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A NOTE ON HOLOMORPHIC MATRIC AUTOMORPHIC FACTORS WITH RESPECT TO A LATTICE IN A COMPLEX VECTOR SPACE

Dedicated to the memory of Taira Honda

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1. A holomorphic $n \times n$-matric automorphic factor with respect to a lattice L in C^g means a system of holomorphic $n \times n$-matrices $\{\mu_\alpha(z) \mid \alpha \in L\}$ such that

$$(1) \qquad \det \mu_\alpha(z) \neq 0 \qquad \text{everywhere on } C^g ,$$

$$(2) \qquad \mu_{\alpha+\beta}(z) = \mu_\alpha(z + \beta)\mu_\beta(z) \qquad (\alpha, \beta \in L) .$$

This is nothing else than the condition of a group action of L on $C^g \times C^n$;

$$(z, u) \longrightarrow (z + \alpha, \mu_\alpha(z)u) \qquad (\alpha \in L) .$$

The quotient $E_\mu = C^g \times C^n/L$ by this group action of L is a holomorphic vector bundle of rank n over the complex torus C^g/L. Holomorphic vector bundles over the complex torus C^g/L are always constructed by this way, since holomorphic vector bundles over C^g are trivial.

Denoting

$$\omega_\alpha(z) = \mu_\alpha(z)^{-1} d\mu_\alpha(z) \qquad (\alpha \in L)$$

we get a system of $n \times n$-matric connections satisfying

$$(3) \qquad d\omega_\alpha(z) + \omega_\alpha(z) \wedge \omega_\alpha(z) = 0 ,$$

$$(4) \qquad \omega_{\alpha+\beta}(z) = \omega_\alpha(z) + \mu_\alpha(z)^{-1}\omega_\beta(z + \alpha)\mu_\alpha(z) \qquad (\alpha, \beta \in L) .$$

In the present short note we shall characterize matric automorphic factors $\{\mu_\alpha(z) \mid \alpha \in L\}$ such that

i) the associated vector bundle E_μ is simple and ii)

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$$\mu_\alpha(z+\beta)\mu_\alpha(z)^{-1} \qquad (\alpha, \beta \in L)$$

are constant matrices.

PROPOSITION 1. *Let $\{\mu_\alpha(z) \mid \alpha \in L\}$ be a holomorphic $n \times n$-matric automorphic factor with respect to a lattice L in C^g, and let $\omega_\alpha(z)$ be the integrable connections given by*

$$\omega_\alpha(z) = \mu_\alpha(z)^{-1} d\mu_\alpha(z) \qquad (\alpha \in L) .$$

Then following three conditions are equivalent each other,

(5) $\omega_\alpha(z) = \sum A_{\alpha\ell} dz_\ell$ *with constant matrices* $A_{\alpha\ell}$ $\quad (\alpha \in L; 1 \leq \ell \leq g)$.

(6) $\mu_\alpha(z) = \mu_\alpha(0) \exp\{\sum A_{\alpha\ell} z_\ell\}$ *with constant matrices* $A_{\alpha\ell}$ *satisfying* $[A_{\alpha\ell}, A_{\alpha h}] = 0 \quad (\alpha \in L;\ 1 \leq \ell,\ h \leq g)$,

(7) $\mu_\alpha(z+\beta)\mu_\alpha(z)^{-1} \quad (\alpha, \beta \in L)$ *are constant matrices* .

Proof. If we assume (5), we have $\omega_\alpha(z) \wedge \omega_\alpha(z) = 0$ and thus $[A_{\alpha\ell}, A_{\alpha h}] = 0$ $(\alpha, \beta \in L; 1 \leq \ell, h \leq g)$. By virtue of this commutativity, putting $\tilde{\mu}_\alpha(z) = \mu_\alpha(0) \exp\{\sum A_{\alpha\ell} z_\ell\}$, we have

$$\tilde{\mu}_\alpha(z)^{-1} d\tilde{\mu}_\alpha(z) = \omega_\alpha(z) = \mu_\alpha(z)^{-1} d\mu_\alpha(z) \qquad (\alpha \in L) .$$

