T. Fukuda
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DEFORMATIONS OF REAL ANALYTIC FUNCTIONS AND THE NATURAL STRATIFICATION OF THE SPACE OF REAL ANALYTIC FUNCTIONS

TAKUO FUKUDA

0. Introduction.

Let A be a real analytic set, M be a compact real analytic manifold and $f: A \times M \to \boldsymbol{R}$ be a real analytic function. Then we have a family of real analytic functions $f_a, a \in A$, on M defined by $f_a(X) = f(a, x)$.

Two functions f_a and f_b are said to be topologically equivalent if there exist homeomorphisms h_1 of M and h_2 of $\boldsymbol{R}$ such that $h_2 \circ f_a = f_b \circ h_1$.

The purpose of the present paper is to prove the following

THEOREM 1. *There is a Whitney stratification of A satisfying the following properties*:

(i) *Each stratum is a smooth subanalytic subset of A. (For the subanalycity, see* § 3.)

(ii) *For any two points a and b belonging to the same stratum, the corresponding functions f_a and f_b are topologically equivalent.*

COMMENT 1. By Theorem 1, we can see that any analytic deformation of a real analytic function on M contains locally only a finite number of topological types of functions: An analytic deformation of an analytic function $g: M \to \boldsymbol{R}$ is an analytic function $f: U \times M \to \boldsymbol{R}$, U being an open set of $\boldsymbol{R}^n$, or the family $\{f_a\}$, $a \in U$, of real analytic functions on M defined by $f_a(x) = f(a, x)$ such that $f_o = g$ where o is the origin of $\boldsymbol{R}^n$. Then the above statement means that there is a neighborhood $U(o)$ of o in U such that the number of the topological equivalence classes of functions f_a, $a \in U(o)$, is finite. This property holds even for deformations of a function of infinite codimension. This is a special phenomena for analytic deformations. In fact, it is known that there is a

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smooth deformation of a smooth function which contains infinitely many topological types of functions.

COMMENT 2. Let $C^{\omega}(M)$ denote the space of real analytic functions on a real analytic manifold M. The topological equivalence relation defined above gives a partition of $C^{\omega}(M)$ into the topological equivalence classes of functions. Let's call provisionally in this paper this partition the natural stratification of $C^{\omega}(M)$.

Of cause, the natural stratification of $C^{\omega}(M)$ is not a stratification in the sense of Cerf [2], for instance the local finiteness fails. Nevertheless, our Theorem 1 gives us a peep of a good and neat structure of this natural stratification: A mapping $F: A \rightarrow C(M)$ of an analytic set A into $C^{\omega}(M)$ is also called an analytic deformation if the corresponding function $f: A \times M \rightarrow \boldsymbol{R}$, defined by $f(a, x) - F(a)(x)$, is real analytic. By Theorem 1, for any analytic deformation $F: A \rightarrow C^{\omega}(M)$, the partition of A into subsets induced by F from the natural stratification is itself a Whitney stratification or it has a subdivision which is a Whitney stratification of A. Equivalently, for any analytic deformation F, the section $F(A)$ ($\subset C^{\omega}(M)$) has a Whitney stratification which is a subdivision of the natural stratification. That is to say, so far as we peep through the analytic deformations, the natural stratification of $C^{\omega}(M)$ has a structure of Whitney stratification.

The main tools for the proof of the Theorem 1 are the notion of subanalytic subsets obtained by H. Hironaka [3], Thom's second isotopy lemma, a good stratification of holomorphic functions given by Lê-Pham [4] and a stratification of proper real analytic mappings:

THEOREM 2. *A proper real analytic mapping is a stratified map.*

Theorem 1 is an immediate consequence of Thom's second isotopy lemma and the following theorem.

THEOREM 3. *Let A, M and f be as in Theorem 1. Let $F: A \times M \rightarrow A \times \boldsymbol{R}$ be the map defined by $F(a, x) = (a, f(a, x))$. Then there exists a Whitney stratification $S(A)$ of A such that each stratum is subanalytic and such that for any stratum Z of $S(A)$, the restricted map $F|Z \times M: Z \times M \rightarrow Z \times \boldsymbol{R}$ is a Thom mapping over the projection $\pi: Z \times \boldsymbol{R} \rightarrow Z$.*

1. Whitney stratification.

In which we introduce the notion of stratifications which is due to H. Whitney [7]. Here we collect only the definitions and some properties which we need. For the proof of these properties and more details, we are referred to R. Thom [6] and J. Mather [5].

