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HEISENBERG GROUPS AND HOLOMORPHIC VECTOR BUNDLES OVER A COMPLEX TORUS

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Let V be a complex vector space of dimension n, L a lattice of V and $E = V/L$ a complex torus. Let H be a Hermitian form on V. We introduce a multiplication in $L \times C^*$ by

$$(\alpha, a)(\beta, b) = (\alpha + \beta, (\exp \pi H(\alpha, \beta))ab) ,$$

where $\alpha, \beta \in L$ and $a, b \in C^*$. Then $L \times C^*$ becomes a complex Lie group $G_H(L)$ whose identity component is C^*. We call $G_H(L)$ the Heisenberg group associated with a Hermitian form H and a lattice L in V. In general $G_H(L)$ is non-abelian. The group $G_H(L)$ acts on the complex manifold $V \times C^*$ from the right by the rule

$$(u, a)(\beta, b) = (u + \beta, (\exp \pi H(u, \beta))ab) ,$$

where $u \in V, \beta \in L$ and $a, b \in C^*$. The action of $G_H(L)$ is holomorphic and free and we can identify the quotient space with the complex torus E. Thus $V \times C^*$ is a principal fibre bundle over E with structure group $G_H(L)$. If we vary the Hermitian form H, we obtain infinitely many principal holomorphic bundle structures over E in this manner.

The purpose of this article is to study the class of holomorphic vector bundles over E associated with holomorphic representations of the Heisenberg groups $G_H(L)$.

If a representation of $G_H(L)$ is trivial on $\{0\} \times C^*$, the representation is nothing but a representation of the lattice L. The vector bundles over E associated with representations of L have been studied in our previous paper [2]. In this paper we shall show first that every holomorphic line bundle over E is always associated with a holomorphic representation of degree 1 of the Heisenberg group $G_H(L)$ for a suitable Hermitian form H. This result is nothing but an interpretation of the

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"normalized" theta factor associated with a line bundle. Then we show that if the Heisenberg group $G_H(L)$ admits an irreducible holomorphic representation which is not trivial on $\{0\}\times C^*$, the imaginary part A of H, which is an alternating form on V, is rational valued on L. If A is integral valued on L, the group $G_H(L)$ is abelian and so holomorphic irreducible representations are of degree 1 and the associated bundles are line bundles. However if A is not integral valued but rational valued on L, there are holomorphic irreducible representations of $G_H(L)$ of degree greater than one. In the sections 5 and 6 we classify the holomorphic irreducible representations of the Heisenberg group $G_H(L)$ assuming that the imaginary part A of H is rational valued on L. To achieve this classification we introduce another multiplication on $L\times C^*$ by

$$(\alpha, a)(\beta, b) = (\alpha + \beta, (\exp \pi i A(\alpha, \beta))ab) .$$

Then $L \times C^*$ becomes also a complex Lie group $G_A(L)$ with respect to this new multiplication. We call also $G_A(L)$ the Heisenberg group associated with A and L. It is easy to show that $G_H(L)$ and $G_A(L)$ are isomorphic (see § 2) and consequently there is a one-to-one correspondence between holomorphic representations of $G_H(L)$ and those of $G_A(L)$. We classify holomorphic irreducible representations of $G_A(L)$ in the sections 5 and 6. Among the holomorphic irreducible representations of the Heisenberg group $G_A(L)$ there is a distinguished representation D_A which we call the Schrödinger representation. The representation D_H of $G_H(L)$ which corresponds to D_A is also called the Schrödinger representation of $G_H(L)$. In the section 7 we shall show that a holomorphic vector bundle over E associated with a holomorphic irreducible representation of $G_H(L)$ is isomorphic to a holomorphic vector bundle associated with a representation of $G_{kH}(L)$ of the form $\sigma \otimes D_{kH}$, where σ is a 1-dimensional representation of the lattice L, D_{kH} is the Schrödinger representation of $G_{kH}(L)$ and k is a suitable integer.

In the section 8 we study some properties of the vector bundle F associated with the Schrödinger representation D_H. The vector bundle F is simple and hence indecomposable. We study the mechanism to construct the vector valued theta functions associated with F. It will be shown for example, that if the Hermitian form H is positive ($\geqq 0$), then the number of linearly independent theta functions is given by

$$\prod_{i=1}^{\ell} e_i(e_i, d)^{-1} ,$$

whereas the rank of the vector bundle F is

$$d^{\ell} \prod_{i=1}^{\ell} (e_i, d)^{-1} ,$$

where d is the smallest positive integer such that dA is integral valued on L and $e_1, \cdots, e_\ell$ are non-zero elementary divisors of the integral alternating form dA on L.

In the final section we study the properties of tensor products of vector bundles associated with indecomposable holomorphic representations of $G_H(L)$ for variable H and we compute Chern classes. We shall see that, if F_1 and F_2 are vector bundles associated with the Schrödinger representations of $G_{H_1}(L)$ and $G_{H_2}(L)$, then we have the splitting $F_1 \otimes F_2 \cong F_{\sigma_1 \otimes D_H} \oplus \cdots \oplus F_{\sigma_s \otimes D_H}$, where $H = H_1 + H_2$, D_H is the Schrödinger representation of $G_H(L)$, $\sigma_1, \cdots, \sigma_s$ are 1-dimensional representations of L and $F_{\sigma_i \otimes D_H}$ denotes the vector bundle associated with the representation $\sigma_i \otimes D_H$ of $G_H(L)$.

The group $G_A(V)$ appeared already in a paper of Murakami [8] in a similar context as ours and this article is also closely related with the works of Morikawa [3] and Oda [5]. The author wishes to thank J. Hano for his useful comments. J. Hano also proved recently that the class of vector bundles studied by Morikawa and Oda is identical with the one associated with irreducible representations of $G_H(L)$.

§ 1. The nilpotent Lie group G_B.

Let V be a finite dimensional vector space over $\boldsymbol{R}$ and B a complex valued bilinear form on V. We define a multiplication in the product $V \times \boldsymbol{C}^*$ by

$$(1.1) \qquad (u, a)(v, b) = \left(u, v, \varepsilon\left\{\frac{1}{2}B(u, v)\right\}ab\right) ,$$

where $u, v \in V$ and $a, b \in \boldsymbol{C}^*$ and

$$\varepsilon(z) = \exp 2\pi i z$$

for all $z \in \boldsymbol{C}$.

With respect to the multiplication (1.1) $V \times \boldsymbol{C}^*$ forms a Lie group which we shall denote by G_B. The element (0,1) is the identity element and the inverse of (u, a) is given by the formula

$$(1.2) \qquad (u, a)^{-1} = \left(-u, \varepsilon\left\{\frac{1}{2}B(u, u)\right\}a^{-1}\right).$$

Let $B = S + A$, where $S(u, v) = \frac{1}{2}\{B(u, v) + B(v, u)\}$ is a symmetric bilinear form and $A(u, v) = \frac{1}{2}\{B(u, v) - B(v, u)\}$ is an alternating bilinear form. From (1.1) we get the following commutation rule:

$$(1.3) \qquad (u, a)(v, b) = (v, b)(u, a)(0, \varepsilon\{A(u, v)\}).$$

The subset $\{0\} \times C^*$ form a closed normal subgroup contained in the center of G_B and the quotient of G_B by $\{0\} \times C^*$ is an abelian Lie group isomorphic to V. Hence G_B is a connected nilpotent Lie group and G_B is not abelian unless $A = 0$.

We can also define a nilpotent Lie group G_A by introducing another multiplication in $V \times C^*$ by

$$(1.4) \qquad (u, a)(v, b) = \left(u + v, \varepsilon\left\{\frac{1}{2}A(u, v)\right\}ab\right).$$

We can prove easily the following lemma.

LEMMA 1.1. *The map φ from G_B onto G_A given by*

$$\varphi(u, a) = \left(u, \varepsilon\left\{-\frac{1}{4}B(u, u)\right\}a\right)$$

is an isomorphism of G_B onto G_A.

§ 2. The Heisenberg group $G_H(L)$.

We now assume that V is a complex vector space of complex dimension n and let H be a Hermitian form on V. Let

$$H(u, v) = S(u, v) + iA(u, v),$$

for $u, v \in V$, where $S(u, v)$ and $A(u, v)$ are the real part and the imaginary part of $H(u, v)$. Then S is symmetric and A is alternating, both are $\boldsymbol{R}$-bilinear on V. Let

$$B = \frac{1}{i} H$$

and we define the nilpotent Lie group G_B as in § 1 which we shall denote by G_H. Thus the multiplication in the group G_H is defined by

$$(u, a)(v, b) = \left(u, v, \varepsilon\left\{\frac{1}{2i} H(u, v)\right\} ab\right), \tag{2.1}$$

where $u, v \in V$ and $a, b \in \boldsymbol{C}^*$.

Since H is Hermitian, the multiplication in G_H is holomorphic in the variable u, a and b and anti-holomorphic in the variable v. Hence G_H is not a complex Lie group.

The alternating part of $B = \frac{1}{i} H$ is equal to the imaginary part A of H and by Lemma 1.1. the map $\varphi: G_H \to G_A$ defined by

$$\varphi(u, a) = \left(u, \varepsilon\left\{-\frac{1}{4i} H(u, u)\right\} a\right) \tag{2.2}$$

is an isomorphism of G_H onto G_A.

Let L be a lattice of V and we define

$$G_H(L) = L \times \boldsymbol{C}^* .$$

Then $G_H(L)$ is a closed subgroup of G_H and $G_H(L)$ is a *complex Lie group* with the identity component isomorphic to $\boldsymbol{C}^*$.

Analogously

$$G_A(L) = L \times \boldsymbol{C}^*$$

is also a subgroup of G_A and $G_A(L)$ is also a complex Lie group. Moreover the isomorphism φ of G_H onto G_A defined by (2.2) maps $G_H(L)$ onto $G_A(L)$ and induces an isomorphism of complex Lie group.

We call $G_H(L)$ (resp. $G_A(L)$) the *Heisenberg group* associated with a Hermitian form H and a lattice L (resp. A and L).

§ 3. Principal bundle structures over a complex torus associated with Heisenberg groups.

Since $G_H(L)$ is a subgroup of G_H, it acts on $G_H = V \times \boldsymbol{C}^*$ freely by right multiplication

$$(u, a)(\beta, b) = \left(u + \beta, \varepsilon\left\{\frac{1}{2i}H(u, \beta)\right\}ab\right),$$

where $u \in V, \beta \in L$ and $a, b \in \boldsymbol{C}^*$. This is a holomorphic and free action of the complex Lie group $G_H(L)$ on the complex manifold $V \times \boldsymbol{C}^*$ and we can identify the quotient space canonically with the complex torus

$$E = V/L .$$

Thus $L \times \boldsymbol{C}^*$ is a holomorphic principal fibre bundle over the complex torus E with the structure group $G_H(L)$.

A G_H-theta factor $\tilde{J}$ of rank m is a *holomorphic map*

$$\tilde{J} : G_H(L) \times V \times \boldsymbol{C}^* \to GL_m(\boldsymbol{C})$$

such that

$$\tilde{J}(\gamma\gamma', g) = \tilde{J}(\gamma, T_{\gamma'}g)\tilde{J}(\gamma', g)$$

for $\gamma, \gamma' \in G_H(L)$ and $g \in V \times \boldsymbol{C}^*$, where we define $T_\gamma g = g\cdot\gamma^{-1}$.

