M. Aschbacher and G. M. Seitz
Nagoya Math. J.
Vol. 63 (1976), 1-91

INVOLUTIONS IN CHEVALLEY GROUPS OVER FIELDS OF EVEN ORDER

MICHAEL ASCHBACHER* AND GARY M. SEITZ**

Let $G = G(q)$ be a Chevalley group defined over a field F_q of characteristic 2. In this paper we determine the conjugacy classes of involutions in Aut(G) and the centralizers of these involutions. This study was begun in the context of a different problem. Namely, we wanted to find those groups H containing a standard component A satisfying $A/Z(A) \cong G$ and $m_2(C_H(A)) > 1$. Such groups H are determined in [3], where the results of this paper are crucial. In dealing with groups H as above a very important consideration is the tightly embedded subgroups in Aut(G). Consequently we study such subgroups in this paper.

The classical groups and the exceptional Chevalley groups are treated separately. Finding the involutions in the classical groups is accomplished using the underlying vector space and regarding the group as a group of matrices. The results give an explicit matrix representation for the involutions in G and their centralizers. For the exceptional groups the (B, N)-structure is used. Representatives for the classes of involutions are given as explicit products of elements of root groups. The centralizers are determined completely; one could obtain precise generators and relations if needed. One useful piece of information is the complete list of parabolic subgroups containing a given centralizer of an involution.

The analysis of outer automorphisms is carried out in the context of the (B, N)-structure, with the exception of the orthogonal groups where certain information is already available from the linear algebra.

We remark that for t an involution in $G, C_G(t)$ is known to be 2-constrained with $O_{2'}(C_G(t)) = 1$. It turns out that $O_2(C_G(t))$ is always of class at most 3; for the classical groups the class is at most 2.

Received June 3, 1975.
* Partial support supplied by the Alfred P. Sloan Foundation and NSF GD-35678.
** Partial support supplied by NSF GRANT 37982XI.

The paper is organized as follows. § 2 contains preliminary results and § 3 contains a discussion of groups with (B, N)-pairs, giving commutator relations and root systems for the exceptional Chevalley groups. Sections 4-8 are concerned with the involutions in the classical groups. Included is a description of the underlying geometric configuration. Sections 9-11 give technical information needed in the analysis of the structure of the centralizers and for finding the tightly embedded subgroups. The involutions in the rank 2 Chevalley groups have been previously determined, so in § 12 we find the involutions in the remaining exceptional Chevalley groups. Then Sections 13-18 contain the precise information on the centralizers of these involutions. § 19 is concerned with the outer automorphisms. In § 20 we prove that certain types of tightly embedded subgroups in $\mathrm{Aut}(G)$ have all their involutions in G. Finally in § 21 and § 22 we return to the classical groups with a discussion of exceptional Schur multipliers and further analysis of tightly embedded subgroups.

Section 2. Preliminaries.

In this section we record some preliminary lemmas to be used in later sections, especially Section 20.

(2.1) *Let $G = PQA$ be the semi-direct product of the product of the p-group $P \trianglelefteq G$ and the group QA. Assume that A is a cyclic p-group acting fixed-point-freely on Q, and that $C_P(Q) = 1$. Then all complements to PQ in G are conjugate.*

Proof. This is contained in Theorem 3.3 of [20].

(2.2) *Let t be an involution in a group D. Then $L(C_D(t)) \leq L(D)$ and if $F^*(D)$ is a 2-group then $O(C_D(t)) = 1$.*

Proof. See 2.6 and 2.7 of [2].

(2.3) *Let G be a finite Chevalley group (normal or twisted) defined over a field of characteristic p. Then p-local subgroups of G are contained in proper parabolic subgroups.*

Proof. Borel-Tits, 3.12 of [4].

(2.4) *Let G be a finite Chevalley group (normal or twisted) defined over* *a field of characteristic p. Then for any p-element* $x \neq 1, F^*(C_G(x))$ *is a p-group.*

Proof. This follows from 2.3 and the fact that for proper parabolic subgroups, P of $G, F^*(P)$ is a p-group.

(2.5) *If X is a 2-group of rank* >1 *acting on a group Y of odd order, then* $Y = \langle C_Y(x) : x \in X^{\#} \rangle = \Gamma_{1,X}(Y)$.

Proof. See Gorenstein [12].

Section 3. The (*B, N*)-structure of the exceptional groups.

Throughout most of this paper we will be concerned with determining the involutions, and their centralizers, in the Chevalley groups G defined over fields of characteristic 2. In the case of the classical groups the representation of G on its natural module is used to obtain this information. However in the case of the exceptional groups we must utilize the (B, N)-structure of G and work entirely within local subgroups. Moreover in the case of the classical groups our results are stated in terms of the natural modules while in the case of the exceptional groups our results are given explicitly in terms of the (B, N)-structure.

Therefore in this section we record notation and basic facts concerning the (B, N)-structure of the exceptional groups.

We write $G = G(q)$ to mean that G is defined over a field F_q of characteristic 2. Fix a Tit's system for G. So let U be a Sylow 2-subgroup of G with $B = N_G(U)$ and $U = O_2(B)$. Then $B = UH$ where H is an abelian $2'$-group normal in N. Write $W = N/H$, the Weyl group, $W = \langle s_1, \cdots, s_n \rangle$, where the s_i are the fundamental reflections. Associated with W is a root system Δ in $\boldsymbol{R}^n$ with positive roots Δ^+ and fundamental system $\{\alpha_1, \cdots, \alpha_n\}$. We label the Dynkin diagram of G and the roots in Δ as in Bourbaki [5].

For each root $r \in \Delta$ there corresponds a subgroup (called a "root group") $U_r \leq G$ such that $G = \langle U_r : r \in \Delta \rangle$ and $U = \prod_{r \in \Delta^+} U_r$ in some fixed ordering of Δ^+. Then H normalizes each U_r and the permutation representation of W on Δ and on $\{U_r : r \in \Delta\}$ are equivalent under the correspondence $r \leftrightarrow U_r$. For $i = 1, \cdots, n$ let $s_i = U_{-\alpha_i}(1)U_{\alpha_i}(1)U_{-\alpha_i}(1)$. Here we are labeling the elements of U_r by using the base field (or other fields for the twisted groups) and we are identifying s_i as both a coset of H in W and as an element in G. This will not cause difficulty and it is true that $U_{-\alpha_i}(1)U_{\alpha_i}(1)U_{-\alpha_i}(1)H = s_i \in W$.

For $i = 1, \cdots, n$ let $\mathscr{L}^i$ be the set of all roots in Δ with positive coefficient of α_i. Then $\mathscr{L}^i \subseteq \Delta^+$. If $i = 1, \cdots, n$ and $m \geq 0$, let $\mathscr{L}^i_m$ denote the set of long roots with coefficient of α_i equal to m (respectively $\mathscr{S}^i_m$ for the short roots). We write $Q_i = \langle U_r : r \in \mathscr{L}^i \rangle$ and set $Q^i_j = \{U_s : s \in \mathscr{L}^i_j\}$.

For $S \subset \{1, \cdots, n\}$ we let $P_S = \langle B, s_i : i \in S \rangle$. The conjugates of the subgroups P_S are called the parabolic subgroups of G, and some information concerning the structure of P_S can be obtained simply by looking at the Dynkin diagram. Namely $P_S = Q_S L_S H$, where $Q_S = O_2(P_S)$, $L_S = \langle U_{\pm\alpha_i} : i \notin S \rangle = \langle U_{\alpha_i}, s_i : i \notin S \rangle$ and the structure of L_S as a Chevalley group can be read off from the Dynkin diagram. The group Q_S is the product of the groups U_r for $r \in \bigcup_{i \in \mathscr{S}} \mathscr{L}^i$.

We will need the following cases of the Chevalley commutator relations.

(3.1) *Let* $\alpha, \beta \in \Delta$.

i) *If* $\alpha + \beta \notin \Delta$, *then* $[U_\alpha, U_\beta] = 1$.

ii) *If* $\alpha, \beta, \alpha + \beta$ *are all in* Δ *and of the same length, then* $[U_\alpha(s), U_\beta(t)] = U_{\alpha+\beta}(st)$.

iii) *If* $G = F_4(q)$, α *and* β *short roots, and* $\alpha + \beta$ *a long root in* Δ, *then* $[U_\alpha, U_\beta] = 1$.

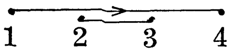
iv) *If* $G = {}^2E_6(q)$, α *and* β *short roots, and* $\alpha + \beta$ *a long root in* Δ, *then* $[U_\alpha(s), U_\beta(t)] = U_{\alpha+\beta}(s^q t + st^q)$, *for* $s, t \in \boldsymbol{F}_{q^2}$.

v) *If* $G = F_4(q), q > 2$, α *is a long root,* β *a short root such that* $\alpha + \beta \in \Delta$, *then* $[U_\alpha, U_\beta] = U_{\alpha+\beta}U_{\alpha+2\beta}$.

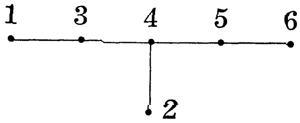
vi) *If* $G = {}^2E_6(q), \alpha$ *a long root,* β *a short root and* $\alpha + \beta \in \Delta$, *then* $[U_\alpha(c), U_\beta(d)] = U_{\alpha+\beta}(cd)U_{\alpha+2\beta}(cdd^q)$, *for* $c \in \boldsymbol{F}_q$ *and* $d \in \boldsymbol{F}_{q^2}$.

Basic facts involving parabolic subgroups and the Bruhat decomposition of G will be used repeatedly. Also we will need a list of some of the roots in Δ^+ for Δ of type F_4, E_6, E_7, and E_8. These are given in the following tables and are based on the following labeling of the Dynkin diagram

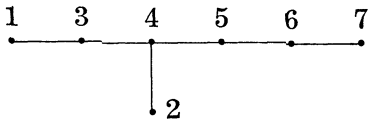
F_4



E_6



E_7



E_8

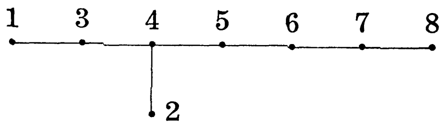


TABLE 1

Roots in Δ^+ for Δ of type F_4

$\mathscr{L}_1^1$		$\mathscr{S}_1^1$		$\mathscr{L}_0^1$		$\mathscr{S}_0^1$	
1.	1000	9.	1110	15.	0100	18.	0010
2.	1100	10.	1111	16.	0120	19.	0110
3.	1120	11.	1121	17.	0122	20.	0111
4.	1220	12.	1221			21.	0121
5.	1122	13.	1231			22.	0001
6.	1222	14.	1232			23.	0011
7.	1242						
8.	1342						

$\mathscr{L}_2^1$	
24.	2342

Let $r = r_{24}$

$s = r_{14}$

$\alpha = r_{12}$

$\beta = r_7$

TABLE 2

Roots in Δ^+ for Δ of type E_6

$\mathscr{L}_1^1$		$\mathscr{L}_1^2 - \mathscr{L}_1^1$		$\Delta^+ - \mathscr{L}_1^1 - \mathscr{L}_1^2$	
1.	10000 0	17.	00000 1	27.	01000 0

2.	11000 0	18.	00100 1	28.	01100 0
3.	11100 0	19.	01100 1	29.	01110 0
4.	11100 1	20.	00110 1	30.	01111 0
5.	11110 0	21.	00111 1	31.	00100 0
6.	11110 1	22.	01110 1	32.	00110 0
7.	11111 1	23.	01111 1	33.	00111 0
8.	11111 0	24.	01210 1	34.	00010 0
9.	11210 1	25.	01211 1	35.	00011 0
10.	11211 1	26.	01221 1	36.	00001 0
11.	12210 1				
12.	12211 1				
13.	11221 1				
14.	12221 1				
15.	12321 1				
16.	12321 2				

$r = r_{16}$ $\quad \gamma = r_{26}$

$\alpha = r_{12}$ $\quad \delta = r_{10}$

$\beta = r_{13}$ $\quad \varepsilon = r_{11}$

TABLE 3

Roots in Δ^+ having non-zero coefficient of α_2 or α_7, for Δ of type E_7.

$\mathscr{L}_1^7 \cap \mathscr{L}_0^2$		$\mathscr{L}_1^7 \cap \mathscr{L}_1^2$		$\mathscr{L}_1^7 \cap \mathscr{L}_2^2$	
1.	000001 0	7.	001111 1	22.	123211 2
2.	000011 0	8.	011111 1	23.	123221 2
3.	000111 0	9.	111111 1	24.	123321 2
4.	001111 0	10.	012111 1	25.	124321 2

5.	011111 0
6.	111111 0

$r = r_{27}$

$\alpha = r_{21}$

$\beta = r_{23}$

$\gamma = r_{48}$

$\delta = r_{18}$

$\varepsilon = r_{19}$

$\varphi = r_{13}$

$\psi = r_{14}$

$\theta = r_{15}$

11.	112111 1
12.	012211 1
13.	112211 1
14.	122111 1
15.	012221 1
16.	122211 1
17.	112221 1
18.	122221 1
19.	123211 1
20.	123221 1
21.	123321 1

26.	134321 2
27.	234321 2

$\mathscr{L}_0^7 \cap \mathscr{L}_1^2 \cap \mathscr{L}_0^1$

28.	000000 1
29.	001000 1
30.	011000 1
31.	001100 1
32.	001110 1
33.	011100 1
34.	011110 1
35.	012100 1
36.	012110 1
37.	012210 1

$\mathscr{L}_0^7 \cap \mathscr{L}^2 \cap \mathscr{L}_1^1$

38.	111000 1	41.	112100 1	44.	112210 1	47.	123210 1
39.	111100 1	42.	112110 1	45.	122110 1	48.	123210 2
40.	111110 1	43.	122100 1	46.	122210 1		

TABLE 4

Roots in $\mathscr{L}^8$, for Δ of type E_8.

$\mathscr{L}^8 \cap \mathscr{L}_0^7$

1.	0000001 0

$\mathscr{L}^8 \cap \mathscr{L}_1^7$

2.	0000011 0	9.	1111111 0	16.	0122211 1	23.	1232111 2

3.	0000111 0	10.	1111111 1	17.	1222111 1	24.	1232211 2
4.	0001111 0	11.	0121111 1	18.	1122211 1	25.	1233211 2
5.	0011111 0	12.	1121111 1	19.	1222211 1	26.	1243211 2
6.	0011111 1	13.	0122111 1	20.	1232111 1	27.	1343211 2
7.	0111111 0	14.	1122111 1	21.	1232211 1	28.	2343211 2
8.	0111111 1	15.	1221111 1	22.	1233211 1		

$\mathscr{L}^8 \cap (\mathscr{L}^7_2 \cup \mathscr{L}^7_3)$

29.	1343221 2	37.	1233221 2	44.	2343321 2	51.	2454321 2
30.	1243321 2	38.	1233321 1	45.	1344321 2	52.	2454321 3
31.	0122221 1	39.	1243221 2	46.	1354321 2	53.	2464321 3
32.	0122221 1	40.	1233321 2	47.	2344321 2	54.	2465321 3
33.	1222221 1	41.	2343221 2	48.	1354321 3	55.	2465421 3
34.	1232221 1	42.	1343321 2	49.	2354321 2	56.	2465431 3
35.	1232221 2	43.	1244321 2	50.	2354321 3	57.	2465432 3
36.	1233221 1						

$r = r_{57}$ $\quad \beta = r_{50}$ $\quad \delta = r_{42}$ $\quad \varphi = \begin{smallmatrix}2343210\\2\end{smallmatrix}$ $\quad \theta = r_{40}$

$\alpha = r_{51}$ $\quad \gamma = r_{41}$ $\quad \varepsilon = r_{43}$ $\quad \psi = r_{27}$ $\quad \omega = r_{39}$

Section 4. The unimodular group.

Let V be an n-dimensional vector space over $GF(q)$, where $n \geq 3$, and q is even. $SL(V)$ is the group of all linear transformations of V of determinant 1. Given an ordered basis $\{x_i\}$ of V and $g \in SL(V)$, associate with g the $n \times n$ matrix (g_{ij}) where

$$x_i^g = \sum_j g_{ij} x_j \, .$$

In considering such matrices we follow the notation of M. Suzuki in [22].

Given an involution a in $SL(V)$, define the *rank* of a to be the dimension of the commutator space $[V, a]$ of a. The rank of a is also the number of Jordan blocks of (a_{ij}) of size 2 with respect to a basis of V in which a is in Jordan form. Hence

(4.1) *Let a and b be involutions in $SL(V)$. Then a and b are conjugate in $SL(V)$ if and only if they have the same rank.*

Fix an ordered basis for V and associate each element of $SL(V)$ with its corresponding matrix. Following Suzuki, given an integer ℓ in the range $1 \leq \ell \leq n/2$, define the involution j_ℓ of $SL(V)$ by

$$j_\ell = \begin{bmatrix} I_\ell & & \\ & I_{n-2\ell} & \\ I_\ell & & I_\ell \end{bmatrix}.$$

Here I_m is the $m \times m$ identity matrix and I_0 is taken to be void.

j_ℓ has rank ℓ and is referred to as the *Suzuki form* of its class. We conclude

(4.2) *The involutions $j_\ell, 1 \leq \ell \leq n/2$, are a set of representatives for the conjugacy classes of involutions of $SL(V)$.*

(4.3) *The centralizer C_ℓ in $SL(V)$ of the involution j_ℓ consists of those matrices g of the form*

$$g = \begin{bmatrix} X(g) & & \\ P(g) & Y(g) & \\ Q(g) & R(g) & X(g) \end{bmatrix} \text{ such that } (\det(X))^2 \det(Y) = 1\ ,$$

where $X(g)$ and $Q(g)$ are of size $\ell \times \ell$, $Y(g)$ has size $(n-2\ell) \times (n-2\ell)$, $P(g)$ has size $(n-2\ell) \times \ell$, and $R(g)$ has size $\ell \times (n-2\ell)$. Further the map $g \to (X(g), Y(g))$ is a homomorphism of C_ℓ into $GL_\ell(q) \times GL_{n-2\ell}(q)$ with the image containing $SL_\ell(q) \times SL_{n-2\ell}(q)$, and covering both factors if $n \neq 2\ell$. The kernel T_ℓ is equal to $O_2(C_\ell)$.

Proof. See Suzuki [22], pages 1048 and 1049.

The remaining three lemmas in this section follow by direct computation.

(4.4) *For $g, h \in C_\ell$*

$$gh = \begin{bmatrix} X(g)X(h) & & \\ P(g)X(h) + Y(g)P(h) & Y(g)Y(h) & \\ Q(g)X(h) + R(g)P(h) + X(g)Q(h) & R(g)Y(h) + X(g)R(h) & X(g)X(h) \end{bmatrix}.$$

(4.5) *Let* $g, h \in C_\ell$ *with*

$$h = \begin{bmatrix} D & & \\ A & W & \\ B & C & D \end{bmatrix} \qquad g = \begin{bmatrix} X & & \\ P & Y & \\ Q & R & X \end{bmatrix}.$$

Then

$$h^g = \begin{bmatrix} D^X & & \\ Y^{-1}(PD^X + AX + WP) & W^Y & \\ Q(h^g) & X^{-1}(RW^Y + CY + DR) & D^X \end{bmatrix}$$

where

$$Q(h^g) = X^{-1}[RY^{-1}(PD^X + AX + WP) + BX + CP + DQ + QD^X] .$$

Further if $h \in T_\ell$ *then*

$$h^g = \begin{bmatrix} I & & \\ Y^{-1}AX & I & \\ X^{-1}(BX + CP + RY^{-1}AX) & X^{-1}CY & I \end{bmatrix}.$$

Certain subgroups of C_ℓ are of interest:

$$\begin{aligned} Z_\ell &= \{g \in T_\ell : P(g) = R(g) = 0\} \\ J_\ell &= \{g \in Z_\ell : Q(z) = aI,\ a \in GF(q)\} \\ L_\ell^* &= \{g \in C_\ell : P(g) = Q(g) = R(g) = 0\} \\ L_\ell &= \{g \in L_\ell^* : \det(X(g)) = 1\} . \end{aligned}$$

(4.6) (1) $C_\ell = T_\ell L_\ell^*$.

(2) $Z_\ell = Z(T_\ell)$.

(3) If $n = 2\ell$ *then* $T_\ell = Z_\ell$.

(4) If $n \neq 2\ell$ *then* $Z_\ell = \Phi(T_\ell)$.

Section 5. Unitary, symplectic and orthogonal groups.

We continue the hypothesis and notation of Section 4.

Let θ be an automorphism of $GF(q)$ with $\theta^2 = 1$. A *θ-symmetric bilinear form* on V is a map $(\ ,\ )$ from $V \times V$ to $GF(q)$ such that for each $x, y, z \in V$ and $a \in GF(q)$

$$(x + y, z) = (x, z) + (y, z) ,$$
$$(ax, z) = a(x, z) , \qquad (x, z)^\theta = (z, x) .$$

The form is *symmetric* if $\theta = 1$ and *hermitian symmetric* if θ is an involution. $v \in V^\#$ is *singular* if $(v, v) = 0$. The form is *symplectic* if it is symmetric and each vector of $V^\#$ is singular.

Given a matrix $M = (M_{ij})$ over $GF(q)$ let $M^* = (M_{ji})$ be the transpose of M and $M^\theta = (M_{ij}^\theta)$. Define $\pi = {}^*\theta$.

(5.1) *Let (,) be a θ-symmetric form on V and $\{x_i\}$ an ordered basis of V. Define $J = (J_{ij})$ to be the $n \times n$ matrix with $J_{ij} = (x_i, x_j)$. Then $g \in SL(V)$ preserves the form (,) if and only if*

$$J = gJg^\pi .$$

Proof. Straightforward calculation.

For $v \in V$ and $X \subseteq V$ define

$$v^\perp = \{x \in V : (x, v) = 0\}$$
$$X^\perp = \textstyle\bigcap_X v^\perp .$$

The form is said to be *nondegenerate* if $V^\perp = 0$. A space V with a nondegenerate hermitian symmetric form is called a *unitary space*. A space with a nondegenerate symplectic form is called a *symplectic space*.

It is well known that

(5.2) *Let V be a unitary space. Then there exists a basis of V in which V has form $J(V) = I_n$.*

Define E_{2m} to be the $2m \times 2m$ matrix with 1 in the $(2i, 2i - 1)$ and $(2i - 1, 2i)$ positions and 0 elsewhere.

(5.3) *Let V be a symplectic space. Then n is even and there exists a basis of V in which V has form $J(V) = E_n$.*

5.2 and 5.3 imply that, up to isomorphism, there is a unique unitary space of dimension n over $GF(q^2)$ and a unique symplectic space of dimension $2m$ over $GF(q)$. An ordered basis for V in which V has form I_n or E_n, for V unitary or symplectic, will be referred to as a *unitary* or *symplectic basis*, respectively.

Denote by $SU(V)$ and $Sp(V)$ the subgroup of $SL(V)$ which preserves the corresponding unitary and symplectic form, respectively. $GU(V)$

denotes the full group of automorphisms of the unitary space. As $\det(g) = \det(g^*)$ and E_n is nonsingular, 5.1 and 5.3 imply $Sp(V)$ is the full group of automorphisms of the symplectic space.

A *quadratic* form on a symplectic space V is a map Q of V to $GF(q)$ such that for each $x, y \in V$ and each $a \in GF(q)$

$$Q(ax) = a^2Q(x) , \qquad Q(x + y) = Q(x) + Q(y) + (x, y) .$$

A symplectic space with a quadratic form will be referred to as an *orthogonal space.*

Suppose $\{u, v\}$ is a symplectic basis for V. Define $Q: V \to GF(q)$ by

$$Q(au + bv) = ab .$$

Then Q is a quadratic form. Denote the 2-dimensional orthogonal space with this form by $D = D_+$.

Let $\alpha x^2 + x + \alpha$ be an irreducible polynomial over $GF(q)$. Then

$$Q(au + bv) = \alpha a^2 + ab + \alpha b^2$$

is a quadratic form on V, and the space with this form is denoted by $Q = D_-$.

A *singular point* in an orthogonal space is a 1-dimensional subspace $\langle v \rangle$ with $Q(v) = 0$. It is well known that

(5.4) *Up to isomorphism, D and Q are the only 2-dimensional orthogonal spaces over $GF(q)$. Q has no singular points. D has exactly two singular points.*

Denote by D^mQ^k the orthogonal sum of m copies of D with k copies of Q. It is easy to check

(5.5) *D^2 is isomorphic to Q^2.*

It follows that

(5.6) *Up to isomorphism, D^m and $D^{m-1}Q$ are the only orthogonal spaces of dimension $2m$.*

The orthogonal space D^m is said to have *sign* $+$ and the space $D^{m-1}Q$ *sign* $-$. Denote by $O^\epsilon(V)$ the subgroup of $Sp(V)$ preserving the quadratic form on V of sign ϵ.

Given an orthogonal space V of sign ϵ, an *orthogonal basis* for V is a symplectic basis $\{x_i\}$ such that $Q(x_i) = 0$ for $i < n-1$ and

$$Q(ax_{n-1} + bx_n) = f_\epsilon(a, b)$$
$$f_+(a, b) = ab \qquad f_-(a, b) = \alpha a^2 + ab + \alpha b^2 .$$

By 5.6, V possesses an orthogonal basis.

$$SL_n(q)\ ,\quad SU_n(q)\ ,\quad Sp_n(q) \quad \text{and} \quad O_n^\epsilon(q)$$

denote the special linear, unitary, symplectic and orthogonal groups determined by a space of dimension n over $GF(q)$, or $GF(q^2)$ in the unitary case. $SL_n(q)$ has a center of order $(q-1, n)$ and $SU_n(q)$ has a center of order $(q+1, n)$. $L_n(q)$ and $U_n(q)$ denote the factor groups over these centers. $\Omega_n^\epsilon(q)$ denotes the commutator group of $O_n^\epsilon(q)$. As q is even $\Omega_n^\epsilon(q) = SO^\epsilon(n, q)$.

Section 6. Involutions in the unitary group.

We continue the hypothesis and notation of sections 4 and 5.

(6.1) *Let a and b be involutions in $SU(V)$. Then the following are equivalent.*

(1) *a is conjugate to b in $SU(V)$.*

(2) *a is conjugate to b in $SL(V)$.*

(3) *a and b have the same rank.*

Proof. See [25], page 34.

(6.2) *Let t be an involution in $SU(V)$ of rank ℓ. Then there exists a basis for V in which $t = j_\ell$ is in Suzuki form and V has unitary form*

$$J(V) = \begin{bmatrix} & & I_\ell \\ & I_{n-2\ell} & \\ I_\ell & & \end{bmatrix} .$$

Further $g \in C_\ell$ is in $SU(V)$ if and only if

$$X^\pi = X^{-1} \qquad Y^\pi = Y^{-1} \qquad YR^\pi = PX^\pi$$
$$XQ^\pi + RR^\pi + QX^\pi = 0 .$$

The map $g \to (X(g), Y(g))$ is a homomorphism of $C = C(t) \cap SU(V)$ into $GU_\ell(q) \times GU_{n-2\ell}(q)$ with the image containing $SU_\ell(q) \times SU_{n-2\ell}(q)$, and covering both factors if $n \neq 2\ell$. The kernel is $T_\ell \cap C = O_2(C)$.

Proof. Let $\{x_i\}$ be a unitary basis for V. Define $s \in SL(V)$ by

$$x_i^s = x_{n-\ell+i} \qquad 1 \le i \le \ell$$
$$x_i^s = x_i \qquad \ell < i \le n - \ell .$$

Then s has rank ℓ and by 5.1, $s \in SU(V)$. Hence by 6.1, we may take $t = s$. Choose $\alpha \in F_{q^2}$ such that $\alpha + \alpha^\theta = 1$ and let $\beta = 1 + \alpha$. Define

$$y_i = x_i + x_i^t \qquad 1 \le i \le \ell$$
$$y_i = x_i \qquad \ell < i \le n - \ell$$
$$y_i = \alpha x_{i-n+\ell} + \beta x_i \qquad n - \ell \le i \le n .$$

Then $\{y_i\}$ is the desired basis. Calculating using 5.1, we determine the conditions on g. This together with 4.3 yields the remaining conclusions.

Section 7. Involutions in the symplectic groups.

In this section we assume V to be a symplectic space and t an involution in $Sp(V)$. It is well known that

(7.1) *Let* $U \le V$. *Then* $\dim(V) = \dim(U) + \dim(U^\perp)$. *Therefore if* U *is non-degenerate then* V *is the orthogonal sum of* U *and* $U^\perp$.

Given an involution $t \in Sp(V)$ define

$$V(t) = \{v \in V : (v, v^t) = 0\}$$

$V(t)$ is the kernel of the additive map $v \to (v, v^t)$, so

(7.2) $V(t)$ *is a subspace of* V *of codimension at most* 1 *with* $V(t)^\perp \le V(t)$.

Given subspaces U and W of V with $U \cap W = 0$ and $(U, W) = 0$, write $\langle U, W \rangle = U \oplus W$. That is $U \oplus W$ is the orthogonal sum of U and W.

(7.3) *Assume* $V = V(t)$. *Then*

$$V = \bigoplus_{i=1}^{m} V_i \oplus W$$

where $[W, t] = 0$ *and each* V_i *is a* 4*-dimensional symplectic space with symplectic basis* $\{y_{ij}\}$ *such that* $y_{i1}^t = y_{i3}$ *and* $y_{i2}^t = y_{i4}$. *In particular* t *has even rank* $\ell = 2m$.

Proof. Choose $u \in V$ with $u^t \notin \langle u \rangle$. Set $u = y_1$ and $u^t = y_3$. Let $v \in (u^t)^\perp - u^\perp$. Then $v^t \in u^\perp - (u^t)^\perp$, so $v \neq v^t$. By choosing a suitable multiple of v we may take $(u, v) = 1$. Set $y_2 = v$ and $y_4 = v^t$. Then $V_1 = \langle y_i : 1 \leq i \leq 4 \rangle$ has the properties claimed above. V_1 is nondegenerate, so by 7.1, $V = V_1 \oplus V_1^\perp$, with $V_1^\perp$ a t-invariant symplectic space. Hence the result follows by induction on the dimension of V.

The following lemma is immediate.

(7.4) *Assume $u \in V$ with $(u, u^t) \neq 0$. Then for $a^2 = (u, u^t)^{-1}, x_1 = au$ and $x_2 = x_1^t$ is a symplectic basis for $\langle u, u^t \rangle$.*

(7.5) *Assume $U = V(t) \neq V$ and set $X = U^\perp$. Let ℓ be the rank of t. Then either*
(1) *ℓ is odd, $[V, t] = X \oplus [U, t]$ and $V = Y \oplus Y^\perp$ where Y has symplectic basis $\{y, y^t\}$ and $[Y, t] = X$.*
(2) *ℓ is even and $V = Y \oplus Y^\perp$ for some Y with symplectic basis $\{y_i : 1 \leq i \leq 4\}$ such that $y_1^t = y_2, y_3^t = y_4$. Further $[U, t] = [Y^\perp, t] \oplus X$ has codimension 1 in $[V, t]$ and $X = Y(t)^\perp$.*

Proof. Let P be maximal with respect to $P = \oplus P_i$ where P_i has a symplectic basis $\{x_i, x_i^t\}$. By 7.1, $V = P \oplus P^\perp$. By 7.4 and maximality of $P, P^\perp \leq U$. $[V, t] = [P, t] \oplus [P^\perp, t]$, so $\ell = \dim([P, t]) + \dim([P^\perp, t]) = a + b$. By 7.3, b is even, so $\ell \equiv a \mod 2$. Further $U = P^\perp \oplus P(t)$ and

$$P(t) = \langle x_i + x_j, x_i + x_i^t : 1 \leq i \leq a, 1 \leq j \leq a \rangle .$$

Hence $x = \sum_i (x_i + x_i^t)$ is a generator of X.

Assume ℓ is odd. Then a is odd, so setting $y = \sum_i x_i, \{y, y^t\}$ is a symplectic basis for $Y = \langle y, y^t \rangle$ and $[y, t] = X$. By 7.1, $V = Y \oplus Y^\perp$. Then $Y^\perp \leq X^\perp \leq U$ and $[V, t] = [Y, t] \oplus [Y^\perp, t] = X \oplus [U, t]$.

So assume ℓ is even. Let $y_3 = \sum_{i \neq 1} x_i$ and $y_4 = y_3^t$. ℓ is even so $a - 1$ is odd and hence $(y_3, y_4) = 1$. Let $x_1 = y_1$ and $x_1^t = y_2$. Then $Y = \langle y_i : 1 \leq i \leq 4 \rangle$ is as claimed. By 7.1, $V = Y \oplus Y^\perp$. Then $U = Y^\perp \oplus Y(t)$ and $Y(t) = \langle y_i + y_j : 1 \leq i \leq j \leq 4 \rangle$. So $[U, t] = [Y^\perp, t] \oplus [Y(t), t] = [Y^\perp, t] \oplus X$ has codimension 1 in $[V, t] = [Y^\perp, t] \oplus [Y, t]$.

(7.6) *Let ℓ be the rank of t. Then there exists a basis $\{x_i\}$ of V with form*

$$J = \begin{bmatrix} & & F \\ & E_{n-2\ell} & \\ F & & \end{bmatrix}$$

in which $t = j_\ell$ *and exactly one of the following holds:*

(1) ℓ *is even,* $V = V(t)$ *and* $F = E_\ell$.

(2) ℓ *is odd,* $V(t) = \langle x_i : i \neq n - \ell + 1 \rangle, V(t)^\perp = \langle x_1 \rangle$

$$[V(t), t]^\perp = \langle x_i : 1 \leq i \leq n - \ell + 1 \rangle$$
$$[V(t), t] = \langle x_i : 1 \leq i \leq \ell \rangle$$
$$F = \begin{bmatrix} 1 & \\ & E_{\ell-1} \end{bmatrix}.$$

(3) ℓ *is even,* $V(t) = \langle x_i : 1 \leq i < n \rangle, V(t)^\perp = \langle x_1 \rangle$

$$[V(t), t]^\perp = \langle x_i : 1 \leq i \leq n - \ell + 1 \rangle$$
$$[V(t), t] = \langle x_i : 1 \leq i < \ell \rangle$$
$$F = \begin{bmatrix} & & 1 \\ & E_{\ell-2} & \\ 1 & & 1 \end{bmatrix}.$$

Proof. Assume first $V = V(t)$. Then define V_i, y_{ij}, and $W, 1 \leq i \leq \ell/2, 1 \leq j \leq 4$, as in 7.3. Define

$$\begin{aligned} x_{2i-1} &= y_{i1} + y_{i3} \\ x_{2i} &= y_{i2} + y_{i4} \\ x_{n-\ell+2i-1} &= y_{i1} \\ x_{n-\ell+2i} &= y_{i2} \end{aligned} \qquad 1 \leq i \leq \ell/2$$

and let $\{x_{\ell+i} : 1 \leq i \leq n - 2\ell\}$ be a symplectic basis for W.

Assume next that $V(t) \neq V$. Choose Y and its symplectic basis as in 7.5. If ℓ is odd let $x_1 = y, x_{n-\ell+1} = y^t$, and choose $\{x_i : 1 \neq i \neq n - \ell + 1\}$ to be a basis for $Y^\perp = Y^\perp(t)$ as in the last paragraph. If ℓ is even define

$$x_1 = \textstyle\sum_i y_i \qquad x_\ell = y_3 + y_4$$
$$x_{n-\ell+1} = y_1 + y_4 \qquad x_n = y_4 .$$

Choose the remaining basis vectors as in the last paragraph.

We shall say an involution $t \in Sp(V)$ is in *symplectic Suzuki form* if the basis is chosen as in 7.6. Denote by a_ℓ, b_ℓ, and c_ℓ the Suzuki forms in (1), (2), and (3) of 7.6, respectively. It follows from 7.6 that

(7.7) *Let t and s be involutions in $Sp(V)$. Then the following are equivalent:*

(1) *t is conjugate to s in $Sp(V)$.*

(2) *t and s have the same symplectic Suzuki form.*

(3) *t and s have the same rank ℓ, and if ℓ is even then $V(t)$ and $V(s)$ have the same dimension.*

(7.8) *Let $F = F^*$ be an $\ell \times \ell$ matrix and*

$$J = \begin{bmatrix} & & F \\ & E_{n-2\ell} & \\ F & & \end{bmatrix}.$$

Let $g \in C_\ell$. Then $gJg^* = J$ if and only if

$$XFX^* = F \qquad YEY^* = E \qquad YER^* = PFX^*$$
$$0 = XFQ^* + RER^* + QFX^* .$$

(7.9) *Let $t = a_\ell$ be in Suzuki form, $g \in C_\ell$, and $E = E_\ell$ or $E_{n-2\ell}$. Then $g \in Sp(V)$ if and only if*

$$XEX^* = E \qquad YEY^* = E \qquad YER^* = PEX^*$$
$$0 = XEQ^* + RER^* + QEX^* .$$

The map $g \to (X(g), Y(g))$ is a homomorphism of $C = C_\ell \cap Sp(V)$ onto $Sp_\ell(q) \times Sp_{n-2\ell}(q)$ with kernel $T_\ell \cap Sp(V) = O_2(C)$.

Proof. 4.3, 7.6, and 7.8.

(7.10) *Let $t = b_\ell$ be in Suzuki form, let $g \in C_\ell$, and $E = E_{\ell-1}$ or $E_{n-2\ell}$. Then*

(1) *$g \in Sp(V)$ if and only if*

$$X = \begin{bmatrix} 1 & \\ & W \end{bmatrix} \qquad R = \begin{bmatrix} \alpha \\ A \end{bmatrix} \qquad P = [YE\alpha^* \quad YEA^*EW]$$
$$Q = \begin{bmatrix} b & B \\ \gamma & L \end{bmatrix}$$

with α, β and γ^* row vectors and

$$WEW^* = E \qquad \alpha EA^* = \gamma^* + \beta EW^*$$
$$YEY^* = E \qquad LEW^* + WEL^* = AEA^* .$$

(2) *The map* $g \to (W(g), Y(g))$ *is a homomorphism of* $C = C_\ell \cap Sp(V)$ *onto* $Sp_{\ell-1}(q) \times Sp_{n-2\ell}(q)$ *with kernel* $T_\ell \cap C = O_2(C)$.

