r$-dimensional subspaces of $V'$. Since $GL(n)$ acts homogeneously on each fibre of $\varphi$, there is a one-to-one correspondence between the orbits of a triplet $(G' \times GL(n),\ \rho \otimes \Lambda_1,\ V' \otimes V(n))$ and the orbits of $G'$ on $T$. Therefore, by assumption, a number of $G$-orbits on $\mathrm{Grass}_r(V')$ $(0 \leq r \leq \ell)$ is finite. In particular, for $\ell' = \min(\dim V', k)$, a number of $G$-orbits on $T' = \mathrm{Grass}_{\ell'}(V') \cup \cdots \cup \mathrm{Grass}_0(V')$ is finite, and hence we obtain our assertion. Q.E.D.

PROPOSITION 2. *Let $(G, \rho, V)$ be an irreducible P.V. with finitely many orbits. Then we have $0 \leq k \leq 3$.*

*Proof.* Assume that $k \geq 4$. Then a triplet $(SL(d_1) \times SL(d_2) \times SL(d_3)$

$\times GL(d_4 \cdots d_k)$, $\Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$, $V(d_1) \otimes V(d_2) \otimes V(d_3) \otimes V(d_4 \cdots d_k)$ has finitely many orbits. Therefore by Proposition 1, a triplet $(SL(d_1) \times SL(d_2) \times SL(d_3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(d_1) \otimes V(d_2) \otimes V(d_3) \otimes V(2))$ has also finitely many orbits. By repeating this procedure, a triplet $(SL(2) \times SL(2) \times SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2) \otimes V(2) \otimes V(2))$ has finitely many orbits. This is a contradiction since $\dim SL(2) \times SL(2) \times SL(2) \times GL(2) = 13 < \dim V(2) \otimes V(2) \otimes V(2) \otimes V(2) = 16$. Q.E.D.

Assume that a number of orbits of a triplet $(G, \rho, V)$ is finite. If $k \leq 1$, then it is a reduced P.V. and hence by Theorem 54 in § 5, we obtain the following assertion.

PROPOSITION 3. *If $k \leq 1$, it is equivalent to one of the following P.V.'s.*

I) *A Regular P.V.*
(1) $(GL(1), \square, V(1))$ $(k = 0)$.
(2) $(GL(n), 2\Lambda_1, V(\frac{1}{2}n(n+1)))$ $(n \geq 2)$.
(3) $(GL(2m), \Lambda_2, V(m(2m-1)))$ $(m \geq 3)$.
(4) $(GL(2), 3\Lambda_1, V(4))$.
(5) $(GL(6), \Lambda_3, V(20))$.
(6) $(GL(7), \Lambda_3, V(35))$.
(7) $(GL(8), \Lambda_3, V(56))$.
(8) $(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$.
(9) $(GL(1) \times SO(n), \square \otimes \Lambda_1, V(1) \otimes V(n))$ $(n \geq 3)$.
(10) $(GL(1) \times Spin\,(7), \square \otimes$ *spin rep.*, $V(1) \otimes V(8))$.
(11) $(GL(1) \times Spin\,(9), \square \otimes$ *spin rep.*, $V(1) \otimes V(16))$.
(12) $(GL(1) \times Spin\,(11), \square \otimes$ *spin rep.*, $V(1) \otimes V(32))$.
(13) $(GL(1) \times Spin\,(12), \square \otimes$ *half-spin rep.*, $V(1) \otimes V(32))$.
(14) $(GL(1) \times Spin\,(14), \square \otimes$ *half-spin rep.*, $V(1) \otimes V(64))$.
(15) $(GL(1) \times (G_2), \square \otimes \Lambda_2, V(1) \otimes V(7))$.
(16) $(GL(1) \times E_6, \square \otimes \Lambda_1, V(1) \otimes V(27))$.
(17) $(GL(1) \times E_7, \square \otimes \Lambda_6, V(1) \otimes V(56))$.