For instance a *holomorphic* representation ρ of the complex Lie group $G_H(L)$ is a G_H-theta factor.

Given a G_H-theta factor $\tilde{J}$ of rank m, we define a holomorphic free action of $G_H(L)$ on the complex manifold $V \times \boldsymbol{C}^* \times \boldsymbol{C}^m$ by

$$(g, \xi)\cdot\gamma = (g\gamma, \tilde{J}(\gamma^{-1}, g)\xi) ,$$

where $g \in V \times \boldsymbol{C}^*, \gamma \in G_H(L)$ and $\xi \in \boldsymbol{C}^m$. The quotient space of $V \times \boldsymbol{C}^* \times \boldsymbol{C}^m$ has a structure of a holomorphic vector bundle $F_{\tilde{J}}$ of rank m over the complex torus E.

On the other hand, a *theta factor* J *of rank* m *for the lattice* L is a map

$$J : L \times V \to GL_m(\boldsymbol{C})$$

such that

1) $J(\alpha, u)(\alpha \in L, u \in V)$ is holomorphic in u,

2) $J(\alpha + \beta, u) = J(\alpha, \beta + u)J(\beta, u)$

for $\alpha, \beta \in L$, and $u \in V$.

We define also a holomorphic free action of the lattice L on $V \times \boldsymbol{C}^m$ by

$$(u, \xi)\alpha = (u + \alpha, J(\alpha, u)\xi) ,$$

where $u \in V, \alpha \in L$ and $\xi \in C^m$.

The quotient of $V \times C^m$ is a holomorphic vector bundle F_J of rank m over the complex torus E. It is well-known that every holomorphic vector bundle over E is obtained in this way.

LEMMA 3.1. *Let $\tilde{J}$ be a G_H-theta factor of rank m. For $\alpha \in L$ and $u \in V$ let*

$$J(\alpha, u) = \tilde{J}\left[\left(-\alpha, \varepsilon\left\{\frac{1}{2i}(H(u, \alpha) + H(\alpha, \alpha))\right\}\right), (u, 1)\right] .$$

Then J is a theta factor for the lattice L and the holomorphic vector bundles $F_{\tilde{J}}$ and F_J are isomorphic.

Proof. We can verify readily that $J(\alpha, u)$ is a theta factor for L. We define a map ψ from $V \times C^m$ into $V \times C^* \times C^m$ by

$$\psi(u, \xi) = ((u, 1), \xi)$$

for all $u \in V$ and $\xi \in C^m$. We say that two points of $V \times C^* \times C^m$ are equivalent if they belong to the same orbit of $G_H(L)$. Two points $\psi(u, \xi)$ and $\psi(v, \eta)$ are equivalent if and only if there exists $(\alpha, a) \in G_H(L)$ such that

$$((u, 1)(\alpha, a), \tilde{J}[(\alpha, a)^{-1}, (u, 1)]\xi) = ((v, 1), \eta) .$$

Since $(u, 1)(\alpha, a) = \left(u + \alpha, \varepsilon\left\{\frac{1}{2i}H(u, \alpha)\right\}a\right)$, we get

$$v = u + \alpha \tag{3.1}$$

and

$$a^{-1} = \varepsilon\left\{\frac{1}{2i}H(u, \alpha)\right\} . \tag{3.2}$$

Then we get from (1.2) and (3.2), $(\alpha, a)^{-1} = \left(-\alpha, \varepsilon\left\{\frac{1}{2i}(H(u, \alpha) + H(\alpha, \alpha))\right\}\right)$ and so

$$\eta = J(\alpha, u)\xi . \tag{3.3}$$

Thus if $\psi(u, \xi)$ and $\psi(v, \eta)$ are equivalent, then we have (3.1) and (3.3) and this means that $(u, \xi)\cdot\alpha = (v, \eta)$, that is, (u, ξ) and (v, η) are equivalent according to the action of L on $V \times C^*$.

Conversely let $(u,\xi)\alpha=(v,\eta)$, where $\alpha\in L$. Then defining a by (3.2), we get $\psi(u,\xi)(\alpha,a)=\psi(v,\eta)$. Hence we have shown that $\psi(u,\xi)$ and $\psi(v,\eta)$ are equivalent if and only if (u,ξ) and (v,η) are equivalent. Therefore ψ defines an injective homomorphism $\bar{\Psi}$ of F_J into $F_{\tilde{J}}$. The homomorphism $\bar{\Psi}$ is surjective. For, let $((u,a),\xi')$ be an arbitrary element of $V\times C^*\times C^m$. Then $(0,a^{-1})\in G_H(L)$ and $((u,a),\xi')(0,a^{-1})$ is of the form $((u,1),\xi)$ and hence the orbit of $((u,a),\xi')$ contains an element of the form $\psi(u,\xi)$ and this proves that $\bar{\Psi}$ is surjective.

As a special case of Lemma 3.1 we get the following Proposition 3.1.

PROPOSITION 3.1. *Let ρ be a holomorphic representation of the Heisenberg group $G_H(L)$. Then the holomorphic vector bundle F_ρ over the complex torus $E=V/L$ associated with ρ is isomorphic to the holomorphic vector bundle associated with the theta factor J for the lattice L, where J is given by*

$$J(\alpha,u)=\rho\left(0,\varepsilon\left\{\frac{1}{2i}H(u,\alpha)+\frac{1}{4i}H(\alpha,\alpha)\right\}\right)\Psi(\alpha)\,, \tag{3.4}$$

where

$$\Psi(\alpha)=\rho\left(0,\varepsilon\left\{\frac{1}{4i}H(\alpha,\alpha)\right\}\right)\rho(-\alpha,1)\,. \tag{3.5}$$

From the definition of ψ we get

$$\Psi(\alpha+\beta)=\Psi(\alpha)\Psi(\beta)\rho\left(0,\varepsilon\left\{\frac{1}{2}A(\beta,\alpha)\right\}\right)$$

for $\alpha,\beta\in L$.

THEOREM 3.1. *Every holomorphic line bundle over $E=V/L$ is associated with a holomorphic representation of degree 1 of the Heisenberg group $G_H(L)$ for a suitable Hermitian form H.*

Proof. A line bundle over E is associated with a theta factor j of rank 1 and we may assume that j is in the normalized form (see [6]):

$$j(\alpha,u)=\psi(\alpha)\varepsilon\left\{\frac{1}{2i}H(u,\alpha)+\frac{1}{4i}H(\alpha,\alpha)\right\}\,,$$

where H is a Hermitian form on V such that the imaginary part A of

H is integral valued on L and ψ is a *semi-character* of L, that is, ψ is a map $L \to C_1^* = \{z \in C \mid |z| = 1\}$ such that $\psi(\alpha+\beta) = \psi(\alpha)\psi(\beta)\varepsilon\left\{\frac{1}{2}A(\alpha, \beta)\right\}$ for $\alpha, \beta \in L$. We define $\rho\colon G_H(L) \to C^*$ by

$$\rho(\alpha, a) = \psi(-\alpha)\varepsilon\left\{-\frac{1}{4i}H(\alpha, \alpha)\right\}a . \tag{3.6}$$

It is easily verified that ρ is a holomorphic representation of the Heisenberg group $G_H(L)$ of degree 1. It follows from (3.4) and (3.5) that the theta factor for L corresponding to ρ is precisely equal to j. Then the line bundle is isomorphic to the line bundle associated with the representation ρ of $G_H(L)$.

4. Holomorphic representations of $G_H(L)$ and $G_A(L)$.

Let H be a Hermitian form on V and A the imaginary part of H. Then $\varphi\colon G_H \to G_A$ defined by (2.2) induces an isomorphism of $G_H(L)$ onto $G_A(L)$. Hence there is a one-to-one correspondence between the set of holomorphic representations of $G_A(L)$ and that of $G_H(L)$. If ρ_A is a holomorphic representation of $G_A(L)$, then $\rho_H = \rho_A \circ \varphi$ is the corresponding holomorphic representation of $G_H(L)$ and we have

$$\rho_H(\alpha, a) = \rho_A\left(\alpha, \varepsilon\left\{-\frac{1}{4i}H(\alpha, \alpha)\right\}\cdot a\right) . \tag{4.1}$$

For instance, if $\psi : L \to C^*$ is a semi-character of L (see the proof of theorem 3.1), then $\rho_A(\alpha, a) = \psi(-\alpha)\cdot a$ defines a holomorphic representation of $G_A(L)$ and the corresponding representation ρ_H of $G_H(L)$ is the one given by (3.6) in the proof of Theorem 3.1.

We remark here that an irreducible holomorphic representation σ of the group C^* is always of degree 1 and of the form

$$\sigma(a) = a^k$$

for all $a \in C^*$, where k is an integer (cf. proof of Lemma 9.1 in § 9).

Let ρ be a holomorphic irreducible representation of $G_A(L)$. Then, since $\{0\} \times C^*$ is in the center of $G_A(L)$, every $\rho(0, a)$ is represented by a scalar operator by Schur's Lemma and hence

$$\rho(0, a) = a^k \cdot 1 \tag{4.2}$$

for every $a \in C^*$, where k is an integer and 1 is the identity operator. We call a holomorphic representation ρ of $G_A(L)$ is *homogeneous of order* k if the equation (4.2) holds for all $a \in C^*$. Every holomorphic irreducible representation of $G_A(L)$ is thus homogeneous.

PROPOSITION 4.1. *Suppose that the Heisenberg group $G_A(L)$ has a holomorphic homogeneous representation ρ of order k with $k \neq 0$. Then the alternating form A is rational valued on the lattice L.*

Proof. We have

$$(\alpha, 1)(\beta, 1) = (\alpha + \beta, 1)\Big(0, \varepsilon\Big\{\frac{1}{2}A(\alpha, \beta)\Big\}\Big)$$

and hence

$$\rho(\alpha, 1)\rho(\beta, 1) = \rho(\alpha + \beta, 1)\varepsilon\Big\{\frac{k}{2}A(\alpha, \beta)\Big\}$$

by (4.2). Then we get

$$\det \rho(\alpha, 1) \cdot \det \rho(\beta, 1)(\det \rho(\alpha + \beta, 1))^{-1} = \varepsilon\Big\{\frac{mk}{2}A(\alpha, \beta)\Big\} ,$$

where m is the degree of ρ. Since the left hand side is symmetric in α and β and A is alternating, we should have $\varepsilon\{mkA(\alpha, \beta)\} = 1$ for all $\alpha, \beta \in L$. Then $mkA(\alpha, \beta)$ is an integer for every α and β in L and $mk \neq 0$. Hence A is rational valued on L.

We assume henceforth that A is rational valued on L and we always denote by d the smallest positive integer such that dA is integral valued on L.

We denote by N the subgroup of L consisting of all $\alpha \in L$ such that $A(\alpha, \beta)$ is an integer for every $\beta \in L$.

Then we have

$$dL \subset N$$

and hence L/N is a finite abelian group of exponent d. In particular, N is also a lattice of V. From the commutation rule (1.3) we get

$$(\alpha, a)(\beta, b) = (\beta, b)(\alpha, a)(0, \varepsilon\{A(\alpha, \beta)\}) .$$

It follows from this that (α, a) commutes with every (β, b) if and only if $\alpha \in N$. Thus

$$G_A(N) = N \times C^*$$

is the center of $G_A(L)$.

