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CHARACTER FORMULAS FOR DISCRETE SERIES ON SEMISIMPLE LIE GROUPS

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§ 1. Introduction

Let G be a connected, semisimple real Lie group with finite center, K a maximal compact subgroup of G. Assume $\operatorname{rank} G = \operatorname{rank} K$. Let $\mathfrak{G}$ be the Lie algebra of $G, \mathfrak{G}_C$ its complexification. If G_C is the simply connected complex analytic group corresponding to $\mathfrak{G}_C$, assume G is the real analytic subgroup of G_C corresponding to $\mathfrak{G}$.

In this case, G always has discrete series representations. The characters of these representations are distributions on the group G, realizable as locally integrable functions. Formulas for these characters are known up to certain integer constants which have only been evaluated for a few special cases. The purpose of this paper is to give information on how these constants can be computed in general, and to illustrate the method for several new cases.

For any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{G}$, let $\mathfrak{h}_C$ denote its complexification, $\Phi(\mathfrak{G}_C, \mathfrak{h}_C)$ the set of roots of the pair $(\mathfrak{G}_C, \mathfrak{h}_C)$, and $W(\mathfrak{G}_C, \mathfrak{h}_C)$ the Weyl group generated by the reflections corresponding to the roots. Let $\pi^{\mathfrak{h}}(H) = \prod \alpha(H)$, the product over all α in $\Phi^+(\mathfrak{G}_C, \mathfrak{h}_C)$, H any element of $\mathfrak{h}$.

Denote by $\mathfrak{k}$ the subalgebra of $\mathfrak{G}$ corresponding to K, and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{G}$ such that $\mathfrak{t} \subseteq \mathfrak{k}$. Consider the space $\mathscr{F}$ of all pure imaginary linear functions on $\mathfrak{t}$. Let $\mathscr{F}' = \{\lambda \in \mathscr{F} : \langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Phi(\mathfrak{G}_C, \mathfrak{t}_C)\}$, the regular elements of $\mathscr{F}$. Then for each $\lambda \in \mathscr{F}'$ there exists a unique invariant distribution T_λ on $\mathfrak{G}$ characterized by certain properties [2a), p. 277].

Let W_K be the subgroup of $W(\mathfrak{G}_C, \mathfrak{t}_C)$ generated by reflections corresponding to the compact roots of $(\mathfrak{G}, \mathfrak{t})$. Then for $H \in \mathfrak{t}' = \mathfrak{t} \cap \mathfrak{G}'$, $\mathfrak{G}'$ the set of regular elements of $\mathfrak{G}$,

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(1.1) $$T_\lambda(H) = \pi^{\mathfrak{t}}(H)^{-1} \sum_{w \in W_K} \det w \exp(w\lambda(H)) \ .$$

Given any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{G}$, there exists $y \in G_C$ such that $y(\mathfrak{t}_C) = \mathfrak{h}_C$. Then for any connected component $\mathfrak{h}^+$ of $\mathfrak{h}'(R) = \{H \in \mathfrak{h} : \alpha(H) \neq 0$ for all real roots $\alpha \in \Phi(\mathfrak{G}_C, \mathfrak{h}_C)\}$, there are integers $c_y(w : \lambda : \mathfrak{h}^+)$ such that for $H \in \mathfrak{h}^+ \cap \mathfrak{h}'$, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{G}'$,

(1.2) $$T_\lambda(H) = \pi^{\mathfrak{h}}(H)^{-1} \sum_{w \in W(\mathfrak{G}_C, \mathfrak{t}_C)} \det w c_y(w : \lambda : \mathfrak{h}^+) \exp({}^y(w\lambda)(H)) \ .$$

In § 2 of this paper we outline an inductive procedure by which these integers can be computed, and in § 3 illustrate the method by explicit computations for the first two stages of the induction in the general case, and by giving the complete solution for the case where $\mathfrak{G}$ has exactly $n + 1$ conjugacy classes of Cartan subalgebras, $n = \operatorname{rank}(G/K)$.

The integers have been computed by Harish-Chandra for the case $\operatorname{rank}(G/K) = 1$ ([2b)]) and by H. Ferguson for the case where $\mathfrak{G}_C$ is the simple complex Lie algebra with root system of type G_2 ([1]). The method given in this paper is different from that used by Harish-Chandra for $\operatorname{rank}(G/K) = 1$ (although it relies heavily on his work in [2a)] and [2b)]), and is an extension of the method used by Ferguson. Hirai has computed the integers for the groups $SU(p, q)$ and $Sp(2, R)$ using specific matrix computations.

The unitary character group $\hat{T}$ of T, T the Cartan subgroup of G corresponding to $\mathfrak{t}$, may be identified with a lattice, L_T, in $\mathscr{F}$. To each $\lambda \in L_T$ is associated a central eigendistribution Θ_λ on G ([2a), p. 289], [2b), p. 90]). If $\lambda \in L'_T = L_T \cap \mathscr{F}'$, a constant multiple of Θ_λ is the character of a discrete series representation of G, and all discrete series characters are of this form.

To explicitly describe the Θ_λ it is necessary to evaluate certain constants which are directly related to the integers $c_y(w : \lambda : \mathfrak{h}^+)$. In § 4, using the results of § 3, we give explicit formulas for the Θ_λ on the Cartan subgroups of G having vector part of dimension one or two. The Cartan subgroups in the dimension one case are those corresponding to maximal parabolic subgroups of G. The results for dimension two give a complete solution for the case $\operatorname{rank}(G/K) = 2$. We also give complete formulas in the case where G has exactly $n + 1$ conjugacy classes of Cartan subgroups, $n = \operatorname{rank}(G/K)$.

§ 2. The Constants on $\mathfrak{G}$

We retain the notation of the introduction. Since T_λ is an invariant distribution, it suffices to determine the integers $c_y(w:\lambda:\mathfrak{h}^+)$ for one representative of each conjugacy class of Cartan subalgebras. Let θ be the Cartan involution of $\mathfrak{G}$ with associated Cartan decomposition $\mathfrak{G}=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{k}$ as above. Then each conjugacy class of Cartan subalgebra contains a representative which is θ-stable. Thus we may restrict ourselves to considering θ-stable Cartan subalgebras.

If $\mathfrak{h}$ is a θ-stable Cartan subalgebra of $\mathfrak{G}$, $\mathfrak{h}=\mathfrak{h}_k+\mathfrak{h}_p$, where $\mathfrak{h}_k=\mathfrak{h}\cap\mathfrak{k}$, $\mathfrak{h}_p=\mathfrak{h}\cap\mathfrak{p}$. We determine the integers $c_y(w:\lambda:\mathfrak{h}^+)$ by induction on $r=\dim\mathfrak{h}_p$. To perform the induction we need the following facts.

(2.1) ([2a), p. 277]). Let Γ be a semi-regular element of noncompact type in $\mathfrak{G}$. Let $\mathfrak{h}_1=\mathfrak{G}_\Gamma^+$ and $\mathfrak{h}_2=\mathfrak{G}_\Gamma^-$ be the corresponding Cartan subalgebras constructed as in [4, Volume I, p. 102], $\nu=\exp(-\pi\sqrt{-1}/4\,\mathrm{ad}\,(X_\Gamma^*+Y_\Gamma^*))$, $\Phi^+(\mathfrak{G}_C,\mathfrak{h}_{2C})=\{{}^\nu\beta:\beta\in\Phi^+(\mathfrak{G}_C,\mathfrak{h}_{1C})\}$, and α the unique positive real root of $\mathfrak{h}_1$ satisfying $\alpha(\Gamma)=0$. Let $y\in G_C$ satisfy $y(\mathfrak{t}_C)=\mathfrak{h}_{1C}$. Let $\mathfrak{h}_1^+$ and $\mathfrak{h}_1^-$ be the connected components of $\mathfrak{h}_1'(R)$ satisfying $\Gamma\in\mathrm{cl}\,(\mathfrak{h}_1^\pm)$. Let $\mathfrak{h}_2^+$ be the connected component of $\mathfrak{h}_2'(R)$ containing Γ. Then $\nu y(\mathfrak{t}_C)=\mathfrak{h}_{2C}$, and for $w\in W(\mathfrak{G}_C,\mathfrak{t}_C)$, $\lambda\in\mathscr{F}'$,

$$\begin{aligned}&c_y(w:\lambda:\mathfrak{h}_1^+)+c_y({}^{y^{-1}}s_\alpha w:\lambda:\mathfrak{h}_1^+)\\&=c_y(w:\lambda:\mathfrak{h}_1^-)+c_y({}^{y^{-1}}s_\alpha w:\lambda:\mathfrak{h}_1^-)\\&=c_{\nu y}(w:\lambda:\mathfrak{h}_2^+)+c_{\nu y}({}^{y^{-1}}s_\alpha w:\lambda:\mathfrak{h}_2^+)\,.\end{aligned}$$

(2.2) Suppose $x,y\in G_C$ such that $\mathfrak{h}_C=x(\mathfrak{t}_C)=y(\mathfrak{t}_C)$. Then for some $w_0\in W(\mathfrak{G}_C,\mathfrak{t}_C)$, ${}^x\lambda={}^yw_0{}^y\lambda$ for all $\lambda\in\mathscr{F}$. Then for all $w\in W(\mathfrak{G}_C,\mathfrak{t}_C)$, ${}^x(w\lambda)={}^y(w_0w\lambda)$ and hence $c_x(w:\lambda:\mathfrak{h}^+)=\det w_0c_y(w_0w:\lambda:\mathfrak{h}^+)$.