II) *A Non-regular P.V. without Relative Invariants.*
(1) $(GL(n), \Lambda_1, V(n))$ $(n \geq 2)$.
(2) $(GL(2m+1), \Lambda_2, V(m(2m+1)))$ $(m \geq 2)$.
(3) $(GL(1) \times Sp(n), \square \otimes \Lambda_1, V(1) \otimes V(2n))$ $(n \geq 2)$.
(4) $(GL(1) \times \mathrm{Spin}\,(10), \square \otimes$ *half-spin rep.*, $V(1) \otimes V(16))$.

Now we shall consider the case of $k=2$. Then it is reduced or is a castling transform of a reduced P.V. with $k \leq 2$. First we shall consider a castling transform of a P.V. with $k=1$.

PROPOSITION 4. *The following non-reduced P.V.'s have infinitely many orbits.*

I) *A Regular P.V.*

(1) $(SL(n) \times GL(\frac{1}{2}n(n+1)-1),\ 2\Lambda_1 \otimes \Lambda_1,\ V(\frac{1}{2}n(n+1)) \otimes V(\frac{1}{2}n(n+1) - 1))$ $(n \geq 3)$.

(2) $(SL(2m) \times GL(m(2m-1)-1),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m-1)) \otimes V(m(2m-1)-1))$ $(m \geq 3)$.

(3) $(SL(2) \times GL(3),\ 3\Lambda_1 \otimes \Lambda_1,\ V(4) \otimes V(3))$.

(4) $(SL(6) \times GL(19),\ \Lambda_3 \otimes \Lambda_1,\ V(20) \otimes V(19))$.

(5) $(SL(7) \times GL(34),\ \Lambda_3 \otimes \Lambda_1,\ V(35) \otimes V(34))$.

(6) $(SL(8) \times GL(55),\ \Lambda_3 \otimes \Lambda_1,\ V(56) \otimes V(55))$.

(7) $Sp(3) \times GL(13),\ \Lambda_3 \otimes \Lambda_1,\ V(14) \otimes V(13))$.

(8) $(\mathrm{Spin}\,(7) \times GL(7)$, *spin rep.* $\otimes \Lambda_1,\ V(8) \otimes V(7))$.

(9) $(\mathrm{Spin}\,(9) \times GL(15)$, *spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(15))$.

(10) $(\mathrm{Spin}\,(11) \times GL(31)$, *spin rep.* $\otimes \Lambda_1,\ V(32) \otimes V(31))$.

(11) $(\mathrm{Spin}\,(12) \times GL(31)$, *half-spin rep.* $\otimes \Lambda_1,\ V(32) \otimes V(31))$.

(12) $(\mathrm{Spin}\,(14) \times GL(63)$, *half-spin rep.* $\otimes \Lambda_1,\ V(64) \otimes V(63))$.

(13) $((G_2) \times GL(6),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(6))$.

(14) $(E_6 \times GL(26),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(26))$.

(15) $(E_7 \times GL(55),\ \Lambda_6 \otimes \Lambda_1,\ V(56) \otimes V(55))$.

II) *A Non-regular P.V. without Relative Invariants.*

(1) $(SL(2m+1) \times GL(m(2m+1)-1),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(m(2m+1)-1))$ $(m \geq 2)$.

(2) $(\mathrm{Spin}\,(10) \times GL(15)$, *half-spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(15))$.

*Proof.* Assume that (1) has finitely many orbits. Then by Proposition 1, a triplet $(SL(n) \times GL(3),\ 2\Lambda_1 \otimes \Lambda_1,\ V(\frac{1}{2}n(n+1)) \otimes V(3))$ has also finitely many orbits. Since $\dim SL(n) \times GL(3) < \dim V(\frac{1}{2}n(n+1)) \otimes V(3)$, it is not a P.V. and hence it has infinitely many orbits, i.e., a contradiction. Similarly, we can prove that other P.V.'s in Proposition 4 have infinitely many orbits. Q.E.D.

