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ON BOUNDARIES OF SCHOTTKY SPACES

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0. Introduction.

Let S be a compact Riemann surface and let S_n be the surface obtained from S in the course of a pinching deformation. We denote by Γ_n the quasi-Fuchsian group representing S_n in the Teichmüller space $T(\Gamma)$, where Γ is a Fuchsian group with $U/\Gamma = S$ (U: the upper half plane). Then in the previous paper [7] we showed that the limit of the sequence of Γ_n is a cusp on the boundary $\partial T(\Gamma)$. In this paper we will consider the case of Schottky space $\mathfrak{S}$. Let G_n be a Schottky group with $\Omega(G_n)/G_n = S_n$. Then the purpose of this paper is to show what the limit of G_n is.

We will begin with defining the boundary of the Schottky space. Usually the boundary is considered in $\boldsymbol{C}^{3g-3}$, the complex $(3g-3)$-dimensional space. However, in our approach, it is more convenient to do it in $\hat{\boldsymbol{C}}^{3g}$. This will be illustrated by some examples.

First we treat the hyperelliptic case. Let G be a Schottky group such that $\Omega(G)/G$ is a hyperelliptic surface whose branch points are $a_1, a_2, \cdots, a_{2g-2}, 0, 1, a_{2g-1}, \infty$; $a_j \in \boldsymbol{R}$ $(j = 1, \cdots, 2g-1)$ and whose branch cuts are $(a_1, a_2), \cdots, (a_{2g-3}, a_{2g-2}), (0, 1), (a_{2g-1}, \infty)$ on $\boldsymbol{R}$. We consider the deformatiom obtained by moving a_{2g-1} to ∞ increasingly along the real axis and keeping other branch points and cuts fixed. Then under the deformation there exist sequences of Schottky groups G_n tending to a point on $\partial_3\mathfrak{S}$ (Theorem 1) and a point on $\partial_2\mathfrak{S} \cup \partial_3\mathfrak{S}$ (Theorem 2) (see §1 for the notations). Next let G be a Schottky group such that $\Omega(G)/G$ is a compact Riemann surface of genus $g \geqq 2$. Let S_n be a compact Riemann surface obtained from S in the course of pinching deformation. We denote by G_n a Schottky group with $\Omega(G_n)/G_n = S_n$. Then we show that the limit of subsequence of G_n may be either a cusp (Theorems 3

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and 4), a point on $\partial_3\mathfrak{S}$ (Theorem 3) or a "node" (Theorem 6). Observe a big difference from the case of Teichmüller space.

In §1 we will state two definitions of a Schottky space and the definition of a normalized Schottky space. Then we define the boundary of a Schottky space and show by some examples that it is inconvenient to use a normalized Schottky space. In §2 we will show that under the above deformation there exists a sequence of Schottky groups tending to a point on $\partial_3\mathfrak{S}$ in the hyperelliptic case. We note that Lemmas 3 and 4 would be interesting and the technique of the proofs would be useful for studying relations between locations of branch points and cuts on a hyperelliptic surface and multipliers of generators of Schottky group which represents the surface. In §3 we will show that when we perform a pinching deformation for a compact Riemann surface S, subsequences of Schottky groups G_n, representing the obtained surfaces, may tend to either a cusp, a "node" or a point on $\partial_3\mathfrak{S}$.

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1. Definition of boundaries of Schottky spaces.

In this section we will state two definitions of a Schottky space and the definition of a normalized Schottky space. Then we will define the boundary of a Schottky space and will show by some examples that it is difficult to define the boundary of a normalized Schottky space.

1-1. Definition of a Schottky space. Let $C_1, C_1', \cdots, C_g, C_g'$ be a set of $2g, g \geqq 2$, mutually disjoint Jordan curves (we call them defining curves) on the Riemann sphere which complize the boundary of a $2g$-ply connected region D. Suppose there are g Möbius transformations $A_1, \cdots, A_g$ which have the property that A_j maps C_j onto C_j' and $A_j(D) \cap D = \phi, 1 \leqq j \leqq g$. Then the g necessarily loxodromic transformations A_j generate a Schottky group of genus g with D as a fundamental region.

The first definition of a Schottky space is due to Marden [5]. Given $g \geqq 2$, consider the compact manifold $\boldsymbol{P}_3^g$, where $\boldsymbol{P}_3$ denotes complex projective 3-space, with the natural topology. We represent points of this space by g-tuples of 2×2 complex matrices $(A_1, \cdots, A_g)$ (with the natural equivalence relation). Let X be the variety determined by the equation $\prod \det A_j = 0$ and set $V = \boldsymbol{P}_3^g - X$. Fix a Schottky group G of genus g and a set of free generators $A_1, \cdots, A_g$. This set of generators determines the point $(A_1, \cdots, A_g) \in V$. To any homomorphism $\theta: G \to H$, where H is a group of Möbius transformations, we will associate the point $(\theta(A_1), \cdots, \theta(A_g)) \in V$. For simplicity we will use the notation (H, θ) for this point. Conversely, a point $(B_1, \cdots, B_g) \in V$ can be expressed as (H, θ), where H is the group generated by $B_1, \cdots, B_g$ and θ is the homomorphism determined by $\theta(A_j) = B_j$. The topology of V corresponds to the "pointwise convergence" topology in the group H. Namely $(H_n, \theta_n) \to (H, \theta)$ in V if and only if $\theta_n(A_j) \to \theta(A_j)$ for each $j, 1 \leqq j \leqq g$. Define the Schottky space $\mathfrak{S}_1$ as follows.

$$\mathfrak{S}_1 = \{(H, \theta) \in V : H \text{ is a Schottky group and } \theta \text{ is an isomorphism}\} .$$

Remark. Let $\hat{G}$ be another Schottky group and $\hat{A}_1, \cdots, \hat{A}_g$ be generators of $\hat{G}$. Let $\hat{\mathfrak{S}}_1$ be the Schottky space constructed as above with respect to $\hat{G}$ and $\hat{A}_1, \cdots, \hat{A}_g$. Then it is easily seen that $\mathfrak{S}_1$ and $\hat{\mathfrak{S}}_1$ are essentially the same and that their boundaries defined later coinside. Since we study boundary of Schottky space in this paper, we may ignore the letters G, $A_1, \cdots, A_g$ for the definition of the first Schottky space.

The second definition of a Schottky spaces is as follows. Let H be any Schottky group. We denote by λ_j, p_j and q_j the multiplier, the repelling and the attracting fixed points of B_j, respectively, where $B_1, \cdots, B_g$ are generators of H and $1 < |\lambda_j| < +\infty$. Thus H determines $3g$-tuples of complex numbers

$$(\lambda_1, p_1, q_1, \lambda_2, \cdots, \lambda_g, p_g, q_g) \in \hat{\boldsymbol{C}}^{3g} .$$

For simplicity we denote by τ such $3g$-tuples. Conversely a point τ with $\lambda_j \neq \infty$ $(1 \leqq j \leqq g)$ determines a point $(B_1, \cdots, B_g) \in V$. We define the second Schottky space $\mathfrak{S}_2$ with the natural equivalence relation as follows.

$$\mathfrak{S}_2 = \{\tau \in \hat{\boldsymbol{C}}^{3g} : \tau \text{ determines a Schottky group}\} .$$

Then it is easily seen that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are equivalent. Thus we may denote by $\mathfrak{S}$ instead of $\mathfrak{S}_1$ and $\mathfrak{S}_2$. We note that the dimension of $\mathfrak{S}$ is $3g$.

If in the first definition of $\mathfrak{S}$ we regard as the same point in $\mathfrak{S}_1$, the points $(B_1, \cdots, B_g)$ and $(TB_1T^{-1}, \cdots, TB_gT^{-1})$ with $T \in SL'(2, C)$, then we have a normalized Schottky space $[\mathfrak{S}_1]$ instead of a Schottky space $\mathfrak{S}_1$. Similarly if in $\mathfrak{S}_2$, we regard as the same point $(\lambda_1, p_1, q_1, \cdots, \lambda_g, p_g, q_g)$ and $(\hat{\lambda}_1, \hat{p}_1, \hat{q}_1, \cdots, \hat{\lambda}_g, \hat{p}_g, \hat{q}_g)$, we have a normalized Schottky space $[\mathfrak{S}_2]$, where $\hat{\lambda}_j, \hat{p}_j$ and $\hat{q}_j$ are the multiplier, the repelling and the attracting fixed points of $TBT^{-1}, 1 \leqq j \leqq g$, respectively. Then it is easily seen that $[\mathfrak{S}_1]$ and $[\mathfrak{S}_2]$ are equivalent and so we denote them by $[\mathfrak{S}]$. We note that the dimension of $[\mathfrak{S}]$ is $3g - 3$ and $[\mathfrak{S}]$ is usually called a Schottky space.

1-2. Definition of the boundary of the Schottky space.

We consider the boundary of a Schottky space. We will use the notation $\partial\mathfrak{S}_1$ for the relative boundary of $\mathfrak{S}_1$ in V, that is, for each $(H, \theta) \in \partial\mathfrak{S}_1$, there is a sequence of points $(H_n, \theta_n) \in \mathfrak{S}_1$ converging to (H, θ). *A point* $(H, \theta) \in \partial\mathfrak{S}_1$ *will be called a boundary group of* G. *A point* $(H, \theta) \in \partial\mathfrak{S}_1$ *will be called a cusp if there is a loxodoromic element* $A \in G$ *such that* $\theta(A)$ *is parabolic.* Then Chuckrow [3] showed that $\partial\mathfrak{S}_1$ consists of cusps and non-Kleinian groups.

We consider the boundary of $\mathfrak{S}_2$ in $\hat{C}^{3g}$. We classify the boundary of $\partial\mathfrak{S}_2$ into the following three cases as limits of point sequences of Schottky groups $G_n = \{A_{1n}, \cdots, A_{gn}\}$ (or τ_n).

(1) We call the first boundary point the following $\tau_0 \in \hat{C}^{3g}$. For $\tau_0 \in \partial\mathfrak{S}_2$, g Möbius transformations A_{j0} are determined as the limit of A_{jn} $(1 \leqq j \leqq g)$. We denote by $\partial_1\mathfrak{S}_2$ the set of all such points τ_0. In this case $\partial\mathfrak{S}_1 = \partial_1\mathfrak{S}_2$.