(2.3) ([2a), p. 272]). For $\mathfrak{h},\mathfrak{h}^+$, and y as above, $c_y(w:\lambda:\mathfrak{h}^+)=0$ unless $\mathrm{Re}\,{}^y(w\lambda)(H)\le 0$ for all $H\in\mathfrak{h}^+$.

(2.4) Suppose $\tilde{\mathfrak{h}}=x(\mathfrak{h})$ for some $x\in G$. Let $\tilde{\mathfrak{h}}^+=x(\mathfrak{h}^+)$ and suppose $\Phi^+(\mathfrak{G}_C,\tilde{\mathfrak{h}}_C)=\{{}^x\alpha:\alpha\in\Phi^+(\mathfrak{G}_C,\mathfrak{h}_C)\}$. Then for $H\in\mathfrak{h}^+$,

$$\begin{aligned}T_\lambda(H)=T_\lambda({}^xH)&=\pi^{\mathfrak{h}}(H)^{-1}\sum_{w\in W(\mathfrak{G}_C,\mathfrak{t}_C)}\det wc_y(w:\lambda:\mathfrak{h}^+)\exp({}^y(w\lambda)(H))\\&=\pi^{\tilde{\mathfrak{h}}}({}^xH)^{-1}\sum_{w\in W(\mathfrak{G}_C,\mathfrak{t}_C)}\det wc_y(w:\lambda:\mathfrak{h}^+)\exp({}^{xy}(w\lambda)({}^xH))\,.\end{aligned}$$

Therefore $c_{xy}(w:\lambda:\tilde{\mathfrak{h}}^+)=c_y(w:\lambda:\mathfrak{h}^+)$.

(2.5) ([2a), p. 281]). For $s \in W(\mathfrak{G}_C, \mathfrak{t}_C)$, $t \in W(G, H) = N_G(H)/H$, H the Cartan subgroup of G corresponding to $\mathfrak{h}$, $N_G(H)$ the normalizer of H in G, and $u \in W_K$, $c_y({}^{y^{-1}}tsu^{-1} : u\lambda : t\mathfrak{h}^+) = c_y(s : \lambda : \mathfrak{h}^+)$. If we denote by W_R the subgroup of $W(\mathfrak{G}_C, \mathfrak{h}_C)$ generated by s_α, α real, then $\mathfrak{h}'(R) = \bigcup_{s \in W_R} s\mathfrak{h}^+, \mathfrak{h}^+$ any component of $\mathfrak{h}'(R)$. Since $W_R \subseteq W(G, H)$, $c_y(w : \lambda : s\mathfrak{h}^+) = c_y({}^{y^{-1}}(s^{-1})w : \lambda : \mathfrak{h}^+)$. Thus in each case, it suffices to compute the constants for one component of $\mathfrak{h}'(R)$.

Suppose $\dim \mathfrak{h}_p = 0$. Then $\mathfrak{h} = \mathfrak{h}_k$ is a Cartan subalgebra of $\mathfrak{k}$ and is conjugate to $\mathfrak{t}$ by an element of K, say $\mathfrak{h} = k(\mathfrak{t})$. $\mathfrak{h}'(R) = \mathfrak{h}$ has exactly one connected component since $\mathfrak{h}$ has no real roots. By (2.4) and (1.1)

$$(2.6) \qquad c_k(w : \lambda : \mathfrak{h}) = c_1(w : \lambda : \mathfrak{t}) = \begin{cases} 1, & w \in W_K \\ 0, & w \notin W_K . \end{cases}$$

If y is any element of G_C such that $y(\mathfrak{t}_C) = \mathfrak{h}_C$, then, choosing w_0 as in (2.3),

$$(2.7) \qquad c_y(w : \lambda : \mathfrak{h}) = \det w_0 c_k(w_0 w : \lambda : \mathfrak{h}) = \begin{cases} \det w_0, & w \in w_0^{-1} W_K \\ 0, & w \notin w_0^{-1} W_K . \end{cases}$$

We may now assume inductively that there is an $r \geq 1$ such that for any θ-stable Cartan subalgebra $\mathfrak{h}$ with $\dim \mathfrak{h}_p < r$ we know the value of $c_y(w : \lambda : \mathfrak{h}^+)$ for every component $\mathfrak{h}^+$ of $\mathfrak{h}'(R)$, and every $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$, $\lambda \in \mathscr{F}'$, and $y \in G_C$ with $\mathfrak{h}_C = y(\mathfrak{t}_C)$.

Let $\mathfrak{j} = \mathfrak{j}_k + \mathfrak{j}_p$ be a θ-stable Cartan subalgebra of $\mathfrak{G}$ with $\dim \mathfrak{j}_p = r$, $\mathfrak{j}_C = y(\mathfrak{t}_C)$, some $y \in G_C$. Let Φ_R be the set of those $\alpha \in \Phi(\mathfrak{G}_C, \mathfrak{j}_C)$ which assume real values on $\mathfrak{j}$. Then $\dim \Phi_R = \dim \mathfrak{j}_p$. Let $\mathfrak{j}^+$ be a connected component of $\mathfrak{j}'(R)$. Define Φ_R^+ by $\alpha \in \Phi_R^+$ if $\alpha(H) > 0$ for all $H \in \mathfrak{j}^+$. Then Φ_R^+ is a set of positive roots for Φ_R. $\mathfrak{j}^- = \{-H : H \in \mathfrak{j}^+\}$ is also a component of $\mathfrak{j}'(R)$.

Let $\{\beta_1, \cdots, \beta_r\}$ be a strongly orthogonal system of positive real roots in Φ_R^+. That is, for $1 \leq i \neq j \leq r$, $\beta_i \pm \beta_j \notin \Phi_R$. Such a system of r elements exists because of the correspondence between these systems and conjugacy classes of Cartan subalgebras in $\mathfrak{G}$, and the fact that $\mathfrak{G}$ has a compact Cartan subalgebra. (See [4, Volume I, Section 1.3.1]). The β_i, $1 \leq i \leq r$, are in particular, mutually orthogonal, and the H_{β_i}, $1 \leq i \leq r$, form a basis for $\mathfrak{j}_p$, where for any root α, H_α will always denote the element of $\mathfrak{j}_C$ satisfying $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{j}_C$, B the Cartan Killing form. We also write $H_\alpha^* = 2H_\alpha / \langle \alpha, \alpha \rangle$.

Set $w^Q = s_{\beta_1} \cdots s_{\beta_r}$. Then $w^Q\mathfrak{j}^+ = \mathfrak{j}^-$. Note that if $\lambda \in \mathscr{F}$, $H \in \mathfrak{j}^+$, $H = H_k + \sum_{j=1}^r r_j H_{\beta_j}$ where $H_k \in \mathfrak{j}_k$, $r_j \in \boldsymbol{R}$, $1 \le j \le r$, then $\mathrm{Re}\,\{{}^y\lambda(w^Q H)\} = \mathrm{Re}\,\{{}^y\lambda(H_k - \sum r_j H_{\beta_j})\} = {}^y\lambda(-\sum r_j H_{\beta_j}) = -\mathrm{Re}\,\{{}^y\lambda(H)\}$. Decompose w^Q into a product of reflections corresponding to simple roots, $w^Q = s_m \cdots s_1$, $s_j = s_{\alpha_{i_j}}$ $\{\alpha_1, \cdots, \alpha_r\}$ a set of simple roots for Φ_R^+, $1 \le i_j \le r$ for all $1 \le j \le m$.

For each α_ℓ, $1 \le \ell \le r$, we construct a Cartan subalgebra $\mathfrak{j}_\ell$ of $\mathfrak{G}$ as follows. Let $\Gamma_\ell \in cl(\mathfrak{j}^+)$ be such that $\pm\alpha_\ell(\Gamma_\ell) = 0$, $\alpha(\Gamma_\ell) \ne 0$ for any other $\alpha \in \Phi(\mathfrak{G}_C, \mathfrak{j}_C)$. Then Γ_ℓ is a semi-regular element of noncompact type. Let $\mathfrak{G}_\ell$ be the centralizer of Γ_ℓ in $\mathfrak{G}$. Then $\mathfrak{G}_\ell = \mathfrak{c}_\ell + \mathfrak{l}_\ell$ where $\mathfrak{c}_\ell$ is the center of $\mathfrak{G}_\ell$ and $\mathfrak{l}_\ell = [\mathfrak{G}_\ell, \mathfrak{G}_\ell]$ is semisimple and isomorphic to $sl(2, \boldsymbol{R})$. Denote by $H_\ell^*, X_\ell^*, Y_\ell^*$ the standard basis of $\mathfrak{l}_\ell$ satisfying $[H_\ell^*, X_\ell^*] = 2X_\ell^*$, $[H_\ell^*, Y_\ell^*] = -2Y_\ell^*$, $[X_\ell^*, Y_\ell^*] = H_\ell^*$. Then $\mathfrak{G}_\ell^+ = \mathfrak{c}_\ell + \boldsymbol{R}H_\ell^* = \mathfrak{j}$, and $\mathfrak{G}_\ell^- = \mathfrak{c}_\ell + \boldsymbol{R}(X_\ell^* - Y_\ell^*)$ is a θ-stable Cartan subalgebra of $\mathfrak{G}$ which we call $\mathfrak{j}_\ell$. $\dim(\mathfrak{j}_\ell \cap \mathfrak{p}) = r - 1$. Let $\mathfrak{j}_\ell^+$ be the connected component of $\mathfrak{j}_\ell'(R)$ containing Γ_ℓ, and let $\nu_\ell = \exp(-\pi\sqrt{-1}/4\,\mathrm{ad}\,(X_\ell^* + Y_\ell^*))$. Then $\nu_\ell(\mathfrak{j}_C) = \mathfrak{j}_{\ell C}$.