Since $\tilde{\mu}_\alpha(0) = \mu_\alpha(0)$, we have $\tilde{\mu}_\alpha(z) = \mu_\alpha(z)$ $(\alpha \in L)$. If we assume (6), then

$$\mu_\alpha(z+\beta)\mu_\alpha(z)^{-1} = \mu_\alpha(0) \exp\{\sum A_{\alpha\ell}\beta_\ell\}\mu_\alpha(0)^{-1} \qquad (\alpha, \beta \in L)$$

are constant matrices. Let us show (7) from (5). From the equation $d(\mu_\alpha(z+\beta)\mu_\alpha(z)^{-1}) = 0$ it follows

$$\begin{aligned} \omega_\alpha(z+\beta) - \omega_\alpha(z) &= \mu_\alpha(z+\beta)^{-1} d\mu_\alpha(z+\beta) - \mu_\alpha(z)^{-1} d\mu_\alpha(z) \\ &= \mu_\alpha(z+\beta)^{-1} d(\mu_\alpha(z+\beta)\mu_\alpha(z)^{-1})\mu_\alpha(z) = 0 , \end{aligned}$$

and thus $\omega_\alpha(z+\beta) = \omega_\alpha(z)$ $(\alpha, \beta \in L)$. By virtue of compactness of C^g/L $\omega_\alpha(z)$ can be written

$$\omega_\alpha(z) = \sum A_{\alpha\ell} dz_\ell$$

with constant matrices.

PROPOSITION 2. *Let $\{\mu_\alpha(0) \mid \alpha \in L\}$, $\{A_{\alpha,\ell} \mid \alpha \in L, 1 \leq \ell \leq g\}$ be two systems of constant $n \times n$-matrices such that* $\det \mu_\alpha(0) \neq 0$, $[A_{\alpha\ell}, A_{\alpha h}] = 0$ $(\alpha \in L; 1 \leq \ell, h \leq g)$. *Then $\{\mu_\alpha(0) \exp\{\sum A_{\alpha\ell} z_\ell\} \mid \alpha \in L\}$ is a holomorphic $n \times n$-matric automorphic factor with respect to L, if and only if*

(8) $$[A_{\alpha\ell}, A_{\beta h}] = 0 \qquad (\alpha, \beta \in L; 1 \leq \ell, h \leq g) ,$$

(9) $$A_{\alpha+\beta,\ell} = \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0) + A_{\beta\ell} \qquad (\alpha, \beta \in L; 1 \leq \ell \leq g) ,$$

(10) $$[\mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0), A_{\beta h}] = 0 \qquad (\alpha, \beta \in L; 1 \leq \ell, h \leq g) .$$

Proof. Assume that $\{\mu_\alpha(z) = \mu_\alpha(0) \exp(\sum A_{\alpha\ell}z_\ell)\}$ is an automorphic factor with respect to L. From the relations

$$\mu_{-\alpha}(z + \alpha)\mu_\alpha(z) = I , \qquad \omega_\alpha(z) = \sum A_{\alpha\ell}dz_\ell ,$$
$$\omega_{\alpha+\beta}(z) = \mu_\alpha(z)^{-1}\omega_\beta(z + \alpha)\mu_\alpha(z) + \omega_\alpha(z)$$

it follows

$$\begin{aligned}\omega_{-\alpha+\beta}(0) &= \omega_{-\alpha+\beta}(z + \alpha) = \mu_{-\alpha}(z + \alpha)^{-1}\omega_\beta(z)\mu_{-\alpha}(z + \alpha) + \omega_{-\alpha}(z + \alpha)\\ &= \mu_\alpha(z)\omega_\beta(z)\mu_\alpha(z)^{-1} + \omega_{-\alpha}(z + \alpha)\\ &= \mu_\alpha(z)\omega_\beta(0)\mu_\alpha(z)^{-1} + \omega_{-\alpha}(0) = \mu_\alpha(0)\omega_\beta(0)\mu_\alpha(0)^{-1} + \omega_{-\alpha}(0) ,\end{aligned}$$

and thus

$$\mu_\alpha(0)^{-1}\mu_\alpha(z)\omega_\beta(0) = \omega_\beta(0)\mu_\alpha(0)^{-1}\mu_\alpha(z) \qquad (\alpha, \beta \in L) .$$