(7.11). *Let* $t = c_\ell$ *be in Suzuki form, let* $g \in C_\ell, E = E_{\ell-2}$ *or* $E_{n-2\ell}$, *and*

$$F = \begin{bmatrix} & & 1 \\ & E_{\ell-2} & \\ 1 & & 1 \end{bmatrix}.$$

Then

(1) $g \in Sp(V)$ *if and only if*

$$X = \begin{bmatrix} 1 & & \\ WE\alpha^* & W & \\ x & \alpha & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \beta \\ A \\ \gamma \end{bmatrix}$$
$$Q = \begin{bmatrix} r & \rho & z \\ \mu & L & \eta \\ y & \xi & s \end{bmatrix}$$

where $\beta, \gamma, \rho, \xi, \mu^*$, *and* η^* *are row vectors and*

$$WEW^* = E \qquad YEY^* = E \qquad P = YER^*F^{-1}X$$
$$0 = XFQ^* + RER^* + QFX^* .$$

(2) *The map* $g \to (W(g), Y(g))$ *is a homomorphism of* $C = C_\ell \cap Sp(V)$ *onto* $Sp_{\ell-2}(q) \times Sp_{n-2\ell}(q)$ *with kernel* $O_2(C)$.

(7.12) *Assume the hypothesis of* 7.11. *Then*

(1) *If* $g \in Z_\ell$ *then* $g \in Sp(V)$ *if and only if*

$$r + s + z = 0 \qquad LE = EL^*$$
$$\eta = E\rho^* \qquad \xi E = \mu^* + \eta^* .$$

(2) *If* $g \in Z_\ell L_\ell$ *with* $\alpha = \mu = \rho = \eta = \xi = 0$ *then* $g \in Sp(V)$ *if and only if*

$$YEY^* = E \qquad WEL^* = LEW^*$$
$$WEW^* = E \qquad r + z + s + xz = 0 .$$

(7.13) *Let* Q *and* X *be as in* 7.11. *Then*

(1) *If* $\alpha = 0$ *and* $W = I$ *then*

$$Q^X = \begin{bmatrix} r + zx & \rho & z \\ \mu + x\eta & L & \eta \\ y + (r + s + zx)x & \xi + \rho x & s + zx \end{bmatrix}.$$

(2) *If* $\mu = \eta = \rho = \xi = 0$, $W = I$ *and* $L = aI$ *then*

$$Q^X = \begin{bmatrix} r + zx & z\alpha & z \\ (r + zx + a)E\alpha^* & L + zE\alpha^*\alpha & zE\alpha^* \\ y + (r + s + zx)x & \alpha(a + s + xz) & xz + s \end{bmatrix}.$$

These last three lemmas follow from straight forward calculations.

Section 8. Involutions in the orthogonal groups.

In this section assume that V is an orthogonal space of sign ε. It is well known that

(8.1) (1) $O_2^\varepsilon(q)$ *is dihedral of order* $2(q - \varepsilon)$.
(2) $O_4^+(q)$ *is the wreath product of* $L_2(q)$ *by* Z_2.
(3) $O_4^-(q)$ *is the split extension of* $L_2(q^2)$ *by a field automorphism of order* 2.
(4) $O_6^+(q)$ *is the split extension of* $L_4(q)$ *by a graph automorphism.*
(5) $O_6^-(q)$ *is the split extension of* $U_4(q)$ *by a graph automorphism.*

Given the isomorphisms in 8.1 we assume $n \geq 8$ in this section, although most of the discussion remains valid without this restriction.

(8.2) *Let* t *be an involution in* $O^\varepsilon(V)$ *of type* a_ℓ. *Then there exists a basis* $\{x_i\}$ *of* V *in which* t *is in its symplectic Suzuki form,* $Q(x_i) = 0$ *for* $1 \leq i \leq \ell$ *and* $n - \ell < i \leq n$, $\{x_i : \ell < i \leq n - \ell\}$ *is an orthogonal basis for the space* W *it generates, and* W *has sign* ε.

Proof. Choose $\{x_i\}$ so that t is in Suzuki form. For $1 \leq i \leq \ell$ define

$$U_i = \langle x_{n-\ell+2i-1}, x_{n-\ell+2i} \rangle .$$

Then $V_i = U_i \oplus U_i^t \cong U_i \oplus U_i$, so by 5.5, V_i is isomorphic to D^2. Thus choosing the basis vectors $\{y_{ij}\}$ for V_i in 7.3 to be an orthogonal basis for V_i, we have $Q(x_j) = 0$. Further as V is the orthogonal sum of W and the spaces V_i, and each V_i has sign $+$, $\varepsilon = \mathrm{sgn}(V) = \mathrm{sgn}(W)$. Hence we need only choose the basis vectors x_j in W to form an orthogonal basis for W.

(8.3) *Let* $t \in O^{\epsilon}(V)$ *be the type* b_ℓ. *Then there is a basis* $\{x_i\}$ *of* V *in which* t *is in symplectic Suzuki form,* $Q(x_1) = 1, X = \langle x_1, x_{n-\ell+1}\rangle$ *has sign* ϵ; $Q(x_i) = 0$ *for* $1 < i \leq \ell$ *and* $n - \ell + 1 < i \leq n$, $\{x_i : \ell < i \leq n - \ell\}$ *is an orthogonal basis for the space* W *it generates, and* W *has sign* $+$.

Proof. Choose $\{x_i\}$ so that t is in Suzuki form. Define

$$U = \langle x_i : 1 < i \leq \ell, n - \ell + 1 < i \leq n\rangle .$$

Considering t restricted to U, 8.2 implies U has sign $+$ and we may choose the basis vectors x_j in U to be singular. t restricted to X is the transvection with center x_1, so $Q(x_1) = 1$.

Suppose W has sign $-$. We may assume $Y = \langle x_{\ell+1}, x_{\ell+2}\rangle \cong Q$. Let $a^2 = Q(x_{\ell+1})$, $x'_{\ell+1} = ax_1 + x_{\ell+1}$, and $x'_{n-\ell+1} = x_{n-\ell+1} + ax_{\ell+2}$. Then $Q(x'_{\ell+1}) = 0$ and $\{x_1, x'_{n-\ell+1}, x'_{\ell+1}, x_{\ell+2}\}$ is a symplectic basis for $X + Y$. As $Q(x'_{\ell+1}) = 0$, $Y' = \langle x'_{\ell+1}, x_{\ell+2}\rangle \cong D$, so with this change of basis W has sign ϵ. Hence $\mathrm{sgn}\,(X) = \mathrm{sgn}\,(V) = \epsilon$.

The proof of the next lemma is similar to that of 8.2 and 8.3, and is omitted.

(8.4) *Let* $t \in O^{\epsilon}(V)$ *be of type* c_ℓ. *Then there is a basis* $\{x_i\}$ *of* V *in which* t *is in Suzuki form,* $Y = \langle x_1, x_\ell, x_{n-\ell+1}, x_n\rangle$ *has sign* $\epsilon, Q(x_i) = 0$ *for* $1 < i < \ell$ *and* $n - \ell + 1 < i < n, \{x_i : \ell < i \leq n - \ell\}$ *is an orthogonal basis for the space* W *it generates, and* W *has sign* $+$. *Also*

$$Q(ax_1 + bx_\ell + cx_{n-\ell+1} + dx_n) = ad + b(b + c + d) + (c^2 + d^2)Q(x_n) .$$

We shall say an involution t in $O^{\epsilon}(V)$ is in *orthogonal Suzuki form* if the basis for V is chosen as in 8.2–8.4 for the suitable type of t. In particular if t is in its orthogonal Suzuki form it is also in symplectic Suzuki form. Therefore

(8.5) *Let* t *and* s *be involutions in* $O^{\epsilon}(V)$. *Then the following are equivalent:*

(1) t *is conjugate to* s *in* $O^{\epsilon}(V)$.

(2) t *is conjugate to* s *in* $Sp(V)$.

(3) t *and* s *have the same (orthogonal or symplectic) Suzuki form.*

(8.6) *Let* $t = a_\ell$ *be in orthogonal Suzuki form and* $g \in C_\ell \cap Sp(V)$. *Then*

(1) $g \in O^{\epsilon}(V)$ *if and only if* $Y(g) \in O^{\epsilon}_{n-2\ell}(q)$ *and*

$$\sum_{j=1}^{\ell/2} g_{i(2j-1)}g_{i(n-\ell+2j)} + g_{i(2j)}g_{i(n-\ell+2j-1)} = \sum_{j=1}^{(n-2\ell)/2} g_{i(\ell+2j-1)}g_{i(\ell+2j)} .$$

(2) *The map $g \to (X(g), Y(g))$ is a homomorphism of $C = C_\ell \cap O^\epsilon(V)$ onto $Sp_\ell(q) \times O^\epsilon_{n-2\ell}(q)$ with kernel $T_\ell \cap C = O_2(C)$.*

Proof. Let $G = Sp(V), H = O^\epsilon(V)$, and $g \in C_\ell \cap G$. Then $g \in H$ if and only if $Q(x_i^g) = Q(x_i)$ for each i. Define

$$U = \langle x_i : 1 \leq i \leq \ell, n - \ell < i \leq n \rangle$$
$$W = \langle x_i : \ell < i \leq n - \ell \rangle .$$

Then $V = U \oplus W$ and from the form of $J(V)$ in 7.6 and the value of the quadratic form Q on $\{x_i\}$ given in 8.2 we determine that for $u + w = \sum a_i x_i$

$$Q(u + w) = \sum_{i=1}^{\ell/2} a_{2j-1}a_{n-\ell+2j} + a_{2j}a_{n-\ell+2j-1} + Q(w)$$

and Q restricted to W has sign ϵ.

Hence for $1 \leq i \leq \ell$, $Q(x_i) = 0$ and

$$Q(x_i^g) = Q(\sum g_{ij}x_j) = \sum_{i=1}^{\ell} g_{ij}g_{i(n-\ell+j)} = 0$$

since $g_{ij} = 0$ for $j > \ell$. That is $Q(x_i) = Q(x_i^g)$ for $i \leq \ell$. Similarly for $x_i \in W, Q(x_i^g) = Q(x_i Y(g))$, so $Q(x_i) = Q(x_i^g)$ if and only if $Y(g) \in O^\epsilon(W) = O^\epsilon_{n-2\ell}(q)$. Finally if $Q(g) = P(g) = 0$, then for $i \leq \ell, Q(x_{n-\ell+i}) = 0 = Q(x^g_{n-\ell+1})$. Therefore 7.9 yields the result.

(8.7) *Let $t = b_\ell$ be in orthogonal Suzuki form. Then the map $g \to (W(g), Y(g))$ is a homomorphism of $C = C_\ell \cap O^\epsilon(V)$ onto $Sp_{\ell-1}(q) \times Sp_{n-2\ell}(q)$ with kernel $T_\ell \cap C = O_2(C)$.*

Proof. The proof goes as in 8.6. The only problem is to show that if $\varphi : g \to (W(g), Y(g))$ then $C\varphi$ covers $Sp_{\ell-1}(q) \times Sp_{n-2\ell}(q)$. Proceeding as in 8.6, $C\varphi$ covers $Sp_{\ell-1}(q) \times O^+_{n-2\ell}(q)$. Let $U = \langle x_i : \ell < i \leq n - \ell \rangle$ and Y the transvection in $Sp(U)$ with center $x_{\ell+1}$. Then $Y \notin O^+(U)$. Further $O^+(U)$ is maximal in $Sp(U)$, so $Sp(U) = \langle Y, O^+(U) \rangle$, and it suffices to exhibit $g \in C$ with $Y(g) = Y$. Define g by

$$g = \begin{bmatrix} 1 & & & & \\ & I & & & \\ \alpha & & Y & & \\ & & \beta & 1 & \\ & & & & I \end{bmatrix}$$

where $\alpha^* = (0, 1, 0, \cdots)$ and $\beta = (1, 0, 0, \cdots)$.

(8.8) *Let $t = c_\ell$ be in orthogonal Suzuki form and $g \in C_\ell \cap Sp(V)$. Given a matrix $A = (a_{ij})$ let A_i be the ith row vector of A and define*

$$Q(A_i) = \textstyle\sum_j a_{i(2j-1)} a_{i(2j)}$$
$$(A_i, B_i) = A_i E B_i^* \ .$$

Then
(1) *$g \in O^\epsilon(V)$ if and only if* (i)–(iv) *hold with notation for g as in* 7.11 *and* 7.12.

- (i) $\tau_i^2 = Q(Y_i)$ *for* $1 \le i \le n - 2\ell$.
- (ii) $z(z + 1) = Q(\beta)$.
- (iii) $\eta_i[\eta_i + (WE\alpha^*)_i] + (WE\alpha^*)_i^2 Q(x_n) + (L_i, W_i) = 0$.
- (iv) $y + s(s + x + 1) + x^2 Q(x_n) + (\xi, \alpha) = 0$.

(2) *The map $g \to (W(g), Y(g))$ is a homomorphism of $C = O^\epsilon(V) \cap C_\ell$ onto $Sp_{\ell-2}(q) \times Sp_{n-2\ell}(q)$ with kernel $O_2(C)$.*

Proof. The proof goes as in 8.6 and 8.7. We exhibit an element $g \in C$ such that $Y(g) = Y$ is the transvection with center $x_{\ell+1}$:

$$g = \begin{bmatrix} 1 & & & & & & \\ & I & & & & & \\ & & 1 & & & & \\ \tau & & \tau & & & & \\ & & & Y & & & \\ & & & \beta & 1 & & \\ & & & & & & I \end{bmatrix}$$

where $\tau^* = (0, 1, 0, \cdots)$ and $\beta = (1, 0, 0, \cdots)$.

(8.9) *Let $t = c_\ell$ be in orthogonal Suzuki form and $g \in C_\ell$ with notation as in* 7.11 *and* 7.12. *Then*
(1) *If $g \in Z_\ell$ then $g \in O^\epsilon(V)$ if and only if $z = 0$ or 1, $y = s^2 + s, r + s + z = 0, \eta = E\rho^*, \xi E = \mu^* + \eta^*, L^E = L^*$, and*

$$\eta = (L_{12}, L_{21}, L_{34}, L_{43}, \cdots)^* \ .$$

(2) *If* $g \in Z_\ell L_\ell$ *with* $\alpha = \mu = \rho = \eta = \xi = 0$ *then* $g \in O^\epsilon(V)$ *if and only if* $WEW^* = E, Y \in O^+_{n-2\ell}(q), WEL^* = LEW^*, r + z + s + xz = 0, y + s(s + x + 1) = xQ(x_n), z = 0$ *or* 1, *and* $(L_i, W_i) = 0$ *for each* $1 \leq i \leq \ell - 2$.

Proof. Calculate.

(8.10) *The commutator group* Ω^ϵ *of* O^ϵ *is a simple subgroup of index* 2 *in* O^ϵ. $t \in \Omega^\epsilon$ *if and only if* t *is of type* a *or* c.

Proof. The first statement is well known. The transvections in O^ϵ are not contained in Ω^ϵ. Further if ℓ is the rank of t then t is the product of m transvections where $\ell \equiv m \bmod 2$.

(8.11) *Let* $t = c_2$ *and* z *are involutions in* $Z_\ell \cap \Omega^\epsilon(V)$. *Then*

$$Q(z) = \begin{bmatrix} r & 0 \\ r(r+1) & r \end{bmatrix} .$$

Proof. By 8.9 if a is an involution in $Z_\ell \cap O^\epsilon(V)$ then

$$Q(a) = \begin{bmatrix} s + z & z \\ s(s+1) & s \end{bmatrix}$$

with $z = 0$ or 1. But if $z = 1$ then a has rank 1, so $a \notin \Omega^\epsilon(V)$.

(8.12) *Let* $G = \Omega^\epsilon_n(q)$ *and* t *an involution of* G *in Suzuki form. Then*
(1) t^G *is a conjugacy class in* $O^\epsilon_n(q)$ *unless* $n = 2\ell, \epsilon = +$, *and* $t = a_\ell$.
(2) *Let* $n = 2\ell, \epsilon = +$, *and* $t = a_\ell$. *Then the class of* t *in* $O^\epsilon_n(q)$ *splits into two classes in* G. *If* s *is of type* a_ℓ *then* $s \in t^G$ *exactly when* $\dim([V, s] \cap [V, t])$ *is even. Further* $C_G(t) = \langle t^G \cap C_G(t)\rangle O_2(C_G(t))$.

Proof. Unless $t = a_\ell, n = 2\ell$, and $\epsilon = +$, t centralizes a transvection, so by 8.10, t^G is a conjugacy class of O^ϵ_n. So assume $n = 2\ell, \epsilon = +$, and $t = a_\ell$. Then, under the action of G, there are two classes of totally isotropic ℓ-dimensional subspaces of V. Moreover if U and W are two such subspaces then U is conjugate to W under G exactly when $\dim(U \cap W)$ is even. As $[V, t]$ is such a subspace, the first two remarks of (2) follow.

If $\ell \equiv 0 \bmod 4$ then the involution $s \in L$ with $X(s) = E_\ell$ is in t^G, since $\dim([V, t] \cap [V, s]) = \ell/2$ is even. Similarly if $\ell \equiv 2 \bmod 4$ then the involution $r \in C_\ell$ with

$$X(r) = \begin{bmatrix} E_{\ell-2} & \\ & I_2 \end{bmatrix} \qquad Q(r) = \begin{bmatrix} & \\ & I_2 \end{bmatrix}$$

is in t^G. Also $C(t)/O_2(C(t)) \cong L$ is simple unless $q = 2$ and $n = 4$, in which case $\langle s^L \rangle = L$. So the last part of (2) follows.

Section 9. Two lemmas.

In this section we prove two lemmas which help describe the action of the centralizer C, of an involution in the classical groups, on $O_2(C)$.

Let G be the subgroup of $SL_\ell(q), SU_\ell(q)$, or $Sp_\ell(q)$ generated by all transvections, or a subgroup of $O^\epsilon_\ell(q)$ of index at most 2. Here q is a power of 2, $\ell > 1$, and $\ell \geq 4$ if $G \leq O^+_\ell(q)$. Consider G as a matrix group with respect to a natural basis $\{x_i\}$ for the linear space V corresponding to G.

Let M be the ring of all $n \times \ell$ matrices over the corresponding field, *i.e.* $GF(q)$ unless $G \leq SU(q)$, in which case the field is $GF(q^2)$.

(9.1) *Let G act by right multiplication on M. Then $C_M(G) = 0$ and G has no orbits of length* 2.

Proof. If $P \in M, g \in G$, and $Pg = P$, then g fixes each of the row vectors of P. Consequently $C_M(G) = 0$. If P is in an orbit of length 2, then $C_G(P) = H \leq G$ has index 2 in G and it is easy to check that this forces $P = 0$, a contradiction.

(9.2) *Let $n = \ell$ and exclude the case $G \leq O^\epsilon_\ell(q)$. Let G act by conjugation on M. Then the centralizer in M of G is the set of scalar matrices and if N is a G-invariant 4-group of M containing I with $[G, N] \neq 0$, then one of the following holds*:

(1) $G = L_2(2)$ *and N consists of*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(2) $G = U_2(2)$ *and choosing a to be a generator of $GF(4)$, N consists of either*

(i) $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & a^2 \end{bmatrix}, \begin{bmatrix} a^2 & 0 \\ 0 & a \end{bmatrix}$$

or

(ii) $$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

(3) *$G \leq SU_3(2)$ and $SU_3(2)$ does not act on N.*

Proof. $C_M(G) = \langle \alpha I \rangle$ is well-known and equivalent to the fact that G is absolutely irreducible on V. Assume that N is a 4-group as described. Then G has an orbit of length 2 on N so $q = 2$ and $G = L_2(2)$, $U_2(2)$, $Sp(4, 2)$, or the subgroup of $SU_3(2)$ generated by transvectors.

If $G = Sp(4, 2)$, then $G' \leq C(N)$, contradicting the fact that G' is absolutely irreducible on V. Similarly if $G \leq SU_3(2)$ and $SU_3(2)$ acts on N, then $O_3(SU_3(2)) \leq C(N)$, whereas $O_3(SU_3(2))$ is absolutely irreducible on V.

If $G = L_2(2)$ or $U_2(2)$, we calculate to obtain (i) or (ii).

Section 10. Certain normal subgroups of centralizers.

In this section we assume G is equal to $SL_n(q), SU_n(q), Sp_n(q)$, or $\Omega_n^\epsilon(q), n \geq 4$, and q a power of 2. We exclude the cases $L_4(2) \cong A_8$ and $Sp_4(2) \cong S_6$. If G is orthogonal we take $n \geq 8$.

Let t be an involution of G of rank ℓ, $C = C_G(t)$ and $\Delta = t^G \cap C$. Let u be an involution in $O_2(C)$ such that either $[u, \Delta] = 1$ or $\langle u, t \rangle \trianglelefteq C$. We wish to determine all possibilities for u. This is done in 10.6–10.8.

Let $z = z_\ell \cap G$, $T = T_\ell \cap G$, and $\bar{C} = C/T$. Set $K = O^{2'}(C)$, except if t is of type c_ℓ we take only those matrices in (7.11) (1) having $\alpha = 0$ and $x = 0$. Then $K/T = L_1/T \times L_2/T$, where the factors are unimodular, unitary, symplectic, or orthogonal groups.

Notice $\overline{K \cap L_i}$ is quasisimple unless $\bar{L}_i$ is $O_m^\epsilon(q), L_2(2), U_2(2), SU_3(2)$, or $Sp_4(2)$. Therefore

(10.1) *$\overline{K \cap L_i}$ is generated by any one of its classes of involutions unless $\bar{L}_i \cong Sp_4(2)$ and $\bar{L}_i$ is generated by involutions of type a_2 or b_1, or $\bar{L}_i \cong O_m^\epsilon(q)$ and $\bar{L}_i$ is generated by transvections.*

(10.2) $\bar{K} = \overline{\langle K \cap \Delta \rangle}$.

Proof. Let $D = \langle K \cap \Delta \rangle$. By 10.1 it suffices to exhibit conjugates r and s of t in K such that $1 \neq \bar{r} \in \overline{K \cap L_1}$ and $\bar{s}$ projects nontrivially on $\overline{K \cap L_2}$, with the projections of r and s in a suitable class if $\overline{K \cap L_i}$ is

$Sp_4(2)$ or $O_m^{\epsilon}(q)$. The case where G is unimodular or unitary is left to the reader. Let G be symplectic or orthogonal. r and s are exhibited below for a_ℓ, b_ℓ, and c_ℓ. In each case W is a suitable involution of maximal possible rank. $m = n - 2\ell$. J is the 2×2 matrix all of whose entries are 1. If $n = 2\ell$ and $t = a_\ell$, then (10.2) follows from (8.12)(2).

$$a_\ell \begin{bmatrix} W & & \\ & I_m & \\ & & W \end{bmatrix}, \begin{bmatrix} I_2 & & & & & \\ & W & & & & \\ & & E_2 & & & \\ & & & I_{m-2} & & \\ J & & & & I_2 & \\ & & & & & W \end{bmatrix}$$

$$b_\ell \begin{bmatrix} 1 & & & & \\ & W & & & \\ & & I_m & & \\ 1 & & & 1 & \\ & & & & W \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & \\ & I_2 & & & & & & \\ & & W & & & & & \\ & & & E_2 & & & & \\ & & & & I_{m-2} & & & \\ & & & & & 1 & & \\ & I_2 & & & & & I_2 & \\ & & & & & & & W \end{bmatrix}$$

$$c_\ell \begin{bmatrix} 1 & & & & & & \\ & W & & & & & \\ & & 1 & & & & \\ & & & I_m & & & \\ 1 & & & & 1 & & \\ & & & & & W & \\ & & 1 & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & & & & & & \\ & I_2 & & & & & & & & \\ & & W & & & & & & & \\ & & & 1 & & & & & & \\ & & & & E_2 & & & & & \\ & & & & & I_{m-2} & & & & \\ 1 & & & & & & 1 & & & \\ & I_2 & & & & & & I_2 & & \\ & & & & & & & & W & \\ & & & & & & & & & 1 \end{bmatrix}.$$

Let $g \in C$ and set $X = X(g), Y = Y(g)$, etc. Recall the definition of u.

(10.3) *If $u \notin T$, then $t = c_2$ and*

$$X(u) = \begin{bmatrix} 1 & \\ d & 1 \end{bmatrix}$$

with $d \neq 0$. If G is symplectic then $q = 2$.

Proof. $T = O_2(C)$ unless $t = c_\ell$, so take $t = c_\ell$. Then by (7.11) (1)

$$X = \begin{bmatrix} 1 & & \\ WE\alpha^* & W & \\ x & \alpha & 1 \end{bmatrix}$$

and if $g = u$ then $W = I$. Let U be the set of elements in $O_2(C)$ with $\alpha = 0$. Then by 4.5 the action of g on $O_2(C)/U$ is determined by $\alpha(x^g) = \alpha(x)W$. ℓ is even so by 10.2 and 9.1 we conclude $u \in U$. So

$$X(u) = \begin{bmatrix} 1 & & \\ & I & \\ d & & 1 \end{bmatrix}.$$

Suppose $\ell > 2$. By 7.11 and 7.13, if $g \in Z$ then

$$Q = \begin{bmatrix} r & \rho & z \\ \mu & L & \eta \\ y & \xi & s \end{bmatrix} \qquad Q(u^g) = \begin{bmatrix} r + zd & \rho & z \\ \mu + d\eta & L & \eta \\ y + (r + s + zd)d & \xi + d\rho & s + dz \end{bmatrix}$$

so that $[g, u] = 1$ exactly when $z = \eta = \rho = 0 = r + s$. However if $\ell \neq 2$ then by 8.9 the element $g \in Z$ with

$$Q = \begin{pmatrix} 1 & 1 & 1 & & \\ & & 1 & & 1 \\ & 1 & & & 1 \\ & & & I_{\ell-4} & \\ & 1 & 1 & & 1 \end{pmatrix}$$

is in Δ, so that $[u, \Delta] \neq 1$ and $\langle u, t\rangle \not\trianglelefteq C$. Also by 7.12 and 8.9, $|Z : C_Z(u)| > 2$ only if $\ell = 2$, G is symplectic, and $q > 2$.

(10.4) $R(u) = P(u) = 0$.

Proof. Let $g \in K$. By 4.5 and 10.3 the action of g on uZ is determined by $P(u^g) = Y^{-1}P(u)X$ and $R(u^g) = X^{-1}R(u)Y$. As $n > 3$, either $n - 2\ell$ or ℓ is greater than 1, and if $t = a_\ell$ then $\ell > 1$. Hence by 9.1 and 10.2, $P(u) = R(u) = 0$ if G is unimodular or unitary, or if $t = a_\ell$.

Suppose $t = b_\ell$. Then by 7.10

$$R(u) = \begin{bmatrix} \alpha \\ A \end{bmatrix} \qquad P(u) = ER^*F \qquad X^{-1} = \begin{bmatrix} 1 & \\ & W \end{bmatrix}$$

and then

$$X^{-1}R(u)Y = \begin{bmatrix} \alpha Y \\ WAY \end{bmatrix}$$

As $t = b_\ell, u \in T$, so $R(u) = X^{-1}R(u)Y$ for every $g \in C$. By 10.2, $\bar{K} = \langle \overline{K \cap \Delta} \rangle$. Further $n - 2\ell$ is even so if $n \neq 2\ell$ then $n - 2\ell > 1$. Thus by 9.1, $R(u) = 0$.

A similar argument works if $t = c_\ell$ and $u \in T$. So assume $u \notin T$. Then by 10.3, $t = c_2$. If G is symplectic then by 10.3 $q = 2$, so $n > 4$, and $\bar{L}_2 \cong Sp_{n-4}(q)$. If G is orthogonal then $n > 6$, so $n - 2\ell = n - 4 \geq 4$ and $\bar{L}_2 \cong Sp_{n-4}(q)$. Hence by 9.1, $R(u) = 0$.

(10.5) $u \in Z$.

Proof. Assume not. By 10.4, $R(u) = P(u) = 0$, so $u \notin T$. By 10.3, $t = c_2$ and

$$X(u) = \begin{bmatrix} 1 & \\ d & 1 \end{bmatrix}.$$

As $R(u) = 0$, the image of gZ under u is determined by $R(u^g) = X(u)R$. Further

$$R = \begin{bmatrix} \beta \\ \gamma \end{bmatrix} \qquad X(u)R = \begin{bmatrix} \beta \\ d\beta + \gamma \end{bmatrix}$$

so u centralizes gZ exactly when $\beta = 0$. However as g ranges over C, β ranges over all possible row vectors of length $n - 4$, so unless $n = 4$, $|u^C| > 2$. Also if $n > 4$ then

$$g = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & E_2 & & & \\ & & & I & & \\ 0 & 1 & & & 1 & \\ 0 & 1 & & & & 1 \end{pmatrix}$$

is in Δ by 8.9, while by 4.5, $Q(u^g) \neq Q(u)$.

So $n = 4$ and hence G is symplectic. Now by 10.3, $G = Sp_4(2)$, against our hypothesis.

(10.6) *Let* $t = j_\ell$ *or* a_ℓ. *Then*
(1) *if* $[u, \Delta] = 1$ *then* $u \in J_\ell$, *and*
(2) *if* $u^C = \{u, ut\}$ *then either*
(i) $G = SL_n(2)$ *or* $Sp_n(2)$, $\ell = 2$, *and* $Q(u)$ *is*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(ii) *$G = SU_n(2)$, $\ell = 2$, and $Q(u)$ is*

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad or \quad \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Proof. u^g is determined by $Q(u^g) = Q(u)^X$. Thus the result follows from 9.2 and 10.2.

(10.7) *Let $t = b_\ell$. Then*
(1) *if $[u, \Delta] = 1$, then*

$$Q(u) = \begin{bmatrix}b & \\ & aI\end{bmatrix}.$$

(2) $u^C \neq \{u, ut\}$.

Proof. By 7.10,

$$Q(u) = \begin{bmatrix}b & \beta\\ \gamma & L\end{bmatrix} \qquad X = \begin{bmatrix}1 & \\ & W\end{bmatrix} \qquad Q(u^g) = \begin{bmatrix}b & \beta^W\\ W^{-1}\gamma & L^W\end{bmatrix}$$

where β and γ are respectively row and column vectors of even length $\ell - 1$. So by 9.2 and 10.2, $\gamma = \beta = 0$ and $L = aI$ if $[u, \Delta] = 1$. Also if $u^C = \{u, ut\}$ then

$$Q(u) + Q(u^g) = \begin{bmatrix}0 & *\\ * & *\end{bmatrix} \neq I.$$

(10.8) *Let $t = c_\ell$. Then*
(1) *if $[u, \Delta] = 1$ then*

$$Q(u) = \begin{bmatrix}r & & \\ & rI & \\ y & & r\end{bmatrix}$$

and if G is orthogonal $y = r(r + 1)$.
(2) *If $u^C = \{u, ut\}$ then $G = Sp_n(2)$, $\ell = 2$, and $Q(u)$ is one of*

$$\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}.$$

Proof. By 7.11

$$Q(u) = \begin{bmatrix}r & \rho & z\\ \mu & L & \eta\\ y & \xi & s\end{bmatrix} \qquad X = \begin{bmatrix}1 & & \\ WE\alpha^* & W & \\ x & \alpha & 1\end{bmatrix}.$$

Choosing $\alpha = x = 0$ we find

$$Q(u^g) = \begin{bmatrix} r & \rho^W & z \\ W^{-1}\mu & L^W & W^{-1}\eta \\ y & \xi W & s \end{bmatrix}$$

so by 9.1 and 10.2, $\mu = \rho = \eta = \xi = 0$ and $L = aI$ is scalar. Choosing $W = I$, it follows from 7.13 that

$$Q(u^g) = \begin{bmatrix} r + zx & z\alpha & z \\ (r + zx + a)E\alpha^* & L + zE\alpha^*\alpha & zE\alpha^* \\ y + (r + s + zx)x & \alpha(a + s + xz) & xz + s \end{bmatrix}.$$

So as $C_C(\Delta) \trianglelefteq C$ we conclude either $\ell = 2$ or $z = 0$ and $a = r = s$. Further if G is orthogonal then by 8.9, $y = r(r + 1)$.

So take $\ell = 2$. Then if G is orthogonal, 8.11 implies $r = s$, $z = 0$, and $y = r(r + 1)$. So take G symplectic. Then for $x \neq 1$

$$g = \begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & I & & \\ & & & 1 & \\ 1 & & & x & 1 \end{pmatrix}$$

is in Δ, so if $[u, \Delta] = 1$ then $z = 0$ and $r = s$. Also if $[u, \Delta] \neq 1$ then as $Q(u) = Q(u^g) + I$ for each such $g, q = 2$ and $Q(u)$ is as claimed.

Section 11. Alpha and beta groups.

In this section we continue the hypothesis stated in the first paragraph of Section 10. In addition we take $q > 2$.

A *primary involution* is defined to be a transvection in the unimodular, unitary, or symplectic group, and an involution of type a_2 in the orthogonal group.

Given an involution $t \in G$ define $\alpha(t)$ to be the set of elements g in G such that $g + I = a(t + I)$ for some scalar a. $\alpha(t)$ is the *α-group* of G containing t. In addition if t is of type c_ℓ or of type b_ℓ let $\beta(t)$ be the group generated by all involutions s in G such that $V(s) = V(t) = U$ and

$$(I + s)|_U = a(I + t)|_U$$
$$I + s \equiv a(I + t) \bmod U^\perp$$

for some scalar a. $\beta(t)$ is the *β-group* of G containing t.

In this section we investigate the α and β-groups and the primary involutions. The next two lemmas relate the α and β-groups to the results in Section 10. For it is easy to check that

(11.1) *Let t be in Suzuki form. Then $\alpha(t) = J_\ell \cap G$.*

(11.2) *Let $t = c_\ell \in \Omega^\epsilon$. Then $z \in \beta(t)$ exactly when $z \in Z$ with*

$$Q(z) = \begin{bmatrix} r & & \\ & rI & \\ r(r+1) & & r \end{bmatrix}.$$

Therefore the results in Section 10 imply $\alpha(t)$, or $\beta(t)$ if t is of type b_ℓ or c_ℓ, is the center of the centralizer of t.

(11.3) *Let t be an involution of rank $\ell > 1$ in SL or SU. Then there exists $s \in t^G \cap C$ such that $\alpha(t)\alpha(s)$ contains one α-group of rank* 1, *one of rank $\ell - 1$, and $q - 1$ α-groups of rank ℓ.*

Proof. Take t in Suzuki form and for scalars a and b define $g(a, b) \in C(t)$ by

$$g(a, b) = \begin{bmatrix} & I_\ell & & & \\ & & & I_{n-2\ell} & \\ a & & & & \\ & b & & & \\ & & I_{\ell-2} & & I_\ell \end{bmatrix}.$$

Let $s = g(1, c)$, $0 \neq c \neq 1$. Then

$$\alpha(g(a, b)) = \{g(da, db)\}$$

$g(1, 0)$ is of rank $1, g(0, 1)$ is of rank $\ell - 1$, and the remaining α-groups have rank ℓ.

The next two lemmas are proved in the same manner.

(11.4) *Let $t = a_\ell, \ell > 2$. Then there exists $s \in t^G \cap C$ such that $\alpha(t)\alpha(s)$ contains one α-group of type a_2, one of type $a_{\ell-2}$, and $q - 1$ of type a_ℓ.*

(11.5) *Let $t = c_\ell, \ell > 2$, or $t = b_\ell, \ell > 1$. Then there exists $s \in t^G \cap C$ such that $\beta(t)\beta(s)$ contains one α-group of type $a_{\ell-2}$, or $a_{\ell-1}$, one β-group of type c_2, or b_1, and $q - 1$ β-groups of type c_ℓ, or b_ℓ, respectively.*

(11.6) *Let $t = a_2$ or $t = b_1$. Then there exists $s \in t^G \cap C$ such that $\alpha(t)$*

and $\alpha(s)$ are the unique α-groups of type t in $\alpha(t)\alpha(s)$ and the remaining $q-1$ α or β-groups are of type c_2.

Proof. This can be proved as above for the symplectic groups. In the case of the orthogonal groups $t = a_2$ and it is easier to recall $\Omega_6^+(q) \cong L_4(q)$, $\Omega_6^-(q) \cong U_4(q)$, and argue directly.

(11.7) *Let t be a primary involution. Then $C_G(t)^\infty = TL$.*

Proof. Assume t is a transvection in Suzuki form. For $x \in T$ and $g \in L$, $R(x)$ is a row vector of length $m = n-2$ and $R(x^g) = R(x)Y(g)$. Thus L acts on the space U of row vectors $R(x)$ in its natural representation. L is irreducible on U. Hence $U = [U, L]$, so $T/Z = [T/Z, L]$ and then $T = [T, L]Z$. So as L is quasisimple it suffices to show $Z \leq [T, L]$.

If G is unimodular or unitary then an easy calculation shows $Z = \Phi(T)$, so $Z \leq [T, L]$. If G is symplectic $Z \leq [T, L]$ follows from § 3 of [7].

Next assume G is orthogonal and $t = a_2$. Arguing as above $T/Z = [T/Z, L]$. As $\Omega_6^+(q) \cong L_4(q)$ and $\Omega_6^-(q) \cong U_4(q)$, with a_2 corresponding to a transvection under the respective isomorphism, $Z \leq [T_0, L_0]$ where $T_0 = T \cap G_0$, $L_0 = L \cap T_0$, and $Z \leq G_0 \leq G$ with $G_0 \cong \Omega_6^\epsilon(q)$.