It follows then that $G_A(L)$ is abelian if and only if $N = L$. Now we have $N = L$ if and only if A is integral valued on L. Thus we get that *the Heisenberg group* $G_A(L)$ *and hence* $G_H(L)$ *is abelian if and only if* A *is integral valued on the lattice* L.

Let ρ_A be a holomorphic homogeneous representation of $G_A(L)$ of order k and ρ_H the corresponding representation of $G_H(L)$. Then the theta factor J corresponding to ρ_H in Proposition 3.1 is of the form

$$J(\alpha, u) = \varepsilon\left\{\frac{1}{2i}kH(u, \alpha) + \frac{1}{4i}kH(\alpha, \alpha)\right\}\varepsilon\left\{\frac{1}{4i}kH(\alpha, \alpha)\right\}\rho_H(-\alpha, 1) .$$

However we have

$$\begin{aligned}\rho_H(-\alpha, 1) &= \rho_A\left(-\alpha, \varepsilon\left\{-\frac{1}{4i}H(\alpha, \alpha)\right\}\right)\\ &= \varepsilon\left\{-\frac{1}{4i}kH(\alpha, \alpha)\right\}\rho_A(-\alpha, 1)\end{aligned}$$

and so we obtain

$$J(\alpha, u) = \varepsilon\left\{\frac{1}{2i}kH(u, \alpha) + \frac{1}{4i}kH(\alpha, \alpha)\right\}\rho_A(-\alpha, 1) . \tag{4.3}$$

§ 5. Construction of irreducible representations of $G_A(L)$.

In the sections 5 and 6 we denote by A any $\boldsymbol{R}$-bilinear alternating form on a complex vector space V of complex dimension n which is rational valued on a lattice L of V. We do not assume that A is the imaginary part of an Hermitian form H. We denote by d the smallest positive integer such that dA is integral valued on the lattice L. There exists a basis

$$\{\omega_1, \cdots, \omega_n, \omega_1', \cdots, \omega_n'\}$$

of L such that $dA(\omega_i, \omega_j) = dA(\omega_i', \omega_j') = 0$ and $dA(\omega_i, \omega_j') = e_i\delta_{ij}$ $(i, j = 1, \cdots, n)$, where $e_1, \cdots, e_\ell, e_{\ell+1}, \cdots, e_n$ are integers such that $e_{\ell+1} = \cdots = e_n = 0$, $e_i > 0$ $(1 \leqq i \leqq \ell)$, $e_i \mid e_{i+1}$ $(i = 1, 2, \cdots, \ell - 1)$.

These integers are called the *elementary divisors* of the integral alternating form dA on the lattice L. We have then

$$A(\omega_i, \omega'_j) = \delta_{ij} e_i d^{-1} , \qquad A(\omega_i, \omega_j) = A(\omega'_i, \omega'_j) = 0 .$$

Since d is the smallest positive integer such that dA is integral valued on L and $e_1 | e_i$ for all indices i, d and e_1 have to be relatively prime. Let

$$d_i = d(e_i, d)^{-1} , \qquad (i = 1, 2, \cdots, \ell) , \tag{5.1}$$

where (e_i, d) denotes the greatest common divisor of e_i and d. Then d_i is the smallest positive integer such that $d_i e_i d^{-1}$ is an integer and we have $d_1 = d$ and $d_{i+1} | d_i$ $(i = 1, 2, \cdots, \ell - 1)$.

Let N be the subgroup of L consisting of all $\alpha \in L$ such that $A(\alpha, \beta)$ is an integer for every $\beta \in L$. Then

$$\{d_1\omega_1, \cdots, d_\ell\omega_\ell, \omega_{\ell+1}, \cdots, \omega_n, d_1\omega'_1, \cdots, d_\ell\omega'_\ell, \omega'_{\ell+1}, \cdots, \omega'_n\}$$

is a basis of N.

Let L_1 and L'_1 be the subgroups of L generated by $\{\omega_1, \cdots, \omega_n\}$ and $\{\omega'_1, \cdots, \omega'_n\}$ respectively and let

$$N_1 = L_1 \cap N , \qquad N'_2 = L_2 \cap N .$$

Then N_1 and N'_2 are the subgroups of N generated by

$$\{d_1\omega_1, \cdots, d_\ell\omega_\ell, \omega_{\ell+1}, \cdots, \omega_n\} \quad \text{and} \quad \{d_1\omega'_1, \cdots, d_\ell\omega'_\ell, \omega'_{\ell+1}, \cdots, \omega'_n\}$$

respectively and we have

$$L = L_1 \oplus L'_1 , \qquad N = N_1 \oplus N'_1 \tag{5.2}$$

and

$$L/N \cong L_1/N_1 \oplus L'_1/N'_1 .$$

Let

$$K = L/N , \quad K_1 = L_1/N_1 , \quad K'_1 = L'_1/N'_1 .$$

Then K, K_1 and K'_1 are finite abelian groups and we have

$$K = K_1 \oplus K'_1 , \qquad K_1 \cong K'_1 .$$

Let

$$m = d_1 d_2, \cdots, d_\ell . \tag{5.3}$$

Then m is the order of K_1 and $\{d_1, \cdots, d_\ell\}$ are the invariants of the finite abelian group K_1. We shall denote by $C(K_1)$ the vector space of all complex valued functions on K_1. Then $C(K_1)$ is an m-dimensional complex vector space. $C(K_1)$ is not only a vector space but also has an algebra structure and $C(K_1)$ is called the group algebra of the finite abelian group K_1. We shall denote by R the regular representation of K_1. R is a representation of K_1 defined by

$$(R(g)f)(h) = f(h + g) \tag{5.4}$$

for all $g, h \in K_1$ and $f \in C(K_1)$.

For each $\lambda \in L$, we shall denote by $\bar{\lambda}$ the image of λ in $K = L/N$. In particular if $\alpha \in L_1$ ($\alpha' \in L'_1$), then $\bar{\alpha}(\bar{\alpha}')$ belongs to $K_1(K'_1)$. We now consider $\varepsilon\{A(\alpha, \alpha')\}$, where $\alpha \in L_1$ and $\alpha' \in L'_1$. We have $\varepsilon\{A(\alpha, \alpha')\} = \varepsilon\{A(\beta, \beta')\}$ whenever $\bar{\alpha} = \bar{\beta}$ and $\bar{\alpha}' = \bar{\beta}'$. For we have then $\alpha - \beta \in N$ and $\alpha' - \beta' \in N$ and hence $A(\alpha - \beta, \alpha')$ and $A(\beta, \alpha' - \beta')$ are integers, whence

$$\varepsilon(A(\alpha, \alpha')) = \varepsilon(A(\beta, \alpha')) = \varepsilon(A(\beta, \beta')) .$$

Therefore we can define a pairing of abelian groups K_1 and K'_1 by

$$\langle g, g' \rangle = \varepsilon\{A(\alpha, \alpha')\} , \quad g \in K_1 , \quad g' \in K'_1 , \tag{5.5}$$

where $g = \bar{\alpha}$ and $g' = \bar{\alpha}'$.

For any $g \in K_1$ and $g' \in K'_1$, $\langle g, g' \rangle$ is a d-th root of unity and we have

$$\langle g + h, g' \rangle = \langle g, g' \rangle \langle h, g' \rangle , \qquad \langle g, g' + h' \rangle = \langle g, g' \rangle \langle g, h' \rangle .$$

Then $\chi_{g'} : g \to \langle g, g' \rangle$ is a character of the abelian group K_1. Moreover, if $g' \neq h'$, we have $\chi_{g'} \neq \chi_{h'}$. For let $g' = \bar{\alpha}', h' = \bar{\beta}'$ where $\alpha', \beta' \in L'_1$. If $\chi_{g'} = \chi_{h'}$, then we have $A(\alpha, \alpha' - \beta') \in \boldsymbol{Z}$ for all $\alpha \in L_1$ and hence $\alpha' - \beta' \in N'_1$. Then we get $\bar{\alpha}' = \bar{\beta}'$ and $g' = h'$. Since the order of K'_1 is equal to the order of K_1 we can identify K'_1 with the character group of K_1 by the identification map $g' \to \chi_{g'}$. The group $G_A(L)$ is identified with $L_1 \times L'_1 \times \boldsymbol{C}^*$ with the multiplication

$$(\alpha, \alpha', a)(\beta, \beta', b) = \left(\alpha + \beta, \alpha' + \beta', \varepsilon\left\{\frac{1}{2}(A(\alpha, \beta') - A(\beta, \alpha'))\right\}ab\right) . \tag{5.6}$$

Then $L_1 \times \{0\} \times \{1\}$ and $\{0\} \times L'_1 \times \{1\}$ are subgroups isomorphic to L_1 and L'_1 respectively and $L_1 \times \{0\} \times \boldsymbol{C}^*$ (resp. $\{0\} \times L'_1 \times \boldsymbol{C}^*$) is also a subgroup isomorphic to the direct sum of L_1 and $\boldsymbol{C}^*$ (resp. L'_1 and $\boldsymbol{C}^*$).

For each integer k, let $\rho_A^{(k)}(\alpha, \alpha', a)$ be a linear transformation of $C(K_1)$ such that

$$(5.7) \qquad (\rho_A^{(k)}(\alpha, \alpha', a)f)(x) = a^k \varepsilon\left\{\frac{k}{2}A(\alpha, \alpha') + kA(\xi, \alpha')\right\} f(x + \bar{\alpha})$$

for all $f \in C(K_1)$ and $x \in K_1$, where ξ is an element of L_1 such that $\bar{\xi} = x$. Notice that if ξ and η are in L_1 and if $\bar{\xi} = \bar{\eta}$, then $\xi - \eta \in N$ and hence $\varepsilon\{A(\xi, \alpha')\} = \varepsilon\{A(\eta, \alpha')\}$. Hence the right hand side of (5.7) is independent of the choice of ξ such that $\bar{\xi} = x$. It is easy to verify that $\rho_A^{(k)}$ is a holomorphic representation of $G_A(L)$ and $\rho_A^{(k)}$ is homogeneous of order k in the sense defined in § 4.

From (5.7) we get

$$(5.8) \qquad \begin{aligned} \rho_A^{(k)}(\nu, 0, 1) &= 1\,, & \nu &\in N_1\,, \\ \rho_A^{(k)}(0, \nu', 1) &= 1\,, & \nu' &\in N_1'\,, \\ \rho_A^{(k)}(0, 0, a) &= a^k \cdot 1\,, & a &\in C^*\,, \end{aligned}$$

where 1 denotes the identity operator of $C(K_1)$.

THEOREM 5.1. *Let m_k be the order of the subgroup kK_1 of K_1 and i_k the index of kK_1 in K_1. Then $\rho_A^{(k)}$ splits into a sum of i_k inequivalent irreducible representations $\rho_{A,t}^{(k)}$:*

$$\rho_A^{(k)} = \rho_{A,1}^{(k)} \oplus \cdots \oplus \rho_{A,i_k}^{(k)}\,.$$

The degree of each irreducible representation $\rho_{A,t}^{(k)}$ is m_k. The representation $\rho_A^{(k)}$ is irreducible if and only if $(k, d) = 1$.