Comparing the coefficients of $z_\ell dz_h$ in the both sides of

$$\exp\{\sum A_{\alpha\ell}z_\ell\} \sum A_{\beta h}dz_h = \sum A_{\beta h}dz_h \exp\{\sum A_{\alpha\ell}z_\ell\}$$

we have $[A_{\alpha\ell}, A_{\beta h}] = 0$ $(\alpha, \beta \in L; 1 \leq \ell,\ h \leq g)$. From the relation $\mu_{\alpha+\beta}(z) = \mu_\alpha(z + \beta)\mu_\beta(z)$ we have

$$\begin{aligned}\mu_{\alpha+\beta}(z) &= \mu_{\alpha+\beta}(0) \exp\{\sum A_{\alpha+\beta\ell}z_\ell\}\\ &= \mu_\alpha(0) \exp\{\sum A_{\alpha\ell}\beta_\ell\}\mu_\beta(0) \exp\{\sum A_{\alpha+\beta\ell}z_\ell\}\\ &= \mu_\alpha(z + \beta)\mu_\beta(z)\\ &= \mu_\alpha(0) \exp\{\sum A_{\alpha\ell}(z_\ell + \beta_\ell)\}\mu_\beta(0) \exp\{\sum A_{\beta\ell}z_\ell\}\\ &= \mu_\alpha(0) \exp\{\sum A_{\alpha\ell}\beta_\ell\}\mu_\beta(0)\\ &= \exp\{\sum \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0)z_\ell\} \exp\{\sum A_{\beta\ell}z_\ell\} .\end{aligned}$$

Hence, comparing the coefficients of z_ℓ and $z_\ell z_h$ in the both sides of

$$\exp\{\sum A_{\alpha+\beta\ell}z_\ell\} = \exp\{\sum \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0)z_\ell\} \exp\{\sum A_{\beta\ell}z_\ell\} ,$$

respectively, we have

$$\begin{aligned}A_{\alpha+\beta\ell} &= \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0) + A_{\beta\ell} ,\\ A_{\alpha+\beta\ell}A_{\alpha+\beta h} &= \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0)\mu_\beta(0)^{-1}A_{\alpha h}\mu_\beta(0)\\ &\quad + \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0)A_{\beta h} + \mu_\beta(0)^{-1}A_{\alpha h}\mu_\beta(0)A_{\beta\ell} + A_{\beta\ell}A_{\beta h}\end{aligned}$$

and thus

$$A_{\beta\ell}u_\beta(0)^{-1}A_{\alpha h}\mu_\beta(0) = \mu_\beta(0)^{-1}A_{\alpha h}\mu_\beta(0)A_{\beta\ell} .$$

Namely $[A_{\beta\ell}, \mu_\beta(0)^{-1}A_{\alpha h}\mu_\beta(0)] = 0$ $(\alpha, \beta \in L\,;\, 1 \leq \ell,\, h \leq g)$. Conversely if we assume (8), (9), (10), then putting

$$\mu_\alpha(z) = \mu_\alpha(0) \exp \{\sum A_{\alpha\ell}z_\ell\} \qquad (\alpha \in L) ,$$

we have

$$\begin{aligned}
&\mu_\alpha(z + \beta)\mu_\beta(z) \\
&\quad = \mu_\alpha(0) \exp \{\sum A_{\alpha\ell}(z_\ell + \beta_\ell)\mu_\beta(0)\} \exp \{\sum A_{\beta\ell}z_\ell\} \\
&\quad = \mu_\alpha(0) \exp \{\sum A_{\alpha\ell}\beta_\ell\}\mu_\beta(0) \exp \{\sum \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0)z_\ell\} \exp \{\sum A_{\beta\ell}z_\ell\} \\
&\quad = \mu_\alpha(0)\mu_\beta(0) \exp \{\sum (\mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0) + A_{\beta\ell})z_\ell\} \\
&\quad = \mu_{\alpha+\beta}(0) \exp \{A_{\alpha+\beta\ell}z_\ell\} = \mu_{\alpha+\beta}(z) .
\end{aligned}$$

COROLLARY 1.

$$\begin{aligned}
&\mu_\alpha(z)^{-1}\mu_\beta(z)^{-1}\mu_\alpha(z)\mu_\beta(z) \\
&\quad = \exp \{\sum (A_{\alpha+\beta\ell} - A_{\beta\ell})\alpha_\ell - \sum (A_{\alpha+\beta\ell} - A_{\alpha\ell})\beta_\ell\} \qquad (\alpha, \beta \in L) .
\end{aligned} \tag{11}$$