Let X and Y be differentiable submanifolds of $\boldsymbol{R}^n$. Let y be a point of Y and let $r = \dim X$. In what follows, $T_p(M)$ denotes the tangent space to a manifold M at point p of M.

DEFINITION 1.1. We say that the pair (X, Y) satisfies condition (a) at $y \in Y$ if the following holds: Given any sequence X_i of points in X such that $x_i \to y$ and the tangent space $T_{x_i}(X)$ converges to some r-plane τ, we have $T_y(Y) \subset \tau$.

Here and in what follows "convergence" means convergence in the standard topology on the Grassmannian manifold of r-planes in $\boldsymbol{R}^n$.

For any two distinct points $x, y \in \boldsymbol{R}^n$, the secant $\widehat{xy}$ denotes the line in $\boldsymbol{R}^n$ which is parallel to the line joining x and y and passes through the origin.

Let X, Y be smooth submanifolds of $\boldsymbol{R}^n$. Let $y \in Y$. Let $r = \dim X$.

DEFINITION 1.2. We say that the pair (X, Y) satisfies *condition* (b) *at* y if the following holds. Given any sequences x_i of points in X and y_i of points in Y such that $x_i \neq y_i$, $x_i \to y$ and $y_i \to y$ and such that $T_{x_i}(X)$ converges to some r-plane τ and the secants $\widehat{x_i y_i}$ converge to some line $\ell \subset \boldsymbol{R}^n$, we have $\ell \subset \tau$.

We say the pair (X, Y) satisfies *condition* (a) (resp. (b)) if it satisfies condition (a) (resp. (b)) at every point of Y.

Remark (Mather [3]). If (X, Y) satisfies condition (b) at y, then it satisfies condition (a) at y.

DEFINITION 1.3. A *W-complex* is a set $S = \{X_\alpha\}$ of connected smooth manifolds in $\boldsymbol{R}^n$, called strata of S, satisfying the following conditions:

(i) The strata X_α are pair-wise disjoint.

(ii) (X, Y) satisfies condition (b) for any pair (X, Y) of strata of $S = \{X_\alpha\}$.

(iii) The family $S = \{X_\alpha\}$ is locally finite: each point of $\boldsymbol{R}^n$ has a neighborhood which meets at most finitely many strata.

(iv) If $\overline{X}_\alpha \cap X_\beta \neq \emptyset$, then $\overline{X}_\alpha \supset X_\beta$.

DEFINITION 1.4. A *stratified set* is a subset E of $\boldsymbol{R}^n$ with a W-complex $S(E) = \{X_\alpha\}$ such that $E = \bigcup_\alpha X_\alpha$. We call $S(E)$ a Whitney stratification of E.

NOTATION. Let X, Y be two strata of $S(E)$ with $Y \cap \bar{X} \neq \emptyset$. Then by the condition (iv) in Definition 1.3, we have $Y \subset \bar{X} - X$. We represent this situation by the symbol $Y < X$ and we say Y is *incident* to X.

2. Stratified mappings and Thom's isotopy lemma.

Let $E \subset \boldsymbol{R}^n$ and $F \subset \boldsymbol{R}^m$.

DEFINITION 2.1. We say that a continuous mapping $f: E \to F$ is a *W-morphism* or a *stratified mapping* if there exist stratifications $S(E)$ of E and $S(F)$ of F and the following conditions hold:

(i) f is extendable to a differentiable mapping of a neighborhood of E into $\boldsymbol{R}^m$.

(ii) For any stratum X of $S(E)$, the image $f(X)$ is contained in a stratum Y of $S(F)$ and the restricted mapping $f|X: X \to Y$ is a submersion.

A W-morphism $f: E \to F$ will be said to be *exact* if for any stratum X of $S(E)$, $f(X)$ is a stratum of $S(F)$.

Remark. A proper W-morphism is an exact W-morphism. (See Mather's existence theorem for tubular neighborhoods [5].)

PROPOSITION 2.2 (Thom's first isotopy lemma). *If $f: E \to F$ is a proper stratified mapping, then for each stratum Y of $S(F)$, the restricted mapping $f|f^{-1}(Y): f^{-1}(Y) \to Y$ is a locally trivial fibre bundle.*

For the proof, see Mather [5] or Thom [6].