Proof. From the commutation rule

$$(\alpha, 0, 1)(0, \alpha', 1) = (0, \alpha', 1)(\alpha, 0, 1)(0, 0, \varepsilon\{A(\alpha, \alpha')\})$$

and from (5.5) and (5.8) we obtain

$$(5.9) \qquad \rho_A^{(k)}(\alpha, 0, 1)\rho_A^{(k)}(0, \alpha', 1) = \langle \bar{\alpha}, k\bar{\alpha}' \rangle \rho_A^{(k)}(0, \alpha', 1)\rho_A^{(k)}(\alpha, 0, 1)\,.$$

It also follows from (5.7) that

$$(5.10) \qquad \rho_A^{(k)}(\alpha, 0, 1)f = R(\bar{\alpha})f$$

for all $f \in C(K_1)$ and $\alpha \in L_1$. Since K_1 is abelian, the regular representation R decomposes into a sum of 1-dimensional representations and each 1-dimensional representation is a character of K_1. We know that the regular representation contains every irreducible representation and hence R contains every characters of K_1. On the other hand we have identified K_1' with the character group of K_1 via the pairing (5.5). Hence $C(K_1)$ is a direct sum of 1-dimensional subspaces $W_{g'}(g' \in K_1')$ such that that $W_{g'}$ consists of all $\phi \in C(K_1)$ with

$$\rho_A^{(k)}(\alpha, 0, 1)\phi = \langle \bar{\alpha}, g' \rangle \phi$$

for all $\alpha \in L_1$. Then we get from (5.9) that

$$\rho_A^{(k)}(0, \alpha', 1) W_{g'} = W_{g'+k\alpha'} \ . \tag{5.11}$$

Let $C_1, \cdots, C_{i_k}$ be distinct cosets of K_1' modulo kK_1' and let

$$U_t = \sum_{g' \in C_t} W_{g'} \qquad (1 \leqq t \leqq i_k) \ .$$

The dimension of U_t is equal to the order m_k of kK_i, because $K_1 \cong K_1'$. Then U_t is an invariant subspace of $C(K_1)$. We show that U_t is irreducible. Let $U_t' \ (\neq \{0\})$ be an invariant subspace of U_t. Then we have $\rho_A^{(k)}(\alpha, 0, 1)U_t' = U_t'$ for all $\alpha \in L_1$ and hence there exists a $g' \in C_t$ such that $W_{g'} \subset U_t'$. Then by (5.11) we get $W_{h'} \subset U_t'$ for all $h' \in C_t$. Thus $U_t' = U_t$ and U_t is irreducible.

Let $\rho_{A,t}^{(k)}(\alpha, \alpha', a)$ be the restriction of $\rho_A^{(k)}(\alpha, \alpha', a)$ to U_t. Then $\rho_{A,t}^{(k)}$ is an irreducible representation of degree m_k for each $t = 1, 2, \cdots, i_k$ and obviously $\rho_A^{(k)}$ decomposes into sum of these irreducible representations. If $\rho_{A,t}^{(k)}$ and $\rho_{A,s}^{(k)}$ were equivalent, the restriction of these representations to element of the form $(\alpha, 0, 1)$ should yield the same set of characters of K_1 and if $t \neq s$, this is not the case. Hence $\rho_{A,t}^{(k)}$ are inequivalent irreducible representations.

The representation $\rho_A^{(k)}$ is irreducible if and only if $K_1 = kK_1$. K_1 is a direct sum of cyclic groups of order d_i generated by $\bar{\omega}_i \ (i = 1, 2, \cdots, \ell)$ and we have $kK_1 = K_1$ if and only if $(k, d_i) = 1$ for $i = 1, 2, \cdots, \ell$. However $d = d_1$ and $d_i \mid d$ for $i = 1, 2, \cdots, \ell$. Hence we have $(k, d_i) = 1$ for all i if and only if $(k, d) = 1$. This proves that $\rho_A^{(k)}$ is irreducible if and only if k and d are relatively prime.

If follows from (5.8) that $\rho_{A,t}^{(k)}$ is homogenous of order k and $\rho_{A,t}^{(k)}(\nu, 0, 1) = \rho_{A,t}^{(k)}(0, \nu', 1) = 1, \nu \in N_1, \nu' \in N_1'$.

We now prove the following lemma:

LEMMA 5.1. *Let ρ be an irreducible representation of $G_A(L)$ in a* *complex vector space U such that*

$$\rho(\nu, 0, 1) = \rho(0, \nu', 1) = 1 \qquad \text{for } \nu \in N_1,\ \nu' \in N_1'$$

and

$$\rho(0, 0, a) = a^k \cdot 1 \qquad \text{for all } a \in \boldsymbol{C}^* .$$

Then ρ is equivalent to one of the irreducible representations $\rho_{A,t}^{(k)}$.

Proof. By our condition on ρ, we can define a representation of K_1 by $\bar{\alpha} \to \rho(\alpha, 0, 1)(\alpha \in L_1)$. We have also

$$\rho(\alpha, 0, 1)\rho(0, \alpha', 1) = \langle \bar{\alpha}, k\bar{\alpha}' \rangle \rho(0, \alpha', 1) \rho(\alpha, 0, 1) . \tag{5.12}$$

There exists $u \in U$ $(u \neq 0)$ and $g' \in K_1'$ such that

$$\rho(\alpha, 0, 1) = \langle \bar{\alpha}, g' \rangle u$$

for all $\alpha \in L_1$. Let $u_{\beta'} = \rho(0, \beta', 1)u$ for all $\bar{\beta}' \in K_1'$. Then we obtain from (5.12) that

$$\rho(\alpha, 0, 1)u_{\beta'} = \langle \bar{\alpha}, g' + k\bar{\beta}' \rangle u_{\beta'} .$$

Let C_t be the coset of K_1' modulo kK_1' containing g'. Then there is an obvious linear map T from $U_t = \sum_{g' \in C_t} W_{g'}$ onto the subspace of U spanned by $\{u_\beta\}$ such that $T \circ \rho_{A,t}^{(k)}(\alpha, \alpha', a) = \rho(\alpha, \alpha', a) \circ T$ and as $\rho_{A,t}^{(k)}$ and ρ are both irreducible, T is bijective. Thus ρ and $\rho_{A,t}^{(k)}$ are equivalent.

§ 6. Classification of irreducible holomorphic representations of $G_A(L)$.

THEOREM 6.1. *Every irreducible holomorphic representation of $G_A(L)$ is equivalent to a representation of the form*

$$\sigma \otimes \rho_{A,t}^{(k)} ,$$

where σ is a 1-dimensional representation of L, that is, σ is a 1-dimensional representation of $G_A(L)$ such that $\sigma(0, a) = 1$ for all $a \in \boldsymbol{C}^*$.

The following proof of Theorem 6.1 is due to J. Hano. The subgroup $N \times \boldsymbol{C}^*$ is the center of $G_A(L)$ and hence

$$\rho(\alpha, 1) = \rho'(\alpha) \cdot 1$$

for all $\alpha \in N$, where $\rho'(\alpha) \in \boldsymbol{C}^*$. The map $\rho' : N \to \boldsymbol{C}^*$ satisfies

$$\rho'(\alpha)\rho'(\beta) = \rho'(\alpha + \beta)\varepsilon\left\{\frac{1}{2}kA(\alpha, \beta)\right\}$$

for $\alpha, \beta \in N$. We define a $\boldsymbol{R}$-bilinear from $\varPsi$ on $V \times V$ by

$$\begin{aligned}\varPsi(\sum x_k\omega_k + \sum x'_k\omega'_k, \sum y_k\omega_k + \sum y'_k\omega'_k) \\ = \sum A(\omega_k, \omega'_k)x_k y'_k \,.\end{aligned}$$

Then $\varPsi$ is a satellite form of A, namely $A(x,y) = \varPsi(x,y) - \varPsi(y,x)$ and letting $\psi_0(\alpha) = \varepsilon\left\{\frac{1}{2}k\varPsi(\alpha,\alpha)\right\}$, $\alpha \in N$, we obtain a semi-character ψ_0 of N with respect kA which is integral valued on $N \times N$. That is, ψ_0 is a map from N to $\boldsymbol{C}_1^* = \{z \in \boldsymbol{C} \mid |z| = 1\}$ such that

$$\psi_0(\alpha)\psi_0(\beta) = \psi_0(\alpha+\beta)\varepsilon\left\{\frac{1}{2}kA(\alpha,\beta)\right\} \,.$$

From the definition of $\varPsi$, it is clear that

$$\psi_0(\alpha) = 1$$

if either $\alpha \in N_1$ or $\alpha \in N'_1$.

There exists a 1-dimensional representation σ' of N such that

$$\rho'(\alpha) = \sigma'(\alpha)\psi_0(\alpha)$$

for all $\alpha \in N$. We can extend σ' to a 1-dimensional representation σ of L. Then we have $(\sigma^{-1} \otimes \rho)(\alpha, 1) = \sigma'(\alpha)^{-1}\rho'(\alpha)\cdot 1 = \psi_0(\alpha)\cdot 1$ for all $\alpha \in N$ and hence $(\sigma^{-1} \otimes \rho)(\alpha, 1) = 1$ for all $\alpha \in N_1$ and $\alpha \in N'_1$. Then by Lemma 5.1, $\sigma^{-1} \otimes \rho$ is equivalent to $\rho_{A,t}^{(k)}$ for some t and hence ρ is equivalent to $\sigma \otimes \rho_{A,t}^{(k)}$.

§ 7. Vector bundles defined by irreducible holomorphic representations of the group $G_H(L)$.

In this section we assume that A is the imaginary part of an Hermitian form H on V. Then there is a one-to-one correspondence between the set of irreducible holomorphic representations ρ_H of $G_H(L)$ and the set of irreducible holomorphic representations ρ_A of $G_A(L)$ and the one-to-one correspondence is given by (4.1), that is,

$$\rho_H(\alpha, \alpha', a) = \rho_A\left(\alpha, \alpha', \varepsilon\left\{-\frac{1}{4i}H(\alpha+\alpha', \alpha+\alpha')\right\}a\right) \,. \tag{7.1}$$

We denote by $\rho_{H,t}^{(k)}$ the irreducible holomorphic representation of $G_H(L)$ which corresponds to the irreducible representation $\rho_{A,t}^{(k)}$ of $G_A(L)$ constructed in § 5.

Consider now the Hermitian form kH. The imaginary part of kH is kA and our decomposition $L = L_1 \oplus L_1'$ of L which is defined by A and which is used to define the representations $\rho^{(k)}_{A,t}$ is also a decomposition of L with respect to the alternating form kA (although the sublattice N of L defined by A could be different from the one defined by kA). Therefore we can define the irreducible holomorphic representations $\rho^{(j)}_{kA,t}$ of the group $G_{kA}(L)$ using the decomposition $L = L_1 \oplus L_1'$ and then the irreducible holomorphic representations $\rho^{(j)}_{kH,t}$ of the group $G_{kH}(L)$.

We prove the following lemma.