(11.8) *Let $t_i \in G$ be transvections, $\langle v_i \rangle = [t_i, V]$, $V_i = C_V(t_i)$, and $W = \langle \alpha(t_1), \alpha(t_2) \rangle$. Then*
(1) $[t_1, t_2] = 1$ *if and only if* $v_i \in V_j$ for $i \neq j$.
(2) *$W \cong L_2(q)$ if and only if $v_i \notin V_j$ for $i \neq j$.*
(3) *G is transitive on subgroups isomorphic to $L_2(q)$ containing the α-group of a transvection.*
(4) *If $W \cong L_2(q)$ then $C_G(W)$ is isomorphic to $GL_{n-2}(q), GU_{n-2}(q)$, or $Sp_{n-2}(q)$ for G unimodular, unitary, or symplectic, respectively. Further $WC_G(W)$ is not centralized by an involutory automorphism of G. Unless $G = L_4(q)$ or $U_4(q)$, the same holds for $WE(C_G(W))$.*

Proof. (1), (2), and the first part of (4) follow by calculation choosing suitable bases for V. By [21] the automorphism group of G is known, and the remainder of (4) follows by inspection.

A *flag* is an incident point hyperplane pair. The unimodular group is transitive on pairs of flags $((v_1, V_1), (v_2, V_2))$ with $v_i \notin V_j, i \neq j$. If G

is unitary or symplectic then $V_i = v_1^{\perp}$ and G is transitive on pairs (v_1, v_2) with $v_1 \notin V_2$. Hence (2) implies (3).

(11.9) *Let* $G = \Omega_n^{\epsilon}(q), t = a_2$ *and* $s \in t^G$. *Set* $U = [t, V]$ *and* $W = \langle \alpha(t), \alpha(s) \rangle$. *Then*

(1) $Z = \alpha(t)$.

(2) $s \in \alpha(t)$ *if and only if* $U = [s, V]$.

(3) $[s, t] = 1$ *if and only if* s *stabilizes* U.

(4) $[t, t^s] = 1$ *if* $U^{\perp} \cap U^s \neq 0$.

(5) W *is isomorphic to a Sylow* 2-*group of* $L_3(q)$ *if and only if* $0 \neq U^{\perp} \cap U^s \neq U$.

(6) $W \cong L_2(q)$ *if and only if* $0 = U^{\perp} \cap U^s$ *if and only if* $U + U^s \cong D^2$.

(7) G *is transitive on subgroups isomorphic to* $L_2(q)$ *containing the* α-*group of a primary involution.*

(8) *If* $W \cong L_2(q)$ *then* $C_G(W) \cong L_2(q) \times \Omega_{n-4}^{\epsilon}(q)$, *and* $WC_G(W)$ *is not centralized by an involutory automorphism of* G.

Proof. (1) follows from an easy calculation. Assume $U = [s, V]$. Then $C_V(s) = U^{\perp} = C_V(t)$, so $s \in Z$, and (1) implies (2). (3) follows from (2).

Assume $0 \neq u^s \in U^{\perp} \cap U^s$. Let $v \in U - \langle u \rangle$. $0 = (u^s, U) = (u, U^s)$. $s \in t^G$, so $V(s) = V$. Hence $(v, v^s) = 0$. So $U^s = \langle u^s, v^s \rangle \leq v^{\perp}$. Hence $U^s \leq U^{\perp}$, so $[U^s, t] = 0$, and then $[t^s, t] = 1$. This is (4).

Assume $0 \neq U^{\perp} \cap U^s \neq U$. Let $A = [s, V]$. $U + A$ is contained in a nondegenerate 6-dimensional space B. As $U + A \leq B$, B is W-invariant so $W \leq \Omega(B)$. $\Omega(B) \cong L_4(q)$ or $U_4(q)$ with t corresponding to a transvection under the isomorphism, so W is isomorphic to $L_2(q)$ or a Sylow 2-group of $L_3(q)$. As $[t, t^s] = 1$, it is the latter. This proves half of (5).

If $U^s \cap U^{\perp} = 0$ then $[V, s] \leq Y = U + U^s$ is nondegenerate and W-invariant, so $W \leq \Omega(Y)$. Hence $Y \cong D^2$, $\Omega(Y) \cong L_2(q) \times L_2(q)$ and W is a factor of $\Omega(Y)$. Further $C_G(W) = (C(W) \cap \Omega(Y)) \times \Omega(Y^{\perp}) \cong L_2(q) \times \Omega_{n-4}^{\epsilon}(q)$. This completes (5) and shows that (6)–(8) follow from (3) and (5).

(11.10) *Let* t *be a transvection and* s *an involution in* G *such that* $C_G(t)^{\infty} \cong C_G(s)^{\infty}$. *Then either* s *is a transvection or* $G = Sp_4(q)$, s *is of type* a_2, *and* t^G *is fused to* s^G *by a graph automorphism.*

Proof. Let $X = C(t)^{\infty}$ and $Y = C(s)^{\infty}$. $X/O_2(X) \cong G_{n-2}(q)$ where $G = SL, SU$, or Sp. But the factors of $Y/O_2(Y)$ are $G_{n-2\ell}(q)$ and $G_k(q)$

where ℓ is the rank of s and $k = \ell, \ell - 1$ or $\ell - 2$. We may take $\ell > 1$, so $n - 2 \leq \ell \leq n/2$. Thus $n = 4$ and $k = \ell = 2$. However $O_2(C(j_2))$ is abelian while $O_2(C(j_1))$ is not. Hence $G = Sp_4(q)$ and s is of type a_2

(11.11) *Let* $t = a_2 \in \Omega_n^\varepsilon(q)$ *and* s *an involution in* G *with* $C(s)^\infty \cong C(t)^\infty$, *then* s *is primary.*

Proof. $C^\infty / O_2(C) \cong L_2(q) \times \Omega_{n-4}^\varepsilon(q)$. Further unless $n = 8$ and $\varepsilon = +$, $\Omega_{n-4}^\varepsilon(q)$ is simple and only primary involutions have a factor of this kind. If $n = 8$ and $\varepsilon = +$ then $\Omega_{n-4}^\varepsilon(q) \cong L_2(q) \times L_2(q)$ and only primary involutions have 3 factors.

(11.12) *Let* $t = j_1$ *or* b_1. *Let* h *be an involution in* $C - T$ *and* $g \in C$ *with*

$$h = \begin{bmatrix} 1 & & \\ A & H & \\ B & C & 1 \end{bmatrix} \qquad g = \begin{bmatrix} X & & \\ P & Y & \\ Q & R & X \end{bmatrix}.$$

Then

(1) $H^2 = I, HA = A, CH = C$ *and* $CA = 0$.

(2) *If* $[Y, H] = 1$ *then*

$$h^g = \begin{bmatrix} 1 & & \\ Y^{-1}(AX + P + HP) & H & \\ X^{-1}RY^{-1}(P + HP) + RY^{-1}A + X^{-1}CP + B & X^{-1}(CY + RH + R) & 1 \end{bmatrix}.$$

(3) *Assume* $A = (a_1, \cdots, a_m)^*$, $C = (c_1, \cdots, c_m)$, *and*

$$H = \begin{bmatrix} E_2 & \\ & I \end{bmatrix}.$$

Then $a_1 = a_2$ *and* $c_1 = c_2$.

Proof. Calculation using 4.4 and 4.5.

(11.13) *Assume the hypothesis of* 11.12 *with* h *as in* 11.12.1 *and*

$$e = \begin{bmatrix} 1 & & \\ & H & \\ & & 1 \end{bmatrix}.$$

Assume $C_T(e) \cong C_T(h)$ *and* $C^\infty \cap C(e) \cong C^\infty \cap C(h)$. *Then* $h \in eZ$.

Proof. Let $g \in C_T(h)$. Then by 11.12, $P = HP$ and $R = RH$, so that

$C_T(h) \leq C_T(e)$. Thus as $C_T(e) \cong C_T(h)$, even $C_T(e) = C_T(h)$. Hence 11.12 implies $RA = CP$ whenever $g \in C_T(e)$. This implies $a_i = c_i = 0$ for $i > 2$. By 11.12, $a_1 = a_2$ and $c_1 = c_2$. Next as $C^\infty \cap C(e) \cong C^\infty \cap C(h)$, h centralizes an element g with

$$Y = \begin{bmatrix} & b^{-1} & \\ b & & \\ & & I \end{bmatrix}.$$

By 11.12, $C = CY + RH + R$, which implies $c_1 = 0$. Similarly $a_1 = 0$. Therefore $h \in eZ$.

(11.14) *Let t be a transvection and β an automorphism of C^∞. Let*

$$e = \begin{Bmatrix} 1 & & & \\ & E_2 & & \\ & & I & \\ & & & 1 \end{Bmatrix}.$$

Then e^β is a transvection and $\alpha(e)^\beta = \alpha(e^\beta)$.

Proof. Let $h = e^\beta$. By 11.7, $C^\infty = TL$. $C_{C^\infty/T}(e) \cong C_{C^\infty/T}(h)$, so by 11.10 either hT is a transvection in C/T or $C/T \cong Sp_4(q)$ and hT is of type a_2. However in the latter case $q^3 = |C_T(e)|$ while $q^2 = |C_T(h)|$. Hence conjugating if necessary in C^∞ we may take $h \in eT$.

Then by 11.13, $h \in eZ$. Further the involutions in $\alpha(e)Z = X$ are precisely those with the same centralizer as e (or h) and thus β acts on X. But $\alpha(e) = [X, N(X)]$, so $\alpha(e)^\beta = \alpha(e)$. Hence $h \in \alpha(e)$, so h is a transvection and $\alpha(e)^\beta = \alpha(e) = \alpha(h)$.

(11.15) *Let $t = a_2 \in \Omega_n^\epsilon(q)$. Let β be an automorphism of C^∞ and*

$$e = \begin{bmatrix} E_2 & & \\ & I & \\ & & E_2 \end{bmatrix}.$$

Then e^β is of type a_2 and $\alpha(e)^\beta = \alpha(e^\beta)$.

Proof. The proof is similar to that of 11.14 and is omitted.

Section 12. Involutions in Exceptional Chevalley groups of characteristic 2.

Recall the notation for groups with a (B, N)-pair established in Section 3. Let $G = G(q)$ be a Chevalley group with q even, and $G \neq F_4(q)$, $PSp(n, q)$ $n \geq 4$. Let G have root system Δ. For $G \neq PSU(n, q)$ n odd,

${}^2F_4(q), Sz(q), {}^2G_2(q)$ let r be the root of highest height in Δ and U_r the corresponding root subgroup. In the remaining cases let U_r be the root subgroup of order q^3, q^2, q^2, q^3 respectively, such that $\Omega_1(U_r) = Z(U)$. For $s \in \Delta$ write $Z_s = \Omega_1(U_s)$. In all cases $Z_r \leq Z(U)$ and, unless $G = PSL(n, q)$, $P = N_G(Z_r)$ is a maximal parabolic subgroup of G.

The structure of P is described in [7]. In particular $P = QLH$ where $Q = O_2(P)$, L is a Chevalley group (or a direct product of Chevalley groups in case G is an orthogonal group of dimension ≥ 8), $QL \trianglelefteq P$ and $QL \leq C(Z_r) \trianglelefteq P$.

(12.1) *Two conjugates of Z_r generate a 2-group or a conjugate of $\langle Z_r, Z_{-r}\rangle \cong SL(2, q)$.*

Proof. Let $\Omega = \{Z_r^g : g \in G\}$. Then from 2.8 and 4.2 of [7] we see that $(1_P^G, 1_P^G) = m$ is the number of orbits of P on Ω and

$m = 2$ if G has rank 1

$m = 3$ if G is unitary or symplectic of rank ≥ 2.

$m = 4$ if $G = G_2(q)$ or ${}^3D_4(q)$

$m = 5$ if G is an exceptional group of rank ≥ 3 or if $G \cong {}^2F_4(q)$.

$m = 6$ if $G \cong PSO^-(2n, q), n \geq 4, PSO^+(n, q)\ n > 4$, or $PSL(n, q)\ n \geq 3$.

$m = 1$ if $G \cong PSO^+(8, q)$ (there is an error in [7] for this case).

We will do the case $m = 5$, the other cases being quite similar. Consider the following subsets of Ω.

$$\Omega_1 = \{Z_r\}\,, \qquad \Omega_2 = \{Z_r^g : Z_r \neq Z_r^g \leq Q\}\,, \qquad \Omega_3 = \{Z_r^g : Z_r^g \leq P - Q\}\,,$$
$$\Omega_4 = \{Z_r^g : \langle Z_r, Z_r^g\rangle \text{ is isomorphic to a Sylow 2-subgroup of } L_3(q)\}\,,$$
$$\Omega_5 = \{Z_r^g : \langle Z_r, Z_r^g\rangle \cong \langle Z_r, Z_{-r}\rangle \cong SL(2, q)\}\,.$$

We first note that each of these subsets is P-invariant and it is easy to verify that $\Omega_i \neq \emptyset$ for $i = 1, \cdots, 5$. This requires information concerning the root system Δ and commutator relations, but is elementary. Consequently these are the orbits of P in Ω. If $Z_r^g \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ then $\langle Z_r, Z_r^g\rangle$ is a 2-group. The result follows.

We remark that for $G = PSO^{\pm}(2n, q)\ n \geq 4$ $(G \neq PSO^+(8, q))$, we have $L \cong L_1 \times L_2$ with conjugates of Z_r in each of L_1, L_2. If $G = PSO^+(8, q)$, then $L = L_1 \times L_2 \times L_3$ with each $L_i \cong L_2(q)$. This accounts for the extra orbit. In $PSL(n, q)$ $Q = Q_1Q_n$ with $Q_1 \triangleleft P, Q_n \triangleleft P$. This also gives an extra orbit.

(12.2) *Let t be an involution in G. Then there is a conjugate Z_r^g of Z_r*

such that $\langle Z_r^g, Z_r^{gt}\rangle \sim \langle Z_r, Z_{-r}\rangle$.

Proof. If G has rank 1 this is obvious. For G of rank 2 the classes of involutions are known (see Section 18) and we check this directly (keeping in mind $G \not\cong Sp(4, q)$). So we may assume G is of rank $n \geq 3$ and we proceed by induction on n. Since $n \geq 3$ L contains a conjugate of Z_r. We first conjugate, if necessary, replacing t by an element of U. Suppose $t \notin Q$. Then considering the image of t in $LQ/Q = \bar{L}$ we inductively see that there is a conjugate Z_r^g of Z_r such that $\langle Z_r^g, Z_r^{gt}\rangle$ is not a 2-group. So (12.1) yields the result. Consequently we may assume that $t \in Q$.

If $t \in Z_r = Q'$, then we obtain the result by considering $t \in \langle U_{\pm r}\rangle \cong SL(2, q)$. So assume that $t \in Q - Z_r$ and consider the image $\bar{t}$ of t in $\bar{Q} = Q/Z_r$. Write $Q = \prod_{i=1}^{k} U_{\beta_i}$ when the product is over a certain set of roots in Δ^+ such that $\beta_i = r$ for some i and such that $\bar{Q} = \bar{U}_{\beta_1} \times \cdots \times \bar{U}_{\beta_h}$. Also $P = \langle B, s_1, \cdots, s_{j-1}, s_{j+1}, \cdots, s_n\rangle$ for some j or $G \cong PSL(n+1, q)$ and $P = \langle B, s_2, \cdots, s_{n-1}\rangle$. In the latter case let $j = 1$. We choose notation so that $\bar{U}_{\beta_1} = \bar{U}_{\alpha_j}$ and $\bar{U}_{\beta_2} = \bar{U}_{\alpha_j+\alpha_i}$ where i, j are given as follows

i)	$G \cong PSO^{\pm}(\ell, q)$ ℓ even, $\ell \geq 8$	$j = 2$	$i = 1$
ii)	G classical not in (i)	$j = 1$	$i = 2$
iii)	$G \cong F_4(q), {}^2E_6(q)$	$j = 1$	$i = 2$
iv)	$G \cong E_6(q)$	$j = 2$	$i = 4$
v)	$G \cong E_7(q)$	$j = 1$	$i = 3$
vi)	$G \cong E_8(q)$	$j = 8$	$i = 7$.

Suppose that for some ℓ $\bar{t}$ projects non-trivially to $\bar{U}_{\beta_\ell}$ and $\beta_\ell \sim \alpha_j$. Then from (2.4) of [7] and the choice of i we have an element $w \in W \cap L$ such that $\bar{t}^w$ projects non-trivially to $\bar{U}_{\beta_2} = \bar{U}_{\alpha_j+\alpha_i}$. Using (3.1) (i) we can find an element u of $U_{-\alpha_i} \leq L$ such that $\bar{t}^{wu}$ projects trivially to $\bar{U}_{\beta_1}$ and non-trivially to $\bar{U}_{\beta_2}$. Then conjugation by s_j shows that $t^{wus_j} \in U - Q$, reducing to a previous case.

Finally we have the case where no such ℓ exists. This only occurs when all non-trivial factors in $\bar{t}$ correspond to roots of length different from that of α_j. This can only occur when $G = {}^2E_6(q), PSU(\ell, q)$ ℓ odd, $PSO^-(\ell, q)$. In these cases we check the possible roots γ with $U_\gamma \leq Q$ and argue essentially as above, using (3.1).

(12.3) $$O^{2'}(N_G(\langle Z_r, Z_{-r}\rangle)) = L \times \langle Z_r, Z_{-r}\rangle\ .$$

Proof. Let $g \in N_G(\langle Z_r, Z_{-r}\rangle)$. Then setting $X = \langle Z_r, Z_{-r}\rangle$ there is some $x \in X$ such that $gx \in N(Z_r) = P$. Now $P = QLH$ and $LH \leq N(X)$. So write $gx = qy$ where $q \in Q, y \in LH$. We then have $q \in Q \cap N(X)$ and checking commutators we see that $[Z_{-r}, q]$ is a 2-group contained in U^{w_0}, where w_0 is the word of greatest length in the generators $\{s_1, \cdots, s_n\}$. Consequently $[Z_{-r}, q] \leq U^{w_0} \cap X = Z_{-r}$, and $q \in N(Z_{-r})$. Then $q \in P \cap P^{w_0} = LH$ and so $q = 1$. Consequently $g \in XLH$ and the result follows.

(12.4) *Let $t \in G$ be an involution. Then some conjugate of t lies in $LZ_r(1)$.*

Proof. Since t normalizes $\langle Z_r^g, Z_r^{gt}\rangle$ this follows from (12.2) and (12.3).

We define roots $r = t_0, t_1, \cdots, t_k$ as follows. Let L be the Levi factor of $N(Z_r)$. Then L is a Chevalley group if G is not an orthogonal group, and $L = X \times Y$ with $Y \cong SL(2, q)$ and X orthogonal if G is orthogonal. ($X \cong L_2(q) \times L_2(q)$ if $G \cong PSO^+(8, q)$). In the first case let t_1 be the root of highest height in the root system $\Delta_1 \subset \Delta$ of L. In the second case let t_1 be the unique root in Δ with $U_{t_1} \leq Y$ and let t_2 be the highest root in Δ_1. Now continue the selection of t_i's by considering $N_L(Z_{t_1})$ (or $N_X(Z_{t_2})$ for G orthogonal). Then setting $X_i = \langle Z_{t_i}, Z_{-t_i}\rangle$ we have $X \cong SL(2, q)$.

(12.5) a) $\langle X_i : i = 0, \cdots, k\rangle = X_0 \times \cdots \times X_k$.

b) *Each involution in G is conjugate to one in $X_0 \cdots X_k$.*

c) *If $G = PSL(n, q), PSU(n, q), E_6(q), {}^2F_4(q)$ or ${}^2E_6(q)$, then $N_W(X_0 \cdots X_k)$ induces S_{k+1} on $X_0 \cdots X_k$ and each involution in G is conjugate to y_i for some $i = 0, \cdots, k$, where $y_i = Z_r(1)Z_{t_1}(1) \cdots Z_{t_i}(1)$.*

Proof. a) follows from the construction of the U_{t_i}'s, noting that at each stage X is contained in the Levi factor of $N(Z_{i-1})$. Also b) follows easily from (12.4) and construction. It remains to prove c). From b) it suffices to prove that $N_W(X_0 \cdots X_k)$ induces S_{k+1} on $X_0 \cdots X_k$. But this follows using induction and calculating with roots to see that there is an element $w \in W$ stabilizing $\{X_0 \cdots X_k\}$ and interchanging X_0 and X_1.

From (12.5) we obtain just a few possibilities for the conjugacy classes of involutions. It remains to narrow this list still further and to find representatives more convenient for finding the centralizers. We use the notation and labeling or roots as in Table 1, 2, 3, 4 of Section 3.

We remark that the following results give representatives for the classes of involutions, although we wait until later sections to prove that no further fusion takes place among the involutions listed. We now allow the case $G = F_4(q)$.

(12.6) (Guterman [14]) *Each involution in $F_4(q)$, q even, is conjugate to one of the following*:

i) $t = U_r(1)$
ii) $u = U_s(1)$
iii) $tu = U_r(1)U_s(1)$
iv) $v = U_\alpha(1)U_\beta(1)$.

(12.7) *Each involution in ${}^2E_6(q)$, q even, is conjugate to one of the following*:

i) $t = U_r(1)$
ii) $u = U_s(1)$
iii) $v = U_\alpha(1)U_\beta(1)$.

Proof. Here $P = P_1 = Q_1L_1H = N(Z_r)$, where $L_1 = \langle U_{\pm\alpha_2}, U_{\pm\alpha_3}, U_{\pm\alpha_4}\rangle \cong SU(6, q)$. Then $t_1 = \alpha_2 + 2\alpha_3 + 2\alpha_4 = r_{17}$, $t_2 = \alpha_2 + 2\alpha_3$, and $t_3 = t_k = \alpha_2$. Then by (12.5) c) each involution in ${}^2E_6(q)$ is conjugate to one of

$$U_r(1)\ ,\quad U_r(1)U_{r_{17}}(1)\ ,\quad U_r(1)U_{r_{17}}(1)U_{\alpha_2+2\alpha_3}(1)\ ,\quad U_r(1)U_{r_{17}}(1)U_{\alpha_2+2\alpha_3}(1)U_{\alpha_2}(1)\ .$$

By (12.5) b) there is an element $g \in G$ such that g interchanges $U_r(1)$ and $U_{\alpha_2}(1)$, and g interchanges $U_{r_{17}}(1)$ and $U_{\alpha_2+2\alpha_3}(1)$. Then $(U_r(1)U_{r_{17}}(1))^{gU_{\alpha_3}(1)} = (U_{\alpha_2}(1)U_{\alpha_2+2\alpha_3}(1))^{U_{\alpha_3}(1)} = U_{\alpha_2}(1)U_{\alpha_2+\alpha_3}(1)$ and by (3.1) $(U_r(1)U_{r_{17}}(1))^{gU_{\alpha_3}(1)U_{-\alpha_3}(\delta)} = U_{\alpha_2+\alpha_3}(1) \sim U_s(1)$, where $\delta \in F_{q^2}$ satisfies $\delta + \delta^q = 1$. Set $x = gU_{\alpha_3}(1)U_{-\alpha_3}(1)$. We have $(U_r(1)U_{r_{17}}(1)U_{\alpha_2+2\alpha_3}(1))^x = U_{\alpha_2+\alpha_3}(1)U_{r_{17}}(1)$. Conjugating this last element by $s_1s_2s_3s_4s_2s_3s_2$ we obtain $U_\alpha(1)U_\beta(1)$.

Finally consider the involution $y_3 = U_r(1)U_{r_{17}}(1)U_{\alpha_2+2\alpha_3}(1)U_{\alpha_2}(1)$. Conjugating by $U_{\alpha_3}(1)U_{-\alpha_3}(1)s_2$ we have $y_3 \sim U_r(1)U_{r_{17}}(1)U_{\alpha_3}(1) = g$. Next conjugate g by $U_{r_4}(1)U_{r_1}(1)s_2s_1s_2U_{\alpha_2}(1)$ to get

$$g \sim U_{r_9}(1)U_{r_8}(1)U_{r_5}(1)\ .$$

Then conjugate this by $s_2U_{\alpha_3}(1)$ to obtain

$$g \sim U_{r_9}(1)U_{r_6}(1)U_{r_{14}}(1) \qquad \text{(here use (3.1)(iv))}\ .$$

Next conjugate by $U_{r_{17}}(1)U_{r_3}(1)$ to get

$$g \sim U_{r_9}(1)U_{r_6}(1) .$$

Finally conjugation by $s_3s_4s_2s_3s_2$ gives

$$g \sim U_\alpha(1)U_\beta(1) .$$

This completes the proof of (12.7).

(12.8) *Each involution in $E_6(q)$, q even, is conjugate to one of the following*:

i) $x = U_r(1)$

ii) $y = U_\alpha(1)U_\beta(1)$

iii) $z = U_\gamma(1)U_\delta(1)U_\epsilon(1)$.

Proof. We proceed as in (12.7). Here $P = P_2$ and $L = \langle U_{\pm\alpha_i} : i \neq 2\rangle \cong SL(6, q)$. The roots $t_0, \cdots, t_k$ are $t_0 = r, t_1 = r_8, t_2 = r_{29}, t_3 = r_{31}$. Consequently each involution in G is conjugate to one of

$$U_r(1) ,\quad U_r(1)U_{r_8}(1) ,\quad U_r(1)U_{r_8}(1)U_{r_{29}}(1) ,\quad U_r(1)U_{r_8}(1)U_{r_{29}}(1)U_{r_{31}}(1) .$$

Conjugating $U_r(1)U_{r_8}(1)$ by $s_2s_4s_3$ we obtain $U_\alpha(1)U_\beta(1)$ and conjugating $U_r(1)U_{r_8}(1)U_{r_{29}}(1)$ by $s_2s_4s_3s_6s_5$ we obtain $U_\gamma(1)U_\delta(1)U_\epsilon(1)$. It remains to consider

$$g = U_r(1)U_{r_8}(1)U_{r_{29}}(1)U_{r_{31}}(1) .$$

Conjugating g by $s_2s_3s_1s_6s_4s_3s_5$ we obtain

$$g \sim U_{r_6}(1)U_{r_{23}}(1)U_{r_{24}}(1)U_{r_{10}}(1) .$$

Conjugating by $U_{\alpha_1}(1)U_{\alpha_6}(1)$ we have

$$\begin{aligned} g &\sim U_{r_6}(1)U_{r_7}(1)U_{r_{23}}(1)U_{r_7}(1)U_{r_{24}}(1)U_{r_9}(1)U_{r_{25}}(1)U_{r_{10}}(1)U_{r_{10}}(1) \\ &= U_{r_6}(1)U_{r_{23}}(1)U_{r_9}(1)U_{r_{25}}(1)U_{r_{24}}(1) . \end{aligned}$$

Next conjugate by $U_{\alpha_4}(1)$ to obtain

$$\begin{aligned} g &\sim U_{r_6}(1)U_{r_9}(1)U_{r_{23}}(1)U_{r_{25}}(1)U_{r_9}(1)U_{r_{28}}(1)U_{r_{24}}(1) \\ &= U_{r_6}(1)U_{r_{23}}(1)U_{r_9}(1)U_r(1)U_{r_{25}}(1)U_{r_9}(1)U_{r_{25}}(1)U_{r_{24}}(1) \\ &= U_{r_6}(1)U_{r_{23}}(1)U_r(1)U_{r_{24}}(1) . \end{aligned}$$

Then conjugation by $U_{r_{25}}(1)$ we have

$$g \sim U_{r_6}(1)U_{r_{23}}(1)U_{r_{24}}(1) .$$

Finally conjugation by $s_1s_4s_6s_5s_3s_4$ shows that

$$g \sim U_\gamma(1)U_\delta(1)U_\varepsilon(1) .$$

This completes this proof of (12.8).

(12.9) *Each involution in $E_7(q)$, q even, is conjugate to one of the following:*

i) $x = U_r(1)$
ii) $y = U_\alpha(1)U_\beta(1)$
iii) $z = U_\gamma(1)U_\delta(1)U_\varepsilon(1)$
iv) $u = U_\varphi(1)U_\psi(1)U_\theta(1)$
v) $v = U_r(1)U_\theta(1)U_\psi(1)U_\varphi(1)$.

In the proof of (12.9) we will use (12.4) as usual, and consequently we need the involutions in $L = L_1 \cong SO^+(12, q)$. We have

(12.10) *Let $G = E_7(q)$ and $L_1 = \langle U_{\pm\alpha_i} : i \neq 1\rangle$. Then each involution in L_1 is conjugate to one of the following:*

i) $U_{r_{15}}(1)$
ii) $U_{r_{10}}(1)U_{r_{37}}(1)$
iii) $U_{r_5}(1)U_{r_7}(1)$
iv) $U_{r_{36}}(1)U_{r_4}(1)U_{r_8}(1)$
v) $U_{r_{36}}(1)U_{\alpha_{33}}(1)U_{r_5}(1)$
vi) $U_{r_{36}}(1)U_{r_{33}}(1)U_{r_7}(1)$
vii) $U_{\alpha_3}(1)U_{r_{35}}(1)U_{r_{15}}(1)U_{\alpha_5}(1)$.

Proof. L_1 has root diagram of type D_6 with fundamental system $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. The highest root is $U_{r_{15}} = Z(U \cap L_1)$ and $N_{L_1}(U_{r_{15}}) = Q_0L_0H_0$, where $Q_0 = O_2(Q_0L_0H_0), H_0 = H \cap L_1$, and $L_0 = \langle U_{\pm\alpha_i} : i \neq 1, 6\rangle = \langle U_{\pm\alpha_i} : i = 2, 3, 4, 5\rangle \times \langle U_{\pm\alpha_7}\rangle \cong SO^+(8, q) \times SL(2, q)$. Consequently each involution in L_1 is conjugate to an involution in $L_0 \times \langle U_{\pm r_{15}}\rangle$. We first find the involutions in $\langle U_{\pm\alpha_i} : i = 2, 3, 4, 5\rangle \cong SO^+(8, q)$. For this we again use (12.4) noting that the highest root is $U_{r_{35}}$ and the corresponding Levi factor is $\langle U_{\pm\alpha_2}\rangle \times \langle U_{\pm\alpha_3}\rangle \times \langle U_{\pm\alpha_5}\rangle$. Conjugating by elements of $\langle s_2, \cdots, s_5\rangle$ we obtain the fact that each involution in $\langle U_{\pm\alpha_i} : i = 2, 3, 4, 5\rangle$ is conjugate to one of

$$U_{r_{35}}(1)\ ,\quad U_{\alpha_3}(1)U_{r_{35}}(1)\ ,\quad U_{\alpha_5}(1)U_{r_{35}}(1)\ ,$$
$$U_{\alpha_2}(1)U_{r_{35}}(1)\ ,\quad U_{\alpha_3}(1)U_{\alpha_5}(1)U_{r_{35}}(1)\ .$$

(For this use the fact that $(U_{\alpha_2}(1)U_{\alpha_3}(1)U_{\alpha_5}(1)U_{r_{35}}(1))^g = U_{\alpha_2}(1)U_{\alpha_3}(1)U_{\alpha_5}(1)$, where $g = U_{\alpha_4+\alpha_5}(1)U_{\alpha_3+\alpha_4}(1)U_{\alpha_2+\alpha_4}(1)U_{\alpha_3+\alpha_4+\alpha_5}(1)$.) Consequently each involution in L_1 is conjugate to one of

$$\begin{aligned}&U_{r_{35}}(1)\,,\quad U_{\alpha_3}(1)U_{r_{35}}(1)\,,\quad U_{\alpha_5}(1)U_{r_{35}}(1)\,,\quad U_{\alpha_2}(1)U_{r_{35}}(1)\,,\\&U_{\alpha_3}(1)U_{\alpha_5}(1)U_{r_{35}}(1)\,,\quad U_{r_{35}}(1)U_{r_{15}}(1)\,,\quad U_{\alpha_3}(1)U_{r_{35}}(1)U_{r_{15}}(1)\,,\\&U_{\alpha_5}(1)U_{r_{35}}(1)U_{r_{15}}(1)\,,\quad U_{\alpha_2}(1)U_{r_{35}}(1)U_{r_{15}}(1)\,,\\&U_{\alpha_3}(1)U_{\alpha_5}(1)U_{r_{35}}(1)U_{r_{15}}(1)\,.\end{aligned}$$

We easily have $U_{r_{35}}(1) \sim U_{r_{15}}(1)$. Conjugating $U_{\alpha_3}(1)U_{r_{35}}(1)$ by $s_6s_5s_4s_2s_7s_6s_5s_4s_6s_7$, $U_{\alpha_2}(1)U_{r_{35}}(1)$ by $s_6s_7s_5s_4s_3s_6s_5s_4s_6s_5$, and $U_{r_{35}}(1)U_{r_{15}}(1)$ by s_6s_5 we see that each of these elements is conjugate to $U_{r_{10}}(1)U_{r_{37}}(1)$. Conjugating $U_{\alpha_5}(1)U_{r_{35}}(1)$ by $s_6s_7s_4s_3$ we have this element conjugate to $U_{r_5}(1)U_{r_7}(1)$. Next conjugate $U_{\alpha_3}(1)U_{\alpha_5}(1)U_{r_{35}}(1)$ by $s_6s_5s_4s_2s_7s_5s_6s_4$ and obtain $U_{r_{36}}(1)U_{r_4}(1)U_{r_8}(1)$. Conjugate $U_{\alpha_3}(1)U_{r_{35}}(1)U_{r_{15}}(1)$ by $s_6s_5s_4s_2s_5s_4s_6$ and get this element conjugate to $U_{r_{36}}(1)U_{r_{33}}(1)U_{r_5}(1)$. Next conjugate $U_{\alpha_5}(1)U_{r_{35}}(1)U_{r_{15}}(1)$ by $s_6s_7s_4s_5$ and get $U_{r_{36}}(1)U_{r_4}(1)U_{r_8}(1)$, and conjugate $U_{\alpha_2}U_{r_{35}}U_{r_{15}}$ by $s_4s_6s_5s_7s_4s_3s_7$ getting $U_{r_{36}}(1)U_{r_7}(1)U_{r_{33}}(1)$.

We now prove (12.9). Using (12.4) we have each involution in $G = E_7(q)$ conjugate to one in $L_1U_r(1)$. Consequently each involution in G is conjugate to gr for g one of the involutions in (12.10). We must show that each of these is conjugate to one of x, y, z, u, v.

Conjugating $U_{r_{15}}(1)U_r(1)$ by $s_1s_3s_4s_2$ and $U_{r_{10}}(1)U_{r_{37}}(1)U_r(1)$ by $s_1s_3s_4s_5s_2s_6$, we see that $U_{r_{15}}(1)U_r(1) \sim y$ and $U_{r_{10}}(1)U_{r_{37}}(1)U_r(1) \sim z$. Next conjugate $U_{r_5}(1)U_{r_7}(1)U_r(1)$ by $s_1s_3s_4s_2s_5s_4s_6s_5$ to obtain u. Transforming $U_{r_{36}}(1)U_{r_4}(1)U_{r_8}(1)U_r(1)$ and $U_{r_{36}}(1)U_{r_{33}}(1)U_{r_7}(1)U_r(1)$ by $s_1s_3s_5s_4s_2s_5s_4s_5s_6s_3$ and $s_1s_3s_4s_5s_6s_7s_3$, respectively, we check that both of these involutions are conjugate to v.

We claim that $g = U_{r_{36}}(1)U_{r_{33}}(1)U_{r_5}(1)U_r(1)$ is conjugate to z. To see this first conjugate by $s_1s_3s_4s_5s_2s_6s_5s_4$ to get

$$g \sim U_{r_{45}}(1)U_{r_{44}}(1)U_{r_{11}}(1)U_{r_{16}}(1)\,.$$

Conjugate this by $U_{\alpha_7}(1)U_{\alpha_5}(1)$ and obtain

$$g \sim U_{r_{45}}(1)U_{r_{46}}(1)U_{r_{14}}(1)U_{r_{44}}(1)U_{r_{11}}(1)\,.$$

Then conjugate by $U_{\alpha_3}(1)$ to obtain

$$\begin{aligned}g &\sim U_{r_{45}}(1)U_{r_{46}}(1)U_{r_{14}}(1)U_{r_{44}}(1)U_{r_{46}}(1)U_{r_{11}}(1)U_{r_{14}}(1)\\&= U_{r_{45}}(1)U_{r_{46}}(1)U_{r_{44}}(1)U_{r_{14}}(1)U_r(1)U_{r_{46}}(1)U_{r_{11}}(1)U_{r_{14}}(1)\\&= U_{r_{45}}(1)U_{r_{44}}(1)U_r(1)U_{r_{11}}(1)\,.\end{aligned}$$

Finally conjugating by $U_{r_{46}}(1)s_7s_3s_5s_4s_2s_6s_5s_4s_3$ gives $g \sim z$.

The last case is $g = U_{\alpha_3}(1)U_{r_{35}}(1)U_{r_{15}}(1)U_{\alpha_5}(1)U_r(1)$. Conjugate g by the element $s_1s_6s_4s_2s_3s_4s_5s_4s_2s_3s_7s_6s_7s_4s_5s_3s_4s_2$ above to get

$$g \sim vU_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6}(1) .$$

Then conjugating by $s_2s_4U_{\alpha_7}(1)$ we have

$$g \sim U_{r_{42}}(1)U_{r_{11}}(1)U_{r_{13}}(1)U_{r_{14}}(1)U_{r_{15}}(1)U_{r_{46}}(1)U_{r_{16}}(1) .$$

Conjugation by $U_{\alpha_5}(1)$ then gives

$$\begin{aligned} g &\sim U_{r_{42}}(1)U_{r_{44}}(1)U_{r_{11}}(1)U_{r_{13}}(1)U_{r_{13}}(1)U_{r_{14}}(1)U_{r_{16}}(1)U_{r_{15}}(1)U_{r_{46}}(1)U_{r_{16}}(1) \\ &= U_{r_{42}}(1)U_{r_{44}}(1)U_{r_{11}}(1)U_{r_{14}}(1)U_{r_{15}}(1)U_{r_{46}}(1) . \end{aligned}$$

Next conjugate by $U_{\alpha_3}(1)$ and get

$$\begin{aligned} g &\sim U_{r_{42}}(1)U_{r_{45}}(1)U_{r_{44}}(1)U_{r_{46}}(1)U_{r_{11}}(1)_{r_{14}}(1)U_{r_{14}}(1)U_{r_{15}}(1)U_{r_{46}}(1) \\ &= (U_{r_{42}}(1)U_{r_{45}}(1)U_{r_{44}}(1)U_{r_{11}}(1)U_{r_{15}}(1))^{U_{r_{46}}(1)} . \end{aligned}$$

Consequently $g \sim U_{r_{42}}(1)U_{r_{45}}(1)U_{r_{44}}(1)U_{r_{11}}(1)U_{r_{15}}(1)$ and conjugation by $U_{-\alpha_7}(1)$ and then by $s_5s_7s_3s_4s_2$ gives $g \sim v$. This completes the proof of (12.9).