Next we shall consider a trivial P.V. $(G \times GL(m),\ \rho \otimes \Lambda_1,\ V(n) \otimes V(m))$ with $k=2$, $m \geq n$. If a number of orbits of this P.V. is finite, then by Proposition 1, a triplet $(GL(1) \times G, \square \otimes \rho, V(1) \otimes V(n))$ and a triplet $(G \times GL(n-1), \rho \otimes \Lambda_1, V(n) \otimes V(n-1))$ have also finitely many orbits. Therefore, by Proposition 3 and Proposition 4, we have $G = SL(n)$, $SO(n)$, or $Sp(n')$ ($n = 2n'$ is even). Now we shall consider a castling transform of a reduced P.V. with $k = 2$.

PROPOSITION 5. *The following non-reduced P.V.'s have infinitely many orbits.*

I) *A Regular P.V.*

(1) $(SL(3) \times GL(4),\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(4))$.

(2) $(SL(6) \times GL(13),\ \Lambda_2 \otimes \Lambda_1,\ V(15) \otimes V(13))$.

(3) $(\mathrm{Spin}\,(7) \times GL(6),\ \textit{spin rep.} \otimes \Lambda_1,\ V(8) \otimes V(6))$.

(4) $(\mathrm{Spin}\,(7) \times GL(5),\ \textit{spin rep.} \otimes \Lambda_1,\ V(8) \otimes V(5))$.

(5) $(\mathrm{Spin}\,(10) \times GL(14),\ \textit{half-spin rep.} \otimes \Lambda_1,\ V(16) \otimes V(14))$.

(6) $(\mathrm{Spin}\,(10) \times GL(13),\ \textit{half-spin rep.} \otimes \Lambda_1,\ V(16) \otimes V(13))$.

(7) $((G_2) \times GL(5),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(5))$.

(8) $(E_6 \times GL(25),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(25))$.

II) *A Non-regular P.V.*

(9) $(SL(2m+1) \times GL(m(2m+1)-2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(m(2m+1)-2))$.

*Proof.* If (3) or (4) has finitely many orbits, then a triplet $(\mathrm{Spin}\,(7) \times GL(4), \text{spin rep.} \otimes \Lambda_1, V(8) \otimes V(4))$ has also finitely many orbits by Proposition 1. However, this triplet is not a P.V. by Proposition 28 in § 5, i.e., a contradiction and hence (3) and (4) has infinitely many orbits. If (5) or (6) has finitely many orbits, a triplet $(\mathrm{Spin}\,(10) \times GL(4), \text{half-spin rep.} \otimes \Lambda_1, V(16) \otimes V(4))$ has also finitely many orbits. Since $\dim \mathrm{Spin}\,(10) \times GL(4) = 61 < 64 = \dim V(16) \otimes V(4)$, this triplet is not a P.V., i.e., a contradiction. Similarly, we can prove our assertion for (1), (2), (7), (8). Q.E.D.

PROPOSITION 6. *Let $(G, \rho, V)$ be a triplet with finitely many orbits and $k = 2$. Then it is equivalent to one of the following triplets. (Note that we have assumed that the center of $\rho(G)$ is one-dimensional.)*

I) *A Regular P.V.*

(1) $(SL(n) \times GL(n),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(n))$ $\quad (n \geq 2)$.

(2) $(SO(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $\quad (n \geq m \geq 2)$.

(3) $(Sp(n) \times GL(2m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m))$ $(n \geq m \geq 1)$.
(4) $(SL(3) \times GL(2),\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(2))$.
(5) $(SL(6) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(15) \otimes V(2))$.
(6) $(SL(5) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$.
(7) $(SL(5) \times GL(4),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(4))$.
(8) $(Spin\,(7) \times GL(2),$ *spin rep.* $\otimes \Lambda_1,\ V(8) \otimes V(2))$.
(9) $(\mathrm{Spin}\,(7) \times GL(3),$ *spin rep.* $\otimes \Lambda_1,\ V(8) \otimes V(3))$.
(10) $(\mathrm{Spin}\,(10) \times GL(2),$ *half-spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(2))$.
(11) $(\mathrm{Spin}\,(10) \times GL(3),$ *half-spin rep.* $\otimes \Lambda_1,\ V(16) \otimes V(3))$.
(12) $((G_2) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(2))$.
(13) $(E_6 \times GL(2),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(2))$.