LEMMA 7.1. *Let F be the vector bundle over the complex torus $E = V/L$ defined by an irreducible holomorphic representation $\rho^{(k)}_{H,t}$ of the group $G_H(L)$. Then F is isomorphic to a vector bundle $L_\mu \otimes F'$, where the vector bundle F' is defined by the irreducible holomorphic representation $\rho^{(1)}_{kH}$ of the group $G_{kH}(L)$ and L_μ is a line bundle defined by a 1-dimensional representation μ of L.*

Proof. We have $\rho^{(k)}_{A,t}(0, 0, a) = a^k \cdot 1$ and $\rho^{(1)}_{kA}(0, 0, a) = a \cdot 1$. The theta factors J and J' associated with the representations $\rho^{(k)}_{A,t}$ of $G_A(L)$ and with the representation $\rho^{(1)}_{kA}$ of $G_{kA}(L)$ are then given by (4.3) and we have

$$J(\alpha + \alpha', u) = \varepsilon\left\{\frac{1}{2i}kH(u, \alpha + \alpha') + \frac{1}{4i}kH(\alpha + \alpha', \alpha + \alpha')\right\} \cdot \rho^{(k)}_{A,t}(-\alpha, -\alpha', 1) \tag{7.2}$$

and

$$J'(\alpha + \alpha', u) = \varepsilon\left\{\frac{1}{2i}kH(u, \alpha + \alpha') + \frac{1}{4i}kH(\alpha + \alpha', \alpha + \alpha')\right\} \cdot \rho^{(1)}_{kA}(-\alpha, -\alpha', 1) \qquad \text{for } \alpha \in L_1 \text{ and } \alpha' \in L_1' . \tag{7.3}$$

Let

$$\tau(\alpha, \alpha', a) = \rho^{(k)}_{A,t}(\alpha, \alpha', 1) \cdot a$$

for $\alpha \in L_1$, $\alpha' \in L_1'$, $a \in \boldsymbol{C}^*$.

We show that τ is a representation of $G_{kA}(L)$. We have

$$\begin{aligned} &\tau(\alpha, \alpha', a)\tau(\beta, \beta', b) \\ &\qquad = \rho^{(k)}_{A,t}((\alpha, \alpha', 1)(\beta, \beta', 1))ab \end{aligned}$$

$$= \rho_{A,t}^{(k)}\Big(\alpha + \beta, \alpha' + \beta', \varepsilon\Big\{\frac{1}{2}A(\alpha + \alpha', \beta + \beta')\Big\}\Big)ab$$

$$= \rho_{A,t}^{(k)}(\alpha + \beta, \alpha' + \beta', 1)\varepsilon\Big\{\frac{1}{2}kA(\alpha + \alpha', \beta + \beta')\Big\}ab\ .$$

On the other hand we have

$$(\alpha, \alpha', a)(\beta, \beta', b) = \Big(\alpha + \beta, \alpha' + \beta', \varepsilon\Big\{\frac{1}{2}kA(\alpha + \alpha', \beta + \beta')\Big\}ab\Big)$$

in the group $G_{kA}(L)$. Hence we have $\tau((\alpha, \alpha', a)(\beta, \beta', b)) = \rho_{A,t}^{(k)}(\alpha + \beta, \alpha' + \beta', 1)\varepsilon\Big\{\frac{1}{2}kA(\alpha + \alpha', \beta + \beta')\Big\}ab$ and we get $\tau(\alpha, \alpha', a)\tau(\beta, \beta', b) = \tau((\alpha, \alpha', a)(\beta, \beta', b))$. Clearly τ is an irreducible holomorphic representation of $G_{kA}(L)$ such that $\tau(0, 0, a) = a\cdot 1$. Then, by Theorem 6.1, we see that τ is equivalent to a representation $\mu \otimes \rho_{kA}^{(1)}$, where μ is a 1-dimensional representation of L. Since we have $\rho_{A,t}^{(k)}(\alpha, \alpha', 1) = \tau(\alpha, \alpha', 1)$, we see from (7.2) and (7.3) that the factor J is equivalent to the factor $\mu \otimes J'$. Let L_μ be the line bundle over E associated with the representation μ of L. Then we have $F \cong L_\mu \otimes F'$ and this proves Lemma 7.1.

THEOREM 7.1. *Let ρ be an irreducible holomorphic representation of $G_H(L)$ and F_ρ the holomorphic vector bundle over $E = V/L$ defined by ρ. Then there exist a 1-dimensional representation σ of L such that*

$$F_\rho \cong L_\sigma \otimes F_{\rho_{kH}^{(1)}}\ ,$$

where k is the order of homogenuity of ρ in the sense of § 4 and L_σ is the line bundle defined by σ.

Proof. By Theorem 6.1, ρ is equivalent to a representation of $G_H(L)$ of the form $\eta \otimes \rho_{H,t}^{(k)}$ and hence F_ρ is isomorphic to $L_\eta \otimes F$, where F is defined by $\rho_{H,t}^{(k)}$. By Lemma 7.1, we have $F \cong L_\mu \otimes F_{\rho_{kH}^{(1)}}$, whence

$$F_\rho \cong L_\sigma \otimes F_{\rho_{kH}^{(1)}} \quad \text{with} \quad \sigma = \eta \otimes \mu\ .$$

COROLLARY 7.2. *Let F be a holomorphic line bundle over $E = V/L$. Then $F \cong L_\sigma \otimes F_{\rho_H^{(1)}}$, where σ is a 1-dimensional representation of L and H is a Hermitian form on V whose imaginary part is integral valued on L.*

§ 8. Properties of the holomorphic vector bundle defined by the Schrödinger representation of $G_H(L)$.

According to Theorem 7.1, the study of holomorphic vector bundles associated with holomorphic irreducible representations of $G_H(L)$ for varying H is reduced to the study of bundles associated with the irreducible representation $\rho_H^{(1)}$. In this and following section we shall denote the representations $\rho_A^{(1)}$ and $\rho_H^{(1)}$ by D_A and D_H respectively and call D_A and D_H the *Schrödinger representation* of $G_A(L)$ and $G_H(L)$. The representation D_A is an irreducible holomorphic representation of $G_A(L)$ by Theorem 5.1 and it is defined by

$$(8.1)\qquad (D_A(\alpha, \alpha', a)f)(x) = a\cdot\varepsilon\Big\{\frac{1}{2}A(\alpha, \alpha') + A(\xi, \alpha')\Big\}f(x + \bar{\alpha})$$

for every f in the group algebra $C(K_1)$, where ξ is any element in L_1 such that $\bar{\xi} = x \in K_1$. For each $g \in K_1$ let f_g be the function on K_1 such that

$$f_g(x) = \begin{cases} 0\,, & x \neq g \\ 1\,, & x = g\,. \end{cases}$$

Then $\{f_g\}$ form a basis of $C(K_1)$ and we have

$$(8.2)\qquad f_g(x + h) = f_{g-h}(x)$$

for any $g, h \in K_1$.

The function $x \to \varepsilon\{A(\xi, \alpha')\}f_g(x + \bar{\alpha})$ is equal to the function $\varepsilon\{A(\beta - \alpha, \alpha')\}f_{g-\bar{\alpha}}$ where $\bar{\xi} = x$ and $\bar{\beta} = g$. Then we get from (8.1) and (8.2) the following formula:

$$(8.3)\qquad D_A(\alpha, \alpha', a)f_g = a\cdot\varepsilon\Big\{-\frac{1}{2}A(\alpha, \alpha') + A(\beta, \alpha')\Big\}f_{g-\bar{\alpha}}\,,$$

where β is any element in L_1 such that $\bar{\beta} = g$.

The theta factor J_H associated with D_H is given by (4.3) and we get

$$J_H(\alpha + \alpha', u) = \varepsilon\Big\{\frac{1}{2i}H(u, \alpha + \alpha') + \frac{1}{4i}H(\alpha + \alpha', \alpha + \alpha')\Big\}D_A(-\alpha, -\alpha', 1)$$

and we obtain from (8.3):

$$
\begin{aligned}
&J_H(\alpha + \alpha', u)f_g \\
(8.4) \qquad &= \varepsilon\left\{\frac{1}{2i}H(u, \alpha + \alpha') + \frac{1}{4i}H(\alpha + \alpha', \alpha + \alpha')\right\} \\
&\quad \cdot \varepsilon\left\{-\frac{1}{2}A(\alpha, \alpha') - A(\beta, \alpha')\right\}f_{g+\alpha} ,
\end{aligned}
$$

where $g = \bar{\beta}, \beta \in L_1$ and $\alpha \in L_1, \alpha' \in L_1'$.

In particular we have

$$
(8.5) \qquad J_H(\alpha, u)f_g = \varepsilon\left\{\frac{1}{2i}H(u, \alpha) + \frac{1}{4i}H(\alpha, \alpha)\right\}f_{g+\alpha} \qquad (\alpha \in L_1)
$$

and

$$
(8.6) \quad J_H(\alpha', u)f_g = \varepsilon\left\{\frac{1}{2i}H(u, \alpha') + \frac{1}{4i}H(\alpha', \alpha')\right\}\varepsilon\{-A(\beta, \alpha')\}f_g , \qquad (\alpha' \in L_1') .
$$

Let

$$
(8.7) \qquad M = N_1 \oplus L_1' .
$$

Then M is a lattice of V containing $N = N_1 \oplus N_1'$ and contained in $L = L_1 \oplus L_1'$ and A is *integral valued on* M.

Let

$$
(8.8) \qquad E_M = V/M .
$$

Then E_M is a complex torus and there is a homomorphism

$$
\varphi : E_M \to E = V/L
$$

and the kernel of φ is L/M which we identify canonically with K_1.

For each $g \in K_1$ let

$$
(8.9) \qquad \psi_g(\nu + \alpha') = \varepsilon\left\{-\frac{1}{2}A(\nu, \alpha') - A(\beta, \alpha')\right\} \qquad (g = \bar{\beta}, \beta \in L_1)
$$

for all $\nu + \alpha' \in M$ with $\nu \in N_1$ and $\alpha' \in L_1'$. Then ψ_g is a semi-character on the lattice M, i.e. we have

$$
\psi_g(m + m') = \psi_g(m)\psi_g(m')\varepsilon\left\{\frac{1}{2}A(m', m)\right\}
$$

for all $m, m' \in M$.

Let

$$(8.10)\quad j_g(\nu+\alpha', u) = \psi_g(\nu+\alpha')\varepsilon\left\{\frac{1}{2i}H(u, \nu+\alpha') + \frac{1}{4i}H(\nu+\alpha', \nu+\alpha')\right\}.$$

Since $\bar{\nu} = 0$ (the zero element of K_1) for $\nu \in N_1$, we get from (8.4) that

$$(8.11)\quad J_H(\nu+\alpha', u)f_g = j_g(\nu+\alpha', u)f_g$$

for all $g \in K_1, \nu \in N_1, \alpha' \in L_1'$. Then we conclude that j_g is a theta factor for the lattice M and that $J_H(\nu+\alpha', u)$ is the diagonal matrix whose diagonal entries are $j_g(\nu+\alpha', u)$.

Let L_g ($g \in K_1$) be the line bundle over E_M defined by the theta factor j_g. Now $J_H(m, u)$ ($m \in M$) is a theta factor for the lattice M and the vector bundle over E_M defined by this factor is the pull back $\varphi^* F_{D_H}$, where F_{D_H} is the vector bundle over E associated with the Schrödinger representation D_H.