(2) We call the second boundary point the following $\tau_0 \in \hat{C}^{3g}$, that is, $\tau_0 = (\lambda_{10}, p_{10}, q_{10}, \cdots, \lambda_{g0}, p_{g0}, q_{g0})$ with $\lambda_{j0} = \lim_{n\to\infty} \lambda_{jn}, p_{j0} = \lim_{n\to\infty} p_{jn}$ and $q_{j0} = \lim_{n\to\infty} q_{jn}$ $(1 \leqq j \leqq g)$ such that at least one of λ_{j0} $(1 \leqq j \leqq g)$ is infinite and all p_{i0} and q_{j0} $(1 \leqq i, j \leqq g)$ are distinct. We denote by $\partial_2\mathfrak{S}_2$ the set of all such points. Furthermore we call the point $\tau_0 \in \partial_2\mathfrak{S}_2$ a "node" if each λ_{j0} $(\neq \infty)$, p_{j0} and q_{j0} determine a loxodromic transformation. We show an example of $\tau_0 \in \partial_2\mathfrak{S}_2$ which is not a "node". Set

$$A_{1n}(z) = \frac{(n+4)i}{n}z \quad \text{and} \quad A_{2n}(z) = \frac{(n+2)z + (n+4+(3/n))}{nz + (n+2)}.$$

We denote by G_n the Schottky group generated by A_{1n} and A_{2n}. Then

$$\tau_n = ((n+4)i/n, 0, \infty, \lambda_{2n}, -\sqrt{(n+1)(n+3)}/n, \sqrt{(n+1)(n+3)}/n)$$

and

$$\tau_0 = \lim_{n\to\infty} \tau_n = (i, 0, \infty, \infty, -1, 1) .$$

Thus $\lambda_{20} = \infty$ and $A_{10} = \lim_{n\to\infty} A_{1n}$ is an elliptic transformation.

(3) We define the third boundary by setting $\partial\mathfrak{S}_2 - \partial_1\mathfrak{S}_2 - \partial_2\mathfrak{S}_2$, and denote it by $\partial_3\mathfrak{S}_2$. We give an example of a point $\tau_0 \in \partial_3\mathfrak{S}_2$. Set

$$A_{1n}(z) = \frac{(n+7)i}{n} z \quad \text{and} \quad A_{2n}(z) = \frac{(2n+2)z + (3-4n^2)/2n}{2nz - (2n-2)} .$$

Then the group generated by A_{1n} and A_{2n} is a Schottky group. Then

$$\tau_n = ((n+7)i/n, 0, \infty, \lambda_{2n}, (2n - \sqrt{3})/2n, (2n + \sqrt{3})/2n)$$

and

$$\tau_0 = \lim_{n\to\infty} \tau_n = (i, 0, \infty, 7 + 4\sqrt{3}, 1, 1) .$$

Thus $A_{10} = \lim_{n\to\infty} A_{1n}$ is an elliptic transformation and $\tau_0 \in \partial_3\mathfrak{S}_2$.

We write $\partial\mathfrak{S}, \partial_1\mathfrak{S}, \partial_2\mathfrak{S}$ and $\partial_3\mathfrak{S}$ instead of $\partial\mathfrak{S}_2, \partial_1\mathfrak{S}_2, \partial_2\mathfrak{S}_2$ and $\partial_3\mathfrak{S}_2$, respectively.

Now we present an example showing that the normalized Schottky space $[\mathfrak{S}]$ is not convenient for our study.

Examples. Let

$$A_r(z) = \frac{z + 1 - r^2}{z + 1} , \qquad 0 < r < 1$$

and

$$B_r(z) = \frac{7z - 29}{z - 4} .$$

Let G_r be the Schottky group generated by A_r and B_r, that is, $G_r = \{A_r, B_r\}$ and

$$\tau_r = ((2 - r^2 + 2\sqrt{1 - r^2})/r^2, -\sqrt{1 - r^2}, \sqrt{1 - r^2}, (7 + 3\sqrt{5})/2, (11 - \sqrt{5})/2, (11 + \sqrt{5})/2) .$$

Set

$$T_r(z) = \frac{z + \sqrt{1 - r^2}}{z - \sqrt{1 - r^2}}\,,$$

$$\hat{A}_r(z) = T_r A_r T_r^{-1}(z) = \frac{2 - r^2 + 2\sqrt{1 - r^2}}{r^2} z$$

and

$$\hat{B}_r(z) = T_r B_r T_r^{-1}(z) = \frac{(-r^2 - 28 + 3\sqrt{1 - r^2})z + (11\sqrt{1 - r^2} + 30 - r^2)}{(11\sqrt{1 - r^2} - 30 + r^2)z + (3\sqrt{1 - r^2} + 28 + r^2)}\,.$$

Let $\hat{G}_r$ be the Schottky group generated by $\hat{A}_r$ and $\hat{B}_r$, that is, $\hat{G}_r = \{\hat{A}_r, \hat{B}_r\}$ and

$$\hat{\tau}_r = ((2 - r^2 + 2\sqrt{1 - r^2})/r^2, 0, \infty, (7 + 3\sqrt{5})/2, \hat{p}_2, \hat{q}_2)\,.$$

For each real number $r, 0 < r < 1, G_r$ and $\hat{G}_r$ determine the same point in $[\mathfrak{S}]$. It is easily seen that

$$A_1(z) = \lim_{r \to 1} A_r(z) = z/(z + 1)$$

is parabolic and

$$B_1(z) = \lim_{r \to 1} B_r(z) = (7z - 29)/(z - 4)$$

is loxodromic. And

$$\tau_0 = \lim_{r \to 1} \tau_r = (1, 0, 0, (7 + 3\sqrt{5})/2, p_2, q_2)\,.$$

Hence the group generated by $A_1(z)$ and $B_1(z)$ is a cusp on $\partial_1\mathfrak{S}$. On the other hand

$$\hat{A}_1(z) = \lim_{r \to 1} \hat{A}_r(z) = z$$

is the identity and

$$\hat{B}_1(z) = \lim_{r \to 1} \hat{B}_r(z) = (-29z + 29)/(-29z + 29)\,,$$

and

$$\hat{\tau}_0 = \lim_{r \to 1} \hat{\tau}_r = (1, 0, \infty, (7 + 3\sqrt{5})/2, 1, 1)\,.$$

Hence $\hat{\tau}_0$ is in X and on $\partial_3\mathfrak{S}$.

Furthermore

$$A_0(z) = \lim_{r\to 0} A_r(z) = (z+1)/(z+1)$$

and

$$B_0(z) = \lim_{r\to 0} B_r(z) = (7z-29)/(z-4)\ .$$

Hence

$$\tau_0 = \lim_{r\to 0} \tau_r = (\infty, -1, 1, (7+3\sqrt{5})/2, p_2, q_2)$$

is in X and on $\partial_2\mathfrak{S}$. On the other hand

$$\hat{A}_0(z) = \lim_{r\to 0} \hat{A}_r(z) = \infty$$

and

$$\hat{B}_0(z) = \lim_{r\to 0} \hat{B}_r(z) = (-25z+41)/(-19z+31)\ .$$

Hence

$$\hat{\tau}_0 = \lim_{r\to 0} \hat{\tau}_r = (\infty, 0, \infty, (7+3\sqrt{5})/2, p_2, q_2)$$

is on $\partial_2\mathfrak{S}$.

G_r and $\hat{G}_r$ represent the same point of the normalized Schottky space $[\mathfrak{S}]$. However, they behave differently as $r\to 0$ or $r\to 1$. This shows that the Schottky space $\mathfrak{S}$ is more convenient than the normalized space $[\mathfrak{S}]$.

2. The hyperelliptic case.

In this section we will discuss the case where G is a Schottky group such that $\Omega(G)/G$ is a hyperelliptic surface, where $\Omega(G)$ denotes the region of discontinuity of G, and we will consider limits of the Schottky groups obtained under the following deformation.

2-1. Let S be a normalized hyperelliptic surface which has branch points $a_1, \cdots, a_{2g-2}, 0, 1, a_{2g-1}, \infty$ and has branch cuts $(a_1, a_2), (a_3, a_4), \cdots, (a_{2g-3}, a_{2g-2}), (0, 1)$ and (a_{2g-1}, ∞) on the real axis, where $a_1 < a_2 < \cdots < a_{2g-2} < 0 < 1 < a_{2g-1}, |a_{2g-1}| > |a_1|, a_j \in \boldsymbol{R}$ $(j = 1, \cdots, 2g-1)$ (cf, see Fig. 1 in the previous paper [7]). Take g simple loops $\alpha_1, \cdots, \alpha_g$ being disjoint each other on S as follows. Each α_j $(2 \leqq j \leqq g)$ surrounds the cut (a_{2j-3}, a_{2j-2}) and not other cuts in its interior and α_1 surrounds the cut

(a_{2g-1}, ∞) and not other cuts in its interior. Now we consider the deformation under which the branch points $a_1, \cdots, a_{2g-2}, 0, 1, \infty$ and the cuts $(a_1, a_2), \cdots, (a_{2g-3}, a_{2g-2}), (0, 1)$ are fixed, and the point a_{2g-1} increasingly tends to ∞ along the real axis.

Let G be a Schottky group of genus g such that $\Omega(G)/G$ is the above hyperelliptic surface S and S_n be the hyperelliptic surface which has branch points $a_1, a_2, \cdots, a_{2g-2}, 0, 1, a_{2g-1}, \infty$ and has cuts $(a_1, a_2), \cdots, (a_{2g-3}, a_{2g-2}), (0, 1), (a_{2g-1}^{(n)}, \infty)$ on the real axis, where $a_{2g-1} < a_{2g-1}^{(n)}$. Now we may take α_1 as the circle about 0 of the radius r with $|a_1| < r < a_{2g-1}$. On the other sheet we denote by α_1' the circle which has the same projection as α_1. Let D_1 be the ring domain containing ∞ bounded by α_1 and α_1' on S. Furthermore we write α_1 and α_1' for the corresponding loops on S_n. Let D_{1n} be the ring domain containing ∞ bounded by α_1 and α_1' on S_n. To the loops $\alpha_1, \cdots, \alpha_g$ on S we assign Möbius tranformations $A_1, \cdots, A_g$, respectively.