Proof. From the relation $\mu_{\alpha+\beta}(z) = \mu_\alpha(z + \beta)\mu_\beta(z) = \mu_\beta(z + \alpha)\mu_\alpha(z)$ it follows

$$\begin{aligned}
\mu_\alpha(z)\mu_\beta(z) &= \mu_\alpha(0)u_\beta(0) \exp \{\sum (\mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0) + A_{\beta\ell})z_\ell\} \\
&= \mu_\alpha(0)\mu_\beta(0) \exp \{\sum A_{\alpha+\beta\ell}z_\ell\} , \\
\mu_\beta(z)\mu_\alpha(z) &= \mu_\beta(0)\mu_\alpha(0) \exp \{\sum (\mu_\alpha(0)^{-1}A_{\beta\ell}\mu_\alpha(0) + A_{\alpha\ell})z_\ell\} \\
&= \mu_\beta(0)\mu_\alpha(0) \exp \{\sum A_{\alpha+\beta\ell}z_\ell\} , \\
\mu_\alpha(0)\mu_\beta(0) &\exp \{\sum \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0)\beta_\ell\} \\
&= \mu_{\alpha+\beta}(0) = \mu_\beta(0)\mu_\alpha(0) \exp \{\sum \mu_\alpha(0)^{-1}A_{\beta\ell}\mu_\alpha(0)\alpha_\ell\} .
\end{aligned}$$

Hence

$$\begin{aligned}
&\mu_\alpha(z)^{-1}\mu_\beta(z)^{-1}\mu_\alpha(z)\mu_\beta(z) \\
&\quad = \mu_\alpha(0)^{-1}\mu_\beta(0)^{-1}\mu_\alpha(0)\mu_\beta(0) \\
&\quad = \exp \{\sum \mu_\alpha(0)^{-1}A_{\beta\ell}\mu_\alpha(0)\alpha_\ell - \sum \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0)\beta_\ell\} \\
&\quad = \exp \{\sum (A_{\alpha+\beta\ell} - A_{\beta\ell})\alpha_\ell - \sum (A_{\alpha+\beta\ell} - A_{\alpha\ell})\beta_\ell\} .
\end{aligned}$$

COROLLARY 2. *The matric group generated by* $\{\mu_\alpha(z) \mid \alpha \in L\}$ *is a metabelian group whose derived group is a group of constant matrices.*

This is an immediate consequence of Corollary 1.

2. A holomorphic vector bundle is called to be simple, if its endomorphisms are scaler multiplications. The vector bundle associated with $\{\mu_\alpha(z) \mid \alpha \in L\}$ is simple, if and only if scaler matrices are only holomorphic matrices $B(z)$ such that

$$B(z+\alpha)\mu_\alpha(z) = \mu_\alpha(z)B(z) \qquad (\alpha \in L) .$$

Characterizing a simple vector bundle E_μ such that

$$\mu_\alpha(z)^{-1}d\mu_\alpha(z) = \sum_{\ell=1}^{g} A_{\alpha\ell}dz_\ell \qquad (\alpha \in L)$$

with constant matrices $A_{\alpha\ell}$, we need the following Clifford theorem,

CLIFFORD THEOREM. *Let G be a group and H be a normal subgroup of G. Let V be a vector space of finite dimension over a field which is a simple G-module. Then there exists a vector subspace W which is simple H-module and elements $g_{11}, \cdots, g_{1m}, \cdots, g_{r1}, \cdots, g_{rm}$ in G such that*

$$V = (g_{11}W \oplus \cdots \oplus g_{1m}W) \oplus \cdots \oplus (g_{r1}W \oplus \cdots \oplus g_{rm}W)$$

and $g_{i\ell}W$ and $g_{jk}W$ are equivalent H-modules if and only if $i = j$.

Proof. Let W be a vector subspace which is a simple H-module. Since H is normal in G, the images gW $(g \in G)$ are simple H-modules, and V is a sum of gW $(g \in G)$. Hence there exist elements $g'_1, \cdots, g'_p$ of G such that

$$V = g'_1W \oplus \cdots \oplus g'_pW .$$

Let $\{g_{11}, \cdots, g_{1m}\}$ be the largest subset in $\{g'_1, \cdots, g'_p\}$ such that $g_{1\ell}W$ $(1 \leq \ell \leq m)$ are equivalent to W. Then V is a direct sum of the images of $g_{11}W \oplus \cdots \oplus g_{1m}W$ by elements of G. This completes the proof of Clifford theorem.