Then we have

$$(8.12)\quad \varphi^* F_{D_H} = \sum_{g \in K_1} L_g .$$

If $g \neq h$, then L_g and L_h are not isomorphic. For we have

$$(8.13)\quad j_g(\nu+\alpha', u) = \varepsilon\{-A(\beta-\gamma, \alpha')\} j_h(\nu+\alpha', u)$$

where $\bar{\beta} = g$ and $\bar{\gamma} = h$, and the representation of M defined by $\nu+\alpha' \to \varepsilon\{-A(\beta-\gamma, \alpha')\}$ is not trivial except for the case $\bar{\beta} = \bar{\gamma}$. Thus two normalized factors j_g and j_h for the lattice M are distinct and hence L_g and L_h are not isomorphic. However they are algebraical equivalent, namely they have the same Chern class (see [6]).

Let now

$$T_g : E_M \to E_M$$

be the translation of the complex torus E_M by an element $g \in K_1$ ($\subset E_M$). We show that

$$(8.14)\quad T_g^* L_h \cong L_{h-g} ,$$

where $T_g^* L_h$ denotes the pull-back of L_h by T_g. The pull-back $T_g^* L_h$ is defined by the factor

$$\varepsilon\left\{\frac{1}{2i}H(\beta, \nu + \alpha')\right\} j_h(\nu + \alpha', u)$$

where $\bar{\beta} = g$. However

$$\frac{1}{2i}H(\beta, \nu + \alpha') = A(\beta, \nu + \alpha') + \frac{1}{2i}H(\nu + \alpha', \beta)$$

and $\varepsilon\{A(\beta, \nu + \alpha')\} = \varepsilon\{A(\beta, \alpha')\}$. Hence $\varepsilon\left\{\frac{1}{2i}H(\beta, \nu + \alpha')\right\} = \varepsilon\{A(\beta, \alpha')\}$ $\varepsilon\left\{\frac{1}{2i}H(\nu + \alpha', \beta)\right\}$. We have also $\varepsilon\left\{\frac{1}{2i}H(\nu + \alpha', \beta)\right\} = P(u + \nu + \alpha')/P(u)$, where $P(u) = \varepsilon\left\{\frac{1}{2i}H(u, \beta)\right\}$. Thus

$$\varepsilon\left\{\frac{1}{2i}H(\beta, \nu + \alpha')\right\} = \varepsilon\{A(\beta, \alpha')\}P(u + \nu + \alpha')/P(u)$$

and since $P(u)$ is a non-vanishing holomorphic function on V, $P(u + \nu + \alpha')/P(u)$ is a trivial factor. Thus $T_g^* L_h$ is isomorphic to the line bundle defined by the factor $\varepsilon\{A(\beta, \alpha')\} \cdot j_h(\nu + \alpha', u)$ which is equal to j_{h-g} by (8.13). Thus $T_g^* L_h \cong L_{h-g}$.

Summing up we obtain the following theorem.

THEOREM 8.1. *Let F_{D_H} be the holomorphic vector bundle on the complex torus E, associated with the Schrödinger representation D_H of $G_H(L)$. There exist a complex torus E_M and a homomorphism $\varphi: E_M \to E$ of E_M onto E whose kernel is K_1 and there are holomorphic line bundles $\{L_g\}_{g \in K_1}$ on E_M such that $L_g \not\cong L_h$ for $g \neq h$, $T_g^* L_h = L_{h-g}$ and $\varphi^* F_{D_H} \cong \sum_{g \in K_1} L_g$.*

Remark. Theorem 8.1 shows that F_{D_H} is the direct images $\varphi_* L_g$ of any one of line bundles L_g on E_M.

We consider now the vector bundle $\mathrm{End}\,(F_{D_H})$. Then there exists an exact sequence

$$0 \to I \to \mathrm{End}\,(F_{D_H}) \to Q \to 0\ ,$$

where I is the trivial line bundle and Q is the quotient bundle. The homomorphism $I \to \mathrm{End}\,(F_{D_H})$ is defined by associating to each complex number c the multiplication of each fibre of F_{D_H} by c. We get then a homomorphism of cohomologies

$$H^j(E, \mathcal{O}) \to H^j(E, \operatorname{End}(F_{D_H}))$$

induced by $I \to F_{D_H}$.

We have the following theorem due to Oda [5].

THEOREM 8.2 (Oda). *Let F be a holomorphic vector bundle on a complex torus X. Then the following two statements are equivalent.*

(1) *There exist a complex torus Y and a homomorphism φ of Y onto X and a line bundle L on Y such that*

$$T_g^* L \not\cong L \quad \textit{for all} \quad g \in \ker(\varphi) \quad \textit{and} \quad \varphi_* L = F .$$

(2) *The homomorphism $H^j(X, \mathcal{O}) \to H^j(X, \operatorname{End}(F))$ induced by $I \to \operatorname{End}(F)$ is an isomorphism for all j.*

Applying the theorem of Oda for the vector bundle F_{D_H}, we get from Theorem 8.1 the following corollary.

COROLLARY 8.2. *We have*

$$H^j(E, \mathcal{O}) \cong H^j(E, \operatorname{End}(F_{D_H}))$$

for all j. In particular we have

$$\Gamma(E, \operatorname{End}(F_{D_H})) = \boldsymbol{C} .$$

A vector bundle F is said to be simple if $\Gamma(E, \operatorname{End}(F)) = \boldsymbol{C}$. A simple vector bundle is indecomposable. Hence

COROLLARY 8.3. *The vector bundle F_{D_H} is simple and hence it is indecomposable.*

Moreover we have $H^j(E, \mathcal{O}) \cong H^{0,j}(E, \boldsymbol{C})$ and since E is an n-dimensional complex torus, $\dim H^{0,j}(E, \boldsymbol{C}) = \binom{n}{j}$. Thus we get

$$\dim H^j(E, \operatorname{End}(F_{D_H})) = \binom{n}{j} .$$

We now consider a *theta function* θ for the factor J_H. The function θ is a $C(K_1)$-valued holomorphic function on V satisfying the equation

$$\theta(\alpha + \alpha' + u) = J_H(\alpha + \alpha', u)\theta(u)$$

for all $\alpha \in L_1, \alpha' \in L_1'$ and $u \in V$.

We write

$$\theta(u) = \sum_{g \in K_1} \theta_g(u) f_g$$

and we identify $\theta(u)$ with the column vector $(\theta_g(u))$ of holomorphic functions $\theta_g(u)$ on V. We see from (8.4) that the components of the vector θ satisfy the following equation.

$$\begin{aligned} \theta_g(\alpha + \alpha' + u) & \\ = \varepsilon\Big\{\frac{1}{2i}H(u, \alpha + \alpha') &+ \frac{1}{4i}H(\alpha + \alpha', \alpha + \alpha')\Big\} \\ \cdot \varepsilon\Big\{\frac{1}{2}A(\alpha, \alpha') &- A(\beta, \alpha')\Big\}\theta_{g-\bar{\alpha}}(u) , \end{aligned} \tag{8.14}$$

where $\bar{\beta} = g$.

If $\nu \in N_1$, then $A(\nu, \alpha')$ is an integer and $\varepsilon\Big\{\frac{1}{2}A(\nu, \alpha')\Big\} = \pm 1$ and hence $\varepsilon\Big\{\frac{1}{2}A(\nu, \alpha')\Big\} = \varepsilon\Big\{-\frac{1}{2}A(\nu, \alpha')\Big\}$. Then we get from (8.10) that

$$\theta_g(\nu + \alpha' + u) = j_g(\nu + \alpha', u)\theta_g(u)$$

for $\nu \in N_1, \alpha' \in L_1'$. This shows that the component $\theta_g(u)$ is a theta function for the lattice M and the factor j_g. In particular θ_0 (0 is the zero element of K_1) is a theta function for the lattice M and the factor j_0. Letting $\alpha = \beta, \alpha' = 0$ and replacing u by $-\beta + u$ in (8.14) we get

$$\theta_g(u) = \varepsilon\Big\{\frac{1}{2i}H(u, \beta) - \frac{1}{4i}H(\beta, \beta)\Big\}\theta_0(-\beta + u) \tag{8.15}$$

where $\bar{\beta} = g$. This shows that θ_g is uniquely determined by θ_0.

Let h be a theta function for the lattice M and the factor j_0. We show that there exists a theta function θ for the lattice L and the factor J_H such that $h = \theta_0$. To see this we first assert that

$$\begin{aligned} &\varepsilon\Big\{\frac{1}{2i}H(u, \beta) - \frac{1}{4i}H(\beta, \beta)\Big\}h(-\beta + u) \\ &\qquad = \varepsilon\Big\{\frac{1}{2i}H(u, \gamma) - \frac{1}{4i}H(\gamma, \gamma)\Big\}h(-\gamma + u) \end{aligned}$$

for any two elements β and γ in L_1 such that $\bar{\beta} = \bar{\gamma}$. In fact, if $\bar{\beta} = \bar{\gamma}$, then we have $\nu = \beta - \gamma \in N_1$ and hence

$$h(-\gamma + u) = h(\nu - \beta + u) = j_0(\nu, -\beta + u)h(-\beta + u) .$$

We have

$$j_0(\nu, -\beta + u) = \varepsilon\Big\{\frac{1}{2i}H(u - \beta, \beta - \gamma) + \frac{1}{4i}H(\beta - \gamma, \beta - \gamma)\Big\}$$
$$= \varepsilon\Big\{\frac{1}{2i}H(u, \beta) - \frac{1}{4i}H(\beta, \beta)\Big\}\varepsilon\Big\{-\frac{1}{2i}H(u, \gamma) + \frac{1}{4i}H(\gamma, \gamma)\Big\}$$
$$\cdot\varepsilon\Big\{\frac{1}{4i}(H(\beta, \gamma) - H(\gamma, \beta)\Big\} .$$

However, as $A(\beta, \gamma) = 0$, we have $H(\beta, \gamma) = H(\gamma, \beta)$, whence

$$\varepsilon\Big\{\frac{1}{2i}H(u, \gamma) - \frac{1}{4i}H(\gamma, \gamma)\Big\}j_0(\nu, -\beta + u) = \varepsilon\Big\{\frac{1}{2i}H(u, \beta) - \frac{1}{4i}H(\beta, \beta)\Big\}$$

from which our assertion follows.

We can then define θ_g for each $g \in K_1$ by

$$\theta_g(u) = \varepsilon\Big\{\frac{1}{2i}H(u, \beta) - \frac{1}{4i}H(\beta, \beta)\Big\}h(-\beta + u) \tag{8.16}$$

where β is any element in L_1 such that $\bar{\beta} = g$. In particular letting $\beta = 0$ we have $\theta_0 = h$. Then we can verify easily that the vector (θ_g) satisfy the equation (8.14). Thus $\theta = (\theta_g)$ is a theta function for the lattice L and the factor J_H such that $\theta_0 = h$.

We have shown that $h \to \theta$ defines a bijective map from the vector space of all theta functions for M and j_0, which is identified with the vector space $\Gamma(E_M, L_0)$ of all holomorphic sections of the line bundle L_0, onto the vector space of all theta functions for L and J_H. The latter vector space is identified with the vector space $\Gamma(E, F_{D_H})$ of all holomorphic sections of the vector bundle F_{D_H}.