We consider the conformal mapping of the Grötzsch extremal region to the concentric annulus (cf. see Fig. 4 in [7]). We map D_1 and D_{1n} to annuli $K_1: \{\rho_1 < |z| < 1\}$ and $K_{1n}: \{\rho_{1n} < |z| < 1\}$ by conformal mappings Φ and Φ_n, respectively. Then

$$\Phi((1/r)a_{2g-1}) = 1/\sqrt{\rho_1}$$

and

$$\Phi_n((1/r)a_{2g-1}^{(n)}) = 1/\sqrt{\rho_{1n}} \ .$$

We define a q.c. mapping $f_n: S \to S_n$ as follows. Let $\tilde{f}_n$ be an arbitrary quasi-comformal mapping of K_1 onto K_{1n} such that $\Phi_n^{-1}\tilde{f}_n\Phi = \text{id.}$ on ∂D_1. We define f_n by setting

$$f_n = \begin{cases} \Phi_n^{-1}\tilde{f}_n\Phi & \text{on } D_1 \\ \text{identity} & \text{on } S - D_1 \ . \end{cases}$$

LEMMA 1. (Sato [7]).

$$\lim_{n\to\infty} \rho_{1n} = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} a_{2g-1}^{(n)} = \infty \ .$$

LEMMA 2. *For f_n defined above, there uniquely exists a q.c. mapping F_n which satisfies the following conditions*:

(1) *With respect to $G_n = F_nGF_n^{-1}, F_n(\Omega(G))/G_n = S_n$*

(2) *With respect to π_n, the natural projection from $\Omega(G_n)$ onto S_n,*

$\pi_n F_n = f_n \pi$ *and*

(3) $F_n(0) = 0, F_n(1) = 1$ *and* $F_n(\infty) = \infty$,

where π *expresses the natural projection from* $\Omega(G)$ *onto* S.

Proof. We can prove the lemma by the same method as in the proof of Lemma 2 in [7], hence we omit the proof here.

Let A_1 be an element of G with the following property: If a path $\widehat{zz'}$ is a lift of α_1, then $z' = A_1(z)$. Set $A_{1n} = F_n A_1 F_n^{-1}$. We denote by λ_{1n} the multiplier of A_{1n}. Then by a similar method to the proof of Lemma 3 in [7] we have the following lemma, but for the completeness here we give a proof.

LEMMA 3. *If* $\lim_{n\to\infty} a_{2g-1}^{(n)} = \infty$, *then* $\lim_{n\to\infty} \log|\lambda_{1n}| = 0$.

Proof. Let p_{1n} and q_{1n} be the fixed points of A_{1n} and we may assume that $p_{1n} = 0$ and $q_{1n} = \infty$. We denote by Γ_{1n} the set of all simple closed rectifiable curves γ separating 0 and ∞ and denote by M_{1n} the extremal length modulo $\{A_{1n}\}$ (the quantity introduced by Bers [2]), that is,

$$M_{1n} = \sup_{\sigma} \frac{\left(\inf_{\gamma\in\Gamma} \int_{\gamma\in\Gamma} \sigma(z)\,|dz|\right)^2}{\iint_{F_n(\hat{C})/\{A_{1n}\}} \sigma(z)^2 dxdy},$$

where $\sigma(z)$ is a non-negative measurable function which satisfies the identity

$$\sigma(A_{1n}(z))\,|dA_{1n}(z)| = \sigma(z)\,|dz|\,.$$

We call the function $\sigma(z)$ an admissible function. Then (Bers [2])

$$M_{1n} = \frac{2\pi}{\log|\lambda_{1n}|}\,. \tag{1}$$

We denote by ℓ_n the lift of the branch cut $(a_{2g-1}^{(n)}, \infty)$ which joins p_{1n} and q_{1n}, and denote by E_{1n} the lift of the ring domain D_{1n} such that $\ell_n \in E_{1n}$. We denote by $\tilde{\Gamma}_{1n}$ the set of all rectifiable curves joining the boundary $|z| = 1$ and another boundary $|z| = \rho_{1n}$ in the annulus K_{1n} and denote by $\tilde{M}_{1n}$ the extremal length of $\tilde{\Gamma}_{1n}$ in K_{1n}. It is known that

$$\tilde{M}_{1n} = -\log\rho_{1n}/(2\pi)\,. \tag{2}$$

For each curve $\gamma \in \Gamma_{1n}$, there exists a curve $\tilde{\gamma}^*$ in E_{1n} being a lift of $\tilde{\gamma} \in \tilde{\Gamma}_{1n}$ such that $\tilde{\gamma}^*$ is a part of γ. It is not difficult to prove that

$$M_{1n} \geqq \tilde{M}_{1n} . \tag{3}$$

By Lemma 1, if $\lim_{n\to\infty} a_{2g-1}^{(n)} = \infty$, then $\lim_{n\to\infty} \rho_{1n} = 0$. Hence from (1), (2) and (3), we have the desired result. Our proof is now complete.

For each $j = 2, 3, \cdots, g$, let A_j be an element of G with the following property: If a path $\widehat{z_j z'_j}$ be a lift of α_j, then $z'_j = A_j(z_j)$. We consider the variations of $A_2, \cdots, A_g$ under the above deformation. Let $\alpha'_2, \cdots, \alpha'_g$ be the loops on the other sheet which have the same projections as $\alpha_2, \cdots, \alpha_g$, respectively. Let D_j $(j = 2, \cdots, g)$ be the ring domain containing the cut (a_{2j-3}, a_{2j-2}) bounded by α_j and α'_j. Map the ring domain D_j to the annulus $K_j: \{\rho_j < |z| < 1\}$ by a conformal mapping g_j. Let f_n be the q.c. mapping constructed above. We set $\alpha_{jn} = f_n(\alpha_j), \alpha'_{jn} = f_n(\alpha'_j)$ and $D_{jn} = f_n(D_j)$. Let g_{jn} be a conformal mapping from D_{jn} to the annulus $K_{jn}: \{\rho_{jn} < |z| < 1\}$.

Let $\tilde{\Gamma}_j$ be the set of curves joining the boundary $|z| = 1$ of K_j and another boundary $|z| = \rho_j$ in K_j. Let $\tilde{\Gamma}_{jn}$ be the set of all curves joining the boundary $|z| = 1$ of K_{jn} and another boundary $|z| = \rho_{jn}$ in K_{jn}. We denote by $\tilde{M}_j$ and $\tilde{M}_{jn}$ the extremal length of $\tilde{\Gamma}_j$ in K_j and of $\tilde{\Gamma}_{jn}$ in K_{jn}, respectively. Then $f_{jn} = g_{jn} f_n g_j^{-1}: K_j \to K_{jn}$ is conformal, hence

$$\tilde{M}_{jn} = \tilde{M}_j = \frac{-\log \rho_j}{2\pi} .$$

Set $A_{jn} = F_n A_j F_n^{-1}$ $(j = 2, \cdots, g)$. We denote by λ_{jn} the multiplier of A_{jn}. We denote by M_{jn} the extremal length modulo $\{A_{jn}\}$ by the same method as in the proof of Lemma 3. Then

$$M_{jn} = \frac{2\pi}{\log|\lambda_{jn}|} , \qquad |\lambda_{jn}| > 1 .$$

By the same way as in the proof of Lemma 3, we have

$$\frac{2\pi}{\log|\lambda_{jn}|} \geqq \frac{-\log \rho_j}{2\pi} .$$

Hence

$$\log|\lambda_{jn}| \leqq \frac{4\pi^2}{-\log \rho_j} .$$

2-2. Next we consider the "β"-cycles on S. Let $\beta_1, \cdots, \beta_g$ be a basis of "β"-cycles as in the Figure 1 below, that is, β_j are mutually disjoint and $\alpha_j \times \beta_k = \delta_{jk}$ (Kronecker's δ) and β'_j is a loop which bounds a ring domain D^*_j together with β_j for each $j = 1, \cdots, g$. Furthermore we assume that β_j and β'_j $(2 \leqq j \leqq g)$ are contained in $S - D_1$. We set $\beta_{jn} = f_n(\beta_j)$, $\beta'_{jn} = f_n(\beta'_j)$ and $D^*_{jn} = f_n(D^*_j)$ $(j = 1, \cdots, g)$.

We fix $j, 2 \leqq j \leqq g$. We assume that $A_{jn}(z) = \lambda_{jn} z$. Let C_{jn} and C'_{jn} be defining curves of G_n such that $A_{jn}(C_{jn}) = C'_{jn}$ and one of the lifts of D^*_j lies between C_{jn} and C'_{jn}. Then C_{jn} and C'_{jn} both separate 0 and ∞. We denote by ω_{jn} the ring domain bounded by C_{jn} and C'_{jn}. We denote by Γ^*_{jn} the set of all curves γ_θ $(0 \leqq \theta \leqq 2\pi)$ which are the intersections of ω_{jn} and rays emanating from the origin, where each $\gamma_\theta \in \Gamma^*_{jn}$ consists of finitely many line segments and $\arg z = \theta$ for each $z \in \gamma_\theta$. We denote by M^*_{jn} the extremal length of Γ^*_{jn} in ω_{jn}, that is,

$$M^*_{jn} = \sup_\sigma \frac{\left(\inf\limits_{\gamma} \int_\gamma \sigma(z)\,|dz|\right)^2}{\iint_{\omega_{jn}} \sigma(z)^2 dx dy},$$

where $\sigma(z)$ is a non-negative measurable function. Then one of the lifts of the curves β_j is in ω_{jn}, and it is a closed curve which separates 0 and ∞. We denote the curve by β^*_j. Similarly we denote by $\beta^{*\prime}_j$ the closed curve separating 0 and ∞ which is a lift of β'_j in ω_{jn}. By conformal mappings g^*_j and g^*_{jn}, we map D^*_j and D^*_{jn} to the annuli $K^*_j : \{\rho^*_j < |z| < 1\}$ and $K^*_{jn} : \{\rho^*_{jn} < |z| < 1\}$, respectively. Let $\tilde{\Gamma}^*_j$ and $\tilde{\Gamma}^*_{jn}$ be the sets of curves joining $|z| = 1$ and $|z| = \rho^*_j$, and $|z| = 1$ and $|z| = \rho^*_{jn}$, respectively. We denote by $\tilde{M}^*_j$ and $\tilde{M}^*_{jn}$ the extremal length of $\tilde{\Gamma}^*_j$ in K^*_j and of $\tilde{\Gamma}^*_{jn}$ in K^*_{jn}, respectively. Then by the conformal invariance of the extremal length we have

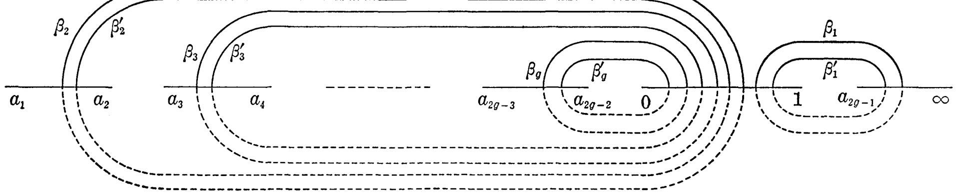

Figure 1.