LEMMA 1. *Let G be a transitive abelian permutation group acting on $\{1, 2, \cdots, r\}$. Then $|G| = r$.*

Proof. Let N be the stabilizer of letter 1. Then $|G/N| = r$ and G acts on G/N. Two permutation groups $(G, \{1, 2, \cdots, r\})$ and $(G, G/N)$ are isomorphic as permutation groups. Since G is abelian, N must be a normal subgroup, and thus any element of N leaves invariant every letter in $\{1, 2, \cdots, r\}$. This shows $|G| = |G/N| = r$.

THEOREM 1. *Let $\{\mu_\alpha(z) \mid \alpha \in L\}$ be a holomorphic $n \times n$-matric automorphic factor with respect to a lattice L in C^g such that* i)

$$\mu_\alpha(z)^{-1} d\mu_\alpha(z) = \sum_{\ell=0}^{g} A_{\alpha\ell} dz_\ell \qquad (\alpha \in L)$$

with constant matrices $A_{\alpha\ell}$, ii) *the associated vector bundle E_μ is simple. Then there exist a sublattice M of L and a line bundle $\mathscr{L}$ over C^g/M such that $[L: M] = n$ and E_μ is the direct image of $\mathscr{L}$ with respect to the natural isogeny $C^g/M \to C^g/L$..*

Proof. We use the following notations:

Λ: the commutative matric algebra generated by $A_{\alpha,\ell}$ $(\alpha \in L; 1 \leq \ell \leq g)$ and identity matrix over C,

G: the group generated by $\{\mu_\alpha(0), \exp\{\sum A_{\alpha\ell}\beta_\ell\}, (\alpha, \beta \in L)\}$ and $GL(n, C) \cap \Lambda$,

G_0: the group generated by $\{\exp\{\sum A_{\alpha\ell}\beta_\ell\}$ $(\alpha, \beta \in L)$ and $GL(n, C) \cap \Lambda$,

$$G_1 = \{g \in G \mid g^{-1}A_{\alpha\ell}g = A_{\alpha\ell} \ (\alpha \in L; 1 \leq \ell \leq g)\}$$
$$L_1 = \{\beta \in L \mid \mu_\beta(0)^{-1}A_{\alpha\ell}\mu_\beta(0) = A_{\alpha\ell} \ (\alpha \in L; 1 \leq \ell \leq g)\} .$$

By virtue of (8), (9) and (11) G_0 is an abelian normal subgroup of G such that G/G_0 is abelian, and G_1 is a normal subgroup containing G_0. Hence G/G_1 is abelian. If a constant matrix B satisfies

$$B\mu_\alpha(0) = \mu_\alpha(0)B , \quad BA_{\alpha\ell} = A_{\alpha\ell}B \qquad (\alpha \in L, 1 \leq \ell \leq g) ,$$

then $B\mu_\alpha(z) = \mu_\alpha(z)B$ $(\alpha \in L)$, because from Proposition 1

$$\mu_\alpha(z) = \mu_\alpha(0) \exp\{\textstyle\sum_\ell A_{\alpha\ell}z_\ell\} \qquad (\alpha \in L) .$$

Since E_μ is simple, such a matrix B must be a scaler matrix. This means that G is an irreducible $n \times n$-matric group. Let V be the vector space of dimension non which G acts. Applying Clifford theorem to the pair (G, G_0), we get a system of inequivalent representations $\chi, \cdots, \chi_n$ of degree one of the abelian group G_0 such that the irreducible G-module V decomposed into a direct sum of G_0-modules

$$V = V_1 \oplus \cdots \oplus V_r$$

with properties that

$$\dim V_i = \frac{n}{r} (1 \leq i \leq r) ,$$

$$g_0 v_i = \chi_i(g_0) v_i \qquad g_0 \in G_0, v_i \in V_i) \ .$$

since $\chi_1, \cdots, \chi_r$ are inequivalent each other, the subspaces $V_1, \cdots, V_r$ are also G_1-modules. Moreover, since G_0 is normal in G, G acts on the set $\{1, \cdots, r\}$ as follows

$$gV_i = V_{i\sigma(g)} \qquad (g \in G) \ ,$$

and the kernel of σ is G_1. Since V is an irreducible G-module and G/G_1 is abelian, by virtue of Lemma 1 we have $|G/G_1| = |\sigma(G)| = r$. Next we shall show that V_i $(1 \leq i \leq g)$ are irreducible G_1-modules. We assume for a moment that at least one of G_1-modules of $\{V_1, \cdots, V_r\}$ is reducible, then dimension of the linear hull of G_1 over $\boldsymbol{C}$ is less than