Since the theta factor j_0 is in the normalized form (8.10), we know that if $\Gamma(E_M, L_0) \neq \{0\}$, then the Hermitian form H is positive ($\geqq$) and $\dim \Gamma(E_M, L_0)$ is equal to the reduced Pfaffian of A relative to M, that is, the product of nonzero elementary divisors of the integral alternating form A on M (see [1], [4], [6]).

The lattice M has a basis

$$\{d_1\omega_1, d_2\omega_2, \cdots, d_\ell\omega_\ell, \omega_{\ell+1}, \cdots, \cdots, \omega_n, \omega_1', \cdots, \omega_n'\}$$

and we have

$$A(\omega_i, \omega_j) = A(\omega_i', \omega_j') = 0$$

$$A(\omega_i, \omega'_j) = \frac{e_i}{d}\delta_{ij}$$

where $e_{\ell+1} = \cdots = e_n = 0$ and $e_1, \cdots, e_\ell$ are non-zero (see § 5). Then

$$A(d_i\omega_i, \omega'_i) = e_i \cdot d_i d^{-1}$$

and by (5.1) we get

$$A(d_i\omega_i, \omega'_i) = e_i(e_i, d)^{-1}$$

for $i = 1, 2, \cdots, \ell$. Therefore $e_i(e_i, d)^{-1}$ $(i = 1, \cdots, \ell)$ are non-zero elementary divisors of the integral alternating form A on M and we get

$$\dim \Gamma(E_M, L_0) = \dim \Gamma(E, F_{D_H}) = \prod_{i=1}^{\ell} e_i(e_i, d)^{-1} .$$

Summing up we get

THEOREM 8.3. *Let F_{D_H} be the holomorphic vector bundle over E defined by the Schrödinger representation of the group $G_H(L)$. The rank of F_{D_H} is equal to $d^\ell \prod_{i=1}^{\ell} (e_i, d)^{-1}$ and the dimension of the vector space $\Gamma(E, F_{D_H})$ of holomorphic sections of F_{D_H} is nonzero if and only if H is positive and $H \neq 0$ and if this is the case, we have*

$$\dim \Gamma(E, F_{D_H}) = \prod_{i=1}^{\ell} e_i(e_i, d)^{-1};$$

here d is the smallest positive integer such that dA is integral valued on L, A being the imaginary part of H, and $e_1, \cdots, e_\ell$ denote the non-zero elementary divisors of the integral alternating form dA on L.

For any complex torus $E = V/L$ and a theta factor J of rank r for L, let us denote by $D^p(E, J)$ the complex vector space of all C^r-valued differential form ω of type $(0, p)$ on V such that

$$(T^*_\alpha \omega)_u = J(\alpha, u)\omega_u$$

for every $\alpha \in L$ and $u \in V$, where $T_\alpha \colon V \to V$ denotes the translation of V by α and ω_u is the value of ω at u. Since $J(\alpha, u)$ is holomorphic in u, we have $d''\omega \in D^{p+1}(E, J)$ for every $\omega \in D^p(E, J)$. Thus $D(E, J) = \sum_p D^p(E, J)$ is a complex with coboundary operator d'' and the cohomology groups $H^p(E, J)$ of this complex is canonically identified with the cohomology groups $H^p(E, F_J)$ of the sheaf of germs of holomorphic sections of the vector bundle F_J defined by J.

Consider now $D^p(E, J_{D_H})$ and let $\omega \in D^p(E, J_{D_H})$. We can represent ω by a column vector $(\omega_g)(g \in K_1)$ and we can show precisely as in the proof of Theorem 8.3 that ω_0 is an element of $D^p(E_M, j_0)$ and that the map from $D^p(E_M, j_0)$ to $D^p(E, J_{D_H})$ defined by $\omega_0 \to \omega$ is a bijective linear map for each p commuting with d''. Thus this induces an isomorphism of cohomologies

$$(8.17) \qquad H^p(E_M, L_0) \cong H^p(E, F_{D_H})$$

for all p.

We now have the following theorem of Mumford [4] (see also [7]). Let E be a complex torus and F a line bundle over E defined by a normalized theta factor $\psi(\alpha)\left\{\frac{1}{2i}H(u, \alpha) + \frac{1}{4i}H(\alpha, \alpha)\right\}$, where $\alpha \in L, u \in V$ with $E = V/L$. Assume that the Hermitian form H is non-degenerate and let $i(H)$ be the number of negative eigenvalues of H. Then $H^p(E, F) = 0$ except for $p = i(H)$ and the dimension of $H^{i(H)}(E, F)$ is equal to the Plaffian of the imaginary part A of H.

Applying this theorem of Mumford to the line bundle L_0 over E_M we obtain from the isomorphism (8.17) the following theorem.

THEOREM 8.4. *The notation being as in Theorem* 8.3, *we assume that H is non-degenerate and let $i(H)$ denote the number of negative eigenvalues of H. Then we have*

$$H^p(E, F_{D_H}) = 0 \,, \qquad p \neq i(H)$$

and

$$\dim H^{i(H)}(E, F_{D_H}) = \prod_{i=1}^{n} e_i(e_i, d)^{-1} \,.$$

§9. Vector bundles associated with indecomposable holomorphic representations of $G_H(L)$, their tensor products and Chern classes.

In §4 we called a holomorphic representation ρ of $G_A(L)$ is *homogeneous of order k*, if $\rho(0, a) = a^k \cdot 1$ for all $a \in C^*$, where k is an integer. Every irreducible holomorphic representation is homogeneous and every representation of L is regarded as a homogeneous representation of degree 0.

LEMMA 9.1. *Let ρ be a holomorphic representation of $G_A(L)$. Then ρ splits into a sum of holomorphic homogeneous representations $\rho_1, \cdots, \rho_s$ having distinct orders.*

As a corollary we get

COROLLARY 9.1. *Every indecomposable holomorphic representation of $G_A(L)$ is homogeneous.*

To prove Lemma 9.1 we first observe that every holomorphic representation $\tilde{\sigma}$ of $\boldsymbol{C}$ is of the form

$$\tilde{\sigma}(z) = \exp zB ,$$

where B is a complex matrix. Now let $\sigma: \boldsymbol{C}^* \to GL_r(\boldsymbol{C})$ be a holomorphic representation and let $p: \boldsymbol{C} \to \boldsymbol{C}^*$ be the covering homomorphism defined by $p(z) = \exp z$. Then $\ker p = \{2\pi im \mid m \in \boldsymbol{Z}\}$. We can lift σ be a representations $\tilde{\sigma}: \boldsymbol{C} \to GL_r(\boldsymbol{C})$ such that $\tilde{\sigma}(z) = \sigma(p(z))$. There is a $r \times r$ complex matrix B such that $\tilde{\sigma}(z) = \exp zB$. Then we have $1 = \sigma(p(2\pi i)) = \exp 2\pi iB$. Then $t \to \exp 2\pi itB$ $(t \in \boldsymbol{R})$ defines a representation of $\boldsymbol{R}/(1)$ and hence we may assume that $\exp 2\pi itB$ are unitary for all $t \in \boldsymbol{R}$. Then iB is skew Hermitian and so B is a Hermitian matrix. Again we may assume that B is diagonal. Then as $\exp 2\pi iB = 1$, the diagonal entries $m_1, \cdots, m_r$ of B are integers and $\exp zB$ is the diagonal matrix with diagonal entries $\exp zm_1, \cdots, \exp zm_r$. Since $p(z) = \exp z$, we see that for each $a \in \boldsymbol{C}^*, \sigma(a)$ is the diagonal matrix whose diagonal entries are $a^{m_1}, \cdots, a^{m_r}$.

Let ρ be a holomorphic representation of $G_A(L)$ in a complex vector space U and let $\sigma(a) = \rho(0, a)$. Then $\sigma(a)$ is a holomorphic representation of $\boldsymbol{C}^*$. We have a basis of U with respect to which every $\sigma(a)$ is represented by a diagonal matrix whose diagonal entries are $a^{m_1}, \cdots, a^{m_r}$. Denote by $\{k_1, \cdots, k_s\}$ the set of distinct integers which appears in $\{m_1, \cdots, m_r\}$ and let

$$U_i = \{u \in U \mid \sigma(a)u = a^{k_i} \cdot u \text{ for all } a \in \boldsymbol{C}^*\} .$$

Then $U = U_1 \oplus \cdots \oplus U_r$. Let $u \in U_i$. Then

$$\sigma(a)\rho(\alpha, 1)u = \rho(0, a)\rho(\alpha, 1)u = \rho(\alpha, 1)\rho(0, a)u$$

$= a^{k_i}\rho(\alpha, 1)u$. Thus we get $\rho(\alpha, 1)u \in U_i$. It follows then that each U_i is an invariant subspace. Let $\rho_i(\alpha, a)$ be the restriction of $\rho(\alpha, a)$ to U_i.

Then ρ_i is a representation of $G_A(L)$ in U_i and $\rho_i(0, a) = \sigma(a) | U_i = a^{k_i} \cdot 1$. Thus ρ_i is homogeneous of order k_i and ρ splits into the sum of $\rho_1, \cdots, \rho_s$.

To a holomorphic representation ρ of $G_A(L)$ corresponds a holomorphic representation ρ_H of $G_H(L)$ and a holomorphic vector bundle over $E = V/L$ associated to ρ_H. In this section we shall denote this vector bundle by F_ρ or by F_{ρ_A}. According to the splitting of the representation ρ, F_ρ also decomposes into direct sum. Hence the important case is the case where ρ is an indecomposable representation. By Corollary 9.1 ρ is then homogeneous.

THEOREM 9.1. *Let H_1 and H_2 be Hermitian forms on V such that the imaginary parts A_1 and A_2 are rational valued on L. Let ρ_{A_1} and ρ_{A_2} be indecomposable representations of $G_{A_1}(L)$ and $G_{A_2}(L)$ of order k_1 and k_2 respectively. Let*

$$H = k_1H_1 + k_2H_2 , \qquad A = k_1A_1 + k_2A_2 .$$

Then there exists a holomorphic representation ρ_A of $G_A(L)$ which is homogeneous of order 1 such that

$$F_{\rho_{A_1}} \otimes F_{\rho_{A_2}} \cong F_{\rho_A} .$$

Proof. Let

$$\sigma(\alpha) = \rho_{A_1}(\alpha, 1) \otimes \rho_{A_2}(\alpha, 1)$$

for all $\alpha \in L$.
Then we have

$$\sigma(\alpha)\sigma(\beta) = \varepsilon\left\{\frac{1}{2}A(\alpha, \beta)\right\}\sigma(\alpha + \beta) .$$

Put

$$\rho_A(\alpha, a) = \sigma(\alpha) \cdot a .$$

Then ρ_A is a holomorphic representation of $G_A(L)$ homogeneous of order 1. The theta factor J corresponding to ρ_A is given by

$$J(\alpha, u) = \varepsilon\left\{\frac{1}{2i}H(u, \alpha) + \frac{1}{4i}H(\alpha, \alpha)\right\}\rho_A(-\alpha, 1)$$

(see (4.3)).