(12.11) *Each involution in $E_8(q)$, q even, is conjugate to one of the following*:

i) $x = U_r(1)$

ii) $y = U_\alpha(1)U_\beta(1)$

iii) $z = U_\gamma(1)U_\delta(1)U_\varepsilon(1)$

iv) $u = U_\varphi(1)U_\psi(1)U_\theta(1)U_\omega(1)$.

Proof. For $G = E_8(q)$ we have $P = P_7$ and $L = L_7 \cong E_7(q)$. By (12.4) each involution in G is conjugate to one in $L_7U_r(1)$, so each involution in G is conjugate to one of

$$\begin{aligned} &U_r(1) , \quad U_{s_{27}}(1)U_r(1) , \quad U_{s_{21}}(1)U_{s_{23}}(1)U_r(1) , \\ &U_{s_{48}}(1)U_{s_{18}}(1)U_{s_{19}}(1)U_r(1) , \quad U_{s_{13}}(1)U_{s_{14}}(1)U_{s_{15}}(1)U_r(1) , \\ &U_{s_{48}}(1)U_{s_{13}}(1)U_{s_{14}}(1)U_{s_{15}}(1)U_r(1) , \end{aligned}$$

where our notation is combining that of Tables 3 and 4. We will write U_{s_i} rather than U_{r_i} to indicate we are using Table 3 for E_7. For example $s_{28} = \alpha_2$.

Conjugating $U_{s_{27}}(1)U_r(1)$ by $s_8s_7s_6s_5s_4s_2$ we see that this involution is conjugate to y. Then conjugate $U_{s_{21}}(1)U_{s_{23}}(1)U_r(1)$ by $s_8s_7s_6s_5s_4s_2s_3s_1s_4s_5$, obtaining z.

Next we claim that $g = U_{s_{13}}(1)U_{s_{14}}(1)U_{s_{15}}(1)U_r(1) \sim z$. To see this first conjugate by $s_8s_7s_6s_5s_4s_3s_1s_2s_4s_5s_6s_3s_4$ getting

$$g \sim U_{r_{22}}(1)U_{r_{34}}(1)U_{r_{24}}(1)U_{r_{37}}(1) .$$

Conjugate this element $U_{\alpha_7}(1)U_{\alpha_2}(1)$ to get

$$g \sim U_{r_{22}}(1)U_{r_{25}}(1)U_{r_{36}}(1)U_{r_{34}}(1)U_{r_{24}}(1) .$$

Then conjugate by $U_{\alpha_5}(1)$ and obtain

$$\begin{aligned} g &\sim U_{r_{22}}(1)U_{r_{25}}(1)U_{r_{34}}(1)U_{r_{24}}(1)U_{r_{25}}(1) \\ &= U_{r_{22}}(1)U_{r_{34}}(1)U_r(1)U_{r_{24}}(1) . \end{aligned}$$

Now conjugate by $U_{r_{35}}(1)$ to get

$$g \sim U_{r_{22}}(1)U_{r_{34}}(1)U_{r_{24}}(1) ,$$

and then by $s_7s_2s_5s_6s_4s_3s_1s_5s_4s_3s_2s_4s_5s_6s_7$ to get the claim.

The last case is $U_{s_{48}}(1)U_{s_{13}}(1)U_{s_{14}}(1)U_{s_{15}}(1)U_r(1) = gU_{s_{48}}(1)$ where g is as in the preceding paragraph. The series of conjugations leading to $g \sim z$ when applied to $gU_{s_{48}}(1)$ yields

$$gU_{s_{48}}(1) \sim zU_{s_{48}}(1) .$$

From here conjugate by $s_7s_6s_5s_4s_3s_1$ and get $gU_{s_{48}}(1) \sim u$. This completes the proof of (12.10).

Section 13. Centralizers of involutions in $F_4(q)$, q even.

Let $G = F_4(q)$ with q a power of 2. Then $W = \langle s_1, s_2, s_3, s_4 \rangle$ and the action on $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is as follows

$$\begin{aligned} (\alpha_j)s_i &= \alpha_j && \text{if } |i-j| \geq 2 . \\ (\alpha_i)s_i &= -\alpha_i . \\ (\alpha_j)s_i &= \alpha_j + \alpha_i && \text{if } |i-j| = 1 \text{ and } (i,j) \neq (3,2) . \\ (\alpha_2)s_3 &= \alpha_2 + 2\alpha_3 . \end{aligned}$$

We will distinguish the roots $r = 2342, s = 1232, \alpha = 1221$, and $\beta = 1242$. Then r and β are long roots (hence conjugate under W) and s and α are short roots (and conjugate under W). Also r is the root of highest height.

In [14] Guterman determined the four conjugacy classes of involutions in $F_4(q)$ and their centralizers. So we simply use his results together with some necessary additional information concerning structural properties of the centralizers. From (12.6) we have

(13.1) *$F_4(q)$ has four conjugacy classes of involutions with representatives given by*

i) $t = U_r(1)$

ii) $u = U_s(1)$

iii) $tu = U_r(1)U_s(1)$

iv) $v = U_\alpha(1)U_\beta(1)$.

We next discuss the centralizers of these involutions. $C_G(t) = O^{2'}(P_1) = Q_1\langle U_{\pm\alpha_i}: i = 2, 3, 4\rangle = Q_1L_1$, where $L_1 \cong Sp(6, q)$. Moreover $Z(C_G(t)) = U_r = Q_1'$, $Q_1 \leq C_G(t)'$, and P_1 contains a subgroup of order $q - 1$ transitive on $U_r^\#$. (The structure of P_1 is discussed in [7], Section 4.)

The graph automorphism interchanges r and s, fusing t and u. Consequently $C_G(u) = O^{2'}(P_4) = Q_4\langle U_{\pm\alpha_i}: i = 1, 2, 3\rangle = Q_4L_4$, where $L_4 \cong Sp(6, q)$. Also $Z(C_G(U)) = U_s = Q_4'$, $Q_4 \leq C_G(u)'$, and P_4 contains a subgroup of order $q - 1$ transitive on $U_s^\#$.

Next consider tu, with centralizer $C_G(tu) = O^{2'}(P_{1,4}) = Q_1Q_4\langle U_{\pm\alpha_2}, U_{\pm\alpha_3}\rangle = Q_1Q_4L_{14}$, where $L_{14} \cong Sp(4, q)$. Using Table 1 together with (3.1) we show that $(Q_1Q_4)' = U_{r_{10}}U_{r_{11}}U_{r_{12}}U_{r_{13}} \times U_rU_s$ and $(Q_1Q_4)' = Z(Q_1Q_4)$.

Now (3.1) shows that $\langle U_{\pm\alpha_3}\rangle = \langle U_{\alpha_3}, s_3\rangle \cong SL(2, q)$ acts on $U_{r_{10}}U_{r_{11}}$ as it does on its natural 2-dimensional module over F_q. Similarly $\langle U_{\pm\alpha_2}\rangle \cong SL(2, q)$ acts irreducibly on $U_{r_{12}}U_{r_{13}}$. Consequently $Z(C_G(tu)) \cap (Q_1Q_4)' = U_rU_s$. Consider the elementary abelian group $Q_1Q_4/(Q_1Q_4)'$, which is acted on by $L_{1,4}$. Using the action of the groups $\langle U_{\pm\alpha_2}\rangle \cong SL(2, q) \cong \langle U_{\pm\alpha_3}\rangle$ we see that in this action the trivial space for $L_{1,4}$ is the image of $\langle U_{r_9}, U_{r_{17}}\rangle$. Moreover $L_{1,4}$ does centralize $\langle U_{r_9}, U_{r_{17}}\rangle$. However $[Q_1Q_4, L_{14}] = Q_1Q_4$.

We make one final observation concerning $C_G(tu)$. Suppose P is parabolic and contains $C_G(tu)$. Then considering $P/O_2(P)$ we see that $P \sim P_1, P_4$, or P_{14}. On the other hand $U \leq C_G(t) \leq P$, so Lemma 1.6 of [19] implies that $P = P_1, P_4$ or P_{14}. So $C_G(tu)$ is contained in only the parabolic subgroups P_1, P_4, P_{14}.

Next we consider $C_G(v)$. Guterman proves that $C_G(v) = \langle U_{\pm\alpha_1}, U_{\pm\alpha_4}, U_r: r > 0, r \neq \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_3 + \alpha_4\rangle$. Then $C_G(v) \leq P_{2,3}$, covers $O^{2'}(P_{2,3}/O_2(P_{2,3}))$, and setting $U_0 = C_G(v) \cap O_2(P_{2,3})$, we have $U_0 = O_2(C_G(v))$, $|O_2(P_{2,3}): U_0| = q^4$, and $C_G(v) = U_0(\langle U_{\pm\alpha_1}\rangle \times \langle U_{\pm\alpha_4}\rangle)$.

We will need information concerning the structure of U_0 and the action of $L_{2,3} = \langle U_{\pm\alpha_1}\rangle \times \langle U_{\pm\alpha_4}\rangle$ on U_0. First we note that $U_0 = \langle U_r : r \in \mathscr{L}^2 \cup \mathscr{L}^3, r \neq \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_3 + \alpha_4\rangle$. Next we use (3.1) and Table 1 to determine U_0' and $Z(U_0)$. For example $[U_{r_{16}}, U_{r_6}] = U_{r_8}, [U_{r_{20}}, U_{r_{11}}] = U_{r_{14}}, [U_{r_3}, U_{r_6}] = U_{r_{24}}, [U_{r_{11}}, U_{r_{19}}] = U_{r_{13}}, [U_{r_5}, U_{r_{16}}] = U_{r_7} = U_\beta, [U_{r_9}, U_{r_{20}}] = U_{r_{12}} = U_\alpha$. So each of $U_\alpha, U_\beta, U_{r_8}, U_{r_{13}}, U_{r_{14}}$, and $U_{r_{24}}$ are in U_0'. Moreover we see that each of these is in $Z(U_0)$ It is then easy to check that $U_0' = Z(U_0) = U_\alpha U_\beta \times U_{r_8}U_{r_{13}}U_{r_{14}}U_{r_{24}}$ and that $[U_0, L_{2,3}] = U_0$. Also (3.1)(iii) shows that $U_\alpha U_\beta \leq Z(C_G(v))$, whereas (3.1)(ii) shows that $\langle U_{\pm\alpha_1}\rangle$ acts irreducibly on $U_{r_8} \times U_{r_{24}}$ (which affords the natural module for $SL(2, q)$), and that $\langle U_{\pm\alpha_4}\rangle$ acts irreducibly on $U_{r_{13}} \times U_{r_{14}}$.

Assume $q > 2$. Then using (3.1) and Table 1 we have $[U_{\alpha_1}, U_{r_{16}}] = U_{r_3}$, so U_{r_3} and $U_{r_3}^{s_4} = U_{r_5}$ are in $C_G(v)'$. Similarly $[U_{\alpha_4}, U_{r_4}] = U_{r_{12}}U_{r_6}$, so $U_{r_{12}}, U_{r_6}, U_{r_6}^{s_4} = U_{r_4}$ are all in $C_G(v)'$. Continuing we see that $U_r \leq C_G(v)'$ for each root $r \in \Delta^+$ satisfying $U_r \leq U_0$. In particular $C_G(v) = O^2(C_G(v)) = C_G(v)'$.

Continue the assumption that $q > 2$. It is easy to see that there is a subgroup $H_0 \leq H$ such that $H_0 = H_1 \times H_2, H_1 \cong H_2$ is cyclic of order $q - 1, H_1$ centralizes U_α and is fixed-point free on U_β, while H_2 centralizes U_β and is fixed-point-free on U_α. For example this can be seen by observing that $\langle U_{\pm\alpha}, U_{\pm\beta}\rangle = \langle U_{\pm\alpha}\rangle \times \langle U_{\pm\beta}\rangle \equiv SL(2, q) \times SL(2, q)$.

We claim that the set $U_\alpha^\# U_\beta^\#$ is uniquely determined by the abstract structure of the group $C_G(v)$. Recall that $Z(U_0) = U_\alpha U_\beta \times Y$ with $Y \trianglelefteq C_G(v)$ and $U_\alpha U_\beta = Z(C_G(v))$. So $U_\alpha U_\beta$ is uniquely determined. Consider $\overline{U}_0 = U_0/Y$ and let bars denote images. Using (3.1) we have $\overline{U}_0 = B_1 \times B_2 \times B_3$, where B_3 is elementary and B_1, B_2 are both special of order q^5 with respective centers $\overline{U}_\alpha$ and $\overline{U}_\beta$. In fact for $i = 1, 2$ $B_i = \langle X_i, Y_i\rangle$, where X_i and Y_i are elementary of order q^2 and for each $x_i \in X_i^\#, |C_{Y_i}(x_i)| = q$. It is then easy to describe the elements of order 4 in $\overline{U}_0$ and to compute their squares. We conclude that the elements in $\overline{U_\alpha^\# U_\beta^\#}$ each have the same number of square roots, and this number is different from the number of square roots of an element in $\overline{U_\alpha^\# \cup U_\beta^\#}$. This proves the claim.

Finally we claim that the only maximal parabolic subgroups of G containing $C_G(v)$ are $P_3, P_4, P_1^{s_1s_2}$, and $P_4^{s_4s_3}$. We first check that $C_G(v) \leq P_1^{s_1s_2}$ and $P_4^{s_4s_3}$. This is easily done by showing that $U_0 \leq Q_1^{s_1s_2} \cap Q_1^{s_4s_3}$ and noting that $L_{2,3} \leq L_1^{s_1s_2} \cap L_4^{s_4s_3}$.

Suppose $C_G(v) \le P_1^g$ some g. First assume that $U_0 \cap O_2(P_1^g)' = 1$. Then $U_0 \cap O_2(P_1^g)$ is abelian. The results in [7] Section 4 show that $|U_0 \cap O_2(P_1^g)| \le q^{10}$, so that $|U_0O_2(P_1^g)/O_2(P_1^g)| \ge q^8$. On the other hand $L_1^g \cong Sp(6, q)$ and $C_G(v)O_2(P_1^g)/O_2(P_1^g)$ is contained in a proper parabolic subgroup of $P_1^g/O_2(P_1^g)$ that has a section isomorphic to $\langle U_{\pm\alpha_1}\rangle \times \langle U_{\pm\alpha_4}\rangle = SL(2, q) \times SL(2, q)$. This is impossible. Consequently $U_0 \cap O_2(P_1^g)' \ne 1$ and $U_r^g \cap U_0 \ne 1$. Now $C_G(v) \le O^{2'}(P_1^g) = C(U_r^g)$, so $U_r^g \cap U_0 \le Z(C_G(v))$ and consequently $U_r^g = U_\beta$. As $r = \beta^{s_2s_1}$, we have $g \in P_1s_1s_2$ and $P_1^g = P_1^{s_1s_2}$. Similarly (by applying a graph automorphism) if $C_G(v) \le P_4^g$, then $P_4^g = P_4^{s_4s_3}$.

Next suppose $C_G(v) \le P_2^g$. We consider two cases. First assume that $U_0 \le O_2(P_2^g)$. From Table 1 and (3.1) we see that Q_2 has class 3 with $[Q_2, Q_2, Q_2] = U_{r_8}U_{r_{24}}$ and in P_2, $\langle U_{\pm\alpha_1}\rangle$ acts irreducibly on $[Q_2, Q_2, Q_2]$. Now $L_2 \cong SL(2, q) \times SL(3, q)$. So $C_G(v)$ must cover $\langle U_{\pm\alpha_1}\rangle^g$ if $q > 2$ and $O_3(\langle U_{\pm\alpha_1}\rangle^g)$ if $q = 2$. In either case $C_G(v)$ acts irreducibly on $[Q_2, Q_2, Q_2]^g$. Since $|U_0| = q^{18}$ and $|Q_2^g| = q^{20}$, we must have $[Q_2, Q_2, Q_2] \le U_0$ and hence in $Z(U_0)$ and normal in $C_G(v)$. It then follows that $[Q_2, Q_2, Q_2]^g = U_{r_8}U_{r_{24}}$, $g \in N(U_{r_8}U_{r_{24}}) = P_2$. Now assume that $U_0 \not\le O_2(P_2^g)$, so that $C_G(v)Q_2^g/Q_2^g$ is contained in a parabolic subgroup of P_2^g/Q_2^g. This group has the form $P_2^g \cap P_i^{gx}$ for $x \in P_2^g$. As $SL(2, q) \times SL(2, q)$ is involved in $C_G(v)$, $i = 3$ or 4. Now $gx = p_2g$ for some $p_2 \in P_2$, so $(P_2^g \cap P_i^{gx}) = (P_2 \cap P_i)^{p_2g}$. If $i = 3$, then since $P_2 \cap P_3 = N_G((O_2(P_2 \cap P_3))') = N_G(U_0')$ and $U_0 \le O_2(P_2^g \cap P_3^{gx})$, we have $P_2 \cap P_3 = (P_2 \cap P_3)^{p_2g}$, $p_2g \in P_2 \cap P_3$, $g \in P_2$, as desired. If $i = 4$, then $C_G(v) \le (P_2 \cap P_4)^{p_2g} \le P_4^{s_4s_3}$. But then $(P_2 \cap P_4)^{p_2g}$ is a parabolic subgroup of $P_4^{s_4s_3}$ and it is easy to see that this must be $(P_2 \cap P_4)^{s_4s_3}$. Consequently $(P_2 \cap P_4)^{p_2g} = (P_2 \cap P_4)^{s_4s_3}$ and $p_2gs_3s_4 \in P_2 \cap P_4$. Again $g \in P_2$. So in all cases $P_2^g = P_2$. Similarly we show that $C_G(v) \le P_3^g$ implies $P_3^g = P_3$. This proves the claim.

We have now proved

(13.2) *The maximal parabolic subgroups of G containing the centralizers of the involutions in* (13.1) *are as follows:*

i) $C_G(t) \le P_1$

ii) $C_G(u) \le P_4$

iii) $C_G(tu) \le P_1, P_4$

iv) $C_G(v) \le P_2, P_3, P_1^{s_1s_2}, P_4^{s_4s_3}$.

(13.3) *With notation as in* (13.1) *we have*

i) $C_G(t) = (C_G(t))' = O^{2'}(P_1)$. $Z(C_G(t)) = Q_1' = U_r$.
ii) $C_G(u) = (C_G(u))' = O^{2'}(P_4)$. $Z(C_G(u)) = Q_4' = U_s$.
iii) $C_G(tu) = O^{2'}(P_{14}) = Q_1Q_4L_{1,4}$ *and* $Q_1Q_4 = [Q_1Q_4, L_{1,4}]$.

$$Z(Q_1Q_4) = (Q_1Q_4)' \quad and \quad Z(C_G(tu)) = U_rU_s\ .$$

iv) $C_G(v) = \langle U_{\pm\alpha_1}, U_{\pm\alpha_4}, U_r : r > 0, r \neq \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_3 + \alpha_4\rangle = U_0L_{2,3}$, *where* $U_0 = O_2(C_G(v)) = C_G(v) \cap Q_2Q_3$ *and* $L_{2,3} \cong SL(2, q) \times SL(2, q)$. $U_0' = Z(U_0)$, $U_\alpha U_\beta = Z(C_G(v))$, $U_0 = [U_0, L_{2,3}]$, *and for* $q > 2$ $C_G(v) = C_G(v)' = O^2(C_G(v))$.

(13.4) *For* $q > 2$ *there is a subgroup* $H_0 = H_1 \times H_2 \leq H$ *such that* H_1, H_2 *are cyclic of order* $q - 1$, H_1 *centralizes* U_α *and is fixed-point-free on* U_β, *and* H_2 *centralizes* U_β *and is fixed-point-free on* U_α. *Also* $q > 2$ *implies that* $U_\alpha^\# U_\beta^\#$ *is determined uniquely by the abstract structure of the group* $C_G(v)$.

For the case $q > 2$ we consider the centralizer of a certain subgroup of H.

(13.5) *Let* $q > 2$ *and set* $W_0 = H \cap \langle U_{\pm r}\rangle$, $W_1 = H \cap \langle U_{\pm s}\rangle$. *Then* $C_G(W_i) = W_i \times L_i$, *when* $L_0 = \langle U_{\pm\alpha_2}, U_{\pm\alpha_3}, U_{\pm\alpha_4}\rangle$ *and* $L_1 = \langle U_{\pm\alpha_1}, U_{\pm\alpha_2}, U_{\pm\alpha_3}\rangle$. *Moreover* $L_0 \cong L_1 \cong Sp(6, q)$.

Proof. We deal with W_0, the proof for W_1 being similar. Let $P = N_G(U_r) = Q(L_0 \times W_0)$, where $Q = O_2(P)$. Then W_0 is fixed-point-free on Q. Similarly W_0 is fixed-point-free on $O_2(N_G(U_{-r}))$. We conclude that $L_0 = \langle U_\gamma : \gamma \in \Delta, U_\gamma \leq C(W_0)\rangle$. In particular L_0 is invariant under $C_N(W_0)$, where $N/H = W$, the Weyl group. The Bruhat decomposition gives $C_G(W_0) = C_U(W_0)C_N(W_0)C_U(W_0) = \langle C_U(W_0), C_N(W_0)\rangle$. Now $N_G(L_0) \geq L_0 \times \langle U_{\pm r}\rangle$ and $L_0 \times \langle U_{\pm r}\rangle = \langle U_\delta : \delta \in \Delta, U_\delta \leq N(L_0)\rangle$, so $C_N(W_0) \leq N(L_0 \times \langle U_{\pm r}\rangle)$ and it follows that $C_N(W_0) \leq L_0 \times \langle U_{\pm r}\rangle$. The result follows.

Section 14. Centralizers of involutions in ${}^2E_6(q)$.

In this section let $G = {}^2E_6(q)$, q even. Then G has Weyl group of type F_4 and roots will be labeled as in Section 13 and Table 1. The action of W on Δ is as in Section 12.

From (12.7) we have

(14.1) *Each involution in* ${}^2E_6(q)$, q *even, is conjugate to one of the fol-*

lowing

i) $t = U_r(1)$

ii) $u = U_s(1)$

iii) $v = U_\alpha(1)U_\beta(1)$.

We now find the centralizers of t, u, and v. For this we will use the results of Section 4 in [7] for information concerning the parabolic subgroups P_1 and P_4. For P_1 we have Q_1 special of order q^{21} with $Z(Q_1) = U_r = U_{r_{24}}$ and $L_1 \cong SU(6, q)$ acts irreducibly on $Q/Z(Q_1)$. In addition $Q_1L_1 = C_G(U_r) = C_G(t)$, $L_1 = L_1'$, $[L_1, Q_1] = Q_1$, $C_G(t) = C_G(t)'$, and $H_1 = C_H(L_1)$ acts regularly on $U_r^\#$. For P_4 we have $Q_4 = O_2(P_4)$ and $R_4 = Z(Q_4)$ has order q^8. Moreover $L_1 \cong SO^-(8, q)$ acts on R_4 preserving a non-degenerate quadratic form and in this action the isotropic 1-spaces are conjugates of U_α, α long. We will use the following notation for α a short root:

$$V_\alpha = \{U_\alpha(c) : c \in F_q\} .$$

Then V_α has index q in U_α.

Consider $u = U_s(1)$, an anisotropic vector in R_4. Consequently $C_{L_4}(u) \cong SO(7, q)$. From (3.1)(iv) we see that $C_{U_{\alpha_3}}(u) = V_{\alpha_3}$. Moreover $s_3 \in \langle V_{\alpha_3}, V_{-\alpha_3}\rangle \cong SL(2, q)$.

Then $U_{r_1}U_{r_{15}}V_{r_{18}}U_{r_2}V_{r_{19}}U_{r_{16}}V_{r_9}U_{r_3}U_{r_4}$ has order q^7 and hence equals $U \cap C_{L_4}(u)$. The Bruhat decomposition can now be used to show that $\langle U_{\alpha_1}, U_{\alpha_2}, V_{\alpha_3}, s_1, s_2, s_3\rangle = C_{L_4}(u)$.

Suppose $P = P_i^g$ is parabolic and $C_G(u) \leq P$. Since $C_G(u)$ involves $SO(7, q)$, $i = 1$ or 4. If $i = 1$, then since $|Q_4| = q^{24} > q^{21} = |Q_1|$, $Q_4 \not\leq Q_i^g$ and $(C_G(u))'Q_i^g/Q_i^g$ is in a proper parabolic subgroup of $L_4 \cong SU(6, q)$. But again there is no room for $SO(7, q)$. So $i = 4$. The above arguments easily imply that $Q_4 = Q_4^g$, so $g \in N(Q_4) = P_4$. Then 2.3 guarantees that $C_G(u) \leq P$ for some proper parabolic subgroup P, so together with the above we have $C_G(u) = C_{P_4}(u) = Q_4(K_4 \times H_1)$.

Next we determine $C_G(v)$, where $v = U_\alpha(1)U_\beta(1)$. We first note that $(\alpha)s_1 = \alpha = (\alpha)s_4$ and $(\beta)s_1 = \beta = (\beta)s_4$. Also U_{α_1} centralizes $U_\alpha U_\beta$. However, $U_{\alpha_4} \cap C(v) = U_{\alpha_4} \cap C(U_\alpha(1)) = V_{\alpha_4}$ a subgroup of order q. Indeed if $\gamma \in \Delta$ and $\alpha + \gamma \in \Delta$, then $C(v) \cap U_\gamma = V_\gamma$ (see (3.1)(vi)). Then the Chevalley relations imply that $s_4 \in \langle V_{\pm\alpha_4}\rangle = \langle V_{\alpha_4}, s_4\rangle \cong SL(2, q)$. So $\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle = C_{L_{2,3}}(v)$.

Next use (3.1) and Table 1 to verify that

$$C_U(U_\alpha) = \langle U_r : r \neq \alpha_3, \alpha_3 + \alpha_4, \alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4\rangle$$
$$C_U(U_\beta) = \langle U_r : r \neq \alpha_2, \alpha_1 + \alpha_2\rangle .$$

But also v is centralized by $V_{\alpha_4}, V_{\alpha_2+2\alpha_3+\alpha_4}$, and $V_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}$. Consequently if we set $U_0 = C(v) \cap Q_2Q_3 = C(v) \cap O_2(P_{2,3})$ we can use (3.1) to check that

$$C_U(v) = U_0(U_{\alpha_1} \times V_{\alpha_4})$$

and

$$U_0 = \langle U_r : r \in \mathscr{L}_2 \cup \mathscr{L}_3, r \neq r_{15}, r_{18}, r_2, r_{23}, r_{21}, r_{11}\rangle V_{r_{21}} V_{r_{11}} .$$

In particular $|U : C_U(v)| = q^9$ and $|U_0| = q^{25}$. We first determine the structure of U_0 and the action of $\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle$ on U_0. Using (3.1) and Table 1 we have $[U_{r_{14}}, U_{r_9}] = U_{r_{24}}, [U_{r_5}, U_{r_{16}}] = U_{r_7} = U_\beta, [U_{r_5}, U_{r_{19}}] = U_{r_{14}}U_{r_8}$, $[U_{r_9}, U_{r_{20}}] = U_{r_{12}} = U_\alpha, [U_{r_{20}}, U_{r_3}] = U_{r_{13}}U_{r_8}$, $[U_{r_{19}}, U_{r_9}] = U_{r_4}$, and $[U_{r_{10}}, U_{r_{20}}] = U_{r_6}$. It is easy to see that

$$U_0' = (U_\alpha U_\beta) \times (U_{r_8}U_{r_{24}}) \times (U_{r_{13}}U_{r_{14}}) \times (U_{r_4}U_{r_6}) .$$

If we set $U_1 = \langle U_r : r \in \mathscr{L}_2 \cup \mathscr{L}_3, r \neq r_{15}, r_{18}, r_2, r_{23}, r_{21}, r_{11}\rangle$, then we also have

$$U_1' = U_0', [U_1, U_1, U_1] = U_{r_8}U_{r_{24}} , \quad \text{and} \quad U_1' \cap Z(U_1) = U_\alpha U_\beta U_{r_8} U_{r_{24}} .$$

We then find that

$$[U_0, U_0, U_0] = U_{r_8}U_{r_{24}} \quad \text{and} \quad U_\beta U_{r_8} U_{r_{24}} \leq Z(U_0) \cap U_0' .$$

It is easy to see that $\langle U_{\pm\alpha_1}\rangle$ acts irreducibly on $U_{r_8}U_{r_{24}}$ and $\langle V_{\pm\alpha_4}\rangle$ acts without fixed points on $U_{r_{13}}U_{r_{14}}$. Also $\langle V_{\pm\alpha_4}\rangle$ acts irreducibly on $U_0'/U_\alpha U_\beta U_{r_8} U_{r_{24}} U_{r_{13}} U_{r_{14}}$. From here and (3.1) one easily determines the action of $\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle$ on U_0'. Also it is easy to see that $[\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle, U_0/U_0'] = U_0/U_0'$ and all composition factors for $\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle$ on U_0/U_0' are of order q^2 and isomorphic to the usual module for $SL(2, q) \cong \langle U_{\pm\alpha_1}\rangle \cong \langle V_{\pm\alpha_4}\rangle$. In particular if $X = U_0(\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle)$, then $X' = X$ if $q > 2$ and $X' = U_0(O_3(\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle))$ if $q = 2$.

From (3.1)(vi) we have $V_\alpha = C_{U_\alpha}(V_{r_{11}})$ and $V_\alpha = C_{U_\alpha}(V_{r_{21}})$. Consequently $V_\alpha U_\beta = Z(X)$. Next we note that since some element of W conjugates the pair (α, β) to (α_4, α_1), H contains a subgroup $H_0 = H_1 \times H_2$ such that $H_1 \cong H_2$ is cyclic of order $q - 1$, H_1 is regular on $U_\beta^\#$ and centralizes V_α, while H_2 is regular on V_α and centralizes U_β. Also we see that $|C_H(U_\alpha U_\beta)| = |C_H(v)| = (q - 1)^2(q + 1)$ so there is a subgroup

$H_\infty \leq H$ such that H_∞ is cyclic of order $q+1$ and $XH_{00} = C_{P_{2,3}}(v)$.

The next step is to determine the maximal parabolic subgroups of G containing XH_∞. Once we have this information we will be able to prove that $XH_0 = C_G(v)$. The claim is that the only maximal parabolic subgroups of G containing XH_0 are $P_2, P_3, P_1^{s_1s_2}$, and $P_4^{s_4s_3}$.

Suppose $C_G(v) \leq P_1^g$. As in the case of $F_4(q)$ we first assume that $U_0 \cap O_2(P_1^g)' = 1$. Then $U_0 \cap O_2(P_1^g)$ is abelian and the structure of Q_1 forces $|U_0 \cap O_2(P_1^g)| \leq q^{10}$. Hence $|U_0O_2(P_1^g)/O_2(P_1^g) \geq q^{15} = |L_1|_2$. As XH_∞ involves $SL(2,q) \times SL(2,q)$ this is impossible. Consequently $U_0 \cap O_2(P_1^g)' \neq 1$, and so $U_0 \cap U_r^g \neq 1$. As before $X \leq O^{2'}(P_1^g) = C(U_r^g)$ implies $U_r^g \cap U_0 \leq Z(X) = V_\alpha U_\beta$, and consequently $U_r^g = U_\beta$. Also $r = \beta^{s_2s_1}$, so $g \in P_1s_1s_2$ and $P_1^g = P_1^{s_1s_2}$.

Suppose that $XH_\infty \leq P_4^g$. As before we obtain a contradiction if $U_0 \cap O_2(P_4^g)' = U_0 \cap (Q_4^g)' = 1$. Since $\langle U_{\pm\alpha_1}\rangle$ acts irreducibly on $[U_0, U_0, U_0] = U_{r_8}U_{r_{24}}$, either $U_{r_8}U_{r_{24}} \leq (Q_4^g)'$ or $U_{r_8}U_{r_{24}} \cap (Q_4^g)' = 1$. If $U_{r_8}U_{r_{24}} \leq (Q_4^g)'$, then $Q_4^g \leq C(U_{r_8}U_{r_{24}})$. But $N(U_{r_8}U_{r_{24}}) = P_2$, so we have $Q_4^g \leq P_2$. In fact $Q_4^g \leq C(U_{r_8}U_{r_{24}})$ implies that $Q_4^g \leq Q_2L_{12}$. Write $g = bwu$ where $b \in B$, $w \in W, u \in U$. Then $Q_4^w \leq Q_2L_{12} = Q_2\langle U_{\pm\alpha_3}, U_{\pm\alpha_4}\rangle$, and so there is an element $w_1 \in \langle s_3, s_4\rangle$ such that $Q_4^{ww_1} \leq U$. But it is easily checked that for $w_2 \in W, (\mathscr{L}_4)w_2 \subset \Delta^+$ implies $w_2 \in \langle s_1, s_2, s_3\rangle$. Consequently $w \in P_4\langle s_3, s_4\rangle$. As $\langle U_{\pm\alpha_1}\rangle \times \langle V_{\pm\alpha_4}\rangle \leq P_4^g = P_4^{wu}$, we have $w \in P_4s_4s_3$. Using the Bruhat decomposition we may assume that $u \in U_{\alpha_3}U_{\alpha_3+\alpha_4}$ and from here we get $u = 1$. Consequently $P_4^g = P_4^{s_4s_3}$ as claimed. So assume $U_{r_8}U_{r_{24}} \cap (Q_4^g)' = 1$. As $[U_0, U_0, U_0] = U_{r_8}U_{r_{24}}, U_0' \cap (Q_4^g)' \leq Z(U_0)$ and hence $U_0' \cap (Q_4^g)' \leq V_\alpha U_\beta$. As $|U_0| > |Q_4^g|, X/U_0$ is in a proper parabolic subgroup of $(P_4/Q_4)^g$ and $|U_0Q_4^g/Q_4^g| \leq q^9$. Hence $|U_0 \cap Q_4^g| \geq q^{16}$. The commutator relations in (3.1) imply that for $g \in U_0 - U_0'$, there is an element $h \in U_0$ such that $[g,h] \leq V_\alpha U_\beta$. Consequently $g \notin U_0 \cap (Q_4^g)'$ and $U_0 \cap (Q_4^g)' \leq V_\alpha U_\beta$. Thus $|U_0 \cap Q_4^g| \leq q^{18}$, and since $U_0(Q_4^g)' \neq Q_4^g, |U_0 \cap Q_4^g| < q^{18}$ and $|U_0Q_4^g/Q_4^g| > q^7$. As $U_0(Q_4^g)' \neq Q_4^g, V_\alpha U_\beta \cap (Q_4^g)' \neq 1$ and so X fixes a 1-space of the orthogonal space R_4^g. Since XQ_4^g/Q_4^g has Sylow 2-subgroups of order at least $2q^9$ this space must be singular. Then consideration of the parabolic subgroups of $(P_4/Q_4)^g$ contradicts the fact that XQ_4^g/Q_4^g involves $SL(2,q) \times SL(2,q)$.

If $XH_\infty \leq P_2^g$, then we argue as in the case of $F_4(q)$ to get $P_2^g = P_2$. So we are left with the case $XH_\infty \leq P_3^g$. Here the situation is a little different than in $F_4(q)$ as the parabolic subgroups are not permuted by a graph isomorphism centralizing the involution. First we note that $U_0 \not\leq Q_3^g$. Indeed $|U_0/U_0'| = q^{14}$ while $Q_3' = \langle U_r : r \in \mathscr{L}_i^3 \cup \mathscr{S}_i^3 \text{ for } i \geq 2\rangle$ and $|Q_3/Q_3'| = q^{12}$. So XQ_3^g/Q_3^g is contained in a proper parabolic subgroup of $O^{2'}(P_3^g/Q_3^g)$. As X involves $SL(2, q) \times SL(2, q)$ this parabolic subgroup has the form $P_3^g \cap P_i^{gx}$ for some $x \in P_3^g$ and $i = 1$ or 2. At this stage we can use the same argument presented for $F_4(q)$. We have now determined the necessary information on parabolic subgroups.

At this point we can prove that $XH_\infty = C_G(v)$. We know by (2.3) that $C_G(v) \leq P$ for some proper parabolic subgroup P. Consequently $XH_0 \leq C_G(v) \leq P$ and $P = P_2, P_3, P_1^{s_1s_2}$, or $P_4^{s_4s_3}$. If $C_G(v) \leq P_2$, then $g \in C_G(v)$ implies $g = bwu$ where $w \in \langle s_1\rangle \times \langle s_3, s_4\rangle$ and it is easy to check that $w \in \langle s_1\rangle \times \langle s_4\rangle$. Then $C_G(v) \leq P_2 \cap P_4$ and $XH_\infty = C_{P_2 \cap P_3}(v)$. Similarly we are done if $C_G(v) \leq P_3$. If $C_G(v) \leq P_1^{s_1s_2} = N_G(U_r^{s_1s_2}) = N_G(U_\beta)$, then it will follow that $C_G(v) \cap Q_1^{s_1s_2}L_1^{s_1s_2} \leq C_G(U_\beta) \cap C_G(v) \leq C_G(U_\alpha(1)) \leq P_4^{s_4s_3}$. Then $C_G(v) \leq P_1^{s_1s_2} \cap P_4^{s_4s_3}$ and again it is easy to check that $C_G(v) \leq P_2 \cap P_3$ and hence $C_G(v) = XH_\infty$. Say $C_G(v) \leq P_4^{s_4s_3}$. Then we are done if $O_2(C_G(v)) \not\leq Q_4^{s_4s_3}$ by a previous case. So $O_2(C_G(v) = C_G(v) \cap O_2(P_4^{s_4s_3})$ and this is easily seen to be $U_0 \cap O_2(P_4^{s_4s_3})$. Then $(U_0 \cap O_2(P_4^{s_4s_3}))'$ generates a subspace of $R_4^{s_4s_3}$ stabilized by $C_G(v)$. But $(U_0 \cap O_2(P_4^{s_4s_3}))' = U_{r_8}U_{r_{24}}$ is an isotropic 2-space of $R_4^{s_4s_3}$ and hence $C_G(v) \leq P_4^{s_4s_3} \cap P_2$. From here we get $C_G(v) \leq Q_4^{s_4s_3}\langle U_{\pm\alpha_1}, U_{\pm\alpha_4}\rangle H$ and checking we have $C_G(v) = XH_\infty$.