$$\tilde{M}^*_j = \tilde{M}^*_{jn}. \tag{4}$$

Furthermore by the same method as in the proof of Lemma 3, we have

$$\tilde{M}^*_{jn} \leqq M^*_{jn} . \tag{5}$$

We easily see that

$$\tilde{M}^*_j = \frac{-\log \rho^*_j}{2\pi} . \tag{6}$$

Next we show that

$$M^*_{jn} \leqq \frac{\log |\lambda_{jn}|}{2\pi} . \tag{7}$$

Set $m(\sigma) = \inf_{\gamma_\theta} \int_{\gamma_\theta} \sigma(z)\,|dz|$. Then for any function $\sigma(z)$ and for each $\gamma_\theta \in \Gamma^*_{jn}$,

$$m(\sigma) \leqq \int_{\gamma_\theta} \sigma(re^{i\theta})dr , \qquad \text{where } z = re^{i\theta} .$$

Hence

$$\int_0^{2\pi} m(\sigma)d\theta \leqq \int_0^{2\pi}\int_{\gamma_\theta} \sigma(re^{i\theta})drd\theta .$$

By using the Schwarz inequality, we have

$$\begin{aligned} 4\pi^2 m(\sigma)^2 &\leqq \int_0^{2\pi}\int_{\gamma_\theta} \sigma(z)^2 rdrd\theta \int_0^{2\pi}\int_{\gamma_\theta} (1/r)drd\theta \\ &= \iint_{\omega_{jn}} \sigma(z)^2 dxdy \int_0^{2\pi}\int_{\gamma_\theta} (1/r)drd\theta . \end{aligned}$$

Hence

$$\frac{4\pi^2 m(\sigma)^2}{\iint_{\omega_{jn}} \sigma(z)^2 dxdy} \leqq \int_0^{2\pi}\int_{\gamma_\theta} \frac{1}{r}drd\theta .$$

On the other hand let $\tilde{\omega}_{jn}$ be the image region of ω_{jn} under the logarithmic function $\zeta = \log z, \zeta = \xi + i\eta$ (see Fig. 2).

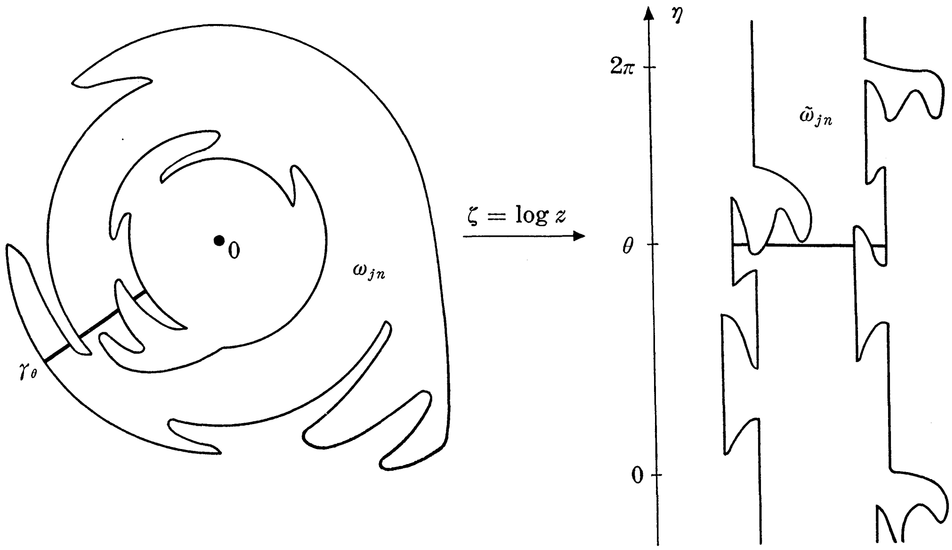

Figure 2.

Then $\int_{\gamma_\theta} (1/r)dr$ expresses the total length of line segments in $\tilde{\omega}_{jn} \cap \{\zeta \mid \operatorname{Im}\zeta = \theta\}$. Hence

$$\int_0^{2\pi}\int_{\gamma_\theta} (1/r)drd\theta$$

is the area of $\tilde{\omega}_{jn}$. Since

$$\int_0^{2\pi}\int_{\gamma_\theta} (1/r)drd\theta = 2\pi \log|\lambda_{jn}| \ ,$$

we have

$$\frac{m(\sigma)^2}{\iint_{\omega_{jn}} \sigma(z)^2 dxdy} \leqq \frac{\log|\lambda_{jn}|}{2\pi} \ .$$

By the arbitrariness of σ, we have (7).

By (4), (5), (6) and (7) we have

$$\frac{\log|\lambda_{jn}|}{2\pi} \geqq \frac{-\log \rho_j^*}{2\pi} \ ,$$

hence

$$|\lambda_{jn}| \geqq 1/\rho_j^* \ .$$

Thus we have the following

LEMMA 4. *Under the same deformation as in Lemma* 3,

$$\frac{1}{\rho_j^*} \leqq |\lambda_{jn}| \leqq \exp\left(\frac{4\pi^2}{-\log \rho_j}\right) \qquad (2 \leqq j \leqq g) .$$

Remark. It would be interesting to compare this with a result of Abikoff [1].

2-3. Now we have

THEOREM 1. *Let G be introduced at the beginning of* 2–1. *Let $G_n = \{A_{1n}, \cdots, A_{gn}\}$ be the Schottky group constructed in Lemma* 2. *Then*

(1) *if $G_0 \in \partial_1\mathfrak{S}$ is the limit of $T_{n_j}G_{n_j}T_{n_j}^{-1}$, whose $\{n_j\} \subset \{n\}$ and T_{n_j} are Möbius transformations, then G_0 is a cusp.*

(2) *There exists a subsequence $\{n_j\} \subset \{n\}$ and Möbius transformations T_{n_j} such that the limit G_0 of the sequence $T_{n_j}G_{n_j}T_{n_j}^{-1}$ is on $\partial_3\mathfrak{S} \cap X$.*

Proof. (1) If the limit G_0 is a point on $\partial_1\mathfrak{S}$, then by Lemma 3, $A_{10} = \lim_{n_j\to\infty} T_{1n_j}A_{1n_j}T_{1n_j}^{-1}$ is parabolic, elliptic or the identity and by Lemma 4, $A_{j0} = \lim_{n_j\to\infty} T_{jn_j}A_{jn_j}T_{jn_j}^{-1}$ is loxodromic for each $j, 2 \leqq j \leqq g$. Hence by Chuckrow [3], A_{10} must be parabolic. Thus G_0 is a cusp on $\partial_1\mathfrak{S}$.

(2) We denote by p_{jn} and q_{jn} the repelling and the attracting fixed points of A_{jn} ($j = 1, \cdots, g$). Let T_n be the Möbius transformation such that $T_n(p_{1n}) = 0, T_n(q_{1n}) = \infty$ and $T_n(p_{2n}) = 1$. Then

$$\lim_{n\to\infty} \hat{A}_{1n} = \lim_{n\to\infty} T_nA_{1n}T_n^{-1} = \text{id.} \qquad \text{or elliptic}$$

since $\hat{p}_{1n} = 0, \hat{q}_{1n} = \infty$ and $\lim_{n\to\infty}|\lambda_{1n}| = 1$, where $\hat{p}_{1n}$ and $\hat{q}_{1n}$ are the repelling and the attracting fixed points of $\hat{A}_{1n}$, respectively.

If $\hat{p}_{20} \neq \hat{q}_{20}$, then by Lemma 4, $\hat{A}_{20} = \lim_{n\to\infty} T_nA_{2n}T_n^{-1}$ is loxodromic, where $\hat{p}_{20} = \lim_{n\to\infty} \hat{p}_{2n}$ and $\hat{q}_{20} = \lim_{n\to\infty} \hat{q}_{2n}$ and $\hat{p}_{2n}$ and $\hat{q}_{2n}$ are the repelling and the attracting fixed points of $T_nA_{2n}T_n^{-1}$. But by Lemma 4 and its corollary in Chuckrow [3] this case does not occur. Hence $\hat{p}_{20} = \hat{q}_{20} = 1$. Set

$$\hat{A}_{2n} = \begin{pmatrix} \hat{a}_{2n} & \hat{b}_{2n} \\ \hat{c}_{2n} & \hat{d}_{2n} \end{pmatrix}, \qquad \hat{a}_{2n}\hat{d}_{2n} - \hat{b}_{2n}\hat{c}_{2n} = 1 .$$

Then by Lemma 4,

$$\hat{c}_{2n} = \frac{\lambda_{2n}^{1/2} - \lambda_{2n}^{-1/2}}{\hat{p}_{2n} - \hat{q}_{2n}} \to \infty \ (n \to \infty) \ .$$

Since

$$\hat{a}_{2n} = \hat{c}_{2n}\hat{q}_{2n} - \lambda_{2n}^{-1/2}$$
$$\hat{b}_{2n} = -\hat{c}_{2n}\hat{p}_{2n}\hat{q}_{2n}$$

and

$$\hat{d}_{2n} = -\hat{c}_{2n}\hat{p}_{2n} - \lambda_{2n}^{-1/2} \ ,$$

we have that

$$\lim_{n\to\infty} \hat{A}_{2n}(z) = (z-1)/(z-1) \ .$$

Hence $\hat{G}_0 = \lim_{n\to\infty} \hat{G}_n$ is in X. Furthermore let $\tau_n \in \mathfrak{S}$ be the associated element with G_n. Then

$$\tau_0 = \lim_{n\to\infty} \tau_n = (1, 0, \infty, \lambda_{20}, 1, 1, \cdots, \lambda_{g0}, p_{g0}, q_{g0}) \ .$$

Hence $\tau_0 \in \partial_3\mathfrak{S}$. Our proof is now complete.

2-4. Next we consider "β"-cycles. Let $\beta_1, \cdots, \beta_g$ be a basis of "β"-cycles on S. We denote by $\tilde{\beta}_j$ the symmetric loop of β_j with respect to the real axis ($j = 1, \cdots, g$). We denote by $\tilde{D}_j^*$ ($1 \leqq j \leqq g$) the ring domain bounded by β_j and $\tilde{\beta}_j$. Let G^* be a Schottky group generated by Möbius transformations $B_1, \cdots, B_g$ assigned to the loops $\beta_1, \cdots, \beta_g$, respectively, in a similar sense for "α"-cycles. Let S_n be the Riemann surface constructed in front of Lemma 1 and let f_n be the same q.c. mapping from S to S_n defined there. Then by the same method as in Lemma 2, we have

LEMMA 5. *There exists a unique q.c. mapping F_n^* which satisfies the following conditions:*

(1) *With respect to $G_n^* = F_n^* G^* F_n^{*-1}$, $F_n^*(\Omega(G^*))/G_n^* = S_n$,*

(2) *with respect to the natural projection $\pi_n^* : \Omega(G_n^*) \to S_n$, $\pi_n^* F_n^* = f_n \pi^*$ and*

(3) *$F_n^*(0) = 0, F_n^*(1) = 1$ and $F_n^*(\infty) = \infty$,*
where $\pi^* : \Omega(G^*) \to S$ is the natural projection.