DEFINITION 2.3 (Thom's condition a_f). Let X and Y be smooth submanifolds of R^n and let N be a smooth manifold. Let $f: U \to N$ be a differentiable mapping defined on a neighborhood U of $X \cup Y$ in $\boldsymbol{R}^n$. Suppose that $f|X$ and $f|Y$ are of constant rank. Then we say *the pair (X, Y) satisfies condition a_f at a point $y \in Y$* if the following holds: Given any sequence x_i of points in X converging to y such that the sequence of planes $\ker(f|X)_{x_i}$ converges to a plane τ in the appropriate Grassmannian manifold, we have

$$\ker(f|Y)_y \subset \tau ,$$

where $\ker(f\,|\,X)_x$ denotes the kernel of the differential

$$(df\,|\,X)_x\colon T_x(X) \longrightarrow T_{f(x)}(N)$$

of $f\,|\,X\colon X \to N$.

We say that the pair (X, Y) *satisfies condition* a_f if it satisfies condition a_f at every point of Y.

DEFINITION 2.4 (*Thom mapping*). Let $f\colon E \to F$ and $g\colon F \to V$ be stratified mappings. Suppose that V is a connected smooth manifold and it is considered as a stratified set with its trivial stratification $S(V) = \{V\}$. Then we say that f is *a Thom mapping over* g if for each point p of V and any pair (X, Y) of strata of $S(E)$, the pair $(X \cap (g\circ f)^{-1}(p), Y \cap (g\circ f)^{-1}(p))$ satisfies condition a_f.

Let $f\colon E \to F$ be a Thom mapping over $g\colon F \to V$. For a point p of V, set $E_p = (g\circ f)^{-1}(p)$ and $F_p = g^{-1}(p)$.

PROPOSITION 2.5 (*Thom's second isotopy lemma*). *Let $f\colon E \to F$ be a proper Thom mapping over a proper stratified mapping $g\colon F \to V$. Then for any two points p and q of V, the restricted mappings $f\,|\,E_p\colon E_p \to F_p$ and $f\,|\,E_q\colon E_q \to F_q$ are of same topological type: there exist homeomorphisms $h_1\colon E_p \to E_q$ and $h_2\colon F_p \to F_q$ such that the following diagram commutes:*

$$\begin{array}{ccc} E_p & \xrightarrow{h_1} & E_q \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ F_p & \xrightarrow{h_2} & F_q \end{array}\ .$$

For the proof of this proposition, see Mather [5].

3. Subanalytic subsets.

In which we introduce the notion of "subanalycity" that is due to H. Hironaka [3]. All the properties are stated without proof. For the proof, more details or examples, see [3].

DEFINITION 3.1. Let Ω be an open set in $\boldsymbol{R}^n$. *An analytic set* $A \subset \Omega$ is a set such that for any point a of Ω, there is a neighborhood U of a in Ω and analytic functions $f_1, \cdots f_k$ in U such that

$$A \cap U = \{x \in U \mid f_1(x) = \cdots = f_k(x) = 0\} .$$

DEFINITION 3.2 (*Analytic mappings*). Let A_i, $i = 1, 2$, be analytic sets in open sets $\Omega_i \subset \boldsymbol{R}^{n_i}$. A continuous mapping $f: A_1 \to A_2$ is said to be *analytic at a point* $a \in A_1$ if there exist a neighborhood U of a in Ω_1 and an analytic mapping $F: U \to \boldsymbol{R}^{n_2}$ with

$$F \mid A_1 \cap U = f \mid A_1 \cap U .$$

An analytic mapping is a continuous mapping of an analytic set A_1 into another analytic set which is analytic at every point of A_1.

DEFINITION 3.3 (*Subanalytic subsets*). Let $X \subset \Omega$ be an analytic subset of an open set in $\boldsymbol{R}^n$. *A subanalytic subset* $A \subset X$ is a set such that for any point a of X there exist an open neighborhood U of a in X and a finite system of analytic sets Y_{ij} and proper real analytic mappings $f_{ij}: Y_{ij} \to X$, $1 \leqq i \leqq p$ and $j = 1, 2$, such that

$$A \cap U = \bigcup_{i=1}^{p} (p_{i1}(Y_{i1}) - f_{i2}(Y_{i2})) .$$

PROPOSITION 3.4. *Let A, B be subanalytic subsets of an analytic set X. Then so are $A \cup B$, $A \cap B$ and $A - B$.*

PROPOSITION 3.5. *Let $f: X \to Y$ be a proper real-analytic mapping.*

(i) *If B is a subanalytic subset of Y, then so is $f^{-1}(B)$ in X.*

(ii) *If A is a subanalytic subset of X, then so is $f(A)$ in Y.*

DEFINITION 3.6. Let A be a subanalytic subset of $X \subset \Omega \subset \boldsymbol{R}^n$. A point $a \in A$ is called *a regular point of A of dimension k* if there is a neighborhood U of a, $U \subset \Omega$, such that $A \cap U$ is an analytic submanifold of dimension k of U. A point $a \in A$ is called *singular* if it is not regular.