By (2.1), for all $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$ and $\lambda \in \mathscr{F}'$,

$$c_y(w : \lambda : \mathfrak{j}^+) + c_y({}^{y^{-1}}(s_{\alpha_\ell})w : \lambda : \mathfrak{j}^+) = c_{\nu_\ell y}(w : \lambda : \mathfrak{j}_\ell^+) + c_{\nu_\ell y}({}^{y^{-1}}(s_{\alpha_\ell})w : \lambda : \mathfrak{j}_\ell^+)$$

where the terms on the right side are known by the induction hypothesis.

Let $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$, $\lambda \in \mathscr{F}'$. Suppose there is $H \in \mathfrak{j}^+$ such that $\mathrm{Re}\,\{{}^y(w\lambda)(H)\} > 0$. By (2.2), $c_y(w : \lambda : \mathfrak{j}^+) = 0$. Thus we may assume $\mathrm{Re}\,\{{}^y(w\lambda)(H)\} \le 0$ for all $H \in \mathfrak{j}^+$. Since λ is regular, there is $H_0 \in \mathfrak{j}^+$ such that $\mathrm{Re}\,\{{}^y(w\lambda)(H_0)\} < 0$. Then $\mathrm{Re}\,\{{}^y({}^{y^{-1}}w^Q w\lambda)(H_0)\} = \mathrm{Re}\,\{{}^y(w\lambda)(w^Q H_0)\} = -\mathrm{Re}\,\{{}^y(w\lambda)(H_0)\} > 0$. Thus $c_y({}^{y^{-1}}w^Q w : \lambda : \mathfrak{j}^+) = 0$. By using (2.1) repeatedly, we obtain

$$\begin{aligned}
c_y(w : \lambda : \mathfrak{j}^+) &= c_{\nu_{i_1}y}(w : \lambda : \mathfrak{j}_{i_1}^+) + c_{\nu_{i_1}y}({}^{y^{-1}}s_1 w : \lambda : \mathfrak{j}_{i_1}^+) \\
&\quad - c_{\nu_{i_2}y}({}^{y^{-1}}s_1 w : \lambda : \mathfrak{j}_{i_2}^+) - c_{\nu_{i_2}y}({}^{y^{-1}}(s_2 s_1)w : \lambda : \mathfrak{j}_{i_2}^+) \\
&\quad \vdots \\
&\quad + (-1)^{j+1}\{c_{\nu_{i_j}y}({}^{y^{-1}}(s_{j-1} \cdots s_1)w : \lambda : \mathfrak{j}_{i_j}^+) \\
&\qquad\qquad + c_{\nu_{i_j}y}({}^{y^{-1}}(s_j \cdots s_1)w : \lambda : \mathfrak{j}_{i_j}^+)\} \\
&\quad \vdots \\
&\quad + (-1)^{m+1}\{c_{\nu_{i_m}y}({}^{y^{-1}}(s_{m-1} \cdots s_1)w : \lambda : \mathfrak{j}_{i_m}^+) \\
&\qquad\qquad + c_{\nu_{i_m}y}({}^{y^{-1}}w^Q w : \lambda : \mathfrak{j}_{i_m}^+)\}\,.
\end{aligned} \tag{2.8}$$

§ 3. The Constants for Special Cases

We will now illustrate the method outlined in § 2 by computing the integers $c_y(w:\lambda:\mathfrak{h}^+)$ for the cases $\dim \mathfrak{h}_p = 1, 2$, using the notation of § 2.

$\dim \mathfrak{h}_p = 1$: Suppose $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ is a θ-stable Cartan subalgebra with $\dim \mathfrak{h}_p = 1$. Let $\mathfrak{h}^+$ be a connected component of $\mathfrak{h}'(R)$. Then Φ_R^+ has exactly one element, call it α, and $\mathfrak{h}^+ = \{H_k + rH_\alpha^*: H_k \in \mathfrak{h}_k,\ r \in \boldsymbol{R},\ r > 0\}$. $\mathfrak{h}^- = \{H_k + rH_\alpha^*: H_k \in \mathfrak{h}_k,\ r \in \boldsymbol{R},\ r < 0\}$ is the only other component of $\mathfrak{h}'(R)$, and $w^Q = s_\alpha$ satisfies $s_\alpha \mathfrak{h}^+ = \mathfrak{h}^-$. For Γ a semi-regular element of $\mathfrak{h}$ corresponding to α, we have $\mathfrak{h} = \mathfrak{h}_\Gamma^+$, and $\mathfrak{h}_\Gamma^- \subseteq \mathfrak{k}$ is a compact Cartan subalgebra of $\mathfrak{G}$. Thus $k(\mathfrak{h}_\Gamma^-) = \mathfrak{t}$ for some $k \in K$, and $k\nu_\alpha(\mathfrak{h}_C) = \mathfrak{t}_C$.

Suppose $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$, $\lambda \in \mathscr{F}'$ such that $\operatorname{Re}\{{}^{\nu_\alpha^{-1}k^{-1}}(w\lambda)(H)\} \leq 0$ for all $H \in \mathfrak{h}^+$. Then by (2.8), $c_{\nu_\alpha^{-1}k^{-1}}(w:\lambda:\mathfrak{h}^+) = c_{k^{-1}}(w:\lambda:\mathfrak{h}_\Gamma^-) + c_{k^{-1}}({}^{(k\nu_\alpha)}s_\alpha w:\lambda:\mathfrak{h}_\Gamma^-)$ where by (2.6),

$$c_{k^{-1}}(w:\lambda:\mathfrak{h}_\Gamma^-) = \begin{cases} 1, & w \in W_K \\ 0, & w \notin W_K \end{cases}$$

and

$$c_{k^{-1}}({}^{(k\nu_\alpha)}s_\alpha w:\lambda:\mathfrak{h}_\Gamma^-) = \begin{cases} 1, & {}^{(k\nu_\alpha)}s_\alpha w \in W_K \ \text{ iff } \ w \in {}^{(k\nu_\alpha)}s_\alpha W_K \\ 0, & \text{otherwise} \end{cases}$$

$W_K \neq {}^{(k\nu_\alpha)}s_\alpha W_K$ as cosets of W_K in $W(\mathfrak{G}_C, \mathfrak{t}_C)$ since ${}^{\nu_\alpha}\alpha$ is a singular imaginary root of $\mathfrak{h}_\Gamma^-$ and hence ${}^{(k\nu_\alpha)}\alpha$ is a singular imaginary root of $\mathfrak{t}$.

For any $H \in \mathfrak{h}^+$, $H = H_k + rH_\alpha^*$, $\operatorname{Re}\{{}^{(k\nu_\alpha)^{-1}}(w\lambda)(H)\} = {}^{(k\nu_\alpha)^{-1}}(w\lambda)(rH_\alpha^*) = r\,{}^{(k\nu_\alpha)^{-1}}(w\lambda)(H_\alpha^*)$. Thus we have:

$$c_{(k\nu_\alpha)^{-1}}(w:\lambda:\mathfrak{h}^+) = \begin{cases} 1, & {}^{(k\nu_\alpha)^{-1}}(w\lambda)(H_\alpha^*) < 0,\ w \in W_K \cup {}^{(k\nu_\alpha)}s_\alpha W_K \\ 0, & \text{otherwise}\,. \end{cases} \tag{3.1}$$

For arbitrary $y \in G_C$, $y(\mathfrak{t}_C) = \mathfrak{h}_C$, we can use (2.2) to determine $c_y(w:\lambda:\mathfrak{h}^+)$.

$\dim \mathfrak{h}_p = 2$: Suppose $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ is a θ-stable Cartan subalgebra with $\dim \mathfrak{h}_p = 2$. Then Φ_R is a two-dimensional root system containing a pair of strongly orthogonal roots. Thus Φ_R is of type $A_1 \times A_1$, B_2, or G_2. We compute the constants separately for each of these three cases.

$A_1 \times A_1$: Let $\mathfrak{h}^+$ be a connected component of $\mathfrak{h}'(R)$. Thus Φ_R^+ has exactly two elements, α_1 and α_2, which are orthogonal and form a set of simple roots. Then $\mathfrak{h}_p = \{r_1H_{\alpha_1}^* + r_2H_{\alpha_2}^*: r_i \in \boldsymbol{R},\ i = 1, 2\}$.

$$\mathfrak{h}^+ = \{H_k + r_1H_{\alpha_1}^* + r_2H_{\alpha_2}^*: H_k \in \mathfrak{h}_k,\ r_i \in \boldsymbol{R},\ r_i > 0,\ i = 1, 2\}$$

and

$$\mathfrak{h}^- = \{H_k + r_1 H^*_{\alpha_1} + r_2 H^*_{\alpha_2} : H_k \in \mathfrak{h}_k,\ r_i \in \boldsymbol{R},\ r_i < 0,\ i = 1, 2\} = s_{\alpha_2} s_{\alpha_1} \mathfrak{h}^+ .$$

Thus $w^Q = s_{\alpha_2} s_{\alpha_1}$ is a decomposition of w^Q into simple reflections. Let $\mathfrak{j}_1$ and $\mathfrak{j}_2$ be the Cartan subalgebras constructed as in § 2.