As in the case of $F_4(q)$ we show that $V_\alpha^\# U_\beta^\#$ is determined uniquely by the abstract structure of $C_G(v)$. Summarizing

(14.2) *The maximal parabolic subgroups of G containing the centralizers of the involutions in* (14.1) *are as follows*:

i) $C_G(t) \leq P_1$

ii) $C_G(u) \leq P_4$

iii) $C_G(v) \leq P_2, P_3, P_1^{s_1s_2}, P_4^{s_4s_3}$.

In particular G has 3 *classes of involutions.*

(14.3) *With notation as in* (14.1) *we have*

i) $C_G(t) = (C_G(t))' = O^{2'}(P_1), Z(C_G(t)) = Q_1' = U_r$.

ii) $C_G(u) = Q_4(K_4 \times H_1)$. *Here* $H_1 = C_G(Q_4')$ *is cyclic of order* $q + 1$, $K_4 = \langle U_{\alpha_1}, U_{\alpha_2}, V_{\alpha_3}, s_1, s_2, s_3\rangle \cong SO(7, q)$, *where* $V_{\alpha_3} = C(u) \cap U_{\alpha_3}$ *and* $|V_{\alpha_3}| = q$. *Also* $[K_4, Q_4] = Q_4, C_G(u)' = C_G(u)$, *and* $Z(Q_4L_4) = V_s = Z(C_G(u)) = \{U_s(c) : c \in F_q\}$.

iii) $C_G(v) = U_0(\langle U_{\alpha_1}, s_1\rangle \times \langle V_{\alpha_4}, s_4\rangle H_\infty$, *where* $U_0 = \langle U_r : r \in \mathscr{L}_2 \cup \mathscr{L}_3$,

$r \neq r_2, r_{11}, r_{15}, r_{18}, r_{21}, r_{23}\rangle V_{r_{11}} V_{r_{21}}$. *$V_r = U_r \cap C(v)$ is elementary of order q, and H_∞ is cyclic of order $q+1$. Moreover $[U_0, U_0, U_0] = U_{r_8} U_{r_{24}}$ and $Z(C_G(v)) = V_\alpha U_\beta$, where $V_\alpha = C_{U_\alpha}(V_{r_{11}})$. Also $O^{2'}(C_G(v)/U_0) \cong SL(2,q) \times SL(2,q)$ and $[U_0, C_G(v)] = U_0$. If $q > 2$ then $C_G(v)' = U_0(\langle U_{\alpha_1}, s_1\rangle \times \langle V_{\alpha_4}, s_4\rangle)$. If $q = 2$, then $C_G(v)' = U_0(O_3(\langle U_{\alpha_1}, s_1\rangle \times \langle V_{\alpha_4}, s_4\rangle)) \trianglelefteq C_G(v)$.*

(14.4) *For $q > 2$ there is a subgroup $H_0 = H_1 \times H_2 \leq H$ such that H_1, H_2 are cyclic of order $q-1$, H_1 centralizes V_α and is fixed-point-free on U_β, while H_2 centralizes U_β and is fixed-point-free on V_α. The set $V_\alpha^\# U_\beta^\#$ is uniquely determined by the abstract structure of $C_G(v)$.*

We complete this section by finding the centralizer of a certain subgroup of H.

(14.5) *Let $q > 2$ and set $W_0 = H \cap \langle U_{\pm r}\rangle$, $\hat{W}_1 = H \cap \langle U_{\pm s}\rangle$, and $W_1 = (\hat{W}_1)^{q+1}$. Then $C_G(W_0) = W_0 L_0$ and $C_G(W_1) = \hat{W}_1 L_1$, when $L_0 = \langle U_{\pm\alpha_2}, U_{\pm\alpha_3}, U_{\pm\alpha_4}\rangle$ and $L_1 = \langle U_{\pm\alpha_1}, U_{\pm\alpha_2}, U_{\pm\alpha_3}\rangle$. Moreover $L_0 \cong SU(6,q)$ and $L_1 \cong SO^-(8,q)$.*

Proof. The proof is similar to the proof of (13.5).

Section 15. Centralizers of involutions in $E_6(q)$, q even.

Let $G = E_6(q)$ with $q = 2^a$. Then $|H| = (q-1)^6(3, q-1)^{-1}$, $W = \langle s_1, \cdots, s_6\rangle$ and the action of W on Δ is determined by

$$(\alpha_3)s_1 = (\alpha_1)s_3 = \alpha_1 + \alpha_3$$
$$(\alpha_4)s_2 = (\alpha_2)s_4 = \alpha_2 + \alpha_4$$
$$(\alpha_j)s_i = \alpha_i + \alpha_j \text{ if } |i-j| = 1 \text{ and } \{i,j\} \neq \{1,2\}, \{2,3\}$$
$$(\alpha_i)s_i = -\alpha_i\,,$$

with $(\alpha_j)s_i = \alpha_j$ for all other pairs (i,j). We remark that in computations with commutators (3.1)(ii) is the only relevant commutator, making such calculations particularly easy.

We use the notion of Table 2, distinguishing the roots $r, \alpha, \beta, \gamma, \delta, \varepsilon$ in Δ^+. By (12.8) we have

(15.1) *Each involution in $E_6(q)$, q even, is conjugate to one of the following*

i) $x = U_r(1)$
ii) $y = U_\alpha(1)U_\beta(1)$
iii) $z = U_\gamma(1)U_\delta(1)U_\varepsilon(1)$.

We will see in (15.4) that no two of x, y, z are conjugate. This will be clear from the structure of the centralizers of these involutions.

Let $x = U_r(1)$. We use the results in Section 4 of [7] to see that $C_G(x) = O^{2'}(P_2) = P'_2 = Q_2L_2$. Also $P_2 = Q_2L_2W_0$, where $[L_2, W_0] = 1$, W_0 is cyclic of order $q-1$, and $W_0 \cap L_2 = Z(L_2)$. $C_G(W_0) = W_0L_2$.

The method for finding the other centralizers is as follows, and will be used repeatedly in Section 16 and Section 17. Say $t = U_{\beta_1}(1) \cdots U_{\beta_k}(1)$ is one of the involutions in (15.1), (16.1), or (17.1) with $k > 1$. We will exhibit a certain parabolic subgroup $P \geq B$ with Levi factor L such that $C_L(t) \geq L_0$, where L_0 is generally obtained from L as the fixed points of a graph automorphism. Usually L_0 will be maximal in L and since $L \not\leq C(t)$, we obtain $L_0 = C_L(t)$. Then we find $C_{O_2(P)}(t)$, $C_U(t)$, and hence $C_P(t)$.

By (2.3) we know that $C_G(t)$ is contained in a proper parabolic subgroup of G. We find all parabolic subgroups of G containing $C_P(t)$. There are very few of these and we are able to conclude that $C_G(t) = C_P(t)$.

We obtained L_0 as follows. From the action of W on the set of root groups we know that $C_W(t)$ is just the stabilizer in W of $\{t_1, \cdots, t_k\}$. This can be found by first finding $C_W(t_1) \cap \cdots \cap C_W(t_k)$ and then studying the induced group. The roots $t_1, \cdots, t_k$ have been chosen so that $C_W(t)$ is clearly the set of fixed points of a parabolic subgroup of W under a graph automorphism. From here P, L, L_0 are obvious.

Consider $C_G(y)$ where $y = U_\alpha(1)U_\beta(1)$. The root groups $U_s \leq U$ not centralizing $U_\alpha(1)$ and $U_\beta(1)$ are as follows:

$$\begin{array}{lll} U_\alpha(1) & \{s = r_{34}, r_{32}, r_{20}\} & \{s + \alpha = r_{14}, r_{15}, r_{16}\} \\ U_\beta(1) & \{s = r_{27}, r_{28}, r_{19}\} & \{s + \beta = r_{14}, r_{15}, r_{16}\} . \end{array}$$

It is then straightforward to check that

$$C_U(U_\alpha U_\beta) = \langle U_r : r \in \Delta^+, r \neq r_{19}, r_{20}, , r_{27}, r_{28}, r_{32}, r_{34} \rangle$$
$$C_U(y) = \langle C_U(U_\alpha U_\beta), U_{r_{34}}(c)U_{r_{27}}(c), U_{r_{32}}(c)U_{r_{28}}(c), U_{r_{20}}(c)U_{r_{19}}(c) : c \in F_q \rangle .$$

In particular $|C_U(U_\alpha U_\beta)| = q^{30}, |C_U(y)| = q^{33}$, and $|U : C_U(y)| = q^3$.

Let $L_0 = \langle U_{\alpha_3}(c)U_{\alpha_5}(c), U_{\alpha_4}, U_{\alpha_2}, s_3s_5, s_4, s_2 : c \in F_q \rangle$. Then $L_0 \leq L$, where L is the Levi factor of $P_{1,6}$. Moreover it is straightforward to check that $L_0 \leq C(y)$ and $L \not\leq C(y)$. Now $L \cong D_4(q)$ and L_0 is obtained as the fixed group of a graph automorphism of L. It follows that $L_0 \cong B_3(q)$.

Let $w = s_1s_6s_3s_5s_4s_2s_3s_5s_4s_3s_5$. Then $(U_\alpha U_\beta)^w = U_{\alpha_1} \times U_{\alpha_6} \le \langle U_{\pm\alpha_1}\rangle \times \langle U_{\pm\alpha_6}\rangle$, so H is transitive on $U_\alpha^\# U_\beta^\#$. Setting $H_1 = C_H(y)$ we have $|H_1| = (q-1)^4(3, q-1)^{-1}$.

Next we set $U_0 = Q_1Q_6$ and $H_0 = C_{H_1}(L)$. Then $L_0 \times H_0$ normalizes U_0 and we set $X = U_0(L_0 \times H_0)$. The first observation is that $U_0L_0 = \langle C_U(y), C_W(y)\rangle$. Consider the action of L_0 on Q_1Q_6. Since Q_1, Q_6 are elementary it is easy to check that $Z(U_0) = Q_1 \cap Q_6 = U_0'$ and that $|Z(U_0)| = q^8$. Using Table 2 and (3.1) we also check that $[L_0, U_0] = U_0$. As $U_\alpha^\# U_\beta^\#$ is fused by H we see that $(U_\alpha U_\beta)^\#$ contains $(q-1)^2$ conjugates of y and $2(q-1)$ conjugates of x. Set $P = \langle U_\alpha(c)U_\beta(c) : c \in F_q\rangle$. Then choosing $h \in H$ such that h centralizes U_β but not U_α we have $PP^h = U_\alpha U_\beta$. Also $P = Z(X)$.

We will show that $X = C_G(y)$ and that P_1, P_6 are the only parabolic subgroups of G containing X. Suppose that $\hat{P} = P_i^g \ge X$. Since $X/O_2(X)$ involves $Sp(6, q)$ we must have $i = 1$ or 6. Then $O_2(\hat{P})$ is abelian and so $O_2(X) \not\le O_2(\hat{P})$ and X is contained in a proper parabolic subgroup of P. This parabolic subgroup has the form $P_i^g \cap P_j^{gk}$, where $j \in \{1, 6\} - \{i\}$ and $k \in P_i^g$. This implies that $X \le (P_1 \cap P_6)^{p_ig}$, for some $p_i \in P_i$. On the other hand at this point we must have $O_2(X) \le O_2((P_1 \cap P_6)^{p_ig})$ and by orders, equality holds. Since $P_1 \cap P_6 = N_G(Q_1Q_6), p_ig \in P_1 \cap P_6$ and $P_i^g = P_i$.

To see that $X = C_G(y)$ we note that $C_G(y)$ is contained in a parabolic subgroup of G and the graph automorphism of G centralizes y. Consequently $C_G(y) \le P_1 \cap P_6$. Now $L_{1,6}$ operates on $Z(Q_1Q_6) = (Q_1Q_6)'$ and using Table 2 it is easy to see that $L_{1,6}$ preserves a non-degenerate quadratic form on $Z(Q_1Q_6)$ in which $Z(Q_1Q_6)$ is the orthogonal direct sum of hyperbolic planes $U_\alpha U_\beta, U_{r_{10}}U_{r_{14}}, U_{r_7}U_{r_{15}}$, and $U_{r_8}U_{r_{16}}$. In this action the elements of $U_s^\#$ for $s \in \Delta^+$ are singular vectors, and y is an anisotropic vector. So $C_{L_{1,6}}(y) \cong SO(7, q) \cong Sp(6, q) \cong L_0$ and it follows that $X = C_G(y)$.

Now we consider $C_G(z), z = U_\gamma(1)U_\delta(1)U_\varepsilon(1)$. We proceed as before by first finding $C_U(z)$ and $C_W(z)$. The root groups U_s for $s \in \Delta^+$ not centralized by U_γ, U_δ, and U_ε are as follows:

$$\begin{array}{lll} U_\gamma & \{s = r_1, r_2, r_3, r_4\} & \{s + \gamma = r_{13}, r_{14}, r_{15}, r_{16}\} \\ U_\delta & \{s + r_{27}, r_{34}, r_{29}, r_{22}\} & \{s + \delta = r_{12}, r_{13}, r_{15}, r_{16}\} \\ U_\varepsilon & \{s = r_{36}, r_{35}, r_{33}, r_{21}\} & \{s + \varepsilon = r_{12}, r_{14}, r_{15}, r_{16}\}\ . \end{array}$$

It is then easy to check that $C_U(U_\gamma U_\delta U_\varepsilon) = \langle U_s : s \in \Delta^+$ and $s \neq r_1, r_2, r_3, r_4, r_{21}, r_{22}, r_{27}, r_{29}, r_{33}, r_{34}, r_{35}, r_{36}\rangle$. Also it is easily seen that z is centralized by each of $U_\varphi(c)U_\psi(c)$ for $(\varphi, \psi) \in C = \{(r_1, r_{34}), (r_{27}, r_{36}), (r_2, r_{35}), (r_3, r_{29}), (r_{29}, r_{33}), (r_4, r_{22}), (r_4, r_{21})\}$. We then have $C_U(z) = \langle C_U(U_\gamma U_\delta U_\varepsilon), U_\varphi(c)U_\psi(c) : c \in F_q, (\varphi, \psi) \in C\rangle$. In particular $|C_U(z)| = q^{31}$.

Let $L_0 = \langle U_{\pm\alpha_2}, s_1s_5, s_3s_6, U_{\alpha_1}(c)U_{\alpha_5}(c), U_{\alpha_3}(c)U_{\alpha_6}(c) : c \in F_q\rangle$. Then $L_0 \leq L$, where L is the Levi factor of P_4, and it is easy to see that $L_0 = C_L(z)$.

$$\begin{aligned} L_0 &= \langle U_{\pm\alpha_2}, s_1s_5, s_3s_6, U_{\alpha_1}(c)U_{\alpha_5}(c), U_{\alpha_3}(c)U_{\alpha_6}(c) : c \in F_q\rangle \\ &\cong SL(2, q) \times SL(3, q) . \end{aligned} \tag{15.2}$$

Now we set

$$\begin{aligned} U_0 &= \langle C_{Q_4}(U_\gamma U_\delta U_\varepsilon), U_{r_3}(c)U_{r_{29}}(c), U_{r_{29}}(c)U_{r_{33}}(c), U_{r_4}(c)U_{r_{22}}(c), \\ &\qquad U_{r_4}(c)U_{r_{21}}(c) : c \in F_q\rangle \\ X &= U_0L_0 = C_{P_4}(z) . \end{aligned} \tag{15.3}$$

Next consider the structure of U_0 and the action of L_0 on U_0. We have $|U_0| = q^{27}$ and $X \leq P_4$. Checking Table 2 and (3.1) we easily see that $U_0' = P_4' = Q_2^4Q_3^4$, where $Q_2^4 = \langle U_s : s \in \mathscr{L}_2^4\rangle$ and $Q_3^4 = U_{r_{15}} \times U_{r_{16}}$. Also we check that $[U_0, U_0, U_0] = Q_3^4 \leq Z(U_0)$ and that $\langle U_{\pm\alpha_2}\rangle$ acts irreducibly on $[U_0, U_0, U_0]$. Next we can use Table 2 and (3.1) to verify that $[L_0, U_0/U_0'] = U_0/U_0'$, so that $[L_0, U_0] = U_0$. In doing so it is useful to note that modulo U_0' we have

$$\begin{aligned} &\langle U_{r_3}(c)U_{r_{29}}(c), U_{r_{29}}(c)U_{r_{33}}(c), U_{r_4}(c)U_{r_{22}}(c), U_{r_4}(c)U_{r_{21}}(c) : c \in F_q\rangle \\ &\quad = \langle U_{r_3}(c)U_{r_{29}}(c), U_{r_4}(c)U_{r_{22}}(c) : c \in F_q\rangle \times \langle U_{r_3}(c)U_{r_{33}}(c), U_{r_4}(c)U_{r_{21}}(c) : c \in F_q\rangle \end{aligned}$$

and that $\langle U_{\pm\alpha_2}\rangle$ acts on both factors as on the natural module for $\langle U_{\pm\alpha_2}\rangle \cong SL(2, q)$.

Say $q > 2$. For $w = s_4s_3s_5s_2s_4s_3s_5s_1s_6s_2s_3s_5s_4$, we have $\{\gamma, \delta, \varepsilon\}^w = \{\alpha_2, \alpha_3, \alpha_5\}$, so $(U_\gamma U_\delta U_\varepsilon)^w = U_{\alpha_2} \times U_{\alpha_3} \times U_{\alpha_5} \leq \langle U_{\pm\alpha_2}\rangle \times \langle U_{\pm\alpha_3}\rangle \times \langle U_{\pm\alpha_5}\rangle$. Therefore there is a subgroup of H of the form $H_\gamma \times H_\delta \times H_\varepsilon$ such that H_γ is regular on $U_\gamma^\#$ and centralizes $U_\delta U_\varepsilon$, and H_δ is regular on $U_\delta^\#$ and centralizes $U_\gamma U_\varepsilon$, and H_ε is regular on $U_\varepsilon^\#$ and centralizes $U_\gamma U_\delta$. Let $P = \langle U_\gamma(c)U_\delta(c)U_\varepsilon(c) : c \in F_q\rangle$. We check that $P = Z(X)$. Let $h \in H_\gamma^\#$. Then $PP^h = \langle U_\gamma(c)U_\delta(c)U_\varepsilon(c), U_\gamma(dc)U_\delta(c)U_\varepsilon(c) : c \in F_q\rangle$ for some fixed $1 \neq d \in F_q^\#$. It follows that $PP^h = U_\gamma \times P$ and that PP^h contains $q - 1$ conjugates of x, $q - 1$ conjugates of y, and $(q - 1)^2$ conjugates of z.

We must still show that $X = C_G(z)$ and find the parabolic subgroups of G that contain X. Suppose that $X \leq P_i^g$. We note that for $i \neq 4$, $\mathscr{L}_3^i = \emptyset$ and consequently Q_i has class at most 2. Therefore $i \neq 4$ implies that $U_0 \not\leq Q_i^g$. First assume that $i = 4$. If $U_0 \leq Q_4^g$, then $[U_0, U_0, U_0] = [Q_4^g, Q_4^g, Q_4^g]$ and $P_4 = N_G([U_0, U_0, U_0]) = P_4^g$. Say $U_0 \not\leq Q_4^g$. Then XQ_4^g/Q_4^g is contained in a parabolic subgroup of P_4^g/Q_4^g. Say XQ_4^g/Q_4^g is contained in P_3^{gp} for $p \in P_4^g$. Since $U_0 = [U_0, L_0]$, U_0 is contained in a conjugate Q^e of $Q = Q_3Q_4$. Using Table 2 and (3.1) we check that $[Q, Q, Q] = Q_2^4Q_3^4$ and $[Q, Q, Q, Q] = Q_3^4$. Now X/U_0 involves $SL(3, q)$ and consequently X covers the conjugate of $L_{1,2,3,4} = \langle U_{\pm\alpha_5}, U_{\pm\alpha_6}\rangle$ in $P_4^g \cap P_3^{gp}/O_2(P_4^g \cap P_3^{gp})$. However it is easy to see that this group acts without fixed points on $[Q^e, Q^e, Q^e]/[Q^e, Q^e, Q^e, Q^e]$, while $[U_0, U_0, U_0]$ is centralized by the corresponding factor isomorphic to $SL(3, q)$. It follows that $[U_0, U_0, U_0] \leq [Q^e, Q^e, Q^e, Q^e]$ and by orders these must be equal. Then $P_4 = N_G(Q_3^4) = N_G([Q^e, Q^e, Q^e, Q^e]) = P_4^g$. But then $g \in P_4$ and $U_0 \leq Q_4^g$, a contradiction. By symmetry we have XQ_4^g/Q_4^g not contained in P_5^{gp} for $p \in P$. Also this argument works to eliminate the cases of $X \leq P_4^g \cap P_1^{gp}$ or $P_4^g \cap P_6^{gp}$. So we are left with the case of $X \leq P_4^g \cap P_2^{gp}$. As the other parabolics have been eliminated we must have $U_0 \leq O_2(P_4^g \cap P_2^{gp})$ but $U_0 \not\leq O_2(P_4^g)$. Since $P_4/O_2(P_4) \cong \langle U_{\pm\alpha_2}\rangle \times SL(3, q) \times SL(3, q)$, we must have $O^{2'}(X) < X$, whereas we have already seen that $O^{2'}(X) = X$. So this is also a contradiction.

We now assume $X \leq P_i^g \cap P_j^{gp} = \hat{P}$ for $i \neq j$ and $i, j \neq 4$. Say $i = 1$, $j = 2$. Then $|Q_{1,2}| = q^{26}$ implies that $U_0 \not\leq O_2(P_i^g \cap P_j^{gp})$ and X is in a proper parabolic subgroup $\bar{P}$ of $P_i^g \cap P_j^{gp}$. By the above $\bar{P} \sim P_{1,2,5}$ and X covers $O^{2'}(\bar{P}/O_2(\bar{P}))$. Then $U_0 \leq O_2(\bar{P})$ and $z \in U_0' \leq O_2(\bar{P})'$. Let $L_{00} \leq L_0 = SL(2, q) \times SL(3, q)$ be a direct sum of Singer cycles. Then checking the action of L_{00} on $K = O_2(\bar{P})'$ we have $C_K(L_{00}) \sim U_{r_{16}}U_{r_{26}}$. But $(U_{r_{16}}U_{r_{26}})^{\#} \subseteq x^G$ and P_2 is the unique parabolic subgroup of G containing $C_G(x)$. This is a contradiction. Similarly $(i, j) \neq (1, 3), (1, 6)$.

Next assume that $(i, j) = (1, 5)$. By the above $U_0 \leq O_2(\hat{P}) \sim Q_1Q_5$. From Table 2 and (3.1) we have $[Q_1Q_5, Q_1Q_5, Q_1Q_5]$ of order q^4 and centralized by $\langle U_{\pm\alpha_6}\rangle$. Now $O^{2'}(\hat{P}/O_2(\hat{P})) \cong SL(4, q) \times SL(2, q)$, where the factor of $SL(2, q)$ centralizes $[U_0, U_0, U_0]$. As $L_0 \cong SL(3, q) \times SL(2, q)$ we have $\langle U_{\pm\alpha_6}\rangle \leq C([U_0, U_0, U_0])$, which is false. Thus $(i, j) \neq (1, 5)$ and hence $i \neq 1$. By symmetry $i \neq 6$.

Now assume $i = 3$. Then $j \neq 1, 4, 6$ by the above. Since $X/O_2(X) \cong SL(2, q) \times SL(3, q)$ we argue as in the above paragraph to show that $(i, j) \neq (3, 2)$. Thus $j = 5$. Then $U_0 \leq O_2(P_3^g \cap P_5^{gp}) \cong Q_3Q_5$. From Table

2 and (3.1) Q_3Q_5 has class $4, Z(Q_3Q_5) = U_{r_{14}}U_{r_{15}}U_{r_{16}}$ and $\langle U_{\pm\alpha_2}, U_{\pm\alpha_4}\rangle \cong SL(3,q)$ acts irreducibly on $Z(Q_3Q_5)$. Since $O^{2'}(P_{3,5}/Q_3Q_5) \cong \langle U_{\pm\alpha_1}\rangle \times \langle U_{\pm\alpha_2}, U_{\pm\alpha_4}\rangle \times \langle U_{\pm\alpha_6}\rangle$, the preimage of the $SL(3,q)$ in X covers the factor of $SL(3,q)$ in $P_3^g \cap P_5^{gp}$. However the preimage centralizes $[U_0, U_0, U_0]$. Consequently $[U_0, U_0, U_0] \cap Z(O_2(P_3^g \cap P_5^{gp})) = 1$. As $Q = O_2(P_3^g \cap P_5^{gp}) \sim Q_3Q_5$, $|Q| = |Q_3Q_5| = q^{31}$. Since $U_0 \cap [Q,Q,Q,Q] = 1$, $|U_0\Phi(Q)/\Phi(Q| \leq q$. Also $U_0\Phi(Q) < Q$, as $U_0 < Q$. But then we must have $[X, Q/\Phi(Q)] = U_0\Phi(Q)/\Phi(Q) < Q/\Phi(Q)$. However the preimage of the $SL(3,q)$ in X/U_0 covers the $SL(3,q)$ in $P_3^g \cap P_5^{gp}/Q$ and checking the action of $\langle U_{\pm\alpha_2}, U_{\pm\alpha_4}\rangle$ on $Q_3Q_5/(Q_3Q_5)'$ we see that $[\langle U_{\pm\alpha_2}, U_{\pm\alpha_4}\rangle, Q_3Q_5] = Q_3Q_5$, which is a contradiction. Therefore $i \neq 3$ and by symmetry $i \neq 5$. We have proved that the only parabolic subgroup of G containing X is P_4.

Now $C_G(z)$ is contained in a parabolic subgroup by (2.3), so we have $C_G(z) \leq P_4$. Considering $C_G(z)$ acting on $[Q_4, Q_4]/[Q_4, Q_4, Q_4]$ we have $C_G(z)Q_4/Q_4 < O^{2'}(P_4/Q_4)$. However XQ_4/Q_4 is maximal in $O^{2'}(P_4/Q_4)$, and so $XQ_4 = C_G(z)Q_4$. Since $X \geq C_U(z)$ we have $X = C_G(z)$.

We now summarize our results.

(15.4) *The maximal parabolic subgroups of G containing the centralizers of the involutions in* (15.1) *are as follows*:

i) $C_G(x) \leq P_2$

ii) $C_G(y) \leq P_1, P_6$

iii) $C_G(z) \leq P_4$.

In particular G has precisely three classes of involutions.

(15.5) i) $C_G(x) = O^{2'}(P_2) = Q_2L_2$, *and* $L_2 \cong SL(6,q)$ *acts irreducibly on* $Q_2/Z(Q_2)$. *The group* Q_2 *is special with center* U_r. *There is a cyclic group* $W_0 \leq H$ *of order* $q-1$ *such that* $[L_2, W_0] = 1, L_2 \cap W_0 = Z(L_2)$, $P_2 = C_G(x)W_0$, *and* $C_G(W_0) = W_0L_2$.

ii) $C_G(y) = U_0(L_0 \times H_0)$, *where* $U_0 = Q_1Q_6, L_0 = \langle U_{\alpha_3}(c)U_{\alpha_5}(c), U_{\alpha_4}, U_{\alpha_2}, s_3s_5, s_4, s_2 : c \in F_q\rangle \cong Sp(6,q) \cong SO(7,q)$, *and* H_0 *is cyclic of order* $(q-1)/(3,q-1)$. *We have* $[L_0, U_0] = U_0$. *Also* $U_0' = Z(U_0) = Q_1 \cap Q_6$ *has order* q^8 *and* $L_{1,6}$ *acts on* $Z(U_0)$ *preserving a non-degenerate quadratic form in which* L_0 *is the centralizer of an anistropic* 1-*space containing* y. $Z(C_G(y)) = P = \langle U_\alpha(c)U_\beta(c) : c \in F_q\rangle$. *There is an element* $h \in H$ *such that* $U_\alpha U_\beta = PP^h$. *Also* $(U_\alpha U_\beta)^\#$ *contains* $2(q-1)$ *conjugates of* x *and* $(q-1)^2$ *conjugates of* y.

iii) $C_G(z) = U_0L_0$ *where* U_0, L_0 *are given in* (15.2) *and* (15.3). *The* *group* U_0 *has class* 3 *with* $[U_0, U_0] = Q_2^4 Q_3^4$ *and* $[U_0, U_0, U_0] = Q_3^4 \leq Z(U_0)$. $L_0 \cong SL(2, q) \times SL(3, q), [L_0, U_0] = U_0$ *and* L_0 *acts irreducibly on* $Z(U_0)$. *The involution* $z \in [U_0, U_0] - [U_0, U_0, U_0]$. $Z(C_G(z)) = P = \langle U_\gamma(c) U_\delta(c) U_\varepsilon(c) : c \in F_q \rangle$. *There is an element* $h \in H$ *such that* $PP^h = U_r \times P$. *For this* h, PP^h *contains* $q-1$ *conjugates of* $x, q-1$ *conjugates of* y, *and* $(q-1)^2$ *conjugates of* z.

Section 16. Centralizers of involutions in $E_7(q)$, q even.

Let $G = E_7(q)$ with $q = 2^a$. Then $|H| = (q-1)^7$, $W = \langle s_1, \cdots, s_7 \rangle$ and the action of W on Δ in essentially as in $E_6(q)$:

$$(\alpha_3)s_1 = (\alpha_1)s_3 = \alpha_1 + \alpha_3$$
$$(\alpha_4)s_2 = (\alpha_2)s_4 = \alpha_2 + \alpha_4$$
$$(\alpha_j)s_i = \alpha_i + \alpha_j \quad \text{if} \quad |i - j| = 1, \{i, j\} \neq \{1, 2\}, \{2, 3\},$$

with $(\alpha_j)s_i = \alpha_j$ for all other pairs (i, j). As in $E_6(q)$ the only non-trivial commutator relation is in (3.1)(ii). Distinguish the roots $r, \alpha, \beta, \gamma, \delta, \varepsilon, \varphi, \psi, \theta$ as in Table 3. From (12.9) we have

(16.1) *Each involution in* $E_7(q)$, q *even, is conjugate to one of the following*:

i) $x = U_r(1)$
ii) $y = U_\alpha(1) U_\beta(1)$
iii) $z = U_\gamma(1) U_\delta(1) U_\varepsilon(1)$
iv) $u = U_\varphi(1) U_\psi(1) U_\theta(1)$
v) $v = U_\gamma(1) U_\theta(1) U_\psi(1) U_\varphi(1)$.

We will determine the centralizers of x, y, z, u, and v. For $x = U_r(1)$ we can use the results in Section 4 of [7] to check the following. $C_G(x) = Q_1 L_1$ and $L_1 \cong SO^+(12, q)$ acts irreducibly on $Q_1/Z(Q_1) = Q_1/U_r$. The parabolic subgroup $P_1 = Q_1(L_1 \times W_0)$ where W_0 is cyclic of order $q-1$ and is regular on $U_r^\#$. Finally $C_G(W_0) = W_0 \times L_1$.

Next consider $y = U_\alpha(1) U_\beta(1)$. The root groups $U_s \leq U$ with $s \in \Delta^+$ and such that U_s does not commute with $U_\alpha(1)$ and $U_\beta(1)$ are as follows:

$U_\alpha(1)$ $\{s = \alpha_2, \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$ $\{\alpha + s = r_{24}, r_{25}, r_{26}, r_{27}\}$
$U_\beta(1)$ $\{s = \alpha_5, \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5\}$ $\{\beta + s = r_{24}, r_{25}, r_{26}, r_{27}\}$.

It follows that

(16.2) $C_U(U_\alpha U_\beta) = \langle U_s : s \neq \alpha_2, \alpha_5, r_{29}, r_{31} - \alpha_2, r_{30}, r_{33} - \alpha_2, r_{38}, r_{39} - \alpha_2 \rangle$

(16.3) $$C_U(y) = \langle C_U(U_\alpha U_\beta), U_{\alpha_2}(c)U_{\alpha_5}(c), U_{r_{29}}(c)U_{r_{31}-\alpha_2}(c), U_{r_{30}}(c)U_{r_{33}-\alpha_2}(c), U_{r_{38}}(c)U_{r_{39}-\alpha_2}(c) : c \in F_q \rangle .$$

In particular $|U : C_U(y)| = q^4$ and $|C_U(y)| = q^{59}$.

Let $L \cong D_5(q) \times SL(2, q)$ be the Levi factor of P_6. Then let L_0 be the direct product of the $SL(2, q)$ factor with the fixed point group of the graph automorphism of the $D_5(q)$ factor. Then

$$L_0 = \langle U_{\alpha_1}, U_{\alpha_3}, U_{\alpha_4}, U_{\alpha_2}(c)U_{\alpha_5}(c), s_1, s_3, s_4, s_2s_5 : c \in F_q \rangle \times \langle U_{\pm\alpha_7} \rangle \cong Sp(8, q) \times SL(2, q)$$

and

$$U_0 = Q_6 .$$

Then one checks that $L_0 \leq C(y)$, $L \not\leq C(y)$ and $X = U_0L_0 = C_{P_6}(y)$. We first show that the only parabolic subgroup of G containing X is P_6. To see this first note that $|U_0| = |Q_6| = q^{42}$ so if $P = P_i^g \geq X$ and $|O_2(P)| < q^{42}$, we must have X contained in a parabolic subgroup of the form $P_i^g \cap P_j^{gp}$, for some $p \in P = P_i^g$. Since $X/O_2(X) \cong Sp(8, q) \times SL(2, q)$ the only possible values for i are $i = 6, 7$. If $i = 7$, then j must exist and hence $j = 6$. But by orders $Sp(8, q) \times SL(2, q) \not\leq P_7^g \cap P_6^{gp}$. So $i = 6$ and $U_0 \leq O_2(P) = Q_6^g$. But then $Q_6 = U_0 = Q_6^g$ and $g \in N_G(Q_6) = P_6$.

Consider the structure of $X \leq Q_6L_6$. We check using Table 2 and (3.1) that $U_0' = Q_6' = Q_2^6$. Construct a quadratic form on U_0' as follows. Elements in root groups are isotropic, $U_0' = U_{r_{15}}U_{r_{22}} \times U_{r_{17}}U_{r_{26}} \times U_{r_{18}}U_{r_{25}} \times U_{r_{20}}U_{r_{24}} \times U_{r_{21}}U_{r_{23}}$ is a decomposition into orthogonal hyperbolic planes and in each factor $g(U_{r_i}(s)U_{r_j}(t)) = st$. Then it is easy to verify that $L_{6,7} = \langle U_{\pm\alpha_1}, U_{\pm\alpha_2}, U_{\pm\alpha_3}, U_{\pm\alpha_4}, U_{\pm\alpha_5} \rangle \cong SO^+(10, q)$ preserves this form. Since $y$isa nisotro pic $C_{L_{1,6}}(y) \cong SO(9, q) \cong Sp(8, q)$. Consequently $C_{L_6}(y) = L_0$ and $X = C_G(y)$.

Using Table 3 and (3.1) we see that $[L_0, U_0/U_0^1] = U_0/U_0^1$, so $[L_0, U_0]$ U_0. If $q > 2$, then $C(y) = C(y)'$, while for $q = 2$, $C(y)'$ has index 2 in $C(y)$ (as $\langle U_{\pm\alpha_7} \rangle \cong S_3$). There is an element $w \in W$ such that $(\alpha, \beta)w = (\alpha_1, \alpha_6)$, so $(U_\alpha U_\beta)^w \leq \langle U_{\pm\alpha_1} \rangle \times \langle U_{\pm\alpha_6} \rangle$ and H is transitive $U_\alpha^\# U_\beta^\#$. So setting $P = \langle U_\alpha(c)U_\beta(c) : c \in F_q \rangle$, we have an element $h \in H$ such that $U_\alpha U_\beta = PP^h$. Then PP^h contains $2(q-1)$ conjugates of x and $(q-1)^2$ conjugates of y. Also $P = Z(C_GH)$.