PROPOSITION 3.7. *Let A be a subanalytic subset of an analytic set X. Then we have:*

(i) *The closure $\bar{A}$ of A in X is subanalytic in X.*

(ii) *Every connected component of A is subanalytic in X and A has locally finite connectedness in X, i.e., every point of X has a neighborhood which meets only a finite number of connected components of A.*

(iii) *The set of singular points of A is subanalytic in X. The set of regular points of A of dimension p is subanalytic in X.*

(iv) *Regular points are dense in A.*

DEFINITION 3.8. Thanks to Proposition 3.7 (iv), we can define, as usually, *the local dimension* of a subanalytic set A at a point $a \in A$. And so we can define *the dimension of* A as the max of the local dimensions of A.

NOTATION 3.9. Let X and Y be real analytic submanifolds of $\boldsymbol{R}^n$. $S_b(X, Y)$ will denote the set of points $y \in Y$ such that the pair (X, Y) does not satisfy condition (b) at y.

PROPOSITION 3.10. *Let X and Y be real analytic submanifolds of $\boldsymbol{R}^n$. Assume that $X \cap Y = \emptyset$ and $\bar{X} \supset Y$ and that X and Y are both subanalytic in an open set of $\boldsymbol{R}^n$. Then there exists a subanalytic subset B of Y such that*

(i) *B is closed in Y and* $\dim B < \dim Y$.

(ii) $B \supset S_b(X, Y)$.

4. Stratification of a subanalytic subset.

In which we give a proof of Hironaka's following theorem:

PROPOSITION 4.1 (Hironaka [3]). *Let A be a subanalytic subset of an analytic set $X \subset \Omega \subset \boldsymbol{R}^n$. Then A admits a Whitney stratification whose strate are subanalytic in X.*

DEFINITION 4.2. We say that a W-complex $S = \{Y_\alpha\}$ in $\boldsymbol{R}^n$ is compatible with a submanifold X of $\boldsymbol{R}^n$ if for any stratum Y of S we have $S_b(X, Y) = \emptyset$.

It is clear that in order to prove Proposition 4.1, it is sufficient to prove the following:

PROPOSITION 4.3. *Let A be a subanalytic subset of an analytic set $X \subset \Omega \subset \boldsymbol{R}^n$. Let $X_1, \cdots, X_k$ be submanifolds of $\boldsymbol{R}^n$ which are subanalytic in X. Assume that $A \cap X_i = \emptyset$ for each i. Then A admits a Whitney stratification which is compatible with $X_1, \cdots, X_k$ and such that each stratum is subanalytic in X.*

Proof. We prove the proposition by induction on dimension of A. If $\dim A = 0$, then the proposition is evident. So we assume the proposition holds for every subanalytic set A with $\dim A < m$ and we shall prove it for a subanalytic set A with $\dim A = m$.

Let A_{sp} denote the set of the regular points of A of dimension m. Then by Proposition 3.7, A_{sp} and $A - A_{sp}$ are both subanalytic in X and we have $\dim(A - A_{sp}) < \dim A = m$. Since A_{sp} is subanalytic in X and a submanifold of R^n and since $A_{sp} \cap X_i \subset A \cap X_i = \emptyset$, there exists, by Proposition 3.10, a subanalytic subset B of A_{sp} such that

(i) B is closed is A_{sp} and $\dim B < \dim A_{sp} = m$.

(ii) $B \supset S_b(X_i, A_{sp})$ for each $i = 1, \cdots, k$.

Set $C = B \cup (A - A_{sp})$, $A^0 = A_{sp} - C$ and set $S(A^0) =$ the set of the connected components of A^0. By Proposition 3.7 (ii), $S(A^0)$ is locally finite in Ω, hence $S(A^0)$ is a W-complex which is compatible with $X_1, \cdots, X_k$ and such that every stratum is subanalytic in X and disjoint with C.

Since $\dim C < m$, by the hypothesis of our induction, C admits a Whitney stratification $S(C)$ which is compatible with $X_1, \cdots, X_k$ and with all of strata of $S(A^0)$.