Then $\nu_i(\mathfrak{h}_C) = (\mathfrak{j}_i)_C$, $i = 1, 2$, and $\mathfrak{t}_{1C} = \nu_2\nu_1(\mathfrak{h}_C) = \nu_2(\mathfrak{j}_{1C}) = \nu_1\nu_2(\mathfrak{h}_C) = \nu_1(\mathfrak{j}_{2C})$ is a compact Cartan subalgebra of $\mathfrak{G}_C$. Let $k \in K$ satisfy $k(\mathfrak{t}_1) = \mathfrak{t}$. Then $\mathfrak{h}_C = \nu_2^{-1}\nu_1^{-1}k^{-1}(\mathfrak{t}_C) = \nu_1^{-1}\nu_2^{-1}k^{-1}(\mathfrak{t}_C)$. Denote $\nu_1^{-1}\nu_2^{-1}k^{-1} = \nu_2^{-1}\nu_1^{-1}k^{-1}$ by y.

Suppose $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$, $\lambda \in \mathscr{F}'$, so that $\operatorname{Re}\{{}^y(w\lambda)(H)\} \leq 0$ for all $H \in \mathfrak{h}^+$. Then

$$\begin{aligned} c_y(w : \lambda : \mathfrak{h}^+) &= c_{\nu_2^{-1}k^{-1}}(w : \lambda : \mathfrak{j}_1^+) + c_{\nu_2^{-1}k^{-1}}({}^{y^{-1}}s_{\alpha_1}w : \lambda : \mathfrak{j}_1^+) \\ &\quad - c_{\nu_1^{-1}k^{-1}}({}^{y^{-1}}s_{\alpha_1}w : \lambda : \mathfrak{j}_2^+) - c_{\nu_1^{-1}k^{-1}}({}^{y^{-1}}(s_{\alpha_2}s_{\alpha_1})w : \lambda : \mathfrak{j}_2^+) , \end{aligned}$$

where by (3.1):

$$c_{\nu_2^{-1}k^{-1}}(w : \lambda : \mathfrak{j}_1^+) = \begin{cases} 1, & {}^{\nu_2^{-1}k^{-1}}(w\lambda)(H^*_{\alpha_2}) < 0,\ w \in W_K \cup {}^{y^{-1}}s_{\alpha_2}W_K \\ 0, & \text{otherwise} \end{cases}$$

$$c_{\nu_2^{-1}k^{-1}}({}^{y^{-1}}s_{\alpha_1}w : \lambda : \mathfrak{j}_1^+) = \begin{cases} 1, & {}^{\nu_2^{-1}k^{-1}}({}^{y^{-1}}s_{\alpha_1}w\lambda)(H^*_{\alpha_2}) = {}^{\nu_2^{-1}k^{-1}}(w\lambda)(H^*_{\alpha_2}) < 0, \\ & w \in {}^{y^{-1}}s_{\alpha_1}W_K \cup {}^{y^{-1}}(s_{\alpha_1}s_{\alpha_2})W_K \\ 0, & \text{otherwise} \end{cases}$$

$$c_{\nu_1^{-1}k^{-1}}({}^{y^{-1}}s_{\alpha_1}w : \lambda : \mathfrak{j}_2^+) = \begin{cases} 1, & {}^{\nu_1^{-1}k^{-1}}({}^{y^{-1}}s_{\alpha_1}w\lambda)(H^*_{\alpha_1}) = -{}^{\nu_1^{-1}k^{-1}}(w\lambda)(H^*_{\alpha_1}) < 0, \\ & w \in {}^{y^{-1}}s_{\alpha_1}W_K \cup W_K \\ 0, & \text{otherwise} \end{cases}$$

$$c_{\nu_1^{-1}k^{-1}}({}^{y^{-1}}(s_{\alpha_2}s_{\alpha_1})w : \lambda : \mathfrak{j}_2^+) = \begin{cases} 1, & {}^{\nu_1^{-1}k^{-1}}({}^{y^{-1}}(s_{\alpha_2}s_{\alpha_1})w\lambda)(H^*_{\alpha_1}) = -{}^{\nu_1^{-1}k^{-1}}(w\lambda)(H^*_{\alpha_1}) \\ & < 0,\ w \in {}^{y^{-1}}(s_{\alpha_1}s_{\alpha_2})W_K \cup {}^{y^{-1}}(s_{\alpha_1})W_K \\ 0, & \text{otherwise} . \end{cases}$$

Thus, since ${}^{\nu_i^{-1}k^{-1}}(w\lambda)(H^*_{\alpha_i}) = {}^y(w\lambda)(H^*_{\alpha_i})$, $i = 1, 2$, and the cosets W_K, ${}^{y^{-1}}s_{\alpha_1}W_K$, ${}^{y^{-1}}s_{\alpha_2}W_K$, and ${}^{y^{-1}}(s_{\alpha_1}s_{\alpha_2})W_K$ are distinct,

$$(3.2)\qquad c_y(w : \lambda : \mathfrak{h}^+) = \begin{cases} 1, & {}^y(w\lambda)(H^*_{\alpha_i}) < 0,\ i = 1, 2, \\ & w \in W_K \cup {}^{y^{-1}}s_{\alpha_1}W_K \cup {}^{y^{-1}}s_{\alpha_2}W_K \cup {}^{y^{-1}}(s_{\alpha_1}s_{\alpha_2})W_K \\ 0, & \text{otherwise} . \end{cases}$$

B_2: Let $\mathfrak{h}^+$ be a connected component of $\mathfrak{h}'(R)$. Then Φ_R^+ has a set of simple roots, $\{\alpha_1, \alpha_2\}$. Assume that α_1 is the long root. Then $\Phi_R^+ = \{\alpha_1, \alpha_2, \beta_2 = \alpha_1 + \alpha_2, \beta_1 = \alpha_1 + 2\alpha_2\}$ where $\langle \alpha_i, \beta_i \rangle = 0, i = 1, 2$. $\mathfrak{h}_p = \{rH^*_{\alpha_1} + sH^*_{\beta_1} : r, s \in \boldsymbol{R}\}$, $\mathfrak{h}^+ = \{H_k + rH^*_{\alpha_1} + sH^*_{\beta_1} : H_k \in \mathfrak{h}_k,\ s > r > 0\}$, and $\mathfrak{h}^- = \{H_k + rH^*_{\alpha_1} + sH^*_{\beta_1} : H_k \in \mathfrak{h}_k,\ 0 > r > s\} = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}\mathfrak{h}^+$. Thus $w^Q = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$.

Let $\mathfrak{j}_1$ and $\mathfrak{j}_2$ be constructed as in § 2 corresponding to α_1 and α_2 respectively. Then ${}^{\nu_i}\beta_i$ is the unique positive real root of $\mathfrak{j}_i$, and using a semi-regular element Γ_i of $\mathfrak{j}_i$ corresponding to ${}^{\nu_i}\beta_i$, we obtain compact Cartan subalgebras $\mathfrak{t}_i = \mathfrak{G}_{\Gamma_i}^-$ together with isomorphisms $\mu_i: (\mathfrak{j}_i)_C \to (\mathfrak{t}_i)_C$, $i = 1, 2$. Pick $k_1 \in K$ such that $k_1(\mathfrak{t}_1) = \mathfrak{t}$. Then there is $k_2 \in K$ such that $k_2(\mathfrak{t}_2) = \mathfrak{t}$ and $k_1\mu_1\nu_1 = k_2\mu_2\nu_2$ as isomorphisms from $\mathfrak{h}_C$ to $\mathfrak{t}_C$. Denote this isomorphism by y.

Fix $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$, $\lambda \in \mathscr{F}'$. Let $n = {}^{y^{-1}}(w\lambda)(H_{\alpha_1}^*)$, $m = {}^{y^{-1}}(w\lambda)(H_{\beta_1}^*)$. Then $\mathrm{Re}\,\{{}^{y^{-1}}(w\lambda)(H_k + rH_{\alpha_1}^* + sH_{\beta_1}^*)\} = rn + sm$. If $m > 0$ or $n > -m$, there is $H \in \mathfrak{h}^+$ such that $\mathrm{Re}\,\{{}^{y^{-1}}(w\lambda)(H)\} > 0$. Assume this is not the case. Then

$$\begin{aligned} c_{y^{-1}}(w:\lambda:\mathfrak{h}^+) = {} & c_{\nu_1 y^{-1}}(w:\lambda:\mathfrak{j}_1^+) + c_{\nu_1 y^{-1}}({}^y s_{\alpha_1} w:\lambda:\mathfrak{j}_1^+) - c_{\nu_2 y^{-1}}({}^y s_{\alpha_1} w:\lambda:\mathfrak{j}_2^+) \\ & - c_{\nu_2 y^{-1}}({}^y(s_{\alpha_2}s_{\alpha_1})w:\lambda:\mathfrak{j}_2^+) + c_{\nu_1 y^{-1}}({}^y(s_{\alpha_2}s_{\alpha_1})w:\lambda:\mathfrak{j}_1^+) \\ & + c_{\nu_1 y^{-1}}({}^y(s_{\alpha_1}s_{\alpha_2}s_{\alpha_1})w:\lambda:\mathfrak{j}_1^+) - c_{\nu_2 y^{-1}}({}^y(s_{\alpha_1}s_{\alpha_2}s_{\alpha_1})w:\lambda:\mathfrak{j}_2^+) \\ & - c_{\nu_2 y^{-1}}({}^y w^Q w:\lambda:\mathfrak{j}_2^+)\ . \end{aligned}$$

The constants for $\mathfrak{j}_1^+$ and $\mathfrak{j}_2^+$ can be evaluated using the facts that $\nu_i y^{-1} = (k_i\mu_i)^{-1}$ as isomorphisms from $\mathfrak{t}_C$ to $(\mathfrak{j}_i)_C$, and that the cosets W_K, ${}^y s_{\alpha_1} W_K$, ${}^y s_{\beta_1} W_K$, ${}^y s_{\beta_2} W_K = {}^y(s_{\alpha_1}s_{\beta_1})W_K$ are distinct, but that ${}^y s_{\alpha_2} W_K = W_K$ since ${}^y\alpha_2$ is a compact root of $\mathfrak{t}$. We obtain the following table of values for $c_{y^{-1}}(w:\lambda:\mathfrak{h}^+)$ where n and m are as defined above.