II) *A Non-regular P.V. with Relative Invariants.*

$$(GL(1) \times Sp(n) \times SO(3),\ \square \otimes \Lambda_1 \otimes \Lambda_1,\ V(1) \otimes V(2n) \otimes V(3)).$$

III) *A Non-regular P.V. without Relative Invariants.*
(1) $(SL(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(n < m)$.
(2) $(SO(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(n < m)$,
(3) $(Sp(n) \times GL(2m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m))$ $(n < m)$.
(4) $(Sp(n) \times GL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$.
(5) $(SL(2m+1) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$ $(m \geq 2)$.

*Proof.* From Theorem 54 in § 5, Propositions 4 and 5, we obtain our assertion. Q.E.D.

Finally, we shall consider the case of $k = 3$. Assume that a triplet $(GL(1) \times G_1 \times G_2 \times G_3,\ \square \otimes \rho_1 \otimes \rho_2 \otimes \rho_3,\ V(1) \otimes V(d_1) \otimes V(d_2) \otimes V(d_3))$ has finitely many orbits. Then clearly a triplet $(SL(d_1) \times SL(d_2) \times GL(d_3),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(d_2) \otimes V(d_3))$ has also finitely many orbits and hence by Proposition 1, $(SL(d_2) \times SL(d_2) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_2) \otimes V(d_2) \otimes V(2))$ has also finitely many orbits. By Proposition 16 in § 5, it is not a P.V. if $d_2 \geq 4$, and hence we have $3 \geq d_2 \geq 2$. If $d_2 = 2$, then our triplet is equivalent to a triplet $(GL(1) \times G_1 \times SO(4),\ \square \otimes \rho_1 \otimes \Lambda_1,\ V(1) \otimes V(d_1) \otimes V(4))$. If $G_1 \neq SL(d_1)$, then this is a reduced P.V., i.e., a contradiction in view of Theorem 54 in § 5. Therefore, $G_1 = SL(d_1)$. Assume that $d_2 = 3$. If $d_3 = 3$, then a triplet $(SL(3) \times SL(3) \times GL(3),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(3) \otimes V(3))$ has also finitely many orbits. This is a contradiction since $\dim SL(3) \times SL(3) \times GL(3) = 25 < 27 = \dim V(3) \otimes V(3) \otimes V(3)$, and hence we have $d_3 = 2$, $G_3 = SL(2)$, $\rho_3 = \Lambda_1$. Since $d_2 = 3$, we have $G_2 = SL(2)$, $\rho_2 = 2\Lambda_1$ or $G_2 = SL(3)$, $\rho_2 = \Lambda_1$. Assume that $G_2 = SL(2)$. Then this triplet is equivalent to $(G_1 \times SO(3) \times GL(2),\ \rho_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$ $(d_1 \geq 3)$. If $G_1 \neq SL(d_1)$, then it is a reduced P.V., i.e., a contradiction in view of Theorem 54 in § 5 and we have $(SL(d_1) \times SO(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$. Since $(d_1^2 - 1) + 3 + 4 \geq 6d_1$, we have $d_1 \geq 5$. Note that it is regular if $6 \geq d_1 \geq 5$ and not regular if $d_1 \geq 7$. Assume that $G_2 = SL(3)$. Then a triplet $(G_1 \times GL(3),\ \rho_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3))$ has finitely many orbits by Proposition 1. Therefore, by Proposition 6, $G_1 = SL(d_1)$, $SO(d_1)$, $Sp(d_1')$ $(d_1 = 2d_1'$: even), $SL(5)$ $(d_1 = 10)$, Spin (7) $(d_1 = 8)$ or Spin (10) $(d_1 = 16)$. However, if $G_1 \neq SL(d_1)$, then $(G_1 \times SL(3) \times GL(2),\ \rho_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$ is a reduced P.V., i.e., a contradiction in view of Theorem 54 in § 4. Hence we have $(SL(d_1) \times SL(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$.