We can write this in the form

$$J(\alpha, u) = J_1(\alpha, u) \otimes J_2(\alpha, u) , \qquad (\alpha \in L) ,$$

where

$$J_j(\alpha, u) = \varepsilon\left\{\frac{1}{2i} k_j H_j(u, \alpha) + \frac{1}{4i} k_j H_j(\alpha, \alpha)\right\} \rho_{A_j}(-\alpha, 1)$$

for $j = 1, 2$. Then J_j is the theta factor corresponding to ρ_{A_j}. This proves that $F_{\rho_A} \cong F_{\rho_{A_1}} \otimes F_{\rho_{A_2}}$.

To prove the next theorem we observe that $L_1 \times L'_1 \times U_{2d}$ is a subgroup of $G_A(L)$, where U_{2d} denotes the group of all $2d$-th roots of unity. For, we have

$$(\alpha, 0, 1) \cdot (0, \alpha', 1) = \left(\alpha, \alpha', \varepsilon\left\{\frac{1}{2} A(\alpha, \alpha')\right\}\right) \quad \text{and} \quad dA(\alpha, \alpha') \in \boldsymbol{Z} .$$

Moreover the image of this group by the Schrödinger representation D_A is a finite group, because we have $D_A(\nu, 0, 1) = D_A(0, \nu, 1) = 1$ for $\nu \in N_1$, $\nu' \in N'_1$. Therefore we may assume that $D_A(\alpha, \alpha', 1)$ are unitary transformations of $C(K_1)$ (for a suitable inner product) for all $\alpha \in L_1, \alpha' \in L'_1$.

THEOREM 9.2. *Let H_1 and H_2 be Hermitian forms on V such that the imaginary parts A_1 and A_2 are rational valued on L. Let ρ_{A_1} and ρ_{A_2} be irreducible holomorphic representations of $G_{A_1}(L)$ and $G_{A_2}(L)$ respectively and let k_1 and k_2 be the orders of homogenuity of ρ_{A_1} and ρ_{A_2}. Let*

$$H = k_1 H_1 + k_2 H_2 , \qquad A = k_1 A_1 + k_2 A_2 .$$

Then there exist 1-dimensional representation $\sigma_1, \cdots, \sigma_s$ of L such that

$$F_{\rho_{A_1}} \otimes F_{\rho_{A_2}} \cong (L_{\sigma_1} \otimes F_{D_A}) \oplus \cdots \oplus (L_{\sigma_s} \otimes F_{D_A}) ,$$

where L_{σ_i} denotes the line bundle associated with σ_i.

Proof. By Theorem 7.1., $F_{\rho_{A_1}} \cong L_{\tau_1} \otimes F_{D_{k_1 A_1}}$ and $F_{\rho_{A_2}} \cong L_{\tau_2} \otimes F_{D_{k_2 A_2}}$ and hence $F_{\rho_{A_1}} \otimes F_{\rho_{A_2}} \cong L_{\tau_1 \otimes \tau_2} \otimes F_{D_{k_1 A_1}} \otimes F_{D_{k_2 A_2}}$. Let

$$\sigma(\alpha) = D_{k_1 A_1}(\alpha, 1) \otimes D_{k_2 A_2}(\alpha, 1) \qquad \text{for all } \alpha \in L$$

and

$$\rho_A(\alpha, a) = \sigma(\alpha) \cdot a .$$

Then as in the proof of Theorem 9.1, ρ_A is a homogeneous representation of $G_A(L)$ of order 1 and we have $F_{D_{k_1A_1}} \otimes F_{D_{k_2A_2}} \cong F_{\rho_A}$. Since $D_{k_1A_1}(\alpha, 1)$ and $D_{k_2A_2}(\alpha, 1)$ are unitary for all $\alpha \in L$ as we have seen above, $\sigma(\alpha)$ is also unitary for all $\alpha \in L$. Then the representation ρ_A is completely reducible and we have $\rho_A = \rho_1 \oplus \cdots \oplus \rho_s$, where each ρ_i is irreducible and homogeneous of order 1. Again by Theorem 7.1, we have $F_{\rho_i} \cong L_{\lambda_i} \otimes F_{D_A}$. Then

$$F_{D_{k_1A_1}} \otimes F_{D_{k_2A_2}} \cong (L_{\lambda_1} \otimes F_{D_A}) \oplus \cdots \oplus (L_{\lambda_s} \otimes F_{D_A}) .$$

Letting $\sigma_i = \tau_1 \otimes \tau_2 \otimes \lambda_i$ for $i = 1, 2, \cdots, s$ we get $F_{\rho_{A_1}} \otimes F_{\rho_{A_2}} \cong (L_{\sigma_1} \otimes F_{D_A}) \oplus \cdots \oplus (L_{\sigma_s} \otimes F_{D_A})$.

Remark. From the proof of Theorem 9.2 we obtain the following rather strange results. Let $m(H)$ be the degree of the Schrödinger representation D_H of $G_H(L)$. Then $m(H_1 + H_2)$ divides $m(H_1) \cdot m(H_2)$.

To compute the Chern classes, let $\pi: V \to E = V/L$ be the covering map. We choose an open covering $\{U_i\}$ of E with the following property: U_i are connected and each connected component of $\pi^{-1}(U_i)$ are mapped homeomorphically onto U_i by π. For each U_i, choose a connected component $\tilde{U}_i$ of $\pi^{-1}(U_i)$. Then we have $\pi^{-1}(U_i) = \bigcup_{\alpha \in L} T_\alpha \tilde{U}_i$, where $T_\alpha: V \to V$ is the translation of V by $\alpha \in L$. Let

$$\rho_i: U_i \to \tilde{U}_i \tag{9.1}$$

be the inverse of the homeomorphism $\pi: \tilde{U}_i \to U_i$.

For each pair (i, j) of indices such that $U_i \cap U_j$ is non-empty, there exists a unique $\sigma_{ji} \in L$ such that

$$\rho_i(x) = \rho_j(x) + \sigma_{ji} \tag{9.2}$$

for all $x \in U_i \cap U_j$.

Let J be a $GL_r(C)$-valued theta factor for the torus E and let

$$g_{ij}(x) = J(\sigma_{ji}, \rho_j(x)) \tag{9.3}$$

for all $x \in U_i \cap U_j$. Then $g_{ij}: U_i \cap U_j \to GL_r(C)$ is a holomorphic map and $\{g_{ij}\}$ is a system of transition functions of the vector bundle F over E associated with the factor J.

A connection of the vector bundle F is defined by a connection form $\omega = \{\omega_i\}$. Here each ω_i is a $r \times r$ matrix whose entries are 1-forms defined on U_i and they satisfy the condition

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1}\omega_i g_{ij} \quad \text{on} \quad U_i \cap U_j \; . \tag{9.4}$$

We assume that J is of the form

$$J(\alpha, u) = \varepsilon\left\{\frac{1}{2i}H(u, \alpha)\right\}C(\alpha) \; , \tag{9.5}$$

where $C(\alpha)$ is a constant matrix depending on $\alpha \in L$. Notice that the theta factor associated with a homogeneous representation of $G_A(L)$ has this form. Take a basis of V and identify V with C^n and write

$$H(u, \alpha) = \sum_{a,b=1} h_{ab} u_a \bar{\alpha}_b \; .$$

Then $J(\alpha, u) = \exp\left\{\pi \sum_{a,b} h_{ab} u_a \bar{\alpha}_b\right\} C(\alpha)$ and

$$g_{ij}(x) = \exp\{\pi \sum h_{ab} u_a(\rho_j(x))(\overline{\sigma_{ji}})_b\} C(\sigma_{ji}) \; .$$

Let

$$z_a^{(i)} = u_a \circ \rho_i$$

for each i. Then $\{z_1^{(i)}, \cdots, z_n^{(i)}\}$ are local coordinates of E on U_i and from (9.2) we get

$$dz_a^{(i)} = dz_a^{(j)} \quad \text{on} \quad U_i \cap U_j \; .$$

Let ζ_a be the holomorphic 1-form on E such that $\pi^*\zeta_a = du_a$. Then we have

$$\zeta_a = dz_a^{(i)}$$

on each U_i.
We get

$$g_{ij}^{-1} dg_{ij} = \left(\pi \sum_{a,b} h_{ab} (\overline{\sigma_{ji}})_b \zeta_a\right) \cdot 1_r$$

where 1_r is the $r \times r$ unit matrix.
Let

$$\omega_i = -\left(\pi \sum_{a,b} h_{ab} \bar{z}_a^{(i)} \zeta_a\right) \cdot 1_r$$

on each U_i. Then it is easy to verify that $\omega = \{\omega_i\}$ is a connection form. The curvature form $\Omega = \{\Omega_i\}$ is the system of 2-forms such that

$$\Omega_i = d\omega_i + \omega_i \wedge \omega_i \quad \text{on} \quad U_i \,.$$

However we have $\omega_i \wedge \omega_i = 0$ and hence $\Omega_i = d\omega_i$. Then we get

$$\Omega_i = \Big(\pi \sum_{a,b} h_{ab}\zeta_a \wedge \bar{\zeta}_b\Big)\cdot 1_r \quad \text{on} \quad U_i$$

and since the left hand side is globally defined we have

$$\Omega = \Big(\pi \sum_{a,b} h_{ab}\zeta_a \wedge \bar{\zeta}_b\Big)\cdot 1_r \tag{9.6}$$

globally.

The total Chern class $C(F)$ is defined by

$$C(F) = \det\Big(1_r + \frac{i}{2\pi}\Omega\Big)\,.$$

Since Ω is of the form (9.6) we get

$$C(F) = \Big(1 + \frac{i}{2}\sum_{a,b} h_{ab}\zeta_a \wedge \bar{\zeta}_b\Big)^r\,.$$

Thus we have

$$C_s(F) = \binom{r}{s}\Big(\frac{i}{2}\sum_{a,b} h_{ab}\zeta_a \wedge \bar{\zeta}_b\Big)^s\,.$$

In particular we have

$$C_1(F) = \frac{ri}{2}\sum_{a,b} h_{ab}\zeta_a \wedge \bar{\zeta}_b$$

and

$$C_s(F) = \frac{1}{r^s}\binom{r}{s}C_1^s(F)\,.$$

Now let F_{ρ_A} be the vector bundle associated with a holomorphic representation ρ_A of $G_A(L)$ homogeneous of degree k. Then the theta factor is of the form

$$J(\alpha, u) = \varepsilon\Big\{\frac{1}{2i}kH(\alpha, u)\Big\}C(\alpha)$$

and we get the following theorem.

THEOREM 9.3. *Let F_{ρ_A} be the vector bundle over $E = V/L$ associated with a holomorphic representation ρ_A of degree r of $G_A(L)$ which is homogeneous of order k. Let H be the Hermitian form whose imaginary part is A and let*

$$H(u, v) = \sum_{a,b} h_{ab} u_a \bar{v}_b$$

for $u, v \in V$. Let ζ_a be the holomorphic 1-form on E whose pullback to V is du_a $(a = 1, 2, \cdots, n)$. Then we have

$$C_1(F_{\rho_A}) = kr \cdot \frac{i}{2} \sum_{a,b} h_{ab} \zeta_a \wedge \bar{\zeta}_b, \; C_s(F_{\rho_A}) = \frac{1}{r^s}\binom{r}{s} C_1^s(F_{\rho_A}) \; .$$

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