If we set $B_{jn} = F_n^* B_j F_n^{*-1}$ $(1 \leqq j \leqq g)$, then $G_n^* = \{B_{1n}, \cdots, B_{gn}\}$. We denote by λ_{jn}^* the multiplier of B_{jn}. We set $\beta_{jn} = f_n(\beta_j)$ and $\tilde{\beta}_{jn} = f_n(\tilde{\beta}_j)$ $(2 \leqq j \leqq g)$. Let b_1 be the intersection point of β_1 and the segment $(0, 1)$. Let β_{1n} be a simple closed curve through the points b_1 and $2c_n$ which does not intersect with β_{jn} $(2 \leqq j \leqq g)$.

Let $\tilde{\alpha}_j$ $(j = 2, \cdots, g)$ be mutually disjoint simple loops homotopic to α_j in $S - D_1$ so that each of $\tilde{\alpha}_j$ bounds a ring domain D_j^* together with α_j, and let $\tilde{\alpha}_1$ be a simple loop homotopic to α_1 so that $\tilde{\alpha}_1$ is disjoint from $\tilde{\alpha}_j$ $(2 \leqq j \leqq g)$ and bounds a ring domain D_1^* together with α_1. Then $\tilde{D}_j$ and $\tilde{D}_j^*$ are conformally mapped to the annuli $\tilde{K}_j : \{\tilde{\rho}_j < |z| < 1\}$ and $\tilde{K}_j^* : \{\tilde{\rho}_j^* < |z| < 1\}$, respectively. Then by using similar method to the proofs of Lemma 3 and Lemma 4, we have the following lemmas.

LEMMA 6. *Under the above deformation,*

$$\frac{1}{\tilde{\rho}_j} \leqq |\lambda_{jn}^*| \leqq \exp\left(\frac{4\pi^2}{-\log \tilde{\rho}_j^*}\right)$$

for $j = 2, 3, \cdots, g$.

LEMMA 7. *If*

$$\lim_{n\to\infty} a_{2q-1} = \infty\ , \quad \textit{then} \quad \lim_{n\to\infty} \lambda_{1n}^* = \infty\ .$$

By using Lemma 6 and Lemma 7 we obtain the following theorem. Here we shall omit the proof.

THEOREM 2. *Let* G_n^* *be the Schottky groups constructed above. Then the limit* $G_0^* \in \partial\mathfrak{S}$ *of the sequence* $T_{n_j} G_{n_j}^* T_{n_j}^{-1}$, *whose* $\{n_j\} \subset \{n\}$ *and* T_{n_j} *are Möbius transformations, is always on* $\partial_2\mathfrak{S} \cup \partial_3\mathfrak{S}$ *but not on* $\partial_1\mathfrak{S}$.

Remark. It is not known whether there exists a subsequence $T_{n_j} G_{n_j}^* T_{n_j}^{-1}$ tending to a "node" or not.

3. The general case.

In this section let S be a compact Riemann surface of genus g and let G be a Schottky group with $\Omega(G)/G = S$. Fix the Schottky group G. Here we study limits of subsequence of Schottky groups G_n with $\Omega(G_n)/G_n = S_n$, where S_n is the Riemann surfaces obtained from S in the course of the following pinching deformation.

3-1. Let $\alpha_1, \cdots, \alpha_g$ be a basis of "α"-cycles on S and $D_1, \cdots, D_g$ be mutually disjoint ring domains such that each D_j contains α_j $(j = 1, \cdots, g)$. We will construct the Riemann surface S_n from S as follows. Let $\hat{f}_n$ be a q.c. mapping with a finite maximal dilatation $D(\hat{f}_n) \leqq K$ on S, where K is a fixed positive constant not depending on n. For $j = 1, \cdots, g$, we set $\hat{\alpha}_{jn} = \hat{f}_n(\alpha_j)$, $\hat{D}_{jn} = \hat{f}_n(D_j)$ and $\hat{f}_n(S) = \hat{S}_n$. Map $\hat{D}_{1n}$ to the annulus $\hat{K}_{1n}: \{\hat{\rho}_{1n} < |z| < 1\}$ by a conformal mapping $\hat{g}_{1n}$ such that the image of $\hat{\alpha}_{1n}$ is homotopic to the circle $|z| = \sqrt{\hat{\rho}_{1n}}$ in $\hat{K}_{1n}$. Let K_{1n} be the annulus $\{\rho_{1n} < |z| < 1\}$ and let $\tilde{f}_n$ be an arbitrary q.c. mapping from $\hat{K}_{1n}$ to K_{1n}. Now we let S_n be the Riemann surface obtained by joining $\hat{S}_n - \hat{D}_{1n}$ and K_{1n} so that each point $p \in \partial(\hat{S}_n - \hat{D}_{1n})$ is identified with $\tilde{f}_n \hat{g}_{1n}(p) \in K_{1n}$

We define a q.c. mapping $\hat{\hat{f}}_n: \hat{S}_n \to S_n$ by setting that $\hat{\hat{f}}_n = \tilde{f}_n \hat{g}_{1n}$ on $\hat{D}_{1n}$ and $\hat{\hat{f}}_n$ is a conformal mapping in $\hat{S}_n - \hat{D}_{1n}$ with the given boundary correspondence. We set $\alpha_{jn} = \hat{\hat{f}}_n(\hat{\alpha}_{jn})$ and $D_{jn} = \hat{\hat{f}}_n(\hat{D}_{jn})$. And set $f_n = \hat{\hat{f}}_n \hat{f}_n$. Then f_n is a q.c. mapping from S to S_n and has a maximal dilatation $D(f_n) \leqq K$ on $S - D_1$. We call the above deformation a pinching deformation for α_1 on S if ρ_{1n} tends to zero for $n \to \infty$. We note that by Bers [2], $\lim_{n\to\infty} L(\rho_{1n}) = 0$ in this case, where $L(\rho_{1n})$ is the least length of the loops homotopic to α_{1n} in D_{1n}.

We denote by G a Schottky group generated by Möbius transformations $A_1, \cdots, A_g$ assigned to the loops $\alpha_1, \cdots, \alpha_g$, respectively, in a similar sense in 2–1. We obtain a similar result to Lemma 2. The obtained q.c. mapping is denoted by F_n. Set $G_n = F_n G F_n^{-1}$ and $A_{jn} = F_n A_j F_n^{-1}$ $(j = 1, \cdots, g)$. Then $G_n = \{A_{1n}, \cdots, A_{gn}\}$. We denote by λ_{jn} $(j = 1, \cdots, g)$ the multipliers of A_{jn}. Then we have the following lemma by the same method as in the proof of Lemma 3.

LEMMA 3′. *Under the above pinching deformation for* α_1,

$$\lim_{n\to\infty} \log|\lambda_{1n}| = 0 .$$

Next we take a basis $\beta_1, \cdots, \beta_g$ of "β"-cycles and choose the loops $\beta_1', \cdots, \beta_g'$ as in § 2. We denote by D_j^* the ring domain bounded by β_j and β_j'. By conformal mappings D_j and D_j^* are mapped to the annuli $K_j: \{\rho_j < |z| < 1\}$ and $K_j^*: \{\rho_j^* < |z| < 1\}$, respectively. Then by slightly modyfying the proof of Lemma 4 in § 2, we have the following important lemma.

LEMMA 4′. *Under the above pinching deformation for* α_1,

$$\left(\frac{1}{\rho_j^*}\right)^{1/K} \leqq |\lambda_{jn}| \leqq \exp\left(\frac{4\pi^2 K}{-\log \rho_j}\right)$$

for $j = 2, \cdots, g$.

3-2. Then we have the following main theorems. Theorem 3 is proved by the same method as in the proof of Theorem 1.

THEOREM 3. *Let* G_n *be the Schottky groups constructed above. Then*

(1) *if* $G_0 \in \partial_1 \mathfrak{S}$ *is the limit of* $T_{n_j} G_{n_j} T_{n_j}^{-1}$, *whose* $\{n_j\} \subset \{n\}$ *and* T_{n_j} *are Möbius transformations, then* G_0 *is a cusp.*

(2) *There exist a subsequence* $\{n_j\} \subset \{n\}$ *and Möbius transformations* T_{n_j} *such that the limit* G_0 *of the sequence* $T_{n_j} G_{n_j} T_{n_j}^{-1}$ *is on* $\partial_3 \mathfrak{S} \cap X$.

THEOREM 4. *Set* $A_{jn} = \begin{pmatrix} a_{jn} & b_{jn} \\ c_{jn} & d_{jn} \end{pmatrix}$, $a_{jn} d_{jn} - b_{jn} c_{jn} = 1$ $(1 \leqq j \leqq g)$. *By taking* T_n *suitably, consider the sequence normalized so that* $c_{1n} = 4$, $A_{1n}(0) = 0$ *and* $A_{2n}(2) = 2$. *Furthermore suppose that the following conditions are satisfied.* (1) $c_{jn} \neq 0$, $j = 1, \cdots, g$ *and* $n = 1, 2, \cdots$, *and* (2) *There exist defining curves* C_{jn} *and* C'_{jn} *of* A_{jn} $(j = 1, \cdots, g)$, *respectively such that* C_{jn} *and* C'_{jn} *are the isometric circles* I_{jn} *of* A_{jn} *and* I_{jn}^{-1} *of* A_{jn}^{-1}, *respectively, and* C_{jn} *and* C'_{jn} $(2 \leqq j \leqq g)$ *are all outside the disk* $\{|z| \leqq 1\}$ *and* $\pi_n^{-1}(D_{1n}) \cap \omega_n \subset \{|z| \leqq 1\}$, *where* ω_n *is the* $2g$*-ply connected region bounded by* $C_{1n}, C'_{1n}, \cdots, C'_{gn}$. *Then the limit* G_0 *of an infinite subsequence* $\{G_{n_j}\}$ *with* $\{n_j\} \subset \{n\}$ *is always a cusp.*

Remark. As is seen from the proof, it seems that the assumptions of Theorem 4 would be weakend considerably, although the present one is sufficient for our purpose. It is not known whether Theorem 4 is true or not in the hyperelliptic case.