(3.3)

	$w \in W_K \cup {}^y s_{\beta_2} W_K$	$w \in {}^y s_{\alpha_1} W_K \cup {}^y s_{\beta_1} W_K$
$0 > m > n$	1	1
$0 > n > m$	1	-1
$0 < n < -m$	2	0
$m > 0$ or $n > -m$	0	0

G_2: Let $\mathfrak{h}^+$ be a connected component of $\mathfrak{h}'(R)$. Let $\{\alpha_1, \alpha_2\}$ be the set of simple roots for Φ_R^+, α_2 the long root. Then $\Phi_R^+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2, \beta_2 = 2\alpha_1 + \alpha_2,\ \beta_3 = 3\alpha_1 + \alpha_2,\ \beta_1 = 3\alpha_1 + 2\alpha_2\}$, where $\langle \alpha_i, \beta_i \rangle = 0$, $i = 1, 2, 3$. $\mathfrak{h}_p = \{rH_{\alpha_1}^* + sH_{\beta_1}^*: r, s \in \boldsymbol{R}\}$, $\mathfrak{h}^+ = \{H_k + rH_{\alpha_1}^* + sH_{\beta_1}^*: H_k \in \mathfrak{h}_k,\ s > 3r > 0\}$ and $\mathfrak{h}^- = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}\mathfrak{h}^+$. Thus $w^Q = (s_{\alpha_1}s_{\alpha_2})^3$. Let $\mathfrak{j}_1$ and $\mathfrak{j}_2$ be constructed as in § 2 and define $\mathfrak{t}_i, \mu_i$ as in the B_2 case for $i = 1, 2$. Let $y = k_1\mu_1\nu_1$ where $k_1 \in K$ satisfies $k_1(\mathfrak{t}_1) = \mathfrak{t}$.

Fix $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$, $\lambda \in \mathscr{F}'$. Let $n = {}^{y^{-1}}(w\lambda)(H_{\alpha_1}^*)$, $m = {}^{y^{-1}}(w\lambda)(H_{\beta_1}^*)$. Then $\mathrm{Re}\,\{{}^{y^{-1}}(w\lambda)(H_k + rH_{\alpha_1}^* + sH_{\beta_1}^*)\} = rn + sm$. If $m > 0$ or $n > -3m$, there is $H \in \mathfrak{h}^+$ such that $\mathrm{Re}\,\{{}^{y^{-1}}(w\lambda)(H)\} > 0$. Otherwise, $c_{y^{-1}}(w:\lambda:\mathfrak{h}^+)$

is given as a sum of twelve terms from (2.8). Since $\nu_1 y^{-1} = (k_1\mu_1)^{-1}$, $c_{\nu_1 y^{-1}}(w' : \lambda : \mathfrak{j}_1^+)$ is given directly by (3.1) for any $w' \in W(\mathfrak{G}_C, \mathfrak{t}_C)$.

However, there is no $k_2 \in K$ with $k_2(\mathfrak{t}_2) = \mathfrak{t}$ and $\nu_2 y^{-1} = (k_2\mu_2)^{-1}$. However, there is $k_2 \in K$ such that $k_2(\mathfrak{t}_2) = \mathfrak{t}$ and ${}^{(\nu_2 y^{-1})}\lambda = {}^{(k_2\mu_2)^{-1}}({}^{y}(s_{\beta_1}s_{\alpha_3})\lambda)$ for all $\lambda \in \mathscr{F}'$. Thus, using (2.2) together with (3.1) we obtain

$$c_{\nu_2 y^{-1}}(w' : \lambda : j_2^+) = \begin{cases} \det {}^{y}(s_{\beta_1}s_{\alpha_3}) = 1, & {}^{\nu_2 y^{-1}}(w\lambda)(H_{\beta_2}^*) < 0,\ w \in {}^{y}s_{\alpha_1}W_K \cup {}^{y}s_{\alpha_3}W_K \\ 0, & \text{otherwise}. \end{cases}$$

The cosets W_K, ${}^{y}s_{\alpha_1}W_K = {}^{y}s_{\beta_1}W_K$, and ${}^{y}s_{\alpha_3}W_K = {}^{y}s_{\beta_3}W_K$ are distinct. ${}^{y}s_{\alpha_2}W_K = {}^{y}s_{\beta_2}W_K = W_K$. Using the above information we obtain the following table for the values of $c_{y^{-1}}(w : \lambda : \mathfrak{h}^+)$.

(3.4)

	$w \in W_K$	$w \in {}^{y}s_{\alpha_1}W_K$	$w \in {}^{y}s_{\alpha_3}W_K$
$0 > 3m > n$	2	2	0
$0 > m > n > 3m$	0	2	-2
$m < n < 0$	0	4	0
$0 < n < -m$	-2	2	0
$0 < -m < n < -3m$	0	2	2
$m > 0$ or $n > -3m$	0	0	0

In general, the constants $c_y(w : \lambda : \mathfrak{h}^+)$ become increasingly complicated as $\dim \mathfrak{h}_p$ increases. However, in the case that $\mathfrak{G}$ has exactly one conjugacy class of Cartan subalgebra corresponding to each possible dimension for $\mathfrak{h}_p$, the constants can be computed completely. There are three infinite families of simple real Lie algebras which have this property, including $su(p, q)$, $p \geq q \geq 1$. The others are of types CII and DIII. ([4, Volume I, Section 1.3.1]).

Thus suppose $\mathfrak{G} = \mathfrak{k} + \mathfrak{p}$ is a simple real Lie algebra with split rank n, $\mathfrak{A}$ a Cartan subalgebra of $\mathfrak{G}$ with $\mathfrak{A}_p = \mathfrak{A} \cap \mathfrak{p}$ of dimension n. Assume the set Φ_R of real roots of $(\mathfrak{G}_C, \mathfrak{A}_C)$ is of type $A_1 \times \cdots \times A_1$, n copies. Denote the real roots by $\{\alpha_1, \cdots, \alpha_n\}$. They are mutually orthogonal and form a set of simple roots for the root system Φ_R. We can write $\mathfrak{A} = \mathfrak{A}_k + \sum_{i=1}^n \boldsymbol{R}H_{\alpha_i}^*$, $\mathfrak{A}_k = \mathfrak{A} \cap \mathfrak{k}$. Representatives of each conjugacy class of Cartan subalgebras of $\mathfrak{G}$ are $\mathfrak{A} = \mathfrak{h}_0, \mathfrak{h}_1, \cdots, \mathfrak{h}_n = \mathfrak{t}$, where $\mathfrak{h}_\ell = \mathfrak{A}_k + \sum_{i=1}^\ell \boldsymbol{R}(X_{\alpha_i}^* - Y_{\alpha_i}^*) + \sum_{i=\ell+1}^n \boldsymbol{R}H_{\alpha_i}^*$. (Here $X_{\alpha_i}^*$ and $Y_{\alpha_i}^*$ denote the elements of the root spaces $\mathfrak{G}^{\alpha_i}$ and $\mathfrak{G}^{-\alpha_i}$ respectively which satisfy $[H_{\alpha_i}^*, X_{\alpha_i}^*] = 2X_{\alpha_i}^*$; $[H_{\alpha_i}^*, Y_{\alpha_i}^*] = -2Y_{\alpha_i}^*$; $[X_{\alpha_i}^*, Y_{\alpha_i}^*] = H_{\alpha_i}^*$). Let

$$\nu_i = \exp(-\pi\sqrt{-1}/4 \operatorname{ad}(X^*_{\alpha_i} + Y^*_{\alpha_i}))\,, \qquad \mu_\ell = \nu_n \cdots \nu_{\ell+1}\,.$$

Then $\mu_\ell(\mathfrak{h}_\ell)_C = \mathfrak{t}_C$. Let $\mathfrak{h}^+_\ell = \{H_k + \sum_{i=\ell+1}^n r_i H^*_{\alpha_i} : H_k \in \mathfrak{h}_\ell \cap \mathfrak{k},\ r_i \in \boldsymbol{R},\ r_i > 0$ for $i = \ell+1, \cdots, n\}$.