We next find $C_G(z)$, $z = U_\gamma(1)U_\delta(1)U_\varepsilon(1)$. As usual we begin with $C_U(z)$. The roots $s \in \Delta^+$ such that U_s does not centralize U_γ, U_δ, and U_ε are as follows

$$\begin{array}{lll} U_\gamma & \{s = r_1, r_2, r_3, r_4, r_5, r_6\} & \{\gamma + s = r_{22}, r_{23}, r_{24}, r_{25}, r_{26}, r_{27}\} \\ U_\delta & \{s = \alpha_4, \alpha_4 + \alpha_5, r_{29}, r_{31}, r_{35}, r_{41}\} & \{\delta + s = r_{20}, r_{21}, r_{23}, r_{24}, r_{26}, r_{27}\} \\ U_\varepsilon & \{s = \alpha_6, \alpha_5 + \alpha_6, r_{28}, r_{32}, r_{34}, r_{40}\} & \{\varepsilon + s = r_{20}, r_{21}, r_{22}, r_{25}, r_{26}, r_{27}\} \end{array} \tag{16.4}$$

consequently $C_U(U_\gamma U_\delta U_\varepsilon) = \langle U_s : s \in \Delta^+ \text{ not a root in 16.4}\rangle$. In addition for $c \in F_q$ $C_U(z)$ contains the following elements of U.

$$\begin{array}{l} U_{\alpha_7}(c)U_{\alpha_2}(c), U_{r_2}(c)U_{r_{29}}(c), U_{r_3}(c)U_{r_{31}}(c), U_{r_4}(c)U_{r_{32}}(c), U_{r_5}(c)U_{r_{35}}(c) , \\ U_{r_{35}}(c)U_{r_{34}}(c), U_{r_6}(c)U_{r_{41}}(c), U_{r_{41}}(c)U_{r_{40}}(c), U_{\alpha_4}(c)U_{\alpha_6}(c), \\ \quad U_{\alpha_4+\alpha_5}(c)U_{\alpha_5+\alpha_6}(c) . \end{array} \tag{16.5}$$

We then obtain

$$C_U(z) = C_{Q_3}(z)C_{U\cap L_3}(z) , \qquad \text{where}$$

$$\begin{array}{l} C_{Q_3}(z) = \langle C_U(U_\gamma U_\delta U_\varepsilon), U_{r_5}(c)U_{r_{35}}(c), U_{r_{34}}(c)U_{r_{35}}(c), U_{r_6}(c)U_{r_{40}}(c), \\ \qquad U_{r_6}(c)U_{r_{41}}(c) : c \in F_q\rangle \\ C_{U\cap L_3}(z) = \langle U_{\alpha_1}\rangle \times \langle U_{\alpha_5}, U_{\alpha_4}(c)U_{\alpha_6}(c), U_{\alpha_2}(c)U_{\alpha_7}(c), U_{\alpha_4+\alpha_5+\alpha_6}, \\ \qquad U_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7}, U_{\alpha_4+\alpha_5}(c)U_{\alpha_5+\alpha_6}(c), U_{r_2}(c)U_{r_{29}}(c), \\ \qquad U_{r_3}(c)U_{r_{31}}(c), U_{r_4}(c)U_{r_{32}}(c)\rangle . \end{array} \tag{16.6}$$

It is easy to check from (3.1) that $C_{U\cap L_3}(z)$ is isomorphic in a natural way to a Sylow 2-subgroup of $SL(2, q) \times Sp(6, q)$. We set

$$\begin{array}{l} U_0 = C_{Q_3}(z) \\ L_0 = \langle C_{U\cap L_3}(z), s_1, s_5, s_4s_6, s_2s_7\rangle \\ \quad = \langle U_{\pm\alpha_1}\rangle \times \langle U_{\alpha_5}, U_{\alpha_4}(c)U_{\alpha_6}(c), U_{\alpha_2}(c)U_{\alpha_7}(c), s_5, s_4s_6, s_2s_7 : c \in F_q\rangle . \end{array} \tag{16.7}$$

Then $U_0L_0 \le C_{P_3}(z)$ and $L_0 = \langle U_{\pm\alpha_1}\rangle \times \langle L_{00}\rangle$, where L_{00} is the fixed point group under the graph automorphism of the Levi factor of $P_{1,3}$. It follows that

$$L_0 \cong SL(2, q) \times Sp(6, q) .$$

We use Table 3 and (3.1) to check that $|U_0| = q^{45}$, $U_0' = Q_2^3Q_3^3 = Q_3'$, and $[U_0, U_0, U_0] = Q_3^3 = [Q_3, Q_3, Q_3]$. Also we check that $[L_0, U_0/U_0'] = U_0/U_0'$, so that $[L_0, U_0] = U_0$. We have $\langle U_{\pm\alpha_1}\rangle \le L_0$ acting irreducibly on

$[U_0, U_0, U_0]$. To show that $C_{P_3}(z) = U_0L_0$ we will use the following lemma.

(16.8) *Let* $M = \langle U_{\pm\alpha_1}, \cdots, U_{\pm 2n-1}\rangle \cong PSL(2n, q)$ *and* $M_0 = \langle U_{\alpha_1}(c)U_{\alpha_{2n-1}}(c), \cdots U_{\alpha_{n-1}}(c)U_{\alpha_{n+1}}(c), U_{\alpha_n}, s_1s_{2n-1}, \cdots s_{n-1}s_{n+1}, s_n : c \in F_q\rangle \cong Sp(2n, q)$ *be the natural embedding of* $Sp(2n, q)$ *in* $SL(2n, q)$. *Then* M_0 *is maximal in* M.

Proof. We sketch a proof by induction. Let V be the natural module for $SL(2n, q)$. It is easy to produce a non-degenerate alternating form on V left invariant by M_0. It follows that M_0 is transitive on $V^{\#}$. Suppose that $M_0 < T < M$ and let P_1 be the stabilizer in M of a 1-space in V. Then $M = M_0P_1$ and $T = M_0(P_1 \cap T)$. The structure of P_1 is well-known: $P_1 = QLH_0$, where Q is elementary of order $q^{2n-1}, L \cong SL(2n-1, q)$ acts in the natural way on Q, and H_0 is cyclic of order $q-1$ normalizing L and Q. Also $P_1 \cap M_0 = Q_0(L_0 \times H_1)$ where Q_0 is elementary of order q^{2n-1} and $L_0 \cong Sp(2n-2, q)$ acts on $Q_0/Q_0 \cap Q$ in the natural way. Here $Q_0 \cap Q$ is a 1-space in Q. If $T \cap Q > M_0 \cap Q = Q_0 \cap Q$, then since L_0 is irreducible on $Q/Q_0 \cap Q$ we have $Q \leq T$. But then $P_1 = N_M(Q)$ and $M = M_0P_1 = TP_1$ implies that $M = Q^M = Q^T \leq T$, a contradiction. So $T \cap Q = Q_0 \cap Q$ and $T \cap P_1$ stabilizes a 1-space of Q. Consequently $T \cap P_1 \leq QL_1H$ where L_1 is a maximal parabolic subgroup of L. But $QL_1H = P_{1,2}$ is a parabolic subgroup of M corresponding to the stabilizer of a 1-space of V and a hyperplane containing it. Now $P_{1,2} = QQ_0(SL(2n-2, q)H_{00})$ where H_{00} is abelian. If $T \cap P_1$ covers the $SL(2n-2, q)$ then we have $TB = M$ where $B \geq H_{00}Q_0Q$ is a Borel group of M. This contradicts the main theorem of [19]. Otherwise we have, by induction (or without any argument if $n = 2$), $T \cap P_1 \leq (M_0 \cap P_1)H_{00}$ and hence $T = M_0K$ where K can be chosen as a subgroup of H. From here it is easy to see that $K \leq M_0$ and $T = M_0$, a contradiction.

Since $O^{2'}(P_3/Q_3) \cong SL(2, q) \times SL(6, q)$ we can use (16.8) to verify that $U_0L_0 = C(z) \cap O^{2'}(P_3)$ and from here we obtain $U_0L_0 = C_{P_3}(z)$.

We claim that P_3 is the only parabolic subgroup of G that contains U_0L_0 and that $U_0L_0 = C_G(z)$. The second statement follows from the first since we know, from (2.3), that $C_G(z)$ is contained in a parabolic subgroup of G. So suppose that $X = U_0L_0 \leq P_i^g$. By order considerations for $X/U_0 \cong SL(2, q) \times Sp(6, q)$, we have $i \neq 2, 4, 5$. Also we have $|U_0| = q^{45}$ so $U_0 \not\leq Q_i^g$ unless $i = 3$. If $i = 3$ then $U_0 \leq Q_3^g$ and $P_3 = N_G([Q_3, Q_3, Q_3])$ $= N_G([U_0, U_0, U_0]) = N_G([Q^g, Q^g, Q^g]) = P_3^g$, so the claim holds. Suppose then that $i \neq 3$ as well. If $i = 7$, then $U_0 \not\leq Q_i^g$ and we have $U_0 \leq P_i^g \cap P_j^{gp}$ where $j = 1$ or 6 and $p \in P_i^g$. In either case we still cannot have $U_0 \leq (P_i^g \cap P_j^{gp})$. Consequently we must have $U_0L_0 \leq P_7^g \cap P_6^{gp} \cap P_1^{gp_1}$ for some $p_1 \in P_7^g$. Again order considerations rule this out. So $i \neq 7$. Similarly $i \neq 1, 6$. We now have the claim, so $C_G(z) = U_0L_0$.

We set $P = \langle U_\gamma(c)U_\delta(c)U_\varepsilon(c) : c \in F_q \rangle$. Then $P = Z(C_G(z))$. As in other cases we have H transitive on $U_\gamma^\# U_\delta^\# U_\varepsilon^\#$. Consequently there is an element $h \in H$ such that h centralizes $U_\delta U_\varepsilon$ but not U_γ. Then $PP^h = P \times U_\gamma$ and PP^h contains $q-1$ conjugates of x, $q-1$ conjugates of y, and $(q-1)^2$ conjugates of z. This completes our discussion of $C_G(z)$.

For $u = U_\varphi(1)U_\psi(1)U_\theta(1)$ we first list the roots $s \in \Delta^+$ such that U_s does not commute with U_φ, U_ψ, or U_θ.

$$(16.9)\quad \begin{array}{ll} U_\varphi & \{s = \alpha_6, \alpha_3, \alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, r_{30}, r_{34}, r_{36}, r_{45}\} \\ & \{\varphi + s = r_{17}, r_{16}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, r_{27}\} \\ U_\psi & \{s = \alpha_5, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5, \alpha_4 + \alpha_5 + \alpha_6, r_{31}, r_{32}, r_{37}, r_{44}\} \\ & \{\psi + s = r_{16}, r_{18}, r_{19}, r_{20}, r_{22}, r_{23}, r_{26}, r_{27}\} \\ U_\theta & \{s = \alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, r_{38}, r_{39}, r_{41}, r_{43}\} \\ & \{\varepsilon + s = r_{17}, r_{18}, r_{20}, r_{21}, r_{23}, r_{24}, r_{25}, r_{26}\}\ . \end{array}$$

So $C_U(U_\varphi U_\psi U_\theta) = \langle U_s : s$ not as in 16.9)$\rangle$. In addition, for each $c \in F_q$, $C_U(u)$ contains the elements

$$(16.10)\quad \begin{array}{l} U_{\alpha_6}(c)U_{\alpha_1}(c), U_{\alpha_3}(c)U_{\alpha_5}(c), U_{\alpha_3+\alpha_4}(c)U_{\alpha_4+\alpha_5}(c), U_{\alpha_1+\alpha_3}(c)U_{\alpha_5+\alpha_6}(c)\ , \\ U_{\alpha_4+\alpha_5+\alpha_6}(c)U_{\alpha_1+\alpha_3+\alpha_4}(c), U_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}(c)U_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}(c), U_{r_{30}}(c)U_{r_{31}}(c)\ , \\ U_{32}(c)U_{r_{38}}(c), U_{r_{34}}(c)U_{r_{39}}(c), U_{r_{36}}(c)U_{r_{41}}(c), U_{r_{37}}(c)U_{r_{43}}(c), U_{r_{45}}(c)U_{r_{44}}(c)\ . \end{array}$$

We note that $U_0 = Q_7 \leq C_U(U_\varphi U_\psi U_\varepsilon) \leq C_U(u)$ and that $C_U(u) = \langle C_U(U_\varphi U_\psi U_\theta),$ $U_{r_i}(c)U_{r_j}(c) : c \in F_q, (i, j)$ as in $(16.10)\rangle$. In particular $|U_0| = q^{27}$ and $|C_U(u)| = q^{51}$.

The Levi factor L of P_7 is isomorphic to $E_6(q)$. Let L_0 be the group of fixed points under the graph automorphism of L. Then

$$(16.11)\quad L_0 = \langle U_{\alpha_2}, U_{\alpha_4}, U_{\alpha_3}(c)U_{\alpha_5}(c), U_{\alpha_1}(c)U_{\alpha_6}(c), s_2, s_4, s_3s_5, s_1s_6 : c \in F_q \rangle\ .$$

Also we see that

$$(16.12)\quad \begin{array}{c} L_0 \cong F_4(q) \\ X = U_0L_0 \leq C_{P_7}(u)\ . \end{array}$$

We first claim that P_7 is the only parabolic subgroup of G containing X. Since $X/O_2(X) \cong L_0 \cong F_4(q)$ we see by order considerations that if $X \leq P_i^g$, then $i = 1$ or 7. Also we must have $U_0 \leq O_2(P_i^g)$, for otherwise $X \leq P_i^g \cap P_j^{gp}$ for some $i \neq j, p \in P_i^g$. If $i = 1$, then $O_2(P_i^g) \cong Q_1$ which is special of order q^{33} with center U_r. If M has index 2 in U_r, then Q_1/M is extra-special of order $2q^{32}$ and from here it is clear that $O_2(P_i^g)$ cannot contain an abelian group of order $q^{27} = |Q_7|$. Consequently $i = 7$, $U_0 \leq O_2(P_7^g)$ and by order $Q_7 = U_0 = O_2(P_7^g) = Q_7^g$. Then $g \in N(Q_7) = P_7$ and the claim holds.

Since $C_G(u)$ is contained in a parabolic subgroup of G we must have $C_G(u) \leq P_7$. The following lemma forces $C_G(u) = X$.

(16.13) *Let $L_0 \cong F_4(q)$ be embedded in $M = L_7 \cong E_6(q)$ as in* (16.12). *Then L_0 is maximal in M.*

Proof. We outline a proof in the following way. First check that if $L_0 < T < M$ and if $H \cap T > H \cap L_0$ then we have the contradiction that $T = M$. We will use Guterman [14] to obtain a contradiction. Let t_1, t_2, t_1t_2 be the central involutions given in (13.1) i), ii), iii) respectively. Then check that in $E_6(q)$, $t_1 \sim x$, $t_2 \sim y$, $t_1t_2 \sim y$, where x, y are as in (15.1).

The idea is to show that $C_T(t_1) = C_{L_0}(t_1), C_T(t_2) = C_{L_0}(t_1)$, and $C_T(t_1t_2) = C_{L_0}(t_1t_2)$. For then [14] shows that $T \cong F_4(q)$ and consequently $L_0 = T$, a contradiction. Using the result in § 13 and § 15 we see that in each case either the centralizers behave as desired or $T \geq U_{\alpha_i}$ for $i = 1, 3, 5$, or 6. But this implies that $T = M$. For example if $U_{\alpha_1} \leq T$, then $T \geq \langle U_{\alpha_1}, U_{\alpha_1}^{s_1s_6}\rangle = \langle U_{\pm\alpha_1}\rangle$ and $s_1 \in T$. Also $T \geq \langle U_{\alpha_1}U_{-\alpha_1}\rangle^{s_3s_5s_1s_6} = \langle U_{\pm\alpha_3}\rangle$ and $s_3 \in T$. Then $s_1, \cdots, s_6 \in T$ and clearly this forces $T = M$.

We have now found $C_G(u) = U_0L_0$ and we need the action of L_0 on $U_0 = Q_7$. Set $P = \langle U_\varphi(c)U_\psi(c)U_\theta(c) : c \in F_q\rangle$. Then L_0 centralizes $P \leq Q_7$ and $P = Z(C_G(u))$. Also we argue as in previous situations to find $h \in H$ such that $PP^h = U_\varphi \times P$. Then PP^h contains $q - 1$ conjugates of x, $q - 1$ conjugates of y, and $(q-1)^2$ conjugates of u.

Let $S = \langle U_\varphi(c)U_\psi(c), U_\varphi(c)U_\theta(c) : c \in F_q\rangle$ and set

$$U_1 = \langle S, U_s : s \in \mathscr{L}_7, s \neq \varphi, \psi, \theta\rangle . \tag{16.14}$$

Then checking the action of L_0 on $U_0 = Q_7$ we see that

$$U_0L_0 = U_1L_1 \times P \tag{16.15}$$

and

$$(U_0L_0)' = U_1L_1 .$$

The final centralizer to consider in this section is $C_G(v)$, where $v = U_\gamma(1)U_\varphi(1)U_\psi(1)U_\theta(1) = U_\gamma(1)u$. The roots $s \in \Delta^+$ with $U_s \not\leq C(U_\varphi U_\psi U_\theta)$ are listed in (16.9) and for U_γ in (16.4). Each root s in (16.9) satisfies $U_s \leq C(U_\gamma)$, so the elements in (16.10) centralize v. In addition for each $c \in F_q$ the following are in $C(v)$

$$(16.16) \quad \begin{array}{l} U_{r_1}(c)U_{r_{30}}(c),\ U_{r_2}(c)U_{r_{32}}(c),\ U_{r_3}(c)U_{r_{34}}(c) \\ U_{r_4}(c)U_{r_{36}}(c),\ U_{r_5}(c)U_{r_{37}}(c),\ U_{r_6}(c)U_{r_{45}}(c) . \end{array}$$

It follows that $|C_U(U_\gamma U_\varphi U_\psi U_\theta)| = q^{63-24-6} = q^{33}$ and $|C_U(v)| = q^{33+18} = q^{51}$. Notice that $C_U(v) \geq C_U(U_\gamma U_\varphi U_\psi U_\theta) \geq Q_2 \cap Q_7$, $C_{Q_7}(v) = Q_2 \cap Q_7$, and $C_{Q_2}(U_\gamma U_\varphi U_\psi U_\theta) = (Q_2 \cap Q_7)\langle U_s : s = r_{28}, r_{29}, r_{33}, r_{35}, r_{40}, r_{42}, r_{44}, r_{47}, r_{48}\rangle$. Setting $U_0 = C_{Q_2Q_7}(v)$ we have

$$(16.17) \quad \begin{array}{l} U_0 = C_{Q_2}(U_\gamma U_\varphi U_\psi U_\theta)\langle U_{r_i}(c)U_{r_j}(c) : c \in F_q,\ (i,j) \text{ as in (16.16) or} \\ \qquad (i,j) = (30,31), (32,38), (34,39), (36,41), (37,43), (45,44)\rangle . \end{array}$$

Setting

$$(16.18) \quad L_0 = \langle U_{\alpha_4}, U_{\alpha_3}(c)U_{\alpha_5}(c), U_{\alpha_1}(c)U_{\alpha_6}(c), s_4, s_3s_5, s_1s_6 = c \in F_q\rangle ,$$

we have from the work on $C_G(u)$, that $L_0 \cong B_3(q) \cong Sp(6, q)$ as it occurs naturally in a parabolic subgroup of $\langle U_{\pm\alpha_2}, L_0\rangle$, which was proved to be naturally isomorphic to $F_4(q)$. Now set $X = U_0L_0$. Then $X = C(v) \cap P_{2,7}$.

We note that L_0 normalizes U_0 and we consider the structure of X. First use Table 3 and (3.1) to show that $[U_0, U_0] = \langle U_s, U_\varphi(c)U_\psi(c), U_\varphi(c)U_\theta(c) : c \in F_q, s \in (\mathscr{L}_7 \cap \mathscr{L}_2) \cup \mathscr{L}_2^2, s \neq r_{13}, r_{14}, r_{15}\rangle$ and also that $[U_0, U_0, U_0] = Q_2^2 \cap Q_7 = [Q_2Q_7, Q_2Q_7, Q_2Q_7]$.

We note that $Q_2^2 = U_\gamma \times (Q_2^2 \cap Q_7)$ and that L_0 acts on $Q_2^2 \cap Q_7$ and centralizes U_γ. Let $P = \langle U_\gamma(c)U_\varphi(c)U_\psi(c)U_\theta(c) : c \in F_q\rangle$ and $P_0 = \langle U_\varphi(c)U_\psi(c)U_\theta(c) : c \in F_q\rangle$. Then we check that $[L_0, U_0/U_0'] \times P_0U_0/U_0' = U_0/U_0'$, so that $U_0 = [L_0, U_0] \times P_0$ and $X' = [L_0, U_0]L_0 = X''$.

As in other cases we obtain H transitive on $U_\gamma^\# U_\varphi^\# U_\psi^\# U_\theta^\#$, so there is an element $h \in H$ such that h centralizes $U_\varphi U_\psi U_\theta$ but not U_γ. Then $PP^h = U_\gamma \times P$ and PP^h contains $q-1$ conjugates of x, $q-1$ conjugates of u, and $(q-1)^2$ conjugates of v. Moreover we can check that $P = Z(X)$.

We claim that the only parabolic subgroups of G that contain X are P_2 and P_7. Say $X \leq P = P_i^g$. Since X involves $Sp(6, q)$, $i \neq 4, 5$. Say $i = 1$. Since $|Q_1| = q^{33}$ and $|U_0| = q^{42}$, we have $U_0 \not\leq O_2(P)$ and hence $X \leq P_i^g \cap P_j^{gp}$ for some $j \neq i$ and $p \in P$. The only possible values for j are $j = 2, 3, 6$ or 7. Say $j = 2$. Then we must have $U_0 \not\leq O_2(P_i^g \cap P_j^{gp}) \sim Q_1Q_2$. We observe that $|U_0'| = q^{21}$ and that U_0' is abelian. Consequently, since Q_1 is special with center U_r and $|Q_1| = q^{33}$ we cannot have U_0' contained in Q_1. However Q_2Q_1/Q_1 is abelian, so the case is out. The same argument works for $j = 3, 6, 7$. Thus $i \neq 1$. If $i = 3$, then since $L_3 \cong SL(2, q) \times SL(6, q)$ we must have $U_1 = [U_0, L_0] \leq O_2(P_3^g) \cong Q_3$. But $|[Q_3, Q_3, Q_3]| = q^2$ while $|[U_1, U_1, U_1]| = q^6$. So $i \neq 3$. If $i = 2$ then $X \not\leq O_2(P)$ and $X \leq P_2^g \cap P_j^{gp}$ for some $j \neq 1, 2, 3, 4, 5$. As $X/U_0 \cong Sp(6, q)$ we must have $j = 7$ and $U_0 \leq O_2(P_2^g \cap P_j^{gp}) = Q$. Then order considerations show that $[U_0, U_0, U_0] = [Q, Q, Q]$ so that $g \in N([Q_2Q_7, Q_2Q_7, Q_2Q_7]) = P_{2,7} \leq P_2$ and $P_2^g = P_2$. Next we assume that $i = 6$. Then since $O_2(P) \sim Q_6$ has class 2 we must have $U_0 \not\leq O_2(P)$ and so $X \leq P_6^g \cap P_7^{gp}$, some $p \in P$. Then $U_0 \leq O_2(P_6^g \cap P_7^{gp}) \sim Q_6Q_7$. Since $|Q_6Q_7/Q_6| = q$ and $L_{6,7}$ is trivial on Q_6Q_7/Q_6, we must have $[U_0, L_0] \leq Q_6$. However $[U_0, L_0]$ has class 3. This is impossible, showing that $i \neq 6$. Finally $i = 7$ forces $X \leq P_7^g \cap P_2^{gp}$ and we proved earlier that this forces $gp \in P_2 \cap P_7$ and as $p \in P_7^g$, we have $P = P_7$. At this point the claim is proved.

Since $C_G(v) \leq P$ for some parabolic subgroup P of G, $C_G(v) \leq P_2$ or P_7. We will show that $C_G(v)$ is contained in $P_2 \cap P_7$. If $C_G(v) \leq P_7$ then since $v \notin Q_7$, we have $C_G(v)Q_7/Q_7$, centralizing an involution in P_7/Q_7 and this forces $C_G(v) \leq P_2 \cap P_7$. Suppose that $C_G(v) \leq P_2$. It is easily verified from (3.1) and Table 3 that L_2 acts on Q_2^2 as on the natural module for $SL(7, q)$. Also XQ_2/Q_2, viewed as a subgroup of L_2, fixes the hyperplane $Q_2^2 \cap Q_7$ of Q_2^2 and is transitive on the remaining hyperplanes. If $C_G(v)$ does not fix $Q_2^2 \cap Q_7$, then it follows that $C_G(v)$ is 2-transitive on the set of hyperplanes in Q_2^2 and this forces $(Q_7Q_2/Q_2)^{C_G(v)} = (Q_7Q_2/Q_2)^{P_2} = L_2Q_2/Q_2$ to be in $C_G(v)Q_2/Q_2$. But this is impossible as L_2 does not fix vQ_2'/Q_2'. Thus $C_G(v)$ stabilizes $Q_2^2 \cap Q_7$ and $C_G(v) \leq P_2 \cap P_7$ as desired.

Since $C_G(v) \leq P_2 \cap P_7$ and $L_{2,7} \cong SL(6, q)$ we apply (16.8) to see that $X = C_G(v)$. We have now completed our discussion of $E_7(q)$ and we list our results below.

(16.19) *The maximal parabolic subgroups containing the centralizers of* *the involutions in* (16.1) *are as follows:*

i) $C_G(x) \leq P_1$

ii) $C_G(y) \leq P_6$

iii) $C_G(z) \leq P_3$

iv) $C_G(u) \leq P_7$

v) $C_G(v) \leq P_2, P_7$.

In particular, no two of the involutions x, y, z, u, v *are conjugate.*

(16.20) *The structure of the centralizers of the involutions on* (16.1) *are as follows:*

i) $C_G(x) = Q_1L_1$ *and* $L_1 \cong SO^+(12, q)$ *acts irreducibly on* $Q_1/Z(Q_1) = Q_1/U_r$. *Also* $P_1 = Q_1(L_1 \times W_0)$, *where* $W_0 \leq H$ *is cyclic of order* $q-1$ *and* $C_G(W_0) = W_0 \times L_1$.

ii) $C_G(y) = U_0L_0$, *where* $U_0 = Q_6, L_0 \cong Sp(8, q) \times SL(2, q)$ *and* $L_0 = \langle U_{\alpha_1}, U_{\alpha_3}, U_{\alpha_4}, U_{\alpha_2}(c)U_{\alpha_5}(c), s_1, s_3, s_4, s_2s_5 : c \in \boldsymbol{F}_q\rangle \times \langle U_{\pm\alpha_7}\rangle$. $[L_0, U_0] = U_0, C_G(y) = C_G(y)'$ *if* $q > 2$, *and* $C_G(y)'$ *has index* 2 *in* $C_G(y)$ *if* $q = 2$. *The group* $L_{6,7} \cong SO^+(10, q)$ *acts on* $[U_0, U_0] = Z(U_0)$ *and preserves a non-degenerate quadratic form. In this action* $L_0 \cap L_{6,7}$ *is the stabilizer of the anisotropic* 1*-space* $P = \langle U_\alpha(c)U_\beta(c) : c \in \boldsymbol{F}_q\rangle = Z(C_G(y))$. *There is an element* $h \in H$ *such that* $PP^h = U_\alpha U_\beta$ *contains* $2(q-1)$ *conjugates of* x *and* $(q-1)^2$ *conjugates of* y.

iii) $C_G(z) = U_0L_0$, *where* $U_0 \leq Q_3$ *and* $L_0 \leq L_3$ *are given in* (16.7). *We have* $L_0 \cong SL(2, q) \times Sp(6, q)$, $[L_0, U_0] = U_0$, *and* U_0 *has class* 3 *with* $[U_0, U_0, U_0] = Q_3^3$. $C_G(z)' = C_G(z)$ *if* $q > 2$ *and* $C_G(z)'$ *has index* 2 *in* $C_G(z)$ *if* $q = 2$. $Z(C_G(z)) = P = \langle U_\gamma(c)U_\delta(c)U_\varepsilon(c) : c \in \boldsymbol{F}_q\rangle$ *and there is an element* $h \in H$ *such that* $PP^h = P \times U_\gamma$ *contains* $q-1$ *conjugates of* $x, q-1$ *conjugates of* y, *and* $(q-1)^2$ *conjugates of* z.

iv) $C_G(u) = U_0L_0$, *where* $U_0 = Q_7, F_4(q) \cong L_0 \leq L_7$, *and* $L_0 = \langle U_{\alpha_2}, U_{\alpha_4}, U_{\alpha_3}(c)U_{\alpha_5}(c), U_{\alpha_1}(c)U_{\alpha_6}(c), s_2, s_4, s_3s_5, s_1s_6 : c \in \boldsymbol{F}_q\rangle$. *Let* $P = \langle U_\varphi(c)U_\psi(c)U_\theta(c) : c \in \boldsymbol{F}_q\rangle$. *Then* $P = Z(C_G(u)), U_0 = [L_0, U_0] \times P$, *and* $C_G(v) = [L_0, U_0]L_0 \times P$ *has derived group* $[L_0, U_0]L_0$. *There is an element* $h \in H$ *such that* $PP^h = U_\varphi \times P$ *contains* $q-1$ *conjugates of* $x, q-1$ *conjugates of* y, *and* $(q-1)^2$ *conjugates of* u.

v) $C_G(v) = U_0L_0$, *where* $U_0 \leq Q_2Q_7$ *is given in* (16.17) *and* $Sp(6, q) \cong L_0 \leq L_{2,7}$ *is given in* (16.18). U_0 *has class* 3 *with* $[U_0, U_0, U_0] = Q_7 \cap Q_2^2 = [Q_2Q_7, Q_2Q_7, Q_2Q_7]$. *Set* $P = \langle U_\gamma(c)U_\varphi(c)U_\psi(c)U_\theta(c) : c \in \boldsymbol{F}_q\rangle$ *and* $P_0 = \langle U_\varphi(c)U_\psi(c)U_\theta(c) : c \in \boldsymbol{F}_q\rangle$. *Then* $U_0 = [L_0, U_0] \times P_0$ *and* $C_G(v)' = [L_0, U_0]L_0$.

There is an element $h \in H$ such that $PP^h = U_r \times P$ contains $q-1$ conjugates of x, $q-1$ conjugates of u, and $(q-1)^2$ conjugates of v. Also $P = Z(C_G(v))$.

Section 17. Centralizers of involutions in $E_8(q)$, q even.

Let $G = E_8(q)$, $q = 2^a$. Then $|H| = (q-1)^8$, $W = \langle s_1, \cdots, s_8 \rangle$ and the action of W on Δ is determined by

$$(\alpha_3)s_1 = (\alpha_1)s_3 = \alpha_1 + \alpha_3$$
$$(\alpha_2)s_4 = (\alpha_4)s_2 = \alpha_2 + \alpha_4$$
$$(\alpha_j)s_i = \alpha_i + \alpha_j \text{ if } |i-j| = 1 \text{ and } \{i,j\} \neq \{1,2\}, \{2,3\}$$
$$(\alpha_i)s_i = -\alpha_i ,$$

with $(\alpha_i)s_i = \alpha_j$ for all other pairs (i,j). As in $E_6(q)$ and $E_7(q)$ commutators are simplified as there is but one root length. We use the notation of Table 4 and distinguish the roots $r, \alpha, \beta, \gamma, \delta, \varepsilon, \varphi, \psi, \theta, \omega$. From (12.10) we have

(17.1) *Each involution in $E_8(q)$, q even, is conjugate to one of the following*

i) $x = U_r(1)$

ii) $y = U_\alpha(1)U_\beta(1)$

iii) $z = U_\gamma(1)U_\delta(1)U_\varepsilon(1)$

iv) $v = U_\varphi(1)U_\psi(1)U_\theta(1)U_\omega(1)$.

We will determine the centralizers of x, y, z, v. For $x = U_r(1)$ we use the following information from Section 4 of [7]. $C_G(x) = O^{2'}(P_8) = Q_8L_8$ and $L_8 \cong E_7(q)$ acts irreducibly on Q_8/U_r. Also we check that $P_8 = Q_8(L_8 \times W_0)$, when W_0 is cyclic of order $q-1$, W_0 is regular on $U_r^\#$, and $C_G(W_0) = W_0 \times L_8$.

Next consider $C_G(y)$, $y = U_\alpha(1)U_\beta(1)$. Since $\alpha, \beta \in \mathscr{L}_2^1$ $U_\alpha U_\beta \leq Z(Q_1)$ and $U_0 = Q_1 \leq C_U(y)$. The roots $s \in \Delta^+$ such that U_s does not centralize U_α or U_β are as follows

$$(17.2) \quad \begin{aligned} U_\alpha \quad & \{s = \alpha_2, \alpha_2+\alpha_4, \alpha_2+\alpha_4+\alpha_5, \alpha_2+\alpha_4+\alpha_5+\alpha_6, \\ & \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7, r_6\}\{s+\alpha = r_{52}, r_{53}, r_{54}, r_{55}, r_{56}, r_{57}\} \\ U_\beta \quad & \{s = \alpha_3, \alpha_3+\alpha_4, \alpha_3+\alpha_4+\alpha_5, \alpha_3+\alpha_4+\alpha_5+\alpha_6, \\ & \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, r_7\} \quad \{s+\beta = r_{52}, r_{53}, r_{54}, r_{55}, r_{56}, r_{57}\} . \end{aligned}$$

Consequently $C_U(U_\alpha U_\beta) = \langle U_s : s$ not in (17.2)$\rangle$. In addition for each $c \in \boldsymbol{F}_q$, y is centralized by

$$(17.3)\quad \begin{aligned} &U_{\alpha_3+\alpha_4}(c)U_{\alpha_2+\alpha_4}(c),\ U_{\alpha_3+\alpha_4+\alpha_5}(c)U_{\alpha_2+\alpha_4+\alpha_5}(c),\ U_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}(c)U_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}(c)\\ &U_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}(c)U_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7}(c),\ U_{r_7}(c)U_{r_6}(c)\ . \end{aligned}$$

It now follows that $C(y)$ contains L_0 where

$$(17.4)\quad \begin{aligned} L_0 &= \langle U_{\alpha_2}(c)U_{\alpha_3}(c), U_{\alpha_4}, U_{\alpha_5}, U_{\alpha_6}, U_{\alpha_7}, U_{\alpha_8}, s_2s_3, s_4, s_5, s_6, s_7, s_8 : c \in \boldsymbol{F}_q\rangle\\ &\cong B_6(q) \cong Sp(12, q) \cong SO(13, q)\ . \end{aligned}$$

Now $U_0' = U_1' = Q_2^1$. As in other situations it is easy to define a non-degenerate quadratic form in Q_1' that is preserved by $L_1 \cong SO^+(14, q)$. In this action the root groups are isotropic 1-spaces and $U_\alpha U_\beta$ is a hyperbolic plane. The element y is anisotropic so $C_{L_1}(y) \cong SO(13, q)$. It follows that $X = U_0L_0 = C_{P_1}(y)$. We check that $[L_0, U_0] = U_0$.

We claim that P_1 is the only parabolic subgroup of G that contains X. Say $X \leq P_i^g$. By order considerations with respect to $X/U_0 \cong Sp(12, q)$ we see that $i = 1, 7$, or 8. We can eliminate $i = 7$ since $Sp(12, q) \not\lesssim E_6(q)$. Consequently $i = 1$ or 8 and by orders and (2.3) we have $U_0 \leq O_2(P_i^g)$. Then $i = 8$ is out since $|U_0'| > q = |Q_8'|$. Consequently $i = 1, U_0 = O_2(P_1^g) = Q_1^g$, and $g \in P_1$ as desired.

Since $C_G(y)$ is contained in some parabolic subgroup of G we can now conclude that $X = C_G(y)$. Since H is transitive on $U_\alpha^\# U_\beta^\#$ there is an element $h \in H$ such that h centralizes U_β but not U_α. Setting $P = \langle U_\alpha(c)U_\beta(c) : c \in \boldsymbol{F}_q\rangle$, we have $PP^h = P \times U_\alpha$ contains $2(q-1)$ conjugates of x and $(q-1)^2$ conjugates of y. Also we check that $P = Z(C_G(y))$.

Next we find $C_G(z)$, where $z = U_\gamma(1)U_\delta(1)U_\varepsilon(1)$. We first note that the roots $s \in \mathscr{L}_7$ such that $U_s \not\leq C(U_\gamma U_\delta U_\varepsilon)$ are as follows

$$(17.5)\qquad r_{14} - \alpha_8, r_{15} - \alpha_8, r_{16} - \alpha_8, r_{14}, r_{15}, r_{16}\ .$$

However z is centralized by each of

$$(17.6)\quad \begin{aligned} &U_{r_{14}-\alpha_8}(c)U_{r_{15}-\alpha_8}(c),\ U_{r_{15}-\alpha_8}(c)U_{r_{16}-\alpha_8}(c)\\ &U_{r_{14}}(c)U_{r_{15}}(c),\ U_{r_{15}}(c)U_{r_{16}}(c) \end{aligned}$$

for $c \in \boldsymbol{F}_q$. It follows that

(17.7) $U_0 = \langle U_t, U_{\varphi i}(c)U_{\varphi j}(c) : t \in \mathscr{L}_7$, t not in (17.9) and $U_{\varphi i}(c)U_{\varphi j}(c)$ as in (17.6)$\rangle = C_{Q_7}(z)$.

Moreover $|Q_7: U_0| = q^2$ and $|U_0| = q^{81}$. Next let

$$(17.8)\quad L_{00} = \langle U_{\alpha_2}, U_{\alpha_4}, U_{\alpha_3}(c)U_{\alpha_5}(c), U_{\alpha_1}(c)U_{\alpha_6}(c), s_2, s_4, s_3s_5, s_1s_6 : c \in F_q \rangle .$$

Then using (3.1) we have $L_{00} \leq C(z)$. In addition L_{00} is the fixed point group under the graph automorphism of the Levi factor, L, of $P_{7,8}$ we have $L_{00} \cong F_4(q)$ and L_{00} maximal in L by (16.13). Consequently

$$(17.9)\quad L_0 = \langle L_{00}, U_{\pm\alpha_8} \rangle \cong F_4(q) \times SL(2, q) \qquad X = U_0L_0 = C_{P_7}(z) .$$

Next we use Table 4 and (3.1) to show $U_0' = Q_7' = Q_2^7$ and $[U_0, U_0, U_0] = Q_3^7$. Moreover $z \in U_0'$ and $[L_0, U_0] = U_0$.

We claim that the only parabolic subgroup of G containing X is P_7. For suppose that $X \leq P_i^g$. Since X involves $F_4(q) \times SL(2, q)$ the only possible values for i are $i = 1, 2, 7, 8$. By (2.3) we can choose a suitable parabolic subgroup P of P_i^g such that $U_0 \leq O_2(P)$. Then $P/O_2(P)$ will involve $F_4(q) \times SL(2, q)$. But then in $P/O_2(P)$ some involution contains $F_4(q)$ in its centralizer. Since $|U_0| = q^{81}$ we can easily check that only $i = 7$ is possible and here $P = P_7^g$. Then $Q_2^7 = [U_0, U_0, U_0] = (Q_2^7)^g$ and $g \in N_G(Q_2^7) = P_7$. This proves the claim. Since $C_G(z)$ is contained in a parabolic subgroup we now have $X = C_G(z)$.