Thus we have a Whitney stratification $S(A) = S(A^0) \cup S(C)$ that is wanted. Q.E.D.

5. Stratification of a proper real analytic mapping

In which we prove the following:

THEOREM 5.1. *Let $f: X \to Y$ be a proper real analytic mapping of a real analytic set X into another one Y. Let $A \subset X$ and $B \subset Y$ be subanalytic subsets. Suppose that $f(A) \subset B$. Then $f|A: A \to B$ is a stratified mapping with stratifications $S(A)$ of A and $S(B)$ of B such that any stratum of $S(A)$ (resp. of $S(B)$) is subanalytic in X (resp. in Y).*

DEFINITION 5.2. Let $A \subset \boldsymbol{R}^n$ and $B \subset \boldsymbol{R}^m$ and let X (resp. Y) be a submanifolds of $\boldsymbol{R}^n$ (resp. of $\boldsymbol{R}^m$). Let $f: A \to B$ be a stratified mapping with stratifications $S(A)$ of A and $S(B)$ of B. Then we say that the stratified mapping $f: A \to B$ *is compatible with* X (resp. with Y) if so is $S(A)$ (resp. $S(B)$).

Theorem 5.1 is an immediate consequence of the following proposition.

PROPOSITION 5.3. *Let $f: X \to Y$ and $A \subset X \subset \boldsymbol{R}^n$, $B \subset Y \subset \boldsymbol{R}^m$ be as in Theorem 5.1. Let $X_1, \cdots, X_k$ (resp. $Y_1, \cdots, Y_\ell$) be submanifolds of $\boldsymbol{R}^n$ (resp. of $\boldsymbol{R}^m$) which are subanalytic in X (resp. in Y). Assume that $A \cap X_i = B \cap Y_j = \emptyset$ for each i and j. Then there exist Whitney stratifications $S(A)$ of A and $S(B)$ of B such that*

(i) *$f|A: A \to B$ is a stratified mapping compatible with $X_1, \cdots, X_k$ and $Y_1, \cdots, Y_\ell$.*

(ii) *The strata of $S(A)$ (resp. of $S(B)$) are subanalytic in X (resp. in Y).*

To prove the proposition, we need

LEMMA 5.4 (Bertini-Sard-Hironaka). *Let $f: X \to Y$ be a proper real analytic map, where both X and Y are smooth. Then there exist a subset S of Y such that*

(i) *S is closed and subanalytic in Y and $\dim S < \dim Y$.*

(ii) *for every connected component U of $Y - S$, either $f^{-1}(U) = \emptyset$ or f induces a submersion from $f^{-1}(U)$ to U.*

For the proof see [3].

Proof of Proposition 5.3. We prove the proposition by induction on $\dim B$. The verification for the case $\dim B = 0$ is immediate from Proposition 4.3. So we assume that the proposition holds for subanalytic sets having dimension $< p$ and we shall prove it for a subanalytic set B of dimension p.

Let B_{sp} denote the set of the regular points of B of dimension p. By Proposition 3.10, there is a closed subanalytic subset B_1 of B_{sp} such that $\dim B_1 < \dim B_{sp}$ and $B_1 \supset S_b(Y_j, B_{sp})$ for each $j = 1, \cdots, \ell$. Set $B_0 = B_{sp} - B_1$ and $A_0 = A \cap f^{-1}(B_{sp})$. Then A_0 is subanalytic in X and so is B_0 in Y. By Proposition 4.3, A_0 admits a Whitney stratification $S(A_0)$ which is compatible with $X_1, \cdots, X_k$ and such that each stratum is subanalytic in X.

Now for each stratum W of $S(A_0)$, consider the restricted map $f|W: W \to B_0$ and set $\Sigma_W = \{x \in W \mid \text{the rank of } f|W \text{ at } x < \dim B_{sp} = P\}$. Then Σ_W is subanalytic in X. Since $S(A_0)$ is locally finite, $\Sigma = \bigcup \Sigma_W$ is subanalytic in X and closed in A_0.

Then $f(\Sigma)$ and its closure $\overline{f(\Sigma)}$ are subanalytic in Y and $\dim \overline{f(\Sigma)} < \dim B$. Set $B_{00} = B_0 - \overline{f(\Sigma)}$ and $A_{00} = A \cap f^{-1}(B_{00})$. Set

$$S(A_{00}) = \{W \cap A_{00} \mid W \in S(A_0)\}$$
$$S(B_{00}) = \text{the set of the connected component of } B_{00}.$$

With these stratifications $S(A_{00})$ and $S(B_{00})$, $f: A_{00} \to B_{00}$ is a stratified mapping which is compatible with $X_1, \cdots, X_k$ and $Y_1, \cdots, Y_\ell$.