Proof. First we prove the theorem for the case of genus $g = 2$. Let A_{1n} and A_{2n} be generators of G_n. By the assumption, $A_{1n}(0) = 0$, $A_{2n}(2) = 2$ and $c_{1n} = 4$. We denote by p_{jn} and q_{jn} the repelling and the attracting fixed points of A_{jn} $(j = 1, 2)$. We assume that $q_{1n} = 0$ and $q_{2n} = 2$.

Suppose r_{2n}, the radius of the isometric circle I_{2n} of A_{2n}, tends to zero. Since $1 < \lim_{n \to \infty} |\lambda_{2n}| < +\infty$ by Lemma 4 and

$$c_{2n} = \frac{\lambda_{2n}^{1/2} - \lambda_{2n}^{-1/2}}{p_{2n} - q_{2n}} ,$$

we have $\lim_{n\to\infty} p_{2n} = 2$. We note that by the assumption the 4-ply connected region bounded by $I_{1n}, I_{1n}^{-1}, I_{2n}$ and I_{2n}^{-1} is a fundamental region for G_n. Let γ_{2n} be the circle of radius $|1/c_{2n}| + |(a_{2n} + d_{2n})/c_{2n}|$ centered at a_{2n}/c_{2n}, and let γ_{1n} be the unit circle. Then for large n, γ_{1n} surrounds I_{1n} and I_{1n}^{-1} and is disjoint from γ_{2n}. Let $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ be the inverse image of γ_{1n} and γ_{2n} under the mapping F_n, respectively. Then $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ are disjoint simple closed curves containing the points $0, p_1$ and the points $2, p_2$ in their interiors, respectively, where p_1 and p_2 are the repelling fixed points of A_1 and A_2 (defined in 3–1), respectively. Let $R_3^{(n)}$ be the doubly connected region bounded by $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ and let R_{3n} be the doubly connected region bounded by γ_{1n} and γ_{2n}. We denote by $M(R_3^{(n)})$ and $M(R_{3n})$ the moduli of $R_3^{(n)}$ and R_{3n}, respectively. It is known that there exists a constant M such that $M(R_3^{(n)}) \leqq M, n = 1, 2, \cdots$. By the well-known property of modulus,

$$M(R_3^{(n)})^K \geqq M(R_{3n}) ,$$

since F_n is the q.c. mapping with maximal dilatation $D(F_n) \leqq K$ on $R_3^{(n)}$.

On the other hand it is easily seen that

$$\lim_{n\to\infty} M(R_{3n}) = \infty .$$

Hence

$$\infty = \lim_{n\to\infty} M(R_{3n}) \leqq \lim_{n\to\infty} M(R_3^{(n)})^K \leqq M^K = \text{a finite constant.}$$

This contradiction shows that $\lim_{n\to\infty} r_{2n} \neq 0$.

Since $r_{20} = \lim_{n\to\infty} r_{2n} \neq 0$, $q_{20} = \lim_{n\to\infty} q_{2n} = 2$ and $|\lambda_{20}| = \lim_{n\to\infty} |\lambda_{2n}| > 1$, we have $p_{20} = \lim_{n\to\infty} p_{2n} \neq 2$, that is, $A_{20} = \lim_{n\to\infty} A_{2n}$ is a loxodromic transformation.

We show that $A_{10} = \lim_{n\to\infty} A_{1n}$ is a parabolic transformation. Suppose that $\lim_{n\to\infty} p_{1n} = p_{10} \neq 0$. Since $c_{1n} = 4, q_{1n} = 0$ and $c_{1n} = (\lambda_{1n}^{1/2} - \lambda_{1n}^{-1/2})/(p_{1n} - q_{1n})$, we have

$$4 = (\lambda_{10}^{1/2} - \lambda_{10}^{-1/2})/p_{10} .$$

Then $\lambda_{10} \neq 1$ and so by $|\lambda_{10}| = 1$ we have $\lambda_{10} = e^{i\theta}$ $(\theta \neq 0)$. Thus $A_{10} = \lim_{n\to\infty} A_{1n}$ is an elliptic transformation. This does not occur by Chuckrow

[3], since A_{20} is a loxodromic transformation. Hence $p_{10}=0$, so $\lambda_{10}=1$. Thus A_{10} is a parabolic transformation. Thus $G_0=\{A_{10}, A_{20}\}$ is a cusp.

Next we prove the theorem for the case of genus $g \geqq 3$. Let p_{jn} and q_{jn} be the fixed points of A_{jn} $(1 \leqq j \leqq g)$. Suppose that $\lim_{n\to\infty} p_{kn} = \lim_{n\to\infty} q_{kn}$ for some $k, 2 \leqq k \leqq g$. We denote by I_{jn} and I_{jn}^{-1} the isometric circles of A_{jn} and A_{jn}^{-1} $(1 \leqq j \leqq g)$, respectively. The radius r_{kn} of I_{kn} becomes 0 as n to ∞. By the assumption, I_{jn} and I_{jn}^{-1} $(2 \leqq j \leqq g)$ are mutually disjoint. Let γ_{jn} be mutually disjoint simple closed curves surrounding I_{jn} and I_{jn}^{-1} which lie outside the disk $\{|z| \leqq 1\}, 2 \leqq j \leqq g$. We may take $\{\gamma_{kn}\}$ as a sequence of simple closed curves as follows: (1) each γ_{kn} surrounds I_{kn} and I_{kn}^{-1}, (2) γ_{kn} does not intersect with I_{jn} and I_{jn}^{-1} $(j \neq k, 1 \leqq j \leqq g)$ and (3) γ_{kn} tends to the point $\lim_{n\to\infty} p_{kn}$ for $n \to \infty$. Let γ_{1n} be the unit circle. Then by the assumption I_{1n} and I_{1n}^{-1} are contained in the interior of γ_{1n} and $\omega_n \cap \pi_n^{-1}(D_{1n}) \subset$ (the interior of γ_{1n}) for large n.

Now we consider the g-ply connected region ω_n' bounded by γ_{jn} $(1 \leqq j \leqq g)$. By using the well-known theorem of the theory of conformal mappings, ω_n' is conformally mapped to the following circular slit annulus, that is, γ_{1n} to the circle $|z|=R_{1n}, \gamma_{kn}$ to the circle $|z|=R_{kn}$ and γ_{jn} $(2 \leqq j \leqq g, j \neq k)$ to the circular arc slits on $|z|=R_{jn}$, where $R_{kn} < R_{jn} < R_{1n}$ $(2 \leqq j \leqq g, j \neq k)$. Set $\gamma_j^{(n)} = F_n^{-1}(\gamma_{jn}), 1 \leqq j \leqq g$. We denote by $\omega'^{(n)}$ the g-ply connected region bounded by these g curves. Then $\omega'^{(n)}$ is conformally mapped to the circular slit annulus like above. Thus for the image $|z|=R_1^{(n)}$ of $\gamma_1^{(n)}$ and the image $|z|=R_k^{(n)}$ of $\gamma_k^{(n)}$,

$$(R_{1n}^{(n)}/R_k^{(n)})^K \geqq R_{1n}/R_{kn} ,$$

since F_n is the q.c. mapping with maximal dilatation $D(f_n) \leqq K$ on $\omega'^{(n)}$. But by the above construction

$$\lim_{n\to\infty} R_{1n}/R_{kn} = \infty .$$

On the other hand $\lim_{n\to\infty} R_1^{(n)}/R_k^{(n)}$ is finite. For, $\gamma_j^{(n)}$ $(1 \leqq j \leqq g)$ contains a curve $C_j^{(n)}$ joining the fixed points of A_j in its interior for each n and j. Let $\omega^{*(n)}$ be the g-ply connected region with $C_j^{(n)}$ as the boundaries. If $\omega^{*(n)}$ is mapped to the circular slit annulus, we denote by $R_1^{*(n)}/R_k^{*(n)}$ the ratio of the inner and outer radii of $\omega^{*(n)}$, where $R_j^{*(n)}$ $(j=1, k)$ has similar meanings to the above. Then for each n,

$$R_1^{*(n)}/R_k^{*(n)} \geqq R_1^{(n)}/R_k^{(n)} .$$

It is known that there exists a constant M_{1k} such that $R_1^{*(n)}/R_k^{*(n)} \leqq M_{1k}, n = 1, 2, \cdots$. Thus for all large n we have

$$M_{1k} \geqq R_1^{(n)}/R_k^{(n)} .$$

This contradiction shows that $\lim_{n\to\infty} p_{kn} \neq \lim_{n\to\infty} q_{kn}$ $(2 \leqq k \leqq g)$. Thus by Lemma 4′, $\lambda_{j0} = \lim_{n\to\infty} \lambda_{jn}$, $p_{j0} = \lim_{n\to\infty} p_{jn}$ and $q_{j0} = \lim_{n\to\infty} q_{jn}$ determine loxodromic transformations A_{j0}, $2 \leqq j \leqq g$. As in the case $g = 2$, $A_{10} = \lim_{n\to\infty} A_{jn}$ is parabolic. In this case the fixed points of A_{j0}, $1 \leqq j \leqq g$, are all distinct by Marden [5], since A_{j0} are all Möbius transformations. Hence $G_0 = \{A_{10}, \cdots, A_{g0}\}$ is a cusp. Our proof is now complete.

3-3. To illustlate our result we shall present an example of the sequence $\{A_{jn}\}$ which satisfies the assumptions in Theorem 4. For brevity we consider the case of genus $g = 2$.

Set

$$A_{1n}(z) = \frac{((1/n) + \sqrt{1 + (1/n^2)})z}{4z - ((1/n) - \sqrt{1 + (1/n^2)})}$$

and

$$A_{2n}(z) = \frac{(17/2)z - 13}{4z - 6} .$$

Let $G_n = \{A_{1n}, A_{2n}\}$. Then G_n is a Schottky group and

$$\tau_n = (1 + (2/n^2) + (2/n)\sqrt{1 + (1/n^2)}, 0, 1/(2n), 4, 13/8, 2) .$$

We have

$$A_{10}(z) = \lim_{n\to\infty} A_{1n}(z) = \frac{z}{4z + 1} ,$$

$$A_{20}(z) = \lim_{n\to\infty} A_{2n}(z) = \frac{(17/2)z - 13}{4z - 6}$$

and

$$\tau_0 = \lim_{n\to\infty} \tau_n = (1, 0, 0, 4, 13/8, 2) .$$

Then it is easily seen that A_{1n} and A_{2n} satisfy the assumptions in Theorem 4.