We assume inductively that for $j > \ell$, $\lambda \in \mathscr{F}'$, $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)$,

$$c_{\mu_j^{-1}}(w : \lambda : \mathfrak{h}^+_j) = \begin{cases} 1, & {}^{\mu_j^{-1}}(w\lambda)(H^*_{\alpha_i}) < 0,\ i = j+1, \cdots, n, \\ & w \in \langle s_{\alpha_{j+1}}, \cdots, s_{\alpha_n}\rangle W_K \\ 0, & \text{otherwise} \end{cases}$$

where $\langle s_{\alpha_{j+1}}, \cdots, s_{\alpha_n}\rangle W_K$ denotes the subgroup of $W(\mathfrak{G}_C, \mathfrak{t}_C)$ generated by W_K together with the reflections ${}^{\mu_0}(s_{\alpha_i})$, $i = j+1, \cdots, n$. We have already proved this for the cases $j = n,\ n-1,\ n-2$ in (2.6), (3.1), and (3.2) respectively.

We know $c_{\mu_\ell^{-1}}(w : \lambda : \mathfrak{h}^+_\ell) = 0$ if $\operatorname{Re}\{{}^{\mu_\ell^{-1}}(w\lambda)(H)\} > 0$ for any $H \in \mathfrak{h}^+_\ell$. Write $H = H_k + \sum_{i=\ell+1}^n r_i H^*_{\alpha_i}$, $H_k \in \mathfrak{h}_\ell \cap \mathfrak{k}$, $r_i > 0$, $i = \ell+1, \cdots, n$. Then $\operatorname{Re}\{{}^{\mu_\ell^{-1}}(w\lambda)(H)\} = \sum_{i=\ell+1}^n r_i\, {}^{\mu_\ell^{-1}}(w\lambda)(H^*_{\alpha_i}) \leq 0$ for all $r_i > 0$ if and only if ${}^{\mu_\ell^{-1}}(w\lambda)(H^*_{\alpha_i}) < 0$ for all i. (By the regularity of λ, ${}^{\mu_\ell^{-1}}(w\lambda)(H^*_{\alpha_i}) \neq 0$ for any i). Suppose this is the case. Then $\operatorname{Re}\{s_{\alpha_{\ell+1}}\, {}^{\mu_\ell^{-1}}(w\lambda)(H_k + \sum r_i H^*_{\alpha_i})\} = -r_{\ell+1}\, {}^{\mu_\ell^{-1}}(w\lambda)(H^*_{\alpha_{\ell+1}}) + \sum_{i=\ell+2}^n r_i\, {}^{\mu_\ell^{-1}}(w\lambda)(H^*_{\alpha_i})$, and there are $r_i > 0$, $i = \ell + 1, \cdots, n$ for which this is strictly positive, since by assumption, ${}^{\mu_\ell^{-1}}(w\lambda)(H^*_{\alpha_{\ell+1}}) < 0$.

Thus $c_{\mu_\ell^{-1}}({}^{\mu_\ell}(s_{\alpha_{\ell+1}})w : \lambda : \mathfrak{h}^+_\ell) = 0$, and so

$$\begin{aligned} c_{\mu_\ell^{-1}}(w : \lambda : \mathfrak{h}^+_\ell) &= c_{\mu_{\ell+1}^{-1}}(w : \lambda : \mathfrak{h}^+_{\ell+1}) + c_{\mu_{\ell+1}^{-1}}({}^{\mu_\ell}(s_{\alpha_{\ell+1}})w : \lambda : \mathfrak{h}^+_{\ell+1}) \\ &= \begin{cases} 1, & {}^{\mu_{\ell+1}^{-1}}(w\lambda)(H^*_{\alpha_j}) < 0,\ j = \ell+2, \cdots, n, \\ & w \in \langle s_{\alpha_{\ell+2}}, \cdots, s_{\alpha_n}\rangle W_K \cup {}^{\mu_{\ell+1}}s_{\alpha_{\ell+1}}\langle s_{\alpha_{\ell+2}}, \cdots, s_{\alpha_n}\rangle W_K \\ 0, & \text{otherwise}\,. \end{cases} \end{aligned}$$

Thus we have, by induction, for any $\ell = 0, \cdots, n$,

$$c_{\mu_\ell^{-1}}(w : \lambda : \mathfrak{h}^+_\ell) = \begin{cases} 1, & {}^{\mu_\ell^{-1}}(w\lambda)(H^*_{\alpha_j}) < 0,\ j = \ell+1, \cdots, n, \\ & w \in \langle s_{\alpha_{\ell+1}}, \cdots, s_{\alpha_n}\rangle W_K \\ 0, & \text{otherwise}\,. \end{cases} \tag{3.5}$$

§4. The Constants on G

We use the notation of §1. Let $\tau \in L_T$. Denote the corresponding character of T by ξ_τ. Thus, for $H \in \mathfrak{t}$, $\xi_\tau(\exp H) = \exp(\tau(H))$. For any Cartan subgroup H of G, let Δ_H be defined by

$$\varDelta_H(h) = \xi_\rho(h) \prod_{\alpha \in \Phi^+(\mathfrak{G}_C, \mathfrak{h}_C)} (1 - \xi_\alpha(h^{-1})) \ , \qquad h \in H$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{G}_C, \mathfrak{h}_C)} \alpha$, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{G}$ corresponding to H.

Then for $t \in T'$,

$$\Theta_\tau(t) = \varDelta_T(t)^{-1} \sum_{w \in W_K} \det w \xi_{w\tau}(t) \ . \tag{4.1}$$

To compute the expression for Θ_τ on other Cartan subgroups, we use the method and notation of [2a), § 23].

Let H be a θ-stable Cartan subgroup of G with Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$, $\dim \mathfrak{h}_p = 1$, H^* a connected component of $H'(R)$. $H^* = H_I^* H_R^*$, H_I^* a connected component of $H_I = H \cap K$, $H_R^* \subseteq H_R = \exp(\mathfrak{h}_p)$. Note $H_I = Z(H_R) H_I^0$, H_I^0 the connected component of the identity, $Z(H_R) = \{I, \gamma_\alpha\}$, α the unique positive real root of $(\mathfrak{G}_C, \mathfrak{h}_C)$. (For any real root α, $\gamma_\alpha = \exp(\pi\sqrt{-1} H_\alpha^*) = \exp(\pi(X_\alpha^* - Y_\alpha^*)) \in \exp(\sqrt{-1}\mathfrak{h}_p) \cap K)$. H_I is connected if and only if $\gamma_\alpha \in H_I^0 = \exp(\mathfrak{h}_k)$.

We will assume $H_I^0 \subseteq T$. Let $\mathfrak{Z}$ denote the centralizer of H_I^* in $\mathfrak{G}$ where $H_I^* = H_I^0$ or $\gamma_\alpha H_I^0$. In either case, $\mathfrak{Z} = \mathfrak{h} + \mathfrak{G}^\alpha + \mathfrak{G}^{-\alpha}$, where for any root α, $\mathfrak{G}^\alpha = \mathfrak{G} \cap \mathfrak{G}_C^\alpha$, $\mathfrak{G}_C^\alpha$ the root space of α in $\mathfrak{G}_C$, Then $\mathfrak{Z}$ is a reductive Lie algebra with Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{t}$, where $\mathfrak{t}_C = \nu_\alpha(\mathfrak{h}_C)$, $\nu_\alpha = \exp(-\pi\sqrt{-1}/4 \operatorname{ad}(X_\alpha^* + Y_\alpha^*))$. $(\mathfrak{Z}_C, \mathfrak{h}_C)$ has exactly one positive root, α, and $W(\mathfrak{Z}_C, \mathfrak{h}_C) = \{I, s_\alpha\}$. $W_K \cap W(\mathfrak{Z}_C, \mathfrak{t}_C) = \{I\}$.

Thus if $h_1 \in H_I^*$, $h_2 \in H_R^*$, $h_1 h_2 \in H'$,

$$\begin{aligned} &\varDelta_H(h_1 h_2) \Theta_\tau(h_1 h_2) \\ &= \sum_{w \in W_K} \det w \xi_{w\tau}(h_1) \sum_{s \in \{I, s_\alpha\}} \det s c_\tau(s : w : H^*) \exp(s^{\nu_\alpha^{-1}}(w\tau)(\log h_2)) \\ &= \sum_{w \in W_K} \det w \xi_{w\tau}(h_1) c(w\tau : \mathfrak{h}^*) \exp(-|^{\nu_\alpha^{-1}}(w\tau)(\log h_2)|) \end{aligned} \tag{4.2}$$

where $\mathfrak{h}^*$ is the component of $\mathfrak{h}'(R)$ corresponding to H^* under the exponential map. For $\mathfrak{h}^* = \mathfrak{h}^+ = \{H_k + rH_\alpha^* : H_k \in \mathfrak{h}_k,\ r > 0\}$, $\tau \in L_T$,

$$c(\tau : \mathfrak{h}^+) = \begin{cases} 1, & {}^{\nu_\alpha^{-1}}\tau(H_\alpha^*) < 0 \\ -1, & {}^{\nu_\alpha^{-1}}\tau(H_\alpha^*) > 0 \\ 0, & {}^{\nu_\alpha^{-1}}\tau(H_\alpha^*) = 0 \ . \end{cases}$$

For $\mathfrak{h}^- = s_\alpha \mathfrak{h}^+$, $c(\tau : \mathfrak{h}^-) = -c(\tau : \mathfrak{h}^+)$ for all $\tau \in L_T$.