$$r\left(\frac{n}{r}\right)^2 = \frac{n^2}{r} \ .$$

Hence the dimension of the linear hull of G is less than $r \cdot n^2/r = n^2$. This contradicts with the irreducibility of G-module V. Hence $V_1, \cdots, V_r$ are irreducible G_1-modules. From equations

$$A_{\alpha+\beta,\ell} = u_\alpha(0)^{-1} A_{\beta\ell} \mu_\alpha(0) + A_{\alpha\ell} = \mu_\beta(0)^{-1} A_{\alpha\ell} \mu_\beta(0) + A_{\beta\ell}$$

we observe that

$$\mu_\beta(0)^{-1} A_{\alpha\ell} \mu_\beta(0) = A_{\alpha\ell} \qquad (\alpha \in L, \ 1 \leq \ell \leq g) \ ,$$

if and oly if

$$\mu_\alpha(0)^{-1} A_{\beta\ell} \mu_\alpha(0) = A_{\beta\ell} \qquad (\alpha \in L, \ 1 \leq \ell \leq g) \ .$$

This means that $L_1 = \{\beta \,|\, A_{\beta\ell} \ (1 \leq \ell \leq g)$ are scaler matrices$\}$. This means that there exist a constant matrix F and an irreducible representation ρ of G_1 such that

$$F^{-1} h F = \begin{pmatrix} \rho(h) & & & \\ & \rho(g_1^{-1} h g_1) & & \\ & & \ddots & \\ & & & \rho(g_r^{-1} h g_r) \end{pmatrix} \qquad (h \in G_1) \ ,$$

where $g_i = \mu_{\alpha_i}(0)$ $(1 \leq i \leq r)$ and $\{\alpha_1, \cdots, \alpha_r\}$ is a system of representatives of G/G_1 in G such that $1^{\sigma(\mu_{\alpha_i}(0))} = i$ $(1 \leq i \leq r)$. Since $\rho(g_0)$ $(g_0 \in G_0)$ are scaler $n/r \times n/r$-matrices, the set $\{\rho(\mu_\beta(0)) \,|\, \beta \in L_1\}$ is an irreducible set. In the previous note [1] we proved that a holomorphic $k \times k$-automorphic factor $\{\nu_\beta(z) \mid \beta \in N\}$ with respect to a lattice N is written

$$\nu_\beta(z) = \nu_\beta(0)\xi_\beta(z) \qquad (\beta \in N)$$

with scaler functions $\xi_\beta(z)$ and $\{\nu_\beta(0) \mid \beta \in N\}$ is an irreducible set of $k \times k$-matrices, then there exist a sublattice N_0 such that $[N: N_0] = k$ and $\nu_\gamma(z)$ $(\gamma \in N_0)$ are scaler matrices. Hence putting

$$\begin{aligned} &\nu_\beta(0) = \rho(\mu_\beta(0)) , \qquad A_{\beta\ell} = a_{\beta\ell} I_n , \\ &\nu_\beta(z) = \nu_\beta(0) \exp\{2\pi\sqrt{-1} \sum a_{\beta\ell} z_\ell\} \qquad (\beta \in L_1) , \end{aligned}$$

we find a sublattice M such that

$$[L: M] = [L: L_1][L_1: M] = r \cdot \frac{n}{r} = n$$

and $\nu_\gamma(z)$ $(\gamma \in M)$ are scaler matrices. Since $\rho(\mu_\gamma(0))$ $(\gamma \in M)$ are scaler matrices, $\rho(\mu_{\alpha_i}(0)^{-1}\mu_\gamma(0)\mu_{\alpha_i}(0))$ $(\gamma \in M; 1 \leq i \leq r)$ are also scaler matrices. This means that there exists holomorphic scaler automorphic factors $\{\eta_\gamma^{(1)}(z)\}, \cdots, \{\eta_\gamma^{(n)}(z)\}$ with respect to the lattice M such that

$$F^{-1}\mu_\gamma(z)F = \begin{pmatrix} \eta_\gamma^{(1)}(z) & & \\ & \ddots & \\ & & \eta_\gamma^{(n)}(z) \end{pmatrix} \qquad (\gamma \in M)$$

and $[L: M] = n$. Let $\mathscr{L}^{(1)}, \cdots, \mathscr{L}^{(n)}$ be line bundles over M corresponding to $\{\eta_\gamma^{(1)}(z), \cdots, \{\eta_\gamma^{(n)}(z)\}$, respectively. Then we have isomorphisms of vector bundles