PROPOSITION 7. *If $k = 3$, then it is equivalent to one of the following triplets.*

(1) $(SL(d_1) \times SL(2) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(2) \otimes V(2)) \simeq (SO(4) \times GL(d_1),\ \Lambda_1 \otimes \Lambda_1,\ V(4) \otimes V(d_1))$ $(d_1 \geq 2)$.

(2) $(SL(d_1) \times SO(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$ $(d_1 \geq 5)$.

(3) $(SL(d_1) \times SL(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$ $(d_1 \geq 3)$.

Now consider the case when $G$ is semi-simple. Then $(GL(1) \times G,\ \square \otimes \rho,\ V(1) \otimes V(d))$ is a non-regular P.V. with finitely many orbits and hence by Propositions 3, 6, and 7, it is equivalent to one of the following P.V.'s.

(1) $(SL(n),\ \Lambda_1,\ V(n))$ $(n \geq 2)$.

(2) $(SL(2m+1),\ \Lambda_2,\ V(m(2m+1)))$ $(m \geq 2)$.

(3) $(Sp(n),\ \Lambda_1,\ V(2n))$ $(n \geq 2)$.

(4) (Spin (10), *half-spin rep.*, $V(16)$).

(5) $(SL(n) \times SL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(n < m)$.

(6) $(SO(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(n < m)$.

(7) $(Sp(n) \times SL(2m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m))$ $(n < m)$.

(8) $(Sp(n) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$.

(9) $(SL(2m+1) \times SL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$.

(10) $(SL(d_1) \times SO(3) \times SL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$ $(d_1 \geq 7)$.

(11) $(SL(d_1) \times SL(3) \times SL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(d_1) \otimes V(3) \otimes V(2))$
$(d_1 \geq 7$ or $d_1 = 5)$.

Therefore we obtain the following theorem.

THEOREM 8. (i) *A trivial P.V. with finitely many orbits is equivalent to one of the following P.V.'s.*

(1) $(SL(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(1 \leq n \leq m)$.
(2) $(SO(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(3 \leq n \leq m)$.
(3) $(Sp(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(m))$ $(2n \leq m)$.
(4) $(SO(3) \times SL(2) \times GL(d),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(2) \otimes V(d))$
$(d \geq 6)$.
(5) $(SL(3) \times SL(2) \times GL(d),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(2) \otimes V(d))$
$(d \geq 6)$.
(1′) $(SL(n) \times SL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(1 \leq n < m)$.
(2′) $(SO(n) \times SL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(3 \leq n < m)$.
(3′) $(Sp(n) \times SL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(m))$ $(2n < m)$.
(4′) $(SO(3) \times SL(2) \times SL(d),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(2) \otimes V(d))$
$(d \geq 7)$.
(5′) $(SL(3) \times SL(2) \times SL(d),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(2) \otimes V(d))$
$(d \geq 7)$.

(ii) *A non-trivial non-reduced P.V. with finitely many orbits is equivalent to one of the following P.V.'s.*

(1) $(SO(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(n > m > n/2)$.
(2) $(Sp(n) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(m))$ $(2n > m > n)$.
(3) $(SO(3) \times SL(2) \times GL(5),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(2) \otimes V(5))$.
(4) $(SL(4) \times SL(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(4) \otimes V(3) \otimes V(2))$.
(5) $(SL(5) \times SL(3) \times GL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(5) \otimes V(3) \otimes V(2))$.
(1′) $(Sp(n) \times SL(2m+1),\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$
$(2n > 2m+1 > n)$.
(2′) $(SL(5) \times SL(3) \times SL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(5) \otimes V(3) \otimes V(2))$.

*Remark* 9. Theorem 8 says that a P.V. with finitely many orbits is reduced with few exceptions. However, the orbital decomposition is necessary to prove that such a P.V. has actually finitely many orbits. The orbital decomposition of reduced P.V.'s is in almost all cases completed by the second author (see [12]), J. Igusa (see [2]), I. Ozeki (see [20]), Kawahara (see [13]) and S. J. Harris (see [7]). These orbital structures have very close relations with the $b$-functions and Fourier transforms of the relative invariants.

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