With respect to this example, let us construct explicitly $S, S_n, D_1, D_{1n}, \alpha_1, \alpha_{1n}, F_n$ and f_n, which we constructed at the beginning of 3–1. We define S and S_n by setting $S = \Omega(G_1)/G_1$ and $S_n = \Omega(G_n)/G_n$. We have the isometric circles $I_{1n}, I_{1n}^{-1}, I_{2n}$ and I_{2n}^{-1} of $A_{1n}, A_{1n}^{-1}, A_{2n}$ and A_{2n}^{-1}, respectively, as follows:

$$I_{1n}: |z - (1/4)((1/n) - \sqrt{1 + (1/n^2)})| = 1/4 ,$$
$$I_{1n}^{-1}: |z - (1/4)((1/n) + \sqrt{1 + (1/n^2)})| = 1/4 ,$$
$$I_{2n}: |z - (3/2)| = 1/4$$

and

$$I_{2n}^{-1}: |z - (17/8)| = 1/4 .$$

Let ω_n be the 4-ply connected region bounded by the above 4 isometric circles. Let $\tilde{\alpha}_{1n}$ be the closed interval

$$[(1/4)((1/n) - \sqrt{1 + (1/n^2)} + 1), (1/4)((1/n) + \sqrt{1 + (1/n^2)} + 1)] .$$

Let δ_{1n} and δ'_{1n} be the segment joining $(1/4)((1/n) - \sqrt{1 + (1/n^2)} + i)$ to $(1/4)((1/n) + \sqrt{1 + (1/n^2)} + i)$ and the segment joining $(1/4)((1/n) - \sqrt{1 + (1/n^2)} - i)$ to $(1/4)((1/n) + \sqrt{1 + (1/n^2)} - i)$, respectively. We denote by E_{1n} the simply connected region bounded by $\delta_{1n}, \delta'_{1n}, I_{1n}$ and I_{1n}^{-1}. Set $E_{2n} = \{|z| \geqq 1\} \cap \omega_n$. Then $E_{2n} = E_{21}$ for each n. Set $E_{3n} = \omega_n - E_{1n} \cup E_{2n}$. Then we define D_1, D_{1n}, α_1 and α_{1n} by setting $D_1 = \pi(E_{11})$, $D_{1n} = \pi_n(E_{1n}), \alpha_1 = \pi(\tilde{\alpha}_{11})$ and $\alpha_{1n} = \pi_n(\tilde{\alpha}_{1n})$, where π and π_n are the natural projections from $\Omega(G_1)$ onto S and from $\Omega(G_n)$ onto S_n, respectively. Furthermore we define q.c. mappings F_n and f_n as follows.

First we define a q.c. mapping F_n from ω_1 to ω_n as follows. Let F_n be the identity mapping in E_{21}. If we set $z = x + iy$, then we define F_n in $E_{11} \cap \omega_1$ by setting

$$F_n(z) = \frac{\sqrt{1 + (1/n^2)} - \sqrt{1 - 16y^2}}{\sqrt{2} - \sqrt{1 - 16y^2}}(x - (1/4)) + 1/(4n) + iy .$$

Furthermore it is easily seen that there exists a q.c. mapping F_n from E_{31} to E_{3n} with the following boundary correspondences, which has a maximal dilatation $D(F_n) \leqq K$ for a fixed positive constant not depending on n: $F_n = \text{id.}$ on $|z| = 1$,

$$F_n(z) = z - \frac{1 + \sqrt{2} - \sqrt{1 + (1/n^2)}}{4} + \frac{1}{4n} \quad \text{on} \quad I_{11}^{-1} \cap \partial E_{31} ,$$

$$F_n(z) = z - \frac{1 - \sqrt{2} + \sqrt{1 + (1/n^2)}}{4} + \frac{1}{4n} \quad \text{on} \quad I_{11} \cap \partial E_{31}$$

$$F_n(z) = \frac{\sqrt{1 + (1/n^2)}}{\sqrt{2}}\left(x - \frac{1}{4}\right) + \frac{1}{4n} + \frac{1}{4}i \quad \text{on} \quad \delta_{11}$$

and

$$F_n(z) = \frac{\sqrt{1 + (1/n^2)}}{\sqrt{2}}\left(x - \frac{1}{4}\right) + \frac{1}{4n} - \frac{1}{4}i \quad \text{on} \quad \delta'_{11} \, .$$

Then we extend the mapping F_n to the whole $\Omega(G)$ by using the identity $F_n G F_n^{-1} = G_n$, and denote by the same letter F_n the extended mapping. We define f_n as the projection of F_n, that is, f_n satisfies the identity $f_n \pi = \pi_n F_n$.

It is easily seen that the modulus of the ring domain D_{1n} tends to ∞ as n to ∞, i.e., $\lim_{n\to\infty} \rho_{1n} = 0$ for the annulus $K_{1n}: \{\rho_{1n} < |z| < 1\}$ conformally equivalent to D_{1n}.

3-4. Let $\beta_1, \cdots, \beta_g$ be a basis of "β"-cycles on S. Let G^* be a Schottky group generated by Möbius transformations $B_1, \cdots, B_g$ assigned to $\beta_1, \cdots, \beta_g$, respectively, in a similar sense for "α"-cycles. Similarly to Lemma 5, there exists a q.c. mapping F_n^*. And set $G_n^* = F_n^* G^* F_n^{*-1}$. If we set $B_{jn} = F_n^* B_j F_n^{*-1}$ $(j = 1, \cdots, g)$, then $G_n^* = \{B_{1n}, \cdots, B_{gn}\}$. We denote by λ_{jn}^* the multiplier of B_{jn}. By the same method as before, we have the following lemmas. Here $\tilde{\rho}_j$ and $\tilde{\rho}_j^*$ have similar meanings in § 2.

LEMMA 6′. *Under the pinching deformation for* α_1,

$$\left(\frac{1}{\tilde{\rho}_j}\right)^{1/K} \leqq |\lambda_{jn}^*| \leqq \exp\left(\frac{4\pi^2 K}{-\log \tilde{\rho}_j^*}\right)$$

for $j = 2, \cdots, g$.

LEMMA 7′. *Under the pinching deformation for* α_1,

$$\lim_{n\to\infty} |\lambda_{1n}^*| = \infty \, .$$

3-5. Then we have the following main theorems.

THEOREM 5. *Let* G_n^* *be the Schottky groups constructed above. Then the limit* $G_0^* \in \partial\mathfrak{S}$ *of the sequence* $T_{n_j} G_{n_j}^* T_{n_j}^{-1}$, *whose* $\{n_j\} \subset \{n\}$ *and*

T_{n_j} *are Möbius transformations, is always on* $\partial_2\mathfrak{S} \cup \partial_3\mathfrak{S}$ *but not on* $\partial_1\mathfrak{S}$.

We can prove it by using Lemma 7′.

We consider the sequence $T_n G_n^* T_n^{-1}$ such that $T_n B_{1n} T_n^{-1}(-1) = -1$, $T_n B_{1n} T_n^{-1}(1) = 1$ and $T_n B_{2n} T_n^{-1}(0) = 0$. For brevity we write G_n^* and B_{kn} instead of $T_n G_n^* T_n^{-1}$ and $T_n B_{kn} T_n^{-1}$ $(1 \leqq k \leqq g)$, respectively. By using Lemma 7′, we note that the radii of the isometric circles of B_{1n} tend to zero for $n \to \infty$. Then we have

THEOREM 6. *Set* $R_1 = \{|z+1| \leqq \varepsilon\}$ *and* $R_1' = \{|z-1| \leqq \varepsilon\}$ *for a fixed small positive number* ε. *If for large* n, *there exist the mutually disjoint isometric circles* I_{jn}^* *and* I_{jn}^{*-1} $(j = 1, \cdots, g)$ *of* B_{jn} *and* B_{jn}^{-1}, *respectively such that* I_{jn}^* *and* I_{jn}^{*-1} $(j = 2, \cdots, g)$ *are outside* $R_1 \cup R_1'$ *and* $\pi_n^{*-1}(D_{1n}) \cap \omega_n^* \subset R_1 \cup R_1'$, *where* ω_n^* *is the* $2g$*-ply connected region bounded by the above* $2g$ *isometric circles and* π_n^* *is the natural projection from* $\Omega(G_n^*)$ *to* S_n, *then the limit* G_0^* *of the sequence* G_n^* *is always on* $\partial_2\mathfrak{S}$ *and a "node".*

Proof. First we prove the theorem for the case of genus $g = 2$. Let the fixed points of B_{2n} be 0 and q_{2n}^*. Suppose that $\lim_{n\to\infty} q_{2n}^* = 0$. Then $\lim_{n\to\infty} c_{2n} = \infty$, so the isometric circles I_{2n}^* and I_{2n}^{*-1} of B_{2n} and B_{2n}^{-1}, respectively, are contained in the disk

$$R_{2n} = \{|z| \leqq \delta_n, \delta_n \to 0\}$$

for large n, where $B_{2n} = \begin{pmatrix} a_{2n} & b_{2n} \\ c_{2n} & d_{2n} \end{pmatrix}$, $a_{2n}d_{2n} - b_{2n}c_{2n} = 1$. By Lemma 7′, the radii of the isometric circles I_{1n}^* and I_{1n}^{*-1} of B_{1n} and B_{1n}^{-1}, respectively, are small for large n. Hence for large n, I_{1n}^* and I_{1n}^{*-1} are contained in R_1 and R_1', respectively. By the assumption, the 4-ply connected region bounded by the above four isometric circles is a fundamental region for G_n^*. Set

$$R_\varepsilon = \{1 - \varepsilon < |z| < 1 + \varepsilon\} \cap \{\operatorname{Im} z < \varepsilon\}$$

and let ∂R_ε be the boundary of R_ε. For large n, $R_\varepsilon \supset I_{1n}^* \cup I_{1n}^{*-1}$, $R_\varepsilon \supset \omega_n \cap \pi_n^{*-1}(D_{1n})$ and the complement of R_ε contains R_{2n}. Set $R_\varepsilon^{(n)} = F_n^{*-1}(R_\varepsilon)$ and $R_2^{(n)} = F_n^{*-1}(R_{2n})$. We denote by (R_{2n}, R_ε) and $(R_2^{(n)}, R_\varepsilon^{(n)})$ the ring domains bounded by ∂R_{2n} and ∂R_ε, and bounded by $\partial R_\varepsilon^{(n)}$ and $\partial R_2^{(n)}$, respectively. Let M_n^* and $M^{(n)*}$ be the moduli of (R_{2n}, R_ε) and $(R_2^{(n)}, R_\varepsilon^{(n)})$, respectively. By the well-known fact on modulus property,

$$M_n^* \leqq (M^{(n)*})^K .$$

It is known that there exists a finite positive constant M^* such that $M^{(n)*} \leqq M^*$, $n = 1, 2, \cdots$. Hence

$$M_n^* \leqq (M^*)^K .$$

On the other hand $\lim_{n\to\infty} M_n^* = \infty$. This contradiction shows that $\lim_{n\to\infty} q_{2n}^* \neq 0$. Hence by Lemma 6′, $B_{20} = \lim_{n\to\infty} B_{2n}$ is a loxodromic transformation. Thus by Lemma 7′, $\tau_0^* = \lim_{n\to\infty} \tau_n^*$ is on $\partial_2\mathfrak{S}$, where τ_n^* is the point associated with G_n^*. It is easily seen that τ_0^* is a "node", since the fixed points of B_{20} are outside of $R_1 \cup R_1'$.