Set $P = \langle U_\gamma(c)U_\delta(c)U_\varepsilon(c) : c \in F_q \rangle$. Then $P = Z(C_G(z))$ and there is an element $h \in H$ centralizing $U_\delta U_\varepsilon$ but not U_γ. Consequently $PP^h = U_\gamma \times P$ contains $q - 1$ conjugates of $x, q - 1$ conjugates of y, and $(q - 1)^2$ conjugates of z. We check that $P = Z(X)$.

We still must find $C_G(u)$, where $u = U_\varphi(1)U_\psi(1)U_\theta(1)U_\omega(1)$. We begin with $C_{Q_2}(u)$. Listed below are the roots $s \in \mathscr{L}_2$ such that U_s does not centralize one of the subgroups $U_\varphi, U_\psi, U_\theta, U_\omega$.

$$(17.10)\quad \begin{aligned} &U_\varphi\{s = r_6, r_8, r_{11}, r_{13}, r_{16}, r_{31}\}\{s + \varphi = r_{50}, r_{52}, r_{53}, r_{54}, r_{55}, r_{56}\} \\ &U_\psi\{s = r_6 - \alpha_8, r_{10} - \alpha_8, r_{12} - \alpha_8, r_{14} - \alpha_8, r_{18} - \alpha_8, r_{32}\} \\ &\qquad \{s + \psi = r_{48}, r_{52}, r_{53}, r_{54}, r_{55}, r_{57}\} \\ &U_\theta\{s = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, r_{12} - \alpha_6 - \alpha_7 - \alpha_8, \\ &\qquad r_{15} - \alpha_6 - \alpha_7 - \alpha_8, r_{20} - \alpha_7 - \alpha_8, r_{20} - \alpha_8, r_{20}\} \\ &\qquad \{s + \theta = r_{48}, r_{50}, r_{52}, r_{55}, r_{56}, r_{57}\} \\ &U_\omega\{s = r_8 - \alpha_7 - \alpha_8, r_{10} - \alpha_7 - \alpha_8, r_{15} - \alpha_7 - \alpha_8, \\ &\qquad r_{17} - \alpha_7 - \alpha_8, r_{19} - \alpha_8, r_{19}\} \\ &\qquad \{s + \omega = r_{48}, r_{50}, r_{53}, r_{54}, r_{56}, r_{57}\} . \end{aligned}$$

Consequently $C_{Q_2}(U_\varphi U_\psi U_\theta U_\omega) = \langle U_s : s \in \mathscr{L}_2, s$ not in (17.10)$\rangle$. In addition for each $c \in F_q$ u is centralized by each element of the form

$$(17.11) \qquad U_{\gamma_1}(c)U_{\gamma_2}(c) ,$$

where γ_1, γ_2 appear in (17.10) and $\gamma_1 + \rho_1 = \gamma_2 + \rho_2$ for $\rho_1, \rho_2 \in \{\varphi, \psi, \theta, \omega\}$. Consequently we have

(17.12) $U_0 = C_{Q_2}(u) = \langle U_t, U_{\gamma_1}(c)U_{\gamma_2}(c) : t \in \mathscr{L}_2, t$ *not in* (17.10), γ_1, γ_2 *as in* (17.11)$\rangle$.

Next we set

(17.13) $L_0 = \langle U_{\alpha_5}, U_{\alpha_1}(c)U_{\alpha_8}(c), U_{\alpha_3}(c)U_{\alpha_7}(c), U_{\alpha_4}(c)U_{\alpha_6}(c), s_5, s_1s_8, s_3s_7, s_4s_6 : c \in F_q\rangle$.

Then we check that $L_0 \leq C(u)$. Moreover if L is the Levi factor of P_2 then L_0 is the fixed point group under the graph automorphism. Consequently $L_0 \cong Sp(8, q)$. By (16.8) L_0 is maximal in L, and since $L \not\leq C(u)$ we check the action of H and conclude that $X = U_0L_0 = C_{P_2}(u)$.

We note that $U_0 \leq Q_2$ and $|U_0| = q^{82}$. Using Table 3 and 4 we check that $U_0' = Q_2' = Q_2^2$ and that $[U_0, U_0, U_0] = [Q_2, Q_2, Q_2] = Q_3^2$. Then $u \in U_0'$. Also $|U_0'| = q^{36}$ and $|[U_0, U_0, U_0]| = q^8$. We have L_2 acting on $[U_0, U_0, U_0]$ as on the natural module for $SL(8, q)$ and there is a non-degenerate alternating form preserved by L_0. Also $[L_0, X_0] = X_0$ and we can use the action of L_0 on U_0 and the structure of U_0 to see that $Z(X) = \langle U_\varphi(c)U_\psi(c)U_\theta(c)U_\omega(c) : c \in F_q\rangle$.

We claim that P_2 is the only parabolic subgroup of G containing X. Say $X \leq P_i^g = P$. Then $L_0 \cong Sp(8, q)$ implies that $i \neq 3, 4, 5$. First suppose that $i = 8$. Then $|O_2(P)| = |Q_8| = q^{57}$ and $P' = C_G(Z(O_2(P)))$. If $Z(O_2(P)) \cap X \neq 1$ then $Z(O_2(P)) \cap X$ is properly contained in $Z(X)$.

We first claim that $Z(O_2(P)) \leq U_0$. Otherwise since $O_2(P)$ is special with center U_r^g we have $(U_0 \cap O_2(P))/(U_0 \cap Z(O_2(P)))$ abelian and with the structure of $O_2(P)/U_0 \cap Z(O_2(P))$ this forces $|U_0 \cap O_2(P)| < q^{29}$, and hence $|U_0O_2(P)/O_2(P)| > q^{53}$. But since $P'/O_2(P) \cong E_7(q)$ and $L_0 \cong Sp(8, q)$ this is impossible. Consequently $Z(O_2(P)) \leq U_0$.

Now $|U_0O_2(P)/O_2(P)| \geq q^{82-57} = q^{25}$ and U_0' has order q^{36} and is abelian. Again the structure of $O_2(P)$ forces $|U_0' \cap O_2(P)| \leq q^{29}$. As $Z(O_2(P)) \leq Z(X)$, $Z(X) = U_r^g = Z(O_2(P))$ (which would give $u \sim x$). Considering the possibilities for $U_0O_2(P)/O_2(P)$ we see that $|U_0' \cap O_2(P)| > q$. From here it follows that $[U_0, U_0, U_0] = Q_3^2 \leq O_2(P)$. Indeed L_0 acts irreducibly on $[U_0, U_0, U_0]$ so if the containment were false then $[U_0, U_0, U_0] \cap O_2(P) = 1$. But no parabolic subgroup D of $E_7(q)$ involving $Sp(8, q)$ has $O_2(D)$ of class at least 3 with $|[O_2(D), O_2(D), O_2(D)]| \geq q^8$. The assertion follows. Let $|U_0' \cap U_2(P)| = q^\ell$. Then $9 \leq \ell \leq 29$ by the above. Since $Z(X) \cap [U_0, U_0, U_0] = 1$, $U_0 \cap O_2(P)$ must centralize $U_0' \cap O_2(P)$. The structure of $O_2(P)$ then implies that $|U_0 \cap O_2(P)| \leq q^{57-\ell}$ and $|U_0O_2(P)/O_2(P)| \geq q^{25+\ell} \geq q^{34}$. Next we use the fact that $Q_3^2 \leq O_2(P)$ and Q_3^2 is abelian to get $C_{O_2(P)}(Q_3^2) = q^{49}$ (here we use the theory ofextra special groups). But $C_{O_2(P)}(Q_3^2) \leq N_G(Q_3^2) = P_2$, and this implies that $C_{O_2(P)}(Q_3^2) \leq U_0$ and $|U_0 \cap O_2(P)| \geq q^{49}$. But then $|U_0| \geq q^{49}q^{34} = q^{83}$, a contradiction. Thus $i \neq 8$.

Suppose that $i = 7$. Then $|O_2(P)| = q^{83}$. Since $|[Q_7, Q_7, Q_7]| = |Q_3^7| = q^2$, we cannot have $U_0 \leq O_2(P) \cong Q_7$. The only possibility is $X \leq P_i^g \cap P_j^{gp}$ for $j = 1$ or 6 and $g \in P$. Then $|U_0O_2(P)/O_2(P)| \leq q^{16}$ and $|U_0 \cap U_2(P)| \geq q^{66}$. Also in this case $U_0' \leq O_2(P)$, so $Q_3^2 \leq O_2(P)$ and $Z(O_2(P)) \leq N(Q_3^2) = P_2$. But $O_2(P)$ has center of order q^2 which will be centralized by L_0. This contradicts $|Z(X)| = q$.

Next assume that $i = 6$. Here $(P/O_2(P))' \cong SO^+(10, q) \times SL(3, q)$. As $[L_0, U_0] = U_0$ we must have $U_0 \leq O_2(P)$. Since $O_2(P) \sim Q_6, Z(O_2(P)) \sim Q_4^6$ which is centralized by L_0. As above this contradicts $|Z(X)| = q$.

If $i = 1$, then $|O_2(P)| < |U_0|$ and we have $U_0 \not\leq O_2(P)$. Then we have $X \leq P_1^g \cap P_j^{gp}$ for some j and some $g \in P$. By orders $j \neq 2$, so by our previous work this is impossible.

Finally if $i = 2$ we argue that by the above $U_0 \leq O_2(P)$ and then $Q_3^2 = [U_0, U_0, U_0] = [O_2(P), O_2(P), O_2(P)] = (Q_3^2)^g$. So $g \in N(Q_3^2) = P_2$. This proves the claim. Consequently $C_G(u) = C_{P_2}(u) = X$.

Setting $P = \langle U_\varphi(c)U_\psi(c)U_\theta(c)U_\omega(c) : c \in F_q \rangle$ we have $P = Z(X)$ and there is an element $h \in H$ centralizing $U_\psi U_\theta U_\omega$ but not U_φ. Then $PP^h = P \times U_\varphi$ contains $q-1$ conjugates of z, and $(q-1)^2$ conjugates of u.

We have now completed our analysis of $E_8(q)$ and we summarize our results as follows.

(17.14) *The maximal parabolic subgroups of G containing the centralizers of the involutions in* (17.1) *are as follows.*

i) $C_G(x) \leq P_8$
ii) $C_G(y) \leq P_1$
iii) $C_G(z) \leq P_7$

iv) $C_G(u) \leq P_2$.
In particular no two of the involutions x, y, z, u are conjugate.

(17.15) i) $C_G(x) = O^{2'}(P_8) = Q_8L_8$, *where* $L_8 \cong E_7(q)$ *acts irreducibly on* $Q_8/U_r = Q_8/Q_8'$. *Also* $P_8 = Q_8(L_8 \times W_0)$ *where* W_0 *is cyclic of order* $q-1$ *and* $C_G(W_0) = W_0 \times L_8$.

ii) $C_G(y) = U_0L_0$, *where* $U_0 = Q_1$ *and* $L_0 \cong Sp(12, q)$ *is as in* (17.4). *Also* $y \in U_0' = Z(U_0)$, $[L_0, U_0] = U_0$, *and* $C_G(y)' = C_G(y)$. *Let* $P = \langle U_\alpha(c) U_\beta(c) : c \in \boldsymbol{F}_q \rangle$. *Then* $P = Z(C_G(y))$ *and there is an element* $h \in H$ *such that* $PP^h = P \times U_\alpha \leq C_G(y)$ *contains* $2(q-1)$ *conjugates of* x *and* $(q-1)^2$ *conjugates of* y. *The group* $L_1 \cong SO^+(14, q)$ *acts on* U_0' *preserving a non-degenerate quadratic form. In this action* P *is an anisotropic* 1-*space.*

iii) $C_G(z) = U_0L_0$ *where* $U_0 \leq Q_7$ *is given by* (17.8) *and* $L_0 \cong F_4(q) \times SL(2, q)$ *is given by* (17.9). *We have* $z \in U_0' = Q_2^7$ *and* $[U_0, U_0, U_0] = Q_3^7$. *Also* $P = \langle U_\gamma(c)U_\theta(c)U_\varepsilon(c) : c \in \boldsymbol{F}_q \rangle = Z(C_G(Z))$, $[L_0, U_0] = U_0$ *and* $|C_G(y) : C_G(y)'| = 1, 2$ *depending on whether* $q > 2$ *or* $q = 2$. *There is an element* $h \in H$ *such that* $PP^h = P \times U_\gamma \leq C_G(z)$ *contains* $q-1$ *conjugates of* x, $q-1$ *conjugates of* y, *and* $(q-1)^2$ *conjugates of* z.

iv) $C_G(u) = U_0L_0$, *where* U_0, L_0 *are given in* (17.12) *and* (17.13). *Here* $U_0 \leq Q_2$ *and* $L_0 \cong Sp(8, q)$. *We have* $u \in U_0' = Q_2'$, $[U_0, U_0, U_0] = Q_3^2$, $[L_0, U_0] = U_0$, *and* $C_G(y) = C_G(y)'$. *Setting* $P = \langle U_\varphi(c)U_\psi(c)U_\theta(c)U_\omega(c) : c \in \boldsymbol{F}_q \rangle$ *we have* $P = Z(C_G(u))$. *There is an element* $h \in H$ *such that* $PP^h = P \times U_\varphi \leq C_G(u)$ *contains* $q-1$ *conjugates of* x, $q-1$ *conjugates of* z, *and* $(q-1)^2$ *conjugates of* u.

Section 18. Involutions and centralizers in the exceptional rank 2 groups.

In this section let $G = G_2(q)$ $q > 2$, ${}^3D_4(q)$, ${}^2F_4(q)$ or ${}^2F_4(2)'$, where q is a power of 2. We will simply record the information on centralizers since much of this already appears in the papers of Thomas [23], Thomas [24], and Parrot [18]. For example the involution classes are given in these papers. The centralizers are either explicitly presented in the appropriate reference or easily obtained from the given information and the commutator relations. These results are also easily obtained from Fong-Seitz [10] Sections 9 and 10, using a root diagram and arguing as in earlier sections.

In each case the group G has a (B, N)-pair of rank 2 (except for

${}^2F_4(2)')$ and we can label the roots with respect to a fundamental set $\{\alpha_1, \alpha_2\}$ for Δ. Then $\langle s_1, s_2\rangle \cong D_{12}$ if $G = G_2(q)$ or ${}^3D_4(q)$ and $\langle s_1, s_2\rangle \cong D_{16}$ if $G \cong {}^2F_4(q)$. In the first case we assume that α_1 is a long root and α_2 a short root.

Listing the roots in Δ^+ we have

$$\begin{aligned}\Delta^+ &= \{r_1 = \alpha_1, r_2 = \alpha_2, r_3 = (\alpha_1)s_2, r_4 = (\alpha_2)s_1, r_5 \\ &= (\alpha_1)s_2s_1, r_6 = (\alpha_2)s_1s_2\}\ , \quad \text{or} \\ \Delta^+ &= \{r_1 = \alpha_1, r_2 = \alpha_2, r_3 = (\alpha_1)s_2, r_4 = (\alpha_2)s_1, r_5 \\ &= (\alpha_1)s_2s_1, r_6 = (\alpha_2)s_1s_2, r_7 = (\alpha_1)s_2s_1s_2, r_8 = (\alpha_2)s_1s_2s_1\}\ ,\end{aligned} \tag{18.1}$$

depending on whether $\langle s_1, s_2\rangle \cong D_{12}$ or D_{16}. If $G = G_2(q)$ then U_{r_1} and U_{r_2} are elementary of order q, while if $G = {}^3D_4(q)$, U_{r_1} and U_{r_2} are elementary of order q and q^3 respectively. If $G = {}^2F_4(q)$ then q is an odd power of 2, and we order α_1, α_2 so that U_{r_1} is elementary of order q and U_{r_2} is isomorphic to the Sylow 2-subgroup of $Sz(q)$.

(18.2) *G has 2 conjugacy classes of involutions with representatives z, t. If $G = G_2(q)$ or ${}^3D_4(q)$ we may choose $z = U_{r_5}(1)$ and $t = U_{r_6}(1)$. If $G = {}^2F_4(q)$ or ${}^2F_4(2)'$ we may choose $z \in \Omega_1(U_{r_8})^\#$ and $t = U_{r_7}(1)$. z is a 2-central involution and t is not a 2-central involution.*

Let $P_1 = \langle B, s_2\rangle$ and $P_2 = \langle B, s_1\rangle$ be the proper parabolic subgroups of G if $G \neq {}^2F_4(2)'$, and if $G = {}^2F_4(2)'$ let P_i be the intersection with G of the appropriate parabolic subgroup $\bar{P}_i$ of $\bar{G} = {}^2F_4(2)$.

(18.3) *If $G \neq {}^2F_4(2)'$ then P_1 is the unique proper parabolic subgroup G containing $C_G(z)$ and P_2 is the unique proper parabolic subgroup of G containing $C_G(t)$.*

(18.4) *Let $G = G_2(q)$, $q \geq 4$.*

i) $C_G(z) = O^{2'}(P_1) = U_0L_0$, where $U_0 = O_2(P_1)$ and $L_0 = \langle U_{\pm r_2}\rangle \cong SL(2, q)$. Also $Z(C_G(z)) = U_{r_5}$ and $C_G(z)' = C_G(z)$.

ii) $C_G(t) = U_0L_0$, where $U_0 = U_{r_3} \times U_{r_5} \times U_{r_6}$ and $L_0 = \langle U_{\pm r_1}\rangle \cong SL(2, q)$. Also $U_0L_0 = (U_{r_3} \times U_{r_5})L_0 \times U_{r_6}$ where L_0 acts irreducibly on $U_{r_3} \times U_{r_5}$.

(18.5) *Let $G = {}^3D_4(q)$*

i) $C_G(z) = O^{2'}(P_1) = U_0L_0$, where $U_0 = O_2(P_1)$ and $L_0 = \langle U_{\pm r_2}\rangle \cong SL(2, q^3)$. $U_0' \geq Z(C_G(z)) = U_{r_5}$ and $C_G(z)' = C_G(z)$.

ii) $C_G(t) = U_0L_0$, *where* $L_0 = \langle U_{\pm r_1}\rangle \cong SL(2, q)$ *and* $U_0 = U_{r_3}U_{r_6}U_{r_5}V_{r_2}V_{r_4}$ *where* $V_{r_i} = \{U_{r_i}(c) : c \in \boldsymbol{F}_{q^3}$ *and* $\mathrm{tr}(c) = 0\}$. *Also* $[L_0, U_0] = U_0$ *and* $U_0' \geq Z(C_G(t)) = \{U_{r_6}(c) : c \in \boldsymbol{F}_q\}$.

(18.6) *Let* $G = {}^2F_4(q)$. *If* $q = 2$, *then* ${}^2F_4(q) - {}^2F_4(q)'$ *contains no involutions.*

i) $C_G(z) = O^{2'}(P_1) = U_0L_0$, *where* $U_0 = O_2(P_1)$ *and* $L_0 = \langle U_{\pm r_2}\rangle \cong Sz(q)$. $Z(C_G(z)) = \Omega_1(U_{r_8})$ *and* $[U_0, L_0]U_{r_8} = U_0$. *Also* $\Omega_1(U_{r_8}) \leq [U_0, L_0]'$.

ii) $C_G(t) = U_0L_0$, *where* $L_0 = \langle U_{\pm r_1}\rangle \cong SL(2, q)$ *and* $U_0 = \Omega_1(U_{r_2})U_{r_3}\Omega_1(U_{r_4})U_{r_5}U_{r_6}U_{r_7}U_{r_8}$. *Here* $U_0' = \Omega_1(U_{r_6})U_{r_7}\Omega_1(U_{r_8})$, $Z(C_G(t)) = U_{r_7}$, *and* $[U_0, L_0] = U_0$.

Let U_0L_0 be the centralizer of an involution as described in (18.4), (18.5) or (18.6). Assume that $L_0 \not\cong S_3$ or $Sz(2)$. Let $W_0 = L_0 \cap H$, except in case $G = {}^3D_4(q)$ and $U_0L_0 = C_G(z)$, in which case we set $W_0 = (L_0 \cap H)^{q-1}$, a cyclic group of order $q^2 + q + 1$. Then

(18.7) $E(C_G(W_0)) = O^{2'}(C_G(W_0)) = L$ *with* L *as follows*

i) *If* $G = G_2(q)$ *and* $C_G(z) = U_0L_0$, *then* $L = \langle U_{\pm r_5}\rangle \cong SL(2, q)$ *if* $q > 4$, *and* $L = \langle U_{\pm r_1}, U_{\pm r_3}, U_{\pm r_5}\rangle \cong SL(3, 4)$ *if* $q = 4$.

ii) *If* $G = {}^3D_4(q)$ *and* $C_G(z) = U_0L_0$, *then* $L = \langle U_{\pm r_1}, U_{\pm r_3}, U_{\pm r_5}\rangle \cong PSL(3, q)$.

iii) *If* $C = {}^2F_4(q)$ *and* $C_G(z) = U_0L_0$, *then* $L = \langle U_{\pm r_8}\rangle \cong Sz(q)$.

iv) *If* $G = G_2(q)$ *or* ${}^3D_4(q)$ *and* $C_G(t) = U_0L_0$, *then* $L = \langle U_{\pm r_6}\rangle \cong SL(2, q)$ *or* $SL(2, q^3)$, *respectively.*

v) *If* $G = {}^2F_4(q)$ *and* $C_G(t) = U_0L_0$, *then* $L = \langle U_{\pm r_7}\rangle \cong SL(2, q)$.

Section 19. Outer automorphisms.

Let $G = G(q)$ be a Chevalley group of characteristic 2 with $Z(G) = 1$ having root system Δ, and regard $G \trianglelefteq \mathrm{Aut}(G)$. In this section we will determine the classes of involutions in $\mathrm{Aut}(G) - G$ and their centralizers in G.

First we need the involutions in $\mathrm{Out}(G)$. In any case $\mathrm{Out}(G)$ has at most 3 classes of involutions. If Δ has no double bonds and G is untwisted, then each involution in $\mathrm{Out}(G)$ is conjugate to a graph, field, or graph-field automorphism of G. In this case $\mathrm{Out}(G)$ can have 0, 1, or 3 classes of involutions depending on the existence of field automorphisms of $\boldsymbol{F}_q$ and on whether Δ admits graph automorphisms. When G

is untwisted and Δ has a double bond and admits a graph automorphism ($\Delta = F_4$ or B_2, as G_2 will not occur in even characteristic) a different situation occurs. For this see (19.3). In the twisted groups there will be a unique involution in $\mathrm{Out}(G)$ except for $G = {}^2F_4(q)$ where there is none and $G = {}^3D_4(q)$ where there is one only if q is a square.

If $G = PSO^-(2n, q)$, then $\mathrm{Out}(G)$ has just one involution and the results of Section 8 give the necessary information on involutions in $\mathrm{Aut}(G) - G$. For $G = PSO^+(2n, q)$ Section 8 gives the involutions in the coset of the graph automorphism of Δ. Consequently these cases will not be discussed here.

The first result handles field automorphisms.

(19.1) *Suppose that $G = G(q)$ is an untwisted Chevalley group or that $G \cong {}^3D_4(q)$. Assume that $q = q_0^2$ and that σ is an involutory field automorphism of G. Then $G\sigma$ contains just one class of involutions and we have $C_G(\sigma) \cong G(q_0)$.*

Proof. This follows from Lang's theorem [17]. However an elementary proof is as follows. Let $B = UH$ be a Borel group normalized by σ, chosen such that H is a σ-invariant abelian group of odd order. Write $H = H_0 \times H_1$, where $H_0 = C_H(\sigma)$ and $H_1 = [H, \sigma]$.

Let $x \in G\sigma$ be an involution. As $U\langle\sigma\rangle$ is Sylow in $G\langle\sigma\rangle$ we may assume $x \in U\sigma$. Consider the group $UH_1\langle\sigma\rangle$. We claim that $C_U(H_1) = 1$. Otherwise considering the decomposition of U into a product of root groups we see that $H_1 \le C(U_s)$ for some $s \in \Delta^+$. However σ acts as a field automorphism on $\langle U_{\pm s}\rangle \cong SL(2, q)$ (or possibly $SL(2, q^3)$ if $G \cong {}^3D_4(q)$), so that $H_1 \cap \langle U_{\pm s}\rangle \ne 1$ and acts fixed-point-freely on U_s. This is impossible, proving the claim.

Now (2.1) applies and gives the result.

(19.2) *The following groups G have each involution in* $\mathrm{Aut}(G) - G$ *conjugate to a field automorphism: $G_2(q)$, ${}^3D_4(q)$, $PSp(2n, q)$ $n \ge 3$, $E_7(q)$, and $E_8(q)$.*

Proof. For each of these groups there are no graph automorphisms and all diagonal automorphisms have odd order. The result follows from (19.1) and Steinberg's theorem [21].

(19.3) *Let $G = PSp(4, q)$ or $F_4(q)$. Then* $\mathrm{Out}(G)$ *is cyclic of order* $2a$, *where $q = 2^a$. In particular if q is a square, then each involution in*

Aut $(G) - G$ *is conjugate to a field automorphism of* G.

Proof. If σ is a graph automorphism of G then one checks (see Carter [6], Section 12.3) that σ can be chosen such that $\langle\sigma^2\rangle$ is the group of field automorphisms of G. Consequently Out $(G) = \langle\sigma\rangle$. If σ^2 has even order, then each involution in Out (G) is in $\langle\sigma^2\rangle$ and the result follows from (19.1). This proves (19.3).

The following lemma is easily checked.

(19.4) *Let* τ *be an involution acting on an elementary* 2-*group* M. *If* $|C_M(\tau)| = |M|^{1/2}$, *then each involution in* $M\tau$ *is conjugate to* τ.

(19.5) *Suppose that* $G = PSp(4, q)$ *or* $F_4(q)$ *and let* t *be an involution in* Aut $(G) - G$. *Then all involutions in* Aut $(G) - G$ *are conjugate. Moreover*

i) $C_G(t) \cong PSp(4, q_0)$ *or* $F_4(q_0)$ *if* q *is a square and* $q = q_0^2$.

ii) $C_G(t) \cong Sz(q)$ *or* ${}^2F_4(q)$ *if* q *is not a square.*

Proof. If q is a square we are done by (19.3) and (19.1). So assume that q is not a square. Let σ generate the group of graph automorphisms preserving Δ and normalizing U. Let τ be the involution in $\langle\sigma\rangle$. We may assume that $t \in \tau G$. In fact since $U \in \mathrm{Syl}_2(G)$ we may assume that $t \in \tau U$.

We will give the proof for $F_4(q)$, leaving the easier case of $PSp(4, q)$ to the reader. We define normal subgroups $U_1 < U_2 < U_3$ of U. For each i, U_i will be the product of root groups and these roots occur in pairs, orbits under the graph automorphism of Δ. The roots are

$$\theta_1 = \{\{1122, 1111\}, \{1222, 1121\}, \{1242, 1221\}, \{1342, 1231\}, \{2342, 1232\}\}$$

$$\theta_2 = \theta_1 \cup \{\{1120, 0111\}, \{1220, 0121\}, \{0122, 1100\}\}$$

$$\theta_3 = \theta_2 \cup \{\{1100, 0011\}, \{0120, 0110\}\}\ .$$

Then $U_i = \prod U_s$, where $\{s, s'\} \in \theta_i$ for some s'. Using (3.1) and Table 1 we easily check that each $U_i \trianglelefteq U$. Let $U_0 = 1, U_4 = U$, and consider the quotient $U_{i+1}/U_i = V_i$, $i = 0, 1, 2, 3$. For each such i, V_i is the direct sum of elementary subgroups and these subgroups occur in pairs such that τ interchanges the subgroups in a given pair. Then by (19.4) all involutions in $V_i\tau$ are conjugate. Apply this to V_3, V_2, V_1, and V_0 and obtain $t \sim \tau$.

Finally (ii) holds since we know that some involution in Aut $(G) - G$

has the correct centralizer.

(19.6) *Let $G(q)$ be an untwisted Chevalley group admitting an involutory graph automorphism τ and an involutory field automorphism σ such that $[\sigma, \tau] = 1$. Then all involutions in $G\sigma\tau$ are conjugate to $\sigma\tau$. Moreover $q = q_0^2$ and one of the following holds:*

i) $G = PSL(n, q)$ *and* $C_G(\sigma\tau)' \cong PSU(n, q_0)$

ii) $G = PSO^+(2n, q)'$ *and* $C_G(\sigma\tau)' \cong PSO^-(2n, q_0)'$

iii) $G = E_6(q)$ *and* $C_G(\sigma\tau)' \cong {}^2E_6(q_0)$.

Proof. The information on centralizers is standard, so we need only prove that all involutions in $G\sigma\tau$ are conjugate. We argue as in (19.1). Let $B = UH$ be a Borel group. We may assume that σ and τ normalize U and H, and we write $H = H_0 \times H_1$ with $H_0 = C_H(\sigma\tau)$ and $H_1 = [H, \sigma\tau]$. As in (19.1) it will suffice to show that $C_U(H_1) = 1$.

Let G_0/G be the group of diagonal automorphisms in Out(G). Then G_0/G is cyclic and $G_0 = [G_0, \tau\sigma]C_{G_0}(\tau\sigma)$, so all fusion in $G_0\tau\sigma$ is accomplished under the action of G. Hence we may take $G = G_0$.

Let $\langle\zeta\rangle = F_q^*$ and for each $i = 1, \cdots, n$ let χ_i be the character such that $\chi_i(\alpha_i) = \zeta$ and $\chi_i(\alpha_j) = 1$ for all $j \neq i$. Set $h_i = h(\chi_i)$. Then for $i = 1, \cdots, n$ $[h_i, \sigma\tau] \in H_1$.

If $C_U(H_1) \neq 1$, then $U_r \leq C_U(H_1)$ for some $r \in \Delta^+$. Write $r = \sum \gamma_i\alpha_i$. Then

$$\begin{aligned} U_r(1)^{[h_i, \sigma\tau]} &= (U_r((\zeta^{\gamma_i})^{-1}))^{h_i^{\sigma\tau}} \\ &= U_r(\zeta^{-\gamma_i})^{h_i^{\sigma\tau}} \\ &= U_r(\zeta^{\delta_i q_0 - \gamma_i}) , \end{aligned}$$

where $U_r^\tau = U_s$ and $s = \sum \delta_i\alpha_i$. Therefore $q_0^2 - 1$ divides $\delta_i q_0 - \gamma_i$. We now use information concerning the roots in Δ^+, for which the reference is Bourbaki [5]. Since $G = PSL(n + 1, q)$, $PSO^+(2n, q)$, or $E_6(q)$, some δ_i must equal 1. Then for this value of i the divisibility condition forces $\gamma_i - q_0 \geq q_0^2 - 1$ and hence $\gamma_i \geq 5$. But this does not occur, proving (19.6).

The remaining cases not treated in Section 8 are $G = PSL(n, q)$, $E_6(q)$, $PSU(n, q)$, and ${}^2E_6(q)$. In the first two cases let σ be an involutory graph automorphism preserving Δ and normalizing U, H in a Borel group $B \leq G$. In the second two cases we let σ be the graph automorphism of the non-twisted group defining G and restrict σ to G. Again we assume $B \leq G$ is Borel and normalized by σ.

We must determine the involutions in $G\sigma$ and their centralizers. The methods are similar and we will need the following lemma.

(19.7) *Let G, σ be as above and $t \in C_G(\sigma) \cap Z(U)$. Write $Q = O_2(C_G(t))$ and assume that $C_{Q/Z(Q)}(\sigma) = C_Q(\sigma)/Z(Q)$. Then $C_G(\sigma t) = C_G(\sigma) \cap C_G(t)$.*

Proof. First we note that there is a root $r \in \Delta^+$ such that $Z(U) = Z(U_r)$ and $U_r \leq Q$. If $G \cong PSU(n, q)$, n odd, then U_r is special of order q^3 and σ inverts elements in $U_r - Z(U_r)$. Consequently the covering condition fails and this case is out. Thus $Z(U) = U_r$ and $|U_r| = q$. Then $C_G(U_r) = C_G(t) = Q(R \times H_0)$ where $H_0 \leq H, Q = O_2(C_G(t))$ and $R^\sigma = R$ is a Chevalley group satisfying $U = Q(U \cap R)$.

Since $(U \cap R)^\sigma = U \cap R$ the covering condition on Q forces $C_U(\sigma)$ to cover $U/Z(Q) \cap C(\sigma) = U/U_r \cap C(\sigma)$. Let $x \in C_G(\sigma t)$ and write $x = \ell_1 w \ell_2$ with $\ell_1, \ell_2 \in B$, $w \in N$. Then $\ell_1^{\sigma t} t w^\sigma t \ell_2^{\sigma t} \in B w^\sigma B$. Since $C_N(\sigma)$ covers $C_W(\sigma)$ we may assume that $w \in C_N(\sigma) \leq C_G(\sigma)$. Now we use the Bruhat decomposition and write $BwB = BwU_w^-$, where $U_w^- = U \cap U^{w_0 w}$ and where w_0 is the word of greatest length in $\{s_1, \cdots, s_n\}$. So we may assume $x = h_1 u_1 w u_2$ for $u_1 \in U, h_1 \in H, u_2 \in U_w^-$.

First assume that $t \in U_w^-$. We then apply the uniqueness of the Bruhat decomposition, obtaining $h_1^{\sigma t} u_1^{\sigma t} t = h_1^{\sigma t} u_1^{\sigma} t = h_1 u_1$ and $t u_2^{\sigma t} = t u_2^\sigma = u_2$. Read these equations modulo $U_r \trianglelefteq B$. We conclude that $h_1^\sigma = h_1$ and using the covering condition we also have $u_1^\sigma = u_1, u_2^\sigma = u_2$. So we are done in this case.

Next assume that $t \in U_w = U \cap U^w$. Then $h_1 u_1 t w t u_2 = h_1 u_1 t t^{w^{-1}} w u_2$ and $t^{w^{-1}} \in U$. Conjugating by σt we get the equations $h_1^{\sigma t} = h_1, u_1^\sigma t t^{w^{-1}} = u_1, u_2^\sigma = u_2$. Reading modulo U_r we get $h_1^\sigma = h_1$, so we need only show that $u_1^\sigma = u_1$.

If $t^{w^{-1}} \in U_r$, then we argue as above to get $U_1^\sigma = U_1$. So we may assume that $t^{w^{-1}} \in U_s \neq U_r$. Then s and r have the same length and $|U_r| = |U_s| = q$. We can factor $\overline{U} = U/U_r = \prod \overline{U}r_i$, where $r_i \in \Delta^+ - \{r\}$ and where respresentations of elements are unique. Consequently $t^{w^{-1}} \in U_s$ for s a root with $(U_{s'})^\sigma = U_s$ and $\overline{u}_1$ having non-trivial projection to $\overline{U_{s'}}$. Otherwise $\overline{u_1^\sigma t^{w^{-1}}}$ has more non-trivial factors than $\overline{u}_1$. However the action of σ implies that $U_{s'}(c)^\sigma = U_s(c)$ for each $c \in F_q$. This situation is impossible, proving the lemma.

(19.8) *Let* $G = PSL(n, q), E_6(q), PSU(n, q)$, *or* ${}^2E_6(q)$, *and let* σ *be the graph automorphism as in* (19.7). *Let* r *be in* Δ^+ *such that* $Z(U_r) = Z(U)$ *and let* $t \in Z(U_r)^\#$.

i) *If* $G = PSL(n, q)$ *or* $PSU(n, q)$ *for* n *odd, then all involutions in* $G\sigma$ *are conjugate to* σ.

ii) *Otherwise each involution in* $G\sigma$ *is conjugate to either* σ *or* σt.

(19.9) *Let* G, σ, t *be as in* (19.7).

i) *If* $G = PSL(n, q)$ *or* $PSU(n, q)$ *for* n *odd, then* $C_G(\sigma) \cong PSO^+(n, q)$.

ii) *If* $G = PSL(n, q)$ *or* $PSU(n, q)$ *for* n *even, then* $C_G(\sigma) \cong PSp(n, q)$ *and* $C_G(\sigma t)$ *is isomorphic to the centralizer of a transvection in* $PSp(n, q)$.

iii) *If* $G = E_6(q)$ *or* ${}^2E_6(q)$, *then* $C_G(\sigma) \cong F_4(q)$ *and* $C_G(\sigma t)$ *is isomorphic to the centralizer of a central involution in* $F_4(q)$.

In cases (ii) *and* (iii) $G\sigma$ *contains precisely* 2 *conjugacy classes of involutions in* Aut (G).

We note that (19.9) follows from (19.7) and (19.8) once the covering condition appearing in (19.7) has been checked.

We let r be as in (19.8) and $t \in Z(U_r)^\#$ (recall that $U_r = Z(U_r)$ unless $G = PSU(n, q)$ for n odd). Then $P = N(Z(U_r))$ is a parabolic subgroup of G and $O^{2'}(P) = C_G(t) = QL$ where $Q = O_2(P)$ and $L \cong SL(n-2, q)$, $SL(6, q), SU(n-2, q)$, or $SU(6, q)$, accordingly as $G \cong PSL(n, q), E_6(q)$, $PSU(n, q)$, or ${}^2E_6(q)$ (see [7], Sections 3, 4). Moreover we can choose L such that $U \cap L \in \mathrm{Syl}_2(L)$ and L is σ-invariant.