$B - B_{00}$ is closed in B and $\dim(B - B_{00}) < \dim B = p$. So by the hypothesis of our induction, there exist stratifications $S(B - B_{00})$ and $S(A - A_{00})$ with which $f: A - A_{00} \to (B: B_{00})$ is a stratified mapping such that it is compatible with $X_1, \cdots, X_k, Y_1, \cdots, Y_\ell$ and with all strata of $S(A_{00})$ and $S(B_{00})$.

Thus we have stratifications $S(A) = S(A_{00}) \cup S(A - A_{00})$ and $S(B) = S(B_{00}) \cup S(B - B_{00})$ which satisfy the conditions in the proposition.

Q.E.D.

6. Lemmas for condition a_f.

Let $f: U \to \boldsymbol{R}^m$ be a real analytic mapping of an open set U $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$, and let X, Y be smooth subanalytic submanifolds of U (i.e. smooth submanifolds which are subanalytic in U). We denote by $S(f|X)$ the set of points x of X at which the rank $d(f|X)_x < \max_{y \in X}\{\text{rank } d(f|X)_y\}$.

NOTATION. In the above situation, let $S_a(f: X, Y)$ denote the set of points y of $(Y - S(f|Y)) \cap (X - S(f|X))$ at which the pair $((X - S(f|X)), (Y - S(f|Y)))$ does not satisfy the condition a_f. (For the condition a_f, see Definition 2.3.)

PROPOSITION 6.1.
(i) *$S(f|X)$ is subanalytic in U.*
(ii) *$S_a(f: X, Y)$ is subanalytic in U.*

This proposition can be proved in a similar way to that of whitney [7] where he proved that $S_b(X, Y)$ is an analytic set for analytic manifolds X and Y.

LEMMA 6.2 (*A good stratification of analytic functions by F. Pham D. T. Lê* [4]). *Let be a real analytic function defined on an open set U in $\boldsymbol{R}^n$. Let $Y \subset X \subset U$ be smooth subanalytic submanifolds of U. Suppose that X is open in U and the restriction $f|Y$ is a locally constant function. Then we have*

$$\dim S_a(f: X, Y) < \dim Y .$$

For the proof of Lemma 6.2, see [4]. In [4], this lemma is proved in complex analytic case. However we can easily modify this proof to the real analytic case using the method of complexification of real analytic sets. (For the complexification of real analytic sets, see for instance [1].)

From now on, we concern with the situation of Theorem 3: A is an analytic set, M is a compact real analytic manifold, $f: A \times M \to \boldsymbol{R}$ is an analytic function, $F: A \times M \to A \times \boldsymbol{R}$ is the mapping defined by $F(a, x) = (a, f(a, x))$ and $p: A \times M \to A$ and $\pi: A \times \boldsymbol{R} \to A$ are the canonical projections.

Let Z be a smooth subanalytic subset of A and let X be a subanalytic submanifold of $Z \times M$ such that the restricted function

$$f \mid (X \cap p^{-1}(z)): X \cap p^{-1}(z) \longrightarrow \boldsymbol{R}$$

is a locally constant function for each point z of Z. Then we set

$$S_\beta(f: Z \times M, X) = \bigcup_{z \in Z} S_\alpha(f: z \times M, X \cap (z \times M)) .$$

LEMMA 6.3. *$S_\beta(f: Z \times M, X)$ is a subanalytic subset of X having dimension* $< \dim X$.

Proof. The subanalycity of $S_\beta(f: Z \times M, X)$ can be proved in a similar way to Proposition 6.1, (ii). The inequality of the dimensions follows from the definition of $S_\beta(f: Z \times M, X)$ and Lemma 6.2. Q.E.D.

7.

We prepare in this section the final lemma for the proof of Theorem 3.