Next we prove the theorem for the case of genus $g \geqq 3$. Suppose that $\lim_{n\to\infty} p_{kn}^* = \lim_{n\to\infty} q_{kn}^*$ for some k, $2 \leqq k \leqq g$. Let γ_{kn} be a simple closed curve having the following properties: (1) γ_{kn} contains the isometric circles I_{kn}^* of B_{kn} and I_{kn}^{*-1} of B_{kn}^{-1} in its interior, (2) $\gamma_{kn+1} \subset \gamma_{kn}$ $(n = 1, 2, \cdots)$, (3) γ_{kn} converges to the point $\lim_{n\to\infty} p_{kn}^*$ for $n \to \infty$ and (4) γ_{kn} does not intersect with and not contain the isometric circles I_{jn}^* of B_{jn} and I_{jn}^{*-1} of B_{jn}^{-1} $(1 \leqq j \leqq g,\ j \neq k)$ in its interior. We denote by γ_{jn} $(1 \leqq j \leqq g, j \neq k)$ mutually disjoint simple closed curves which do not intersect with γ_{kn} such that each γ_{jn} $(2 \leqq j \leqq g,\ j \neq k)$ contains the isometric circles of B_{jn} and B_{jn}^{-1} in its interior and γ_{1n} contains R_1 and R_1' in its interior and is apart from γ_{kn} with a constant distance not depending on n. We denote by ω_n^* the g-ply connected region bounded by γ_{jn} $(1 \leqq j \leqq g)$. For ω_n^*, γ_{kn} and γ_{1n}, we use the same argument as in the proof of Theorem 4. Then we arrive at the same contradiction. Hence for $2 \leqq j \leqq g$, $\lim_{n\to\infty} p_{jn}^* \neq \lim_{n\to\infty} q_{jn}^*$. Then by Lemma 6′, $\lambda_{j0}^* = \lim_{n\to\infty} \lambda_{jn}^*$, $p_{j0}^* = \lim_{n\to\infty} p_{jn}^*$ and $q_{j0}^* = \lim_{n\to\infty} q_{jn}^*$ determine loxodromic transformations $(2 \leqq j \leqq g)$, where p_{jn}^* and q_{jn}^* are the fixed points of B_{jn}.

In this case $\tau_0^* = \lim_{n\to\infty} \tau_n^* \in \partial_2\mathfrak{S}$, where τ_n^* is the point associated with G_n^*. For the proof, let $G_n'^* = \{B_{2n}, \cdots, B_{gn}\}$. Then by Chuckrow [3], $G_n'^*$ is a Schottky group for each n. Then since $B_{j0} = \lim_{n\to\infty} B_{jn}$ $(2 \leqq j \leqq g)$ are loxodromic transformations by the above, the fixed points of B_{j0} are all distinct by Marden [5]. Furthermore $\lim_{n\to\infty} \lambda_{1n}^* = \infty$ by Lemma 7′, so $\tau_0^* = \lim_{n\to\infty} \tau_n^*$ is a "node". Our proof is now complete.

3-6. To illustlate our result we shall present an example of the sequence $\{B_{jn}\}$ which satisfies the assumption in Theorem 6. For brevity we consider the case of genus $g = 2$.

Set

$$B_{1n}(z) = \frac{\sqrt{n^2+1}z + n}{nz + \sqrt{n^2+1}}$$

and

$$B_{2n}(z) = \frac{(\sqrt{37}+6)z}{4z + \sqrt{37} - 6} .$$

Let $G_n^* = \{B_{1n}, B_{2n}\}$. Then G_n^* is a Schottky group and

$$\tau_n^* = (2n^2 + 1 + 2n\sqrt{n^2+1}, -1, 1, 73 + 12\sqrt{37}, 0, 3) .$$

Thus

$$\tau_0^* = \lim_{n\to\infty} \tau_n^* = (\infty, -1, 1, 73 + 12\sqrt{37}, 0, 3) .$$

Hence τ_0^* is a "node". Furthermore G_n^* satisfies the assumption in Theorem 6.

With respect to this example, let us construct explicitly $S, S_n, D_1, D_{1n}, F_n^*$ and f_n, which we constructed previously. We define S and S_n by setting $S = \Omega(G_1^*)/G_1^*$ and $S_n = \Omega(G_n^*)/G_n^*$. We have the following isometric circles:

$$I_{1n}^*: |z + (\sqrt{n^2+1}/n)| = 1/n ,$$
$$I_{1n}^{*-1}: |z - (\sqrt{n^2+1}/n)| = 1/n ,$$
$$I_{2n}^*: |z + (\sqrt{37} - 6)/4| = 1/4$$

and

$$I_{2n}^{*-1}: |z - (\sqrt{37} + 6)/4| = 1/4 .$$

Let ω_n^* be the 4-ply connected region bounded by $I_{1n}^*, I_{1n}^{*-1}, I_{2n}^*$ and I_{2n}^{*-1}. Give some fixed small positive number ε. We fix an integer n_0 as $\varepsilon/2 > 2/n_0$. We set

$$E_{1n_0}: [\{1/n_0 < |z + (\sqrt{n_0^2+1}/n_0)|\} \cap \{|z+1| < \varepsilon/2\}]$$
$$\cup [\{1/n_0 < |z - (\sqrt{n_0^2+1}/n_0)|\} \cap \{|z-1| < \varepsilon/2\}]$$

and

$$E_{1n}: [\{1/n < |z + (\sqrt{n^2+1}/n)|\} \cap \{|z+1| < \varepsilon/2\}]$$
$$\cup [\{1/n < |z - (\sqrt{n^2+1}/n)|\} \cap \{|z-1| < \varepsilon/2\}]$$

for $n > n_0$. We define D_{1n} by setting $D_{1n} = \pi_n^*(E_{1n})$, where π_n^* is the natural projection from $\Omega(G_n^*)$ onto S_n.

Next we define $\hat{F}_n^*$ as follows. Let $\hat{F}_n^*$ be the identity in the set

$$[\{|z-1| \geqq \varepsilon/2\} \cup \{|z+1| \geqq \varepsilon/2\}] \cap \omega_{n_0}^* .$$

It is easily seen that there exists a q.c. mapping $\hat{F}_n^*$ in E_{1n_0} with the following boundary correspondences: $\hat{F}_n^* = \text{id.}$ on $|z-1| = \varepsilon/2, \hat{F}_n^* = \text{id.}$ on $|z+1| = \varepsilon/2$,

$$\hat{F}_n^*(z) = (n_0/n)z + (1/n)(\sqrt{n_0^2+1} - \sqrt{n^2+1}) \quad \text{on} \quad I_{1n_0}^*$$

and

$$\hat{F}_n^*(z) = (n_0/n)z - (1/n)(\sqrt{n_0^2+1} - \sqrt{n^2+1}) \quad \text{on} \quad I_{1n_0}^{*-1} .$$

Then we extend the q.c. mapping $\hat{F}_n^*$ to the whole $\Omega(G_{n_0}^*)$ by using the identity $\hat{F}_n^* G_{n_0}^* \hat{F}_n^{*-1} = G_n^*$, and denote by the same letter $\hat{F}_n^*$ the extended mapping. It is easily seen that the modulus of the ring domain D_{1n} tends to ∞ as n to ∞, i.e., $\lim_{n\to\infty} \rho_{1n} = 0$ for the annulus $K_{1n}: \{\rho_{1n} < |z| < 1\}$ conformally equivalent to D_{1n}. Furthermore we define a q.c. mapping $\hat{\hat{F}}_{n_0}^*: \omega_1^* \to \omega_{n_0}^*$ as follows. It is easily seen that there exists a q.c. mapping $\hat{\hat{F}}_{n_0}^*$ with the following boundary correspondences, which has a maximal dilatation $D(\hat{\hat{F}}_{n_0}^*) = K$ for some positive constant $K, \hat{\hat{F}}_{n_0}^* = \text{id.}$ on $I_{21}^*, \hat{\hat{F}}_{n_0}^* = \text{id.}$ on $I_{21}^{*-1}, \hat{\hat{F}}_{n_0}^*(z) = z/n_0 + (\sqrt{2} - \sqrt{n_0^2+1})/n_0$ on I_{11}^* and $\hat{\hat{F}}_{n_0}^*(z) = (z/n_0) - (\sqrt{2} - \sqrt{n_0^2+1})/n_0$ on I_{11}^{*-1}. Then we extend the q.c. mapping to the whole $\Omega(G_1^*)$ by using the identity $G_{n_0}^* = \hat{\hat{F}}_{n_0}^* G_1^* \hat{\hat{F}}_{n_0}^{*-1}$, and denote by the same letter $\hat{\hat{F}}_{n_0}^*$ the extended q.c. mapping. If we set $F_n^* = \hat{F}_n^* \hat{\hat{F}}_{n_0}^*$, then F_n^* is the desired q.c. mapping.

If we denote by π^* the natural projection from $\Omega(G_1^*)$ onto S, then we define f_n as the projection of F_n^*, that is, $f_n \pi^* = \pi_n^* F_n^*$ is satisfied. We define D_1 by setting $\pi^* F_{n_0}^{*-1}(E_{1n_0}) = D_1$.

Remark. As we see from the proof of Theorem 6, it seems that the assumption in Theorem 6 is weakend considerably, although the present one is sufficient for our purpose.

Conclusion. Give a compact Riemann surface S of genus g ($g \geqq 2$). Fix a Schottky group G such that $\Omega(G)/G = S$. When we perform the pinching deformation for S, the limit of a sequence of Schottky groups representing the resulting surface S_n may be either (1) a cusp, (2) a "node" or (3) a point on $\partial_3\mathfrak{S}$.

Remark. For the Teichmüller space $T(\Gamma)$, on performing the pinching deformation, the group we get as the limit of quasi-Fuchsian groups Γ_n is always a cusp (cf. Bers [2] and Sato [7]), where Γ is a fixed Fuchsian group with $U/\Gamma = S$ (U: the upper half plane) and $\Omega(\Gamma_n)/\Gamma_n = S_n$.

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