In each case Q can be factored as a product of root subgroups, $Q = \prod_{i=1}^k U_{r_i}$, and $\bar{Q} = Q/Z(U_r) = \bar{U}_{r_1} \times \cdots \times \bar{U}_{r_k}$. If $|U_{r_i}| = q$ and $U_{r_i}^\sigma = U_{r_i}$, then σ centralizes U_{r_i}. On the other hand if $|U_{r_i}| = q^2$ and $U_{r_i}^\sigma = U_{r_i}$, then $U_{r_i}(c)^\sigma = U_{r_i}(c^q)$. Finally if $|U_{r_i}| = q^3$, then $r_i = r$, $C_{U_r}(\sigma) = Z(U_r)$ and $|C_{\bar{U}_r}(\sigma)| = q$. If $G \neq PSL(n, q)$ for n odd, then $U_{r_i}^\sigma = U_{r_j} \neq U_{r_i}$ implies that $[U_{r_i}, U_{r_j}] = 1$. From this information one can verify the covering condition of (19.7) in each of the relevant cases.

What remains is the

Proof of (19.8) First we note that if $G = PSL(3, q)$ or $PSU(3, q)$ then (19.8)(i) is well-known. For $s \in \Delta$ let $Z_s = \Omega_1(U_s)$ ($Z_s = U_s$ unless $G = PSU(n, q)$, n odd) and set $X = \langle Z_r, Z_{-r} \rangle$.

Let j be an involution in $G\sigma$. As $C_{G\langle\sigma\rangle}(Z_r) = QL\langle\sigma\rangle$, we may assume that j centralizes $Z_r, j \in QL\langle\sigma\rangle$. We claim that for some $g \in G$ $\langle Z_r^g, Z_r^{gj}\rangle \sim X$. To see this it suffices by (12.1) to find $g \in G$ such that Z_r^g, Z_r^{gj} do not generate a 2-group. This we do by induction on the rank of G. If $G = SL(3, q)$ or $SU(3, q)$ we use the fact that (19.8)(i) holds and check this directly. Otherwise consider $j \in QL\langle\sigma\rangle$ and note that σ induces a graph automorphism on L. Consequently we let j act on QL/Q and apply induction.

By the above we may assume that j normalizes X and conjugating by an element of $X \cong SL(2, q)$ and using the fact that $O^{2'}(C_G(X)) = L$ we may assume that $tj \in L\langle\sigma\rangle$. Inductively we have $j \sim tu\sigma$ where $u = 1$ or $U_s(1)$, for s the root of highest height in the room system $\Delta_1 \subset \Delta$ of L. Moreover $u = 1$ if $G = PSL(n, q)$ or $PSU(n, q)$ with n odd.

Say $G = PSL(n, q)$ or $PSU(n, q)$ with n odd. Then $j \sim t\sigma$. If $G = PSU(n, q)$ then σ acts on U_r and inverts a homocyclic subgroup of order q^2 and exponent 4. It follows that $t\sigma \sim \sigma$. If $G = PSL(n, q)$, then $U_r = [U_{s_1}, U_{s_2}]$ where $s_1 = \alpha_1 + \cdots + \alpha_{(n-1)/2}, s_2 = s_1^\sigma = \alpha_{(n+1)/2} + \cdots + \alpha_n$. Then σ acts on $U_{s_1}U_{s_2}U_r$ and again $\sigma \sim \sigma t$.

For the remaining cases we are done if $u = 1$. Consequently we may assume $j \sim tU_s(1)\sigma$. We now perform simple computations to obtain the result. For $G = PSL(n, q)$, $r = \alpha_1 + \cdots + \alpha_{n-1}$ and $s = \alpha_2 + \cdots + \alpha_{n-2}$. Then σ centralizes $U_{\alpha_1}(1)U_{\alpha_{n-1}}(1) = g$ and $(tU_s(1))^g = U_{s+\alpha_1}(1) U_{s+\alpha_{n-1}}(1)U_s(1)$. As σ interchanges $U_{s+\alpha_1}(1)$ and $U_{s+\alpha_{n-1}}(1)$ we have $j \sim (U_r(1)U_s(1)\sigma)^g \sim U_s(1)\sigma \sim t\sigma$ as desired.

Suppose $G = E_6(q)$. Then $s = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. Conjugating $tu\sigma$ by $s_2s_4s_1s_6s_3s_5s_4s_3s_5s_1s_6s_2 \in C(\sigma)$ we have $j \sim U_{\alpha_4}(1)U_{\alpha_3+\alpha_4+\alpha_5}(1)\sigma$. Considering $\langle U_{\pm\alpha_3}, U_{\pm\alpha_4}, U_{\pm\alpha_5}\rangle \cong SL(4, q)$ we use the above to get $j \sim U_{\alpha_3+\alpha_4+\alpha_5}(1)\sigma$ and hence $j \sim t\sigma$.

Finally, suppose that $G = {}^2E_6(q)$ or $PSU(n, q)$ for n even. Then there is an element $w \in W$ such that $\{r^w, s^w\} = \{\alpha_2, \alpha_2 + 2\alpha_3\}, \{\alpha_n, \alpha_n + 2\alpha_{n-1}\}$. Then $w \in C(\sigma)$ and we have $tu\sigma \sim U_\alpha(1)U_{\alpha+2\beta}(1)\sigma$, where $\langle U_\alpha, U_\beta\rangle = U_\alpha U_\beta U_{\alpha+\beta} U_{\alpha+2\beta}$ is isomorphic to a Sylow 2-subgroup of $PSU(4, q)$. Then $(U_\alpha(1)U_{\alpha+2\beta}(1)\sigma)^{U_\beta(1)} = U_\alpha(1)U_{\alpha+\beta}(1)\sigma$ and conjugating by $U_{-\beta}(c)$, for $c \in F_q^2$ with $c + c^q = 1$, we have $tu\sigma \sim U_{-\beta}(1)\sigma$. Finally the action of σ on $U_{-\beta}$ described above together with (19.4) shows that $U_{-\beta}(1)\sigma \sim \sigma$. This completes the proof of (19.8).

Section 20. Hypothesis II for Chevalley groups of characteristic 2.

Let A be a perfect central extension of a Chevalley group $\bar{A} = A/Z(A)$ defined over a field of characteristic 2. We will use our information on involutions in Aut $(\bar{A})$ to prove

(20.1) *Let T be an elementary abelian* 2-*group acting on A and $K \leq TA$ such that*

i) $m_2(T) > 1$.

ii) $T \in \mathrm{Syl}_2(K)$ *and* $K = O(K)T$.

iii) $T \cap C(A) = 1$.

iv) *K is tightly embedded in TA.*

Then $T \leq AC(A)$.

(20.1) is called Hypothesis 2 in [1] and this result will be essential in [3].

Assume (20.1) is false and A is a counterexample. We will use the following notation: $K_0 = O_{2'}(K) \leq A$ and $T_0 = T \cap AC(A)$. Since $AC(A)/C(A) \cong \bar{A}$ we will let bars denote projections of subsets of $AC(A)$ to $\bar{A}$, when no confusion arises from this abuse.

(20.2) *$\bar{A}$ has (B, N)-rank $n > 1$.*

Proof. This can be easily checked from the known structure of Aut (A).

(20.3) $T_0 \neq 1$.

Proof. Suppose $T_0 = 1$. Then $m_2(T) > 1$ implies that Out (A) contains a klein group and the Dynkin diagram of $\bar{A}$ is simply laced. By Griess [13] either $\bar{A} \cong L_3(4)$ or $Z(A)$ has odd order. In the latter case $C_A(t)$ covers $C_{\bar{A}}(t)$ for each $t \in T^\#$. This also holds if $\bar{A} = L_3(4)$ since $C_A(t)$ covers $O^2(C_{\bar{A}}(t)) = C_{\bar{A}}(t)$ for each $t \in T^\#$. Also for $t \in T^\#$ $[C_{\bar{A}}(t), T] \leq C_{\bar{A}}(t) \cap K$, and so T centralizes $C_{\bar{A}}(t)/O_{2',2}(C_{\bar{A}}(t))$ for each involution $t \in T^\#$. Now T contains involutions t_1, t_2 such that t_1 is in the coset of a field automorphism of $\bar{A}$ and t_2 is in the coset of a graph-field automorphism of $\bar{A}$. However (19.1) and (19.5) show this to be impossible.

(20.4) *$N_A(K)$ is* 2-*constrained.*

Proof. $K = O(K)T$. By (20.3) we may choose $t \in T_0^\#$. Then t must centralize $E(N_A(K)/O(N_A(K)))$ and so $O_{2',E}(N_A(K)) = X = O(X)C_X(t)$. As $C_A(t) \leq N_A(K)$, $C_X(t) \trianglelefteq C_A(t)$. But $O_{2',E}(C_A(t)) = O(C(A))$ and so $O_{2',E}(N(K)) = O(N(K))$ proving the lemma.

(20.5) *$C_A(t)$ is* 2-*constrained for each* $t \in T^\#$.

Proof. $L(C(t)) \leq L(N(K)) = 1$ by (2.2) and (20.4).

Knowing (20.5) we can immediately use the information in Section 19 to eliminate many possibilities for $\bar{A}$. Let $j_1 \in T - AC(A)$. We then have

(20.6) *$T = \langle T_0, j_1 \rangle$ and either $\bar{A} \cong PSO^{\pm}(n, q)$ with $n \geq 8$ even and j_1 of type b_ℓ for some ℓ, or letting σ, t be as in* (19.8) (*and replacing G by $\bar{A}$*) *we may assume that $\bar{A} = PSL(n, q)$ n even, $PSU(n, q)$ n even, $E_6(q)$, or ${}^2E_6(q)$, and $j_1 = \sigma t$.*

Proof. Use (20.5) and the results in Section 19.

(20.7) *$T \trianglelefteq K$ and T is tightly embedded.*

Proof. We first claim that $C_{\bar{A}}(T)$ contains a klein group. Let $j \in T_0^{\#}$. Then $C_{\bar{A}}(j)$ is 2-constrained and $O_{2'}(C_{\bar{A}}(j)) = 1$ (by (2.4)). Also $\bar{T}_0 \leq C_{\bar{A}}(j)$ so assume that $T_0 = \langle j \rangle$. $C_A(j)$ covers $O^2(C_{\bar{A}}(j))$. So $[O^2(C_{\bar{A}}(j)), T] \leq O^2(C_{\bar{A}}(j)) \cap T_0K$. Since $O_{2'}(C_{\bar{A}}(j)) = 1$ we must have this commutator contained in $T_0 = \langle j \rangle$. It follows that $O^2(C_{\bar{A}}(j)) \leq C_{\bar{A}}(T)$. So if the claim is false $O^2(C_{\bar{A}}(j))$ has 2-rank ≤ 1 and is 2-constrained. The only possibility is $O^2(C_{\bar{A}}(j)) = 1$ or $SL(2, 3)$. The latter case contradicts the known structure of $C_{\bar{A}}(j)$, and $O^2(C_{\bar{A}}(j)) = 1$ forces $\bar{A} = L_3(2)$ where we also obtain a contradiction.

Let U_0 be a klein group in $C_{\bar{A}}(T)$. Then $O(K) = \Gamma_{1,U_0}(O(K))$ by (2.5), so it suffices to show that $Q = C_{0(K)}(u) = 1$ for each $u \in U_0^{\#}$. We have $Q = \Gamma_{1,T}(Q)$ by (2.5) so we need only show $P = C_Q(t) = 1$ for each $x \in T^{\#}$. But as $C_{AT}(x)$ is 2-constrained (2.4) implies that $P \leq O(C_{AT}(u)) = 1$. This proves (20.7).

At this point we may replace K by T.

(20.8) *If $Z(A)$ has odd order, then $C_{\bar{A}}(x) \leq N_{\bar{A}}(T)$ for each $x \in T^{\#}$. In any case $[O^2(C_{\bar{A}}(x)), T] \leq T$.*

Proof. If $Z(A)$ has odd order, then $C_{\bar{A}}(x) = \overline{C_A(x)}$ and (20.7) gives the result. The second statement is similarly proved using $\overline{O^2(C_A(x))} = O^2(C_{\bar{A}}(x))$.

(20.9) *If $\bar{A} \not\cong PSO^{\pm}(n, q)$, then $t^a \in \bar{T}_0$, where t is as in* (20.6), *for some $\bar{a} \in \bar{A}$ and $C_{\bar{A}}(t) = C_{\bar{A}}(t^a)$.*

Proof. We take $j_1 = \sigma t$ as in (20.6) and consider $C_{\bar{A}}(j_1)$. From (19.8) we have $C_{\bar{A}}(j_1)$ isomorphic to the centralizer of a transvection in $PSp(n, q)$ or to the centralizer of an element of a root group in $F_4(q)$, depending on whether $\bar{A}$ is as in (19.8) ii) or (19.8) iii). In either case $O^2(C_{\bar{A}}(j_1)) = C_{\bar{A}}(j_1)$, so (20.8) implies that $C_{\bar{A}}(j_1) \leq N_{\bar{A}}(T)$. Consequently $\bar{T}_0 \trianglelefteq C_{\bar{A}}(j_1) = Y$.

We consider the structure of $C_{\bar{A}}(j_1)$ (see [7], (3.2) and (4.5)). In either case $Z(Y)$ is a root group of $\bar{A}$. Given any normal subgroup $J \leq Y$ either $Z(Y) \leq J$ or $J \leq Z(Y)$. Consequently $\bar{T}_0 \cap Z(Y) \neq 1$ and the result follows.

(20.10) *$\bar{A} \not\cong PSL(n, q)$ n even, $PSU(n, q)$ n even, $E_6(q)$, or ${}^2E_6(q)$.*

Proof. Let $j_1 = \sigma t$ as in (20.6) and $t^a \in \bar{T}_0$ as in (20.9). Let $v \in T_0^\#$ such that $\bar{v} = t^a$. By (20.8) and (20.9) $O^2(C_{\bar{A}}(v)) = O^2(C_{\bar{A}}(t)) \leq N_{\bar{A}}(T)$. In particular j_1 centralizes $X = E(C_{\bar{A}}(v)/O_2(C_{\bar{A}}(v)))$. However from the definitions of σ and t we see that σ induces an involutory outer automorphism of X. This is a contradiction.

We are left with the use $\bar{A} \cong PSO^{\pm}(n, q)$ $n \geq q$ even and j_1 inducing b_{ℓ_1} for some ℓ_1 (see (20.6)). Consequently we may assume $\bar{A}\langle j_1\rangle = \bar{A}T = O^{\pm}(n, q)$. Accordingly we let M be the associated n-dimensional space over F_q having a non-degenerate quadratic form preserved by $\bar{A}$. Let $j \in T_0^\#$ and P_j the stabilizer in $\bar{A}$ of $[M, j]$. Then P_j is a parabolic subgroup of $\bar{A}$ with $P_j^{j_1} = P_j$. Let $V = O^{2'}(C_{\bar{A}}(j)/O_2(C_{\bar{A}}(j)))$. Then V is the central product of certain linear groups as described in Section 8.

We first note that $V^{j_1} = V$ and also (20.8) implies that $[O^2(V), T] \leq V \cap T = 1$. Consequently j_1 centralizes $O^2(V)$. On the other hand P_j is a maximal parabolic subgroup obtained by omitting the ℓ^{th} node of the Dykin diagram. Consequently $O^{2'}(P_j/O_2(P_j))$ is the central product of $SO^{\pm}(n - 2\ell, q)$ and $SL(\ell, q)$. Suppose $\bar{j}$ is of type a_ℓ. Then from (8.6) we have V the central product of $SO^{\pm}(n - 2\ell, q)$ and $Sp(\ell, q)$. As j_1 centralizes $O^2(V)$, j_1 centralizes the factor isomorphic to $SO^{\pm}(n - 2\ell, q)$ in $O^{2'}(P_j/O^2(P_j))$, and since j_1 is a graph automorphism of G, we must have $n - 2\ell \leq 2$. We also note that since $C_{\bar{A}}(x)$ is 2-constrained for each $x \in T^\#$ we must have j_1 and $j_1 j$ inducing outer automorphisms of $\bar{A}$ of type b_{ℓ_1}, b_{ℓ_2} with $\ell_1 > 1$ and $\ell_2 > 1$.

Now $N_{\bar{A}}(\bar{T}_0) \leq P$ where P is a proper parabolic subgroup of $\bar{A}$, the stabilizer of a singular subspace, M_0, of the natural module M. Using the known structure of $C_{\bar{A}}(j_1)$, and $C_{\bar{A}}(j_1 j)$ given in (8.7) one verifies that

$O^2(C_{\bar{A}}(j_1))$ and $O^2(C_{\bar{A}}(j_1\bar{j}))$ stabilize unique isotropic subspaces of M of dimensions $\ell_1 - 1$, $\ell_2 - 1$, respectively. Consequently $\ell_1 - 1 = \ell_2 - 1 = \dim(M_0)$. Similarly if $\bar{j}$ has type c_ℓ then from (8.9) we see that $O^2(C_{\bar{A}}(\bar{j}))$ stabilizes precisely two isotropic subspaces of M of dimensions 1 and $\ell - 1$. As $\dim(M_0) = \ell_1 - 1$ is even, this is impossible. Consequently $\bar{j}$ has type a_ℓ and from earlier remarks $n - 2\ell \leq 2$.

Next we use (8.6) to check that $C_{\bar{A}}(\bar{j})$ stabilizes precisely one isotropic subspace of M having dimension ℓ. If $n - 2\ell = 2$, then $O^2(C_{\bar{A}}(\bar{j}))$ will stabilize isotropic subspaces of dimensions ℓ and $\ell + 1$. As $O^2(C_{\bar{A}}(\bar{j})) \leq P$ we have $\ell = \ell_1 - 1 = \ell_2 - 1$ in all cases. Note that if $n = 8$, then we must have $\ell_1 = \ell_2 = 3$, so $\ell = 2$. This contradicts $n - 2\ell \leq 2$. Consequently $n > 8$, $Z(A)$ has odd order and $C_{\bar{A}}(\bar{j}), C_{\bar{A}}(j_1), C_{\bar{A}}(j_1\bar{j})$ are all in P. Moreover $\ell \geq 4$ and since $2\ell_1 \leq n$ we have $n = 2\ell_1 = 2\ell + 2$.

Let $\mathscr{B} = \langle x_1, \cdots, x_n \rangle$ be an ordered basis for M as in (8.2), so that $\bar{j}$ is in orthogonal Suzuki form relative to $\mathscr{B}$. Then $M_0 = [M, \bar{j}] = \langle x_1, \cdots, x_\ell \rangle$ is invariant under $C_{\bar{A}}(j_1)$ and it follows from (8.3) that $M_0 = \langle v \in [M, \bar{j}_1] : Q(v) = 0 \rangle$. Then j_1 stabilizes $M_0^\perp = \langle x_1, \cdots, x_\ell, x_{\ell+1}, x_{\ell+2} \rangle = \langle x_1, \cdots, x_{n-\ell} \rangle$. From (8.3) we see that there is another basis $\{x_1, \cdots, x_\ell, x, y\}$ of $M_0^\perp$ such that $(x)j_1 = 1$ and $(y)j_1 = x + y$, $Q(x) = 1$, $(x, y) = 1$, and $\langle x, y \rangle$ is non-degenerate. With respect to the basis $\mathscr{B}' = \{x_1, \cdots, x_\ell, x, y, x_{n-\ell+1}, \cdots, x_n\}$ $\bar{j}$ is still in Suzuki symplectic form and j_1 has matrix form

$$\begin{bmatrix} I_\ell & & & \\ & 1 & & \\ & 1 & 1 & \\ B & \beta & & I_\ell \end{bmatrix}$$

for some $\ell \times \ell$ matrix B and column vector β. We have $C_{\bar{A}}(\bar{j}) \leq N_{\bar{A}}(T)$. Suppose $\beta \neq 0$. Let

$$P = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

be $2 \times \ell$ and $\ell \times 2$ matrices and let $g \in C_{\bar{A}}(\bar{j})$ be the matrix

$$\begin{pmatrix} I_\ell & & \\ P & I_2 & \\ 0 & Q & I_\ell \end{pmatrix}.$$

Then $\bar{j}_2 = \bar{j}_1^g \bar{j}_1$ satisfies $\dim [M, \bar{j}_2] = 2$ and so $C_{\bar{A}}(\bar{j}_2) \not\leq P$. As $\bar{j}_2 \in \bar{T}$, this is a contradiction.

Section 21. Coverings of classical groups.

HYPOTHESIS 21.1. *A is a quasisimple group such that $Z(A)$ is an elementary abelian 2-group and $A/Z(A) = \bar{A}$ is a classical group $L_n(q)$, $U_n(q), Sp_n(q)$, or $\Omega_n^\varepsilon(q), n \geq 4, q$ even. Exclude the cases where $\bar{A}$ is $L_4(2) \cong A_8$ or $Sp_4(2) \cong S_6$. If $\bar{A}$ is orthogonal take $n \geq 8$.*

If $Z(A) \neq 1$ then results of Steinberg and Greiss (eg. [13]) show $\bar{A}$ to be $U_4(2), U_6(2), \Omega_8^+(2)$, or $Sp_6(2)$.

HYPOTHESIS 21.2. *T is a 4-group in A with $T \cap Z(A) = 1$ such that for each $t \in T^\#$, $T \leq O_2(C_A(t))$ and either*

i) $[T, t^A \cap C_A(t)] = 1$, or

ii) $T \trianglelefteq C_A(t)$.

In this section we prove

(21.3) *Let A satisfy hypothesis 21.1 with $Z(A) \neq 1$. Then there exists no 4-group T in A satisfying hypothesis 21.2.*

Assume 21.3 to be false and let T be a counter example in A. Let $t \in T^\#$. We consider the four possibilities for $\bar{A}$ separately.

(21.4) $\bar{A} \not\cong U_4(2)$.

Proof. Assume $\bar{A} \cong U_4(2)$. Then $A \cong Sp(4,3)$. Then there is a unique class t^A of involutions with $t \in A - Z(A)$. Further $\langle t \rangle Z(A)$ is the unique normal 4-group in $C_A(t)$.

(21.5) $\bar{A} \not\cong U_6(2)$.

Assume $\bar{A} \cong U_6(2)$. By Section 6, $\bar{A}$ has 3 classes of involutions with representatives j_1, j_2, and j_3. Next

(21.6) *$|Z(A)| \leq 4$ and A admits a group of automorphisms transitive on $Z(A)^\#$ and fixing each class of involutions in $\bar{A}$.*

So in particular there is a unique covering B of $\bar{A}$ such that $Z(B) = \langle \pi \rangle$ is of order 2. The following facts appear in [16]. B has 6 classes of involutions $\pi, d, d\pi, r, r\pi$, and n, where $\bar{d} = j_1, \bar{r} = j_2$, and $\bar{n} = j_3$. In particular each involution $\bar{x}$ in $\bar{A}$ lifts to an involution x of B and if $\bar{x}$ is fused to j_1 or j_2 then $C_{\bar{A}}(\bar{x}) = \overline{C_B(x)}$ while $|C_{\bar{A}}(\bar{n}) : \overline{C_B(n)}| = 2$. Hence by 21.6 if follows that

(21.7) *Let* $Z(A) = \{z_i : 1 \le i \le m = |Z(A)|\}$. *There are* $2m + 1$ *classes of involutions in* $A - Z(A)$ *with representatives* dz_i, rz_i, *and* $n, 1 \le i \le m$, *where* $\bar{d} = j_1, \bar{r} = j_2$, *and* $\bar{n} = j_3$. *If* x *is fused to* dz_i *or* rz_i *then* $C_{\bar{A}}(\bar{x}) = \overline{C_A(x)}$, *while if* $x \sim n$ *then* $|C_{\bar{A}}(\bar{x}) : \overline{C_A(x)}| = m$.

It follows from 21.7 and 10.6 that we may take $t = n$ and $s \sim n$ for each $s \in T^{\#}$. Next, calculating using Section 6 we find $C_{\bar{A}}(\bar{n}) = \bar{Y}\bar{X}$ where $\bar{Y}$ is elementary of order 2^9 and $\bar{X} \cong U_3(2)$ is the extension of an elementary group $\bar{W}$ of order 9 by a quaternion group $\bar{Q}$. Further $\bar{Y}$ is generated by involutions $\bar{y}$ of rank 1 or 2, so $C_{\bar{A}}(\bar{y}) = \overline{C_A(y)}$ and hence $\Phi(Y) = 1$. As $T \le O_2(C_A(t)) = Y, Y$ centralizes T. As $|C_{\bar{A}}(\bar{n}) : \overline{C_A(n)}| = m \le 4$, each involution u with $\bar{u} \in \bar{Q}$ centralizes t. Thus if v is an involution with $[\bar{v}, \bar{t}] = 1$ then $[v, t] = 1$. So by 10.6 we conclude $T \trianglelefteq C_A(t)$. Then W centralizes T. But $\bar{W}$ acts without fixed points on $\bar{Y}/\langle \bar{t} \rangle$, a contradiction. This completes the proof of 21.5.

(21.8) $\bar{A} \not\cong \Omega_8^+(2)$.

Assume $\bar{A} \cong \Omega_8^+(2)$, Then by Section 8, $\bar{A}$ has five classes of involutions with representatives a_2, c_2, c_4, a_4, and a_4'. Next

(21.9) $|Z(A)| = m \le 4$ *and if* $m = 4$ *then* A *admits a group of automorphisms transitive on* $Z(A)^{\#}$, *fixing* a_2 *and* c_4, *and permuting* c_2, a_4, *and* a_4' *transitively.*

So in particular there is a unique covering B of $\bar{A}$ with $Z(B) = \langle \pi \rangle$ of order 2. From [11] and [15] we find there are 4 classes of involutions in $B - Z(B)$ with representatives a, b, c, and $c\pi$, where $\bar{a} = a_2, \bar{b} = c_4$, and $\bar{c} = c_2$. a_4 and a_4' lift to elements of order 4. Therefore with 21.9 we conclude

(21.10) *If* $Z(A) = \langle \pi \rangle$ *then there are* 4 *classes of involutions in* $A - Z(A)$ *with representatives* a, b, c, *and* $c\pi$, *where* $\bar{a} = a_2, \bar{b} = c_4$, *and* $\bar{c} = c_2$. *If* $|Z(A)| = 4$ *then* a *and* b *are representatives for the classes in* $A - Z(A)$, *with* $\bar{a} = a_2$ *and* $\bar{b} = c_4$.

If $|Z(A)| = 2$, then $C_{\bar{A}}(\bar{c}) = \overline{C_A(c)}$, so by 10.8 we conclude $t \not\sim c$ or $c\pi$. Suppose $t = a$. Calculating using Section 8 we find $O^2(C_{\bar{A}}(\bar{a})) = \bar{X}\bar{Y}$ where $\bar{Y} = O_2(C_{\bar{A}}(\bar{a})) = \langle \bar{Y} \cap \bar{a}^{\bar{A}} \rangle$ is extraspecial of order 2^9 and $\bar{X}$ is of order 27 and acts without fixed points on $\bar{Y}/\langle\bar{a}\rangle$. As $O^2(C_{\bar{A}}(\bar{a})) = O^2(\overline{C_A(a)})$, we have a contradiction. Therefore

(21.11) *$s \sim b$ for each $s \in T^\#$.*

So we may take $t = b$. Let $W = O^2(C_A(t))$ and $Q = O_2(W)$. Then calculating using Section 8 we find

(21.12) *$\bar{Q} = \langle \bar{t}^{\bar{A}} \cap \bar{Q}\rangle$ with $Z(\bar{Q}) = O_2(C_{\bar{A}}(\bar{t})) \cap C(\bar{Q}) = \langle \bar{t}\rangle \times [Z(\bar{Q}), \bar{X}]$ where $X \in \mathrm{Syl}_3(C_A(t))$. Moreover $\bar{t}^{\bar{A}} \cap Z(\bar{Q}) = \bar{t}[Z(\bar{Q}), \bar{X}]$.*

Now if $[t^A \cap C(t), T] = 1$ then by 21.12, $\bar{T} \leq Z(\bar{Q})$. On the other hand if $T \trianglelefteq C_A(t)$ then X centralizes T and then $W = \langle X^{C(t)}\rangle \leq C(T)$. So again $\bar{T} \leq Z(\bar{Q})$. But this is impossible since by 21.11, $\bar{T}^\# \subseteq \bar{t}^{\bar{A}} \cap C(\bar{t})$, while by 21.12, $\bar{t}^{\bar{A}} \cap Z(\bar{Q}) \subseteq \bar{t}[Z(\bar{Q}), \bar{X}]$. This completes the proof of 21.8.

(21.13) *$\bar{A} \not\cong Sp_6(2)$.*

Assume $\bar{A} \cong Sp_6(2)$. By Section 7, $\bar{A}$ has 4 classes of involutions with representatives b_1, b_3, a_2, and c_2. $Sp_6(2)$ has a multiplier of order 2, so A is the covering group of $Sp_6(2)$. Let $\langle \pi\rangle = Z(A)$. By [8], Theorem 3.12

(21.14) *A has 3 classes of involutions with representatives π, a, and b, where $\bar{a} = a_2$ and $\bar{b} = b_3$.*

Next by lemmas 3.16 and 3.17 in [8]

(21.15) *$C_A(b) = \langle b\rangle \times B$ where B is isomorphic to the centralizer in M_{12} of a 2-central involution of M_{12}, and $\pi \in B$. Moreover $O_2(B) = \langle b^A \cap O_2(B)\rangle$.*

In particular $\langle b, \pi\rangle$ is the unique normal 4-subgroup of $C_A(b)$ and the only 4-group centralized by $b^A \cap C(b)$. Therefore

(21.16) *$s \sim a$ for each $s \in T^\#$.*

Thus we may take $t = a$. Let $g, h \in A$ with

$$\bar{g} = \begin{vmatrix} & 1 & & & & \\ 1 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & 1 & 1 \end{vmatrix} \qquad \bar{h} = \begin{vmatrix} & 1 & & & & \\ 1 & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & 1 & \end{vmatrix}.$$

Then $h \in a^A$ and g is an element of order 3 inverted by h. Further $\overline{ah}$ is of type b_3 and hence by 21.14 ah is an involution. Thus $h \in C(a)$. So $g = [g, h]$ must centralize T. However an easy calculation shows $\bar{g}$ centralizes an involution $\bar{u}$ in $O_2(C(\bar{a}))$ exactly when $Q(\bar{u})$ is

$$\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} \quad \begin{vmatrix}1 & 1\\1 & 0\end{vmatrix} \quad \begin{vmatrix}0 & 1\\1 & 1\end{vmatrix}.$$

In particular a and $a\pi$ are the only members of $a^G \cap O_2(C(a))$ centralized by g.

This completes the proof of 21.3.

Section 22. Tightly embedded subgroups of the classical groups.

HYPOTHESIS 22.1. $G = TA = C_G(A)A$ where $A = A(q)$ satisfies hypothesis 21.1, $C_G(A)$ is an elementary abelian 2-group, $C_T(A) = 1 = \Phi(T)$, $m(T) > 1$, and T is tightly embedded in G. Further either

(1) $|T| = 4$, or

(2) $T^g \cap N(T) \leq C(T)$ for each $g \in G$.

Let P be a complement to $Z(A)$ in the preimage in A of the projection of T on $\bar{A} = A/Z(A)$, containing $T \cap A$. Let $t \in T^\#$ with ℓ the rank of t. Set $\{p\} = P \cap tC(A)$.

In this section we prove

(22.2) *Assume hypothesis* 22.1. *Then* $Z(A) = 1, P^\#$ *is fused in* A *and either*

(I) $P \leq J = O_2(C_A(p)) \cap C(p^A \cap C(p)), |T| \leq q$, *and one of the following holds*:

(1) $J = \alpha(p)$ *and* $\mathrm{Aut}_A(J)$ *is cyclic of order* $q-1$ *and regular on* $J^\#$.

(2) $A = Sp_n(q), p = b_\ell, \ell > 1, J = \alpha(a)\alpha(b)$ *where* a *and* b *are of type* $a_{\ell-1}$ *and* b_1, *respectively, and* $\mathrm{Aut}_A(J) \cong Z_{q-1} \times Z_{q-1}$ *is regular on* $J - (\alpha(a) \cup \alpha(b))$.

(3) $A = Sp_n(q), p = c_\ell, J = \alpha(a)\alpha(b)$ *where* a *and* b *are of type* a_ℓ *and* b_1, *respectively, and* $\mathrm{Aut}_A(J)$ *is as in* (2).

(4) $A = \Omega_n^\epsilon(q), p = c_\ell, J = \beta(p)$ *and* $\mathrm{Aut}_A(J)$ *is cyclic of order* $q-1$ *and regular on* $J^\#$.

or,

(II) $T = P$ *has order* 4, *and either*

(5) $A = SL_n(2)$, $t = j_2$, *and* $T \leq \Phi(S)$, $S \in \mathrm{Syl}_2(C_A(t))$.

(6) $A = Sp_n(2)$, *and* $t = c_2$.

Proof. If 22.1.2 holds then $[P, p^A \cap C_A(p)] = [T, p^A \cap C_A(p)] = 1$. Suppose 22.1.2 does not hold. Then 22.1.1 holds so T and P are 4-groups. Moreover $T = \langle t, s\rangle$ and $s^a = st$, some $a \in C_A(t)$. Let $\{u\} = sC(A) \cap P$. Then $s = cu$, $c \in C(A)$, and $cu^a = s^a = st = cut$, so $u^a = ut$. In particular $P \trianglelefteq C_A(t)$. Therefore hypothesis 21.2 is satisfied in A by any 4-group in P. So by 21.3, $Z(A) = 1$.

Now suppose $P \leq J$. Then as hypothesis 21.2 is satisfied, the results in Section 10 and Section 11 imply one of (1)–(4) occurs. Also $\mathrm{Aut}_A(J)$ is transitive on $P^A \cap J$ so has the form indicated.

Next suppose $P \not\leq J$. Then 22.1.2 is not satisfied, so as shown above, T and $P = \langle u, t\rangle$ are of order 4 with $u^{C(t)} = \{u, ut\}$. So by Section 10, P is a known subgroup of A. Suppose $A = U_n(q)$. By 10.6, $A = U_n(2)$, $\ell = 2$, and $u \in Z_\ell$ with

$$Q(u) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

But then u is a transvection, so P is not normal in $C_A(u)$. The same argument eliminates the possibility that $A = Sp_n(2)$ and $t = a_2$ or $t = c_2$ with

$$Q(u) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore Section 10 implies $P^\# = T^\#$ is fused and either $A = SL_n(2)$ and $t = j_2$ or $A = Sp(n, 2)$ and $t = c_2$. By 4.6, if $A = SL_n(2)$ then $T \leq \Phi(S)$, $S \in \mathrm{Syl}_2(C_A(t))$.

In Case II we have shown $P^\#$ to be fused. Moreover in cases (1) and (4), $P^\#$ is fused. Consider cases (2) and (3) and assume $P^\#$ is not fused. Then $P^\#$ contains an element v of type b_1 or a_k. But then by what we have shown, P does not centralize $v^A \cap C(v)$, a contradiction. So in general, $P^\#$ is fused.

BIBLIOGRAPHY

[1] Aschbacher, M., Tightly embedded subgroups of finite groups, (to appear).

[2] Aschbacher, M., On finite groups of component type, Ill. J. Math., **19** (1975), 87–115.

[3] Aschbacher, M. and Seitz, G., On groups with a standard component of known type, (to appear).

[4] Borel, A. and Tits, J., Éléments unipotents et sousgroupes paraboliques de groupes réductifs, I, Inventiones Math., **12** (1971), 95–104.
[5] Bourbaki, N., Groupes et algebres de Lie, Chap. 4, 5, 6 Fascicule XXXIV, Elements de Math., Paris, Hermann 1968.
[6] Carter, R., Simple groups of Lie type, Wiley, New York, 1972.
[7] Curtis, C. Kantor, W., and Seitz, G., The two transitive permutation representations of the finite Chevalley groups, Trans. A. M. S., **218** (1976), 1–59.
[8] Fendel, D., A characterization of Conway's group, 3, J. Alg., **24** (1973), 159–196.
[9] Feit, W., The current situation in the theory of finite simple groups, Actes. Congress Intern. Math., **1** (1974), 55–93.
[10] Fong, P., and Seitz, G., Groups with a (B, N)-pair of rank 2, I, II, Inventiones Math., **21, 24** (1973), (1974), 1–57, 191–239.
[11] Frame, J., The characters of the Weyl group of E_8, Computational Problems in Abstract Algebra, 111–130, Pergamon Press, Oxford, (1970) (Ed. J. Leech).
[12] Gorenstein, D., Finite Groups, Harper and Row, New York, 1968.
[13] Griess, R., Schur multipliers of the finite simple groups of Lie type (to appear
[13] Griess, R., Schur multipliers of the finite simple groups of Lie type, Trans. A. M. S., **183** (1973), 355–421.
[14] Guterman, A characterization of the groups $F_4(2^n)$, J. Alg., **20** (1972), 1–23.
[15] Hamill, C., A collineation group of order $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$, Proc. London Math. Soc., **3** (1953), 54–79.
[16] Hunt, D., A characterization of the finite simple groups M(22), J. Alg., **21** (1972), 103–112.
[17] Lang, S., Algebraic groups over finite field, Amer. J. Math., **18** (1956), 555–563.
[18] Parrot, D., A characterization of the Ree groups ${}^2F_4(q)$, J. Alg., **27** (1973), 341–357.
[19] Seitz, G., Flag-transitive subgroups of Chevalley groups, Annals of Math., **97** (1973), 27–56.
[20] Seitz, G., and Wright, C., On complements of F-residuals in finite solvable groups, Archiv. der Math., **21** (1970), 139–150.
[21] Steinberg, R., Automorphisms of finite linear groups, Canadian J. Math., **12** (1960), 606–615.
[22] Suzuki, M., Characterizations of linear groups, Bull, A. M. S., **75** (1969), 1043–1091.
[23] Thomas, G., A characterization of the group $G_2(2^n)$, J. Alg., **13** (1969), 87–118.
[24] Thomas, G., A characterization of the Steinberg group $D_4^2(q^3)$, $q=2^n$, J. Alg., **14** (1970), 373–385.
[25] Wall, G., On the conjugacy classes in the unitary, symplectic, and orthogonal groups, J. Australian Math. Soc., **3** (1963), 1–62.

California Institute of Technology
University of Oregon