LEMMA 7.1. *Let the notations be the same as in Lemma 6.3. Suppose that the restricted function $f \mid a \times M: a \times M \to \boldsymbol{R}$ is not locally constant for each $a \in A$. Then there exists a Whitney stratification $S(A)$ of A whose strate are all subanalytic in A and such that for each stratum Z of $S(A)$ there exist Whitney stratifications $S(Z \times M)$ of $Z \times M$ and $S(Z \times \boldsymbol{R})$ of $Z \times \boldsymbol{R}$ whose strata are all subanalytic and which satisfy the following conditions:*

(i) *$F \mid Z \times M: Z \times M \to Z \times \boldsymbol{R}$ and the projection $\pi: Z \times \boldsymbol{R} \to Z$ are stratified mappings, Z being a stratified set with its trivial stratification.*

(ii) *$S_\beta(f: Z \times M, X)$ is well defined and $= \emptyset$ for any pair of strata Z of $S(A)$ and X of $S(Z \times M)$ with* $\dim X < \dim (Z \times M)$.

(iii) *If* $\dim F(X) = \dim (Z \times \boldsymbol{R})$ *for a stratum X of $S(Z \times M)$, then* $\dim X = \dim (Z \times M)$.

Proof. We prove the lemma by constructing a sequence

$$C_0 \subset C_1 \subset \cdots \subset C_n = A\ , \qquad n = \dim A\ ,$$

of subanalytic subsets of A satisfying the following conditions (iv)–(viii):

(iv) $\dim C_i \leqq i$ and C_{i-1} is closed in C_i.

(v) $C_i - C_{i-1}$ are subanalytic submanifolds and $S_b(C_j - C_{j-1}, C_i - C_{i-1}) = \emptyset$ if $j > i$.

(vi) There exist Whitney stratifications $S((C_i - C_{i-1}) \times M)$ of $(C_i - C_{i-1}) \times M$ and $S((C_i - C_{i-1}) \times \boldsymbol{R})$ of $(C_i - C_{i-1}) \times \boldsymbol{R}$ whose strata are all subanalytic and with which the restricted mapping $F: (C_i - C_{i-1}) \times M \to (C_i - C_{i-1}) \times \boldsymbol{R}$ and $\pi: (C_i - C_{i-1}) \times \boldsymbol{R} \to C_i - C_{i-1}$ are stratified mappings, where $C_i - C_{i-1}$ is considered as a stratified set with its trivial stratification consisting of the connected components.

(vii) $S_\beta(f: (C_i - C_{i-1}) \times M, X)$ is well-defined and $= \emptyset$ for each stratum X of $S((C_i - C_{i-1}) \times M)$ of dimension $< \dim (C_i - C_{i-1}) \times M$.

(viii) If $\dim F(X) = \dim (C_i - C_{i-1}) \times \boldsymbol{R}$ for a stratum X of $S((C_i - C_{i-1}) \times M)$, then we have $\dim X = \dim (C_i - C_{i-1}) \times M$.

We construct such a sequence C_i by descending induction on the dimension of C_i. We begin the induction setting $C_n = A$, where $n = \dim A$. Supposing we have already constructed C_i, we are going to construct such a C_{i-1}.

If $\dim C_i < i$, then we set $C_{i-1} = C_i$ and this ends the step of the induction. So we suppose $\dim C_i = i$.

Set

$(C_i)_{sp} =$ the set of all regular points of C_i having dimension i.

$C_i^* = (C_i)_{sp} - \bigcup_{j>i} \overline{S_b(C_{j-1}, (C_i)_{sp})}$

$R^* =$ the set of all regular values of the mapping

$F: C_i^* \times M \to C_i^* \times \boldsymbol{R}$.

Then, since f_a is not locally constant for any $a \in A$, R^* is a nonempty open subset of $C_i^* \times \boldsymbol{R}$ and $C_i^* = \pi(R^*)$. And the restricted map $F: F^{-1}(R^*) \to R^*$ is a stratified mapping with their trivial stratifications $S(F^{-1}(R^*))$ and $S(R^*)$ consisting of the connected components of $F^{-1}(R^*)$ and of R^* respectively.

For any subanalytic submanifold Y of $\pi(R^*) \times M - F^{-1}(R^*) = C_i^* \times M - F^{-1}(R^*)$ and for any point a of C_i^*, the restricted function $f \mid Y \cap p^{-1}(a) = f \mid Y \cap (a \times M)$ is a locally constant function, where $p: A$

$\times M \to A$ is the canonical projection. Hence, by Lemma 6.3, the set $S_\beta(f: C_i^* \times M, Y)$ is well-defined for such a manifold Y and it is a subanalytic subset of Y having dimension $< \dim Y$.