Now suppose H is a Cartan subgroup of G with Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$, $\dim \mathfrak{h}_p = 2$, H^* a connected component of $H'(R)$. Again, we must consider the three cases where Φ_R is of type $A_1 \times A_1$, B_2, or G_2, Φ_R the set of real roots in $\Phi(\mathfrak{G}_C, \mathfrak{h}_C)$. We use the notation of § 3. We assume $H^* = H_I^* H_R^*$, $H_I^0 \subseteq T$.

$A_1 \times A_1$: H_I can have one, two, or four connected components, as $Z(H_R) = \{I, \gamma_{\alpha_1}, \gamma_{\alpha_2}, \gamma_{\alpha_1}\gamma_{\alpha_2}\}$. However, in each case, $\mathfrak{Z} = \mathfrak{h} + \sum_{\alpha \in \Phi_R} \mathfrak{G}^\alpha$. The roots of $(\mathfrak{Z}_C, \mathfrak{h}_C)$ are exactly the real roots of $(\mathfrak{G}_C, \mathfrak{h}_C)$, $W(\mathfrak{Z}_C, \mathfrak{h}_C) = \{I, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}\}$, and $W_K \cap W(\mathfrak{Z}_C, \mathfrak{t}_C) = \{I\}$. Thus if $h_1 \in H_I^*$, $h_2 \in H_R^*$, $h_1h_2 \in H'$, $h_2 = \exp(rH_{\alpha_1}^* + sH_{\alpha_2}^*)$,

$$
\begin{aligned}
&\Delta_H(h_1h_2)\Theta_\tau(h_1h_2) \\
&\quad = \sum_{w \in W_K} \det w\xi_{w\tau}(h_1) \sum_{s \in W(\mathfrak{Z}_C, \mathfrak{h}_C)} \det s c_\tau(s : w : H^*) \\
&\qquad \times \exp(s^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(\log h_2)) \qquad (4.3)\\
&\quad = \sum_{w \in W_K} \det w\xi_{w\tau}(h_1) c(w\tau : \mathfrak{h}^*) \exp(-|^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(rH_{\alpha_1}^*)|) \\
&\qquad \times \exp(-|^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(sH_{\alpha_2}^*)|)
\end{aligned}
$$

where $\mathfrak{h}^*$ is the component of $\mathfrak{h}'(R)$ corresponding to H^*. For $\mathfrak{h}^* = \mathfrak{h}^+ = \{H_k + rH_{\alpha_1}^* + sH_{\alpha_2}^* : H_k \in \mathfrak{h}_k,\ r, s > 0\}$, $\tau \in L_T$,

$$
c(\tau : \mathfrak{h}^+) = \begin{cases} 1, & {}^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(H_{\alpha_1}^*) \times {}^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(H_{\alpha_2}^*) > 0 \\ -1, & {}^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(H_{\alpha_1}^*) \times {}^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(H_{\alpha_2}^*) < 0 \\ 0, & {}^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(H_{\alpha_1}^*) \times {}^{\nu_{\alpha_1}^{-1}\nu_{\alpha_2}^{-1}}(w\tau)(H_{\alpha_2}^*) = 0\,. \end{cases}
$$

Otherwise, $c(\tau : s\mathfrak{h}^+) = \det sc(\tau : \mathfrak{h}^+)$, $s \in W(\mathfrak{Z}_C, \mathfrak{h}_C)$.

B_2: If Φ_R is of type B_2, $H_I = H_I^0 \cup \gamma_{\alpha_1}H_I^0 \cup \gamma_{\alpha_2}H_I^0 \cup \gamma_{\alpha_1}\gamma_{\alpha_2}H_I^0$, (the four components not necessarily distinct). The centralizer of H_I^0 and $\gamma_{\alpha_2}H_I^0$ is $\mathfrak{Z} = \mathfrak{A} + \sum_{\alpha \in \Phi_R} \mathfrak{G}^\alpha$. The roots of $(\mathfrak{Z}_C, \mathfrak{h}_C)$ are exactly the real roots of $(\mathfrak{G}_C, \mathfrak{h}_C)$, and so $W(\mathfrak{Z}_C, \mathfrak{h}_C) = W_R(\mathfrak{G}_C, \mathfrak{h}_C)$, $W_K \cap W(\mathfrak{Z}_C, \mathfrak{t}_C) = \{I, {}^{\nu_{\beta_1}\nu_{\alpha_1}}s_{\alpha_2}\}$. For $h_1h_2 \in H'$, $h_1 \in H_I^0 \cup \gamma_{\alpha_2}H_I^0$, $h_2 \in H_R^* = \exp(\mathfrak{h}^* \cap \mathfrak{p})$, $\mathfrak{h}^*$ a component of $\mathfrak{h}'(R)$, we have

$$
\begin{aligned}
&\Delta_H(h_1h_2)\Theta_\tau(h_1h_2) \\
&\quad = \sum_{w \in W_K/\{I, {}^{\nu_{\beta_1}\nu_{\alpha_1}}s_{\alpha_2}\}} \det w\xi_{w\tau}(h_1) \sum_{s \in W(\mathfrak{Z}_C, \mathfrak{h}_C)} \det sc_\tau(s : w : H^*) \qquad (4.4)\\
&\qquad \times \exp(s^{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(w\tau)(\log h_2))
\end{aligned}
$$

where

$$
c_\tau(s : w : H^*) = c_{\mathfrak{Z}}(s : w\tau : \mathfrak{h}^*) = c_{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(s : w\tau : \mathfrak{h}^*)
$$

where $c_{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(s : w\tau : \mathfrak{h}^+)$ is given by table (3.3) for regular τ such that $\langle\tau, \alpha\rangle \neq 0$ for $\alpha \in \Phi_R$. For singular τ for which $\langle\tau, \alpha\rangle = 0$ for some $\alpha \in \Phi_R$, and any $w \in W(\mathfrak{G}_C, \mathfrak{t}_C)c_{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(w : \tau : \mathfrak{h}^+)$ reduces to zero except for the cases $n = 0$, $m < 0$, $w \in W_K \cup {}^{\nu_{\alpha_1}\nu_{\beta_1}}s_{\beta_2}W_K$ where we have $c(w : \tau : \mathfrak{h}^+) = 2$, and $m = n < 0$, $w \in W_K \cup {}^{\nu_{\alpha_1}\nu_{\beta_1}}s_{\beta_2}W_K$, where $c(w : \tau : \mathfrak{h}^+) = 1$. ($n$ and m are as defined in § 3).

The centralizer of $\gamma_{\alpha_1}H_I^0$ and $\gamma_{\alpha_1}\gamma_{\alpha_2}H_I^0$ is $\mathfrak{Z} = \mathfrak{h} + \mathfrak{G}^{\alpha_1} + \mathfrak{G}^{-\alpha_1} + \mathfrak{G}^{\beta_1} + \mathfrak{G}^{-\beta_1}$. The only roots of $(\mathfrak{Z}_C, \mathfrak{h}_C)$ are $\pm\alpha_1$ and $\pm\beta_1$, so that the root system of $\mathfrak{Z}$ is of type $A_1 \times A_1$ rather than of type B_2. $W(\mathfrak{Z}_C, \mathfrak{h}_C) = \{I, s_{\alpha_1}, s_{\beta_1}, s_{\alpha_1}s_{\beta_1}\}$, and $W_K \cap W(\mathfrak{Z}_C, \mathfrak{t}_C) = \{I\}$. Thus for $h_1h_2 \in H'$, $h_1 \in \gamma_{\alpha_1}H_I^0 \cup \gamma_{\alpha_1}\gamma_{\alpha_2}H_I^0$, $h_2 \in H_R^*$,

$$\begin{aligned} &\Delta_H(h_1h_2)\Theta_\tau(h_1h_2) \\ &\quad = \sum_{w\in W_K} \det w\xi_{w\tau}(h_1) \sum_{w\in W(\mathfrak{Z}_C,\mathfrak{h}_C)} \det sc_\tau(s : w : H^*) \\ &\quad\quad \times \exp\left(s^{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(w\tau)(\log h_2)\right) \end{aligned}$$

where $c_\tau(s : w : H^*) = c_{\mathfrak{Z}}(s : w\tau : \mathfrak{h}^*)$. In this case, since the root system of $\mathfrak{Z}$ is of type $A_1 \times A_1$, by [2c), p. 285], we have, for $s \in W(\mathfrak{Z}_C, \mathfrak{h}_C)$,

$$(4.5) \qquad c_{\mathfrak{Z}}(s : \tau : \mathfrak{h}^*) = \begin{cases} 1, & {}^{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(s\tau)(H_{\alpha_1}^*) < 0,\ {}^{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(s\tau)(H_{\beta_1}^*) < 0 \\ 0, & \text{otherwise} \end{cases}$$

for any component $\mathfrak{h}^*$ of $\mathfrak{h}'(R)$ such that $\mathfrak{h}^* \subseteq \mathfrak{h}_{\mathfrak{Z}}^+ = \{H_k + rH_{\alpha_1}^* + sH_{\beta_1}^* : H_k \in \mathfrak{h}_k,\ r, s > 0\}$. As in the case for Φ_R of type $A_1 \times A_1$, the expression for Θ_τ in this case simplifies to (4.3), with β_1 replacing α_2, where $c(\tau : \mathfrak{h}^*)$ is defined as previously for any $\mathfrak{h}^* \subseteq \mathfrak{h}_{\mathfrak{Z}}^+$. Otherwise, $c(\tau : \mathfrak{h}^*) = c({}^{\nu_{\alpha_1}\nu_{\beta_1}}s\tau : s\mathfrak{h}^*)$ where $s \in W(\mathfrak{Z}_C, \mathfrak{h}_C)$ satisfies $s\mathfrak{h}^* \subseteq \mathfrak{h}_{\mathfrak{Z}}^+$.

