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SOME PROPERTIES OF COMPLETE INTERSECTIONS IN "GOOD" PROJECTIVE VARIETIES

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In [10] it was proved that, if X denotes a non singular surface which is a complete intersection in P_k^n (k an algebraically closed field of characteristic 0) and C an irreducible curve on X, which is a set-theoretic complete intersection in X, then C is actually a complete intersection in X; the key point was to show that $\mathrm{Pic}(X)$ modulo the subgroup generated by the class of $\mathcal{O}_X(1)$ is torsion-free.

The first purpose of this paper is to give a local version of the above fact and generalize it in various directions; this is what we get through the theorem of § 3, where, with respect to the theorem of [10], we have a field k which is not necessarily algebraically closed, a "good" variety (see the precise definition at the beginning of the paper) instead of P_k^n and X need not be smooth but it is sufficient that C is locally defined in it by a single equation.

Essential tool is the theorem of § 1, whose proof, although more complicated, is based upon the main idea of [10] (we remark that even in [9] the same kind of technique is used for similar purposes).

The first paragraph is completely devoted to the proof of the theorem which essentially says that if (A, m, k) is a local ring of the type we are interested in, $\mathrm{char}\, k = 0$, $\dim(A) = 3$ and U_A denotes the scheme $\mathrm{Spec}(A) - \{m\}$, then $\mathrm{Pic