Therefore, with a similar argument to Proposition 5.3, we see that there exist Whitney stratifications $S(C_i^* \times M - F^{-1}(R^*))$ of $C_i^* \times M - F^{-1}(R^*)$ and $S(C_i^* \times \boldsymbol{R} - R^*)$ of $C_i^* \times \boldsymbol{R} - R^*$ whose strata are all subanalytic, which are both compatible with all strata of $S(F^{-1}(R^*))$ and of $S(R^*)$ and which satisfy the following conditions (ix) and (x):

(ix) $F: (C_i^* \times M - F^{-1}(R^*)) \to C_i^* \times \boldsymbol{R} - R^*$ is a stratified mapping:

(x) $S_\beta(f: C_i^* \times M, Y) = \emptyset$ for any stratum Y of $S(C_i^* \times M - F^{-1}(R^*))$.

For each stratum X of $S(C_i^* \times \boldsymbol{R} - R^*)$ we denote by Σ_X the set of all critical points of the restricted mapping $\pi | X: X \to C_i^*$. Set

$$C_{i-1} = \text{the closure of the set } (C_i - C_i^*) \cup \bigcup \pi(\Sigma_X) ,$$

where X rans over all strata of $S(C_i^* \times \boldsymbol{R} - R^*)$. Then C_{i-1} is a closed subanalytic subset of C_i having dimension $< i$.

We denote by $S((C_i - C_{i-1}) \times M)$ the set of all connected components of $X - p^{-1}(C_{i-1})$ where X runs over all strata of $S(F^{-1}(R^*)) \cup S(C_i^* \times M - F^{-1}(R^*))$, by $S((C_i - C_{i-1}) \times \boldsymbol{R})$ the set of all connected components of $Y - \pi^{-1}(C_{i-1})$, where Y runs over all strata of $S(R^*) \cup S(C_i^* \times \boldsymbol{R} - R^*)$ and we denote by $S(C_i - C_{i-1})$ the set of all connected components of $C_i - C_{i-1}$.

It is easy to check that the sets $\{C_k\}$, $k \geqq i-1$, and the stratifications thus obtained satisfy the conditions (iv)–(viii). Q.E.D.

8. Proof of Theorem 3.

Let the notations be as in Theorem 3: $f: A \times M \to \boldsymbol{R}$ is a real analytic function, A being a real analytic set and M being a compact real analytic manifold. $F: A \times M \to A \times \boldsymbol{R}$ is the map defined by $F(a, x) = (a, f(a, x))$.

$$p: A \times M \longrightarrow A \quad \text{and} \quad \pi: A \times \boldsymbol{R} \longrightarrow A$$

are the canonical projections.

Let C be the set of points $a \in A$ such that the restricted function $f | a \times M: a \times M \to \boldsymbol{R}$ is a locally constant function. Then C is a closed subanalytic subset.

Then by Lemma 7.1, there exists a Whitney stratification $S(A - C)$

of $A-C$ satisfying the conditions (i)–(iii) of Lemma 7.1. Now, we prove that for each stratum Z of $S(A-C)$, the restricted map $F: Z \times M \to Z \times \boldsymbol{R}$ is a Thom-mapping over $\pi: Z \times \boldsymbol{R} \to Z$.

Let Z be a given stratum of $S(A-C)$ and let X and Y be strata of $S(Z \times M)$ with $F(X) < F(Y)$. (For the symbol $F(X) < F(Y)$, see the notation given at the end of § 1.) Since $\dim Z \times \boldsymbol{R} \geqq \dim F(X) > \dim F(Y) \geqq \dim Z$, by the condition (iii) in Lemma 7.1, X is an open set of $Z \times M$ and hence, by the condition (ii), $S_\beta(f: X, Y)$ is well defined and $=\emptyset$. Hence for each point $a \in Z$, $S_\alpha(f: X \cap p^{-1}(a), Y \cap p^{-1}(a)) = \emptyset$. Therefore $F: Z \times M \to Z \times \boldsymbol{R}$ is a Thom-mapping over $\pi: Z \times \boldsymbol{R} \to Z$.

Since C is a closed subanalytic subset of A, by Proposition 4.3, there is a Whitney stratification $S(C)$ of C compatible with all strata of $S(A-C)$. Since the restricted function $f \mid a \times M: a \times M \to \boldsymbol{R}$ is a locally constant function for each $a \in C$, for each stratum Z of $S(C)$, $F: Z \times M \to Z \times \boldsymbol{R}$ is a Thom-mapping over $\pi: Z \times \boldsymbol{R} \to Z$. Set $S(A) = S(A-C) \cup S(C)$ and $S(A)$ is what we want. Q.E.D.

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Department of Mathematics
Chiba University