$$\begin{aligned} &\varphi^*(E_\mu) \sim \mathscr{L}^{(1)} \oplus \cdots \oplus \mathscr{L}^{(n)} , \\ &\varphi_*\varphi^*(E_\mu) \sim \varphi_*(\mathscr{L}^{(1)}) \oplus \cdots \oplus \varphi_*(\mathscr{L}^{(n)}) , \end{aligned}$$

where $\varphi^*(E_\mu)$ is the reciprocal image of E_μ and $\varphi_*(\mathscr{L})$ is the direct image of $\mathscr{L}$ with respect to φ. Hence, if we put

$$\nu_\beta(0) = \rho(\mu_\beta(0)) , \qquad A_{\beta\ell} = a_{\beta\ell} I_n , \qquad \nu_\beta(z) = \nu_\beta(0) \exp\{\sum a_{\beta\ell} z_\ell\} ,$$

by virtue of the above result, we find a sublattice M such that $[L: M] = [L: L_1][L_1: M] = r \cdot n/r = n$ and $\nu_\gamma(z)$ $(\gamma \in M)$ are scaler $n/r \times n/r$-matrices. Since $\rho(\mu_\gamma(0))$ $(\gamma \in M)$ are scaler matrices, $\rho(\mu_{\alpha_i}(0)^{-1}\mu_\gamma(0)\mu_{\alpha_i}(0))$ $(\gamma \in M; 1 \leq i \leq r)$ are also scaler matrices. This means that there exists holomorphic scaler automorphic factors $\{\eta_\gamma^{(1)}(z) \mid \gamma \in M\}, \cdots, \{\eta_\gamma^{(r)}(z) \mid \gamma \in M\}$ such that

$$F^{-1}\mu_\gamma(z)F = \begin{pmatrix} \eta_\gamma^{(1)}(z)I_{n/r} & & 0 \\ & \ddots & \\ 0 & & \eta_\gamma^{(r)}(z)I_{n/r} \end{pmatrix} \qquad (\gamma \in M) .$$

Let $\mathscr{L}^{(1)}, \cdots, \mathscr{L}^{(r)}$ be the line bundles over C^g/M associated to $\{\eta_\gamma^{(1)}(z) \mid \gamma \in M\}, \cdots, \{\eta_\gamma^{(r)}(z) \mid \gamma \in M\}$, respectively. Then we have an isomorphism of vector bundles

$$\varphi_*\varphi^*(E_\mu) \sim (\varphi\varphi_*\overbrace{(\mathscr{L}^{(1)}) \oplus \cdots \oplus \varphi\varphi_*}^{n/r}(\mathscr{L}^{(1)})) \oplus \cdots \oplus (\varphi\varphi_*\overbrace{(\mathscr{L}^{(n)}) \oplus \cdots \oplus \varphi\varphi_*}^{n/r}(\mathscr{L}^{(n)}))$$

where $\varphi^*(E_\mu)$ is the inverse image of E_μ and $\varphi_*\varphi^*(E_\mu)$ is the direct image of $\varphi^*(E_\mu)$. Since E_μ is simple and $\operatorname{rank} \varphi_*(\mathscr{L}_*^{(i)}) = \operatorname{rank} E_\mu$ $(1 \leq i \leq n)$, E_μ must be isomorphic to one of $\varphi_*(\mathscr{L}^{(i)})$ $(1 \leq i \leq n)$. This completes the proof of Theorem.

THEOREM 2. *Let $\{\mu_\alpha(z) \mid \alpha \in L\}$ be a holomorphic $n \times n$-automorphic factor with respect to a lattice L in C^g, such that* i) *the associated vector bundle E_μ is simple and* ii) *$\mu_\alpha(z + \beta)\mu_\alpha(z)^{-1}$ $(\alpha, \beta \in L)$ are constant matrices. Then there exist a sublattice M of L and a line bundle $\mathscr{L}$ on C^g/M such that E_μ is isomorphic to the direct image of $\mathscr{L}$ with respect to the natural isogeny $C^g/M \to C^g/N$.*

Proof. This an immediate consequence from Theorem 1 and Proposition 1.

REFERENCES

[1] H. Morikawa, A note on holomorphic vector bundles over complex tori, Nagoya Math. J. Vol. **41** (1970), 101–106.

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