G_2: If Φ_R is of type G_2, $H_I = \{I\} \cup \{\gamma_{\alpha_1}\} \cup \{\gamma_{\alpha_2}\} \cup \{\gamma_{\alpha_1}\gamma_{\alpha_2}\}$. The centralizer of $\{I\}$ is $\mathfrak{G}$, so for $h \in H_R^*$,

$$(4.6) \qquad \Delta_H(h)\Theta_\tau(h) = \sum_{s\in W(\mathfrak{Z}_C,\mathfrak{h}_C)} \det sc_\tau(s : I : H_R^*) \exp\left(s^{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}\tau(\log h)\right)$$

where $c_\tau(s : I : H_R^*) = c_{\nu_{\beta_1}^{-1}\nu_{\alpha_1}^{-1}}(s : \tau : \mathfrak{h}^*)$ which for $\mathfrak{h}^* = \mathfrak{h}^+$ is given in table (3.4) for regular τ. For singular τ, using the notation of the table, we have

$$c(w : \tau : \mathfrak{h}^+) = \begin{cases} 4, & m = n < 0,\ w \in {}^{\nu_{\alpha_1}\nu_{\beta_1}}s_{\alpha_1}W_K \\ 2, & m < n = 0,\ m = -n < 0,\ 3m = n < 0,\ w \in {}^{\nu_{\alpha_1}\nu_{\beta_1}}s_{\alpha_1}W_K \\ 2, & 3m = n < 0,\ w \in W_K \end{cases}$$

$$\begin{cases} -2, & m < n = 0,\ w \in W_K \\ 2, & m = -n < 0,\ w \in {}^{\nu_{\alpha_1}\nu_{\beta_1}}s_{\alpha_3} W_K\,. \end{cases}$$

Of course, for other components $\mathfrak{h}^*$, we use (2.5) together with the values of the constants for $\mathfrak{h}^+$.

The centralizer of $\{\gamma_{\alpha_1}\}$ is $\mathfrak{Z} = \mathfrak{h} + \mathfrak{G}^{\alpha_1} + \mathfrak{G}^{-\alpha_1} + \mathfrak{G}^{\beta_1} + \mathfrak{G}^{-\beta_1}$. The only roots of $(\mathfrak{Z}_C, \mathfrak{h}_C)$ are $\pm\alpha_1$ and $\pm\beta_1$. $W(\mathfrak{Z}_C, \mathfrak{h}_C) = \{I, s_{\alpha_1}, s_{\beta_1}, s_{\alpha_1}s_{\beta_1}\}$, and $W_K \cap W(\mathfrak{Z}_C, \mathfrak{t}_C) = \{I, -I\}$. Thus for $h \in H_R^*$,

$$\begin{aligned} &\Delta_H(h)\Theta_\tau(h) \\ &\quad = \sum_{w \in W_K/\{I,-I\}} \det w\xi_{w\tau}(\gamma_{\alpha_1}) \sum_{s \in W(\mathfrak{Z}_C, \mathfrak{h}_C)} \det s c_\tau(s : w : H_R^+) \\ &\qquad \times \exp\left(s^{\nu_{\beta_1}{}^{-1}\nu_{\alpha_1}{}^{-1}}(w\tau)(\log h)\right) \end{aligned}$$

where $c_\tau(s : w : H_R^+) = c_{\mathfrak{Z}}(s : w\tau : \mathfrak{h}^*) + c_{\mathfrak{Z}}(-s : -w\tau : \mathfrak{h}^*)$ and $c_{\mathfrak{Z}}(s : \tau : \mathfrak{h}^*)$ is given as in (4.5). As before, the expression simplifies to (4.3) where $c(\tau : \mathfrak{h}^*)$ is defined as for the B_2 case, and again we replace α_2 by β_1.

$\{\gamma_{\alpha_2}\} \not\subset T$, but there exist $k, k' \in K$ such that ${}^k(\gamma_{\alpha_2}) = \gamma_{\alpha_1}$ and ${}^{k'}(\gamma_{\alpha_1}\gamma_{\alpha_2}) = \gamma_{\alpha_1}$. k and k' correspond to the elements s_{α_3} and s_{α_2} in $W(G, H)$ respectively. Using the invariance of Θ_τ, $\Theta_\tau(\gamma_{\alpha_2}h) = \Theta_\tau(\gamma_{\alpha_1}s_{\alpha_3}h)$ and $\Theta_\tau(\gamma_{\alpha_1}\gamma_{\alpha_2}h) = \Theta_\tau(\gamma_{\alpha_1}s_{\alpha_2}h)$, $h \in H_R^*$, and so can be obtained from the formulas above.

As in § 3, we can give a complete description of Θ_τ in the case that G has exactly $n + 1$ conjugacy classes of Cartan subgroups, $n = \operatorname{rank}(G/K)$. Let H_ℓ be the Cartan subgroup of G corresponding to $\mathfrak{h}_\ell$, $0 \le \ell \le n$, notation as in § 3.

Each component of $(H_\ell)_I$ has as centralizer in $\mathfrak{G}$, $\mathfrak{Z} = \mathfrak{h}_\ell + \sum_{i=\ell+1}^n (\mathfrak{G}^{\alpha_i} + \mathfrak{G}^{-\alpha_i})$, and $(\mathfrak{Z}_C, \mathfrak{h}_{\ell C})$ has roots $\pm\alpha_{\ell+1}, \cdots, \pm\alpha_n$. $W(\mathfrak{Z}_C, \mathfrak{h}_{\ell C})$ is the subgroup of $W(\mathfrak{G}_C, \mathfrak{h}_{\ell C})$ generated by the s_{α_i}, $i = \ell + 1, \cdots, n$, and $W_K \cap W(\mathfrak{Z}_C, \mathfrak{t}_C) = \{I\}$. Thus if $h_1h_2 \in H'_\ell$, $h_1 \in (H_\ell)_I^*$, $h_2 \in (H_\ell)_R^*$, $h_2 = \exp(\sum_{i=\ell+1}^n r_i H_{\alpha_i}^*)$,

$$\begin{aligned} &\Delta_{H_\ell}(h_1h_2)\Theta_\tau(h_1h_2) \\ &\quad = \sum_{w \in W_K} \det w\xi_{w\tau}(h_1) \sum_{s \in W(\mathfrak{Z}_C, \mathfrak{h}_{\ell C})} \det s c_\tau(s : w : H_\ell^*) \\ &\qquad \times \exp\left(s^{\mu_\ell{}^{-1}}(w\tau)(\log h_2)\right) \qquad (4.7) \\ &\quad = \sum_{w \in W_K} \det w\xi_{w\tau}(h_1) c(w\tau : \mathfrak{h}_\ell^*) \exp\left(-|{}^{\mu_\ell{}^{-1}}(w\tau)(r_{\ell+1}H_{\alpha_{\ell+1}}^*)|\right) \\ &\qquad \cdots \exp\left(-|{}^{\mu_\ell{}^{-1}}(w\tau)(r_n H_{\alpha_n}^*)|\right) \end{aligned}$$

where for

$$\mathfrak{h}^* = \mathfrak{h}^+ = \left\{H_k + \sum_{i=\ell+1}^n r_i H_{\alpha_i}^* : H_k \in (\mathfrak{h}_\ell)_k,\ r_i > 0,\ i = \ell + 1, \cdots, n\right\},$$

$$c(\tau:\mathfrak{h}_\ell^+)=\begin{cases}1 & (-1)^{n-\ell}\prod_{i=\ell+1}^{n}{}^{\mu^{\ell-1}}(\tau)(H_{\alpha i}^*)>0\\ -1 & \qquad\qquad '' \qquad\qquad <0\\ 0 & \qquad\qquad '' \qquad\qquad =0\,.\end{cases}$$

For $\mathfrak{h}^*=s\mathfrak{h}_\ell^+$, $s\in W(\mathfrak{Z}_C,\mathfrak{h}_{\ell C})$, $c(\tau:s\mathfrak{h}^+)=\det sc(\tau:\mathfrak{h}_\ell^+)$.

For $\tau\in L'_T$, let

$$\varepsilon(\tau)=\operatorname{sign}\left\{\prod_{\alpha\in\Phi^+(\mathfrak{G}_C,\mathfrak{t}_C)}\langle\alpha,\tau\rangle\right\}.$$

Let $s=\frac{1}{2}\dim(G/K)$. Then $T_\tau=(-1)^s\varepsilon(\tau)\Theta_\tau$ is the character of a discrete series representation of G, and all discrete series characters are of this form. $T_{\tau_1}=T_{\tau_2}$ if and only if τ_1 and τ_2 are conjugate by W_K.

For singular τ,Θ_τ has no known character theoretic interpretation in general. If $w\tau=\tau$ for some $w\neq 1$ in W_K, $\Theta_\tau\equiv 0$. However for other singular τ,Θ_τ need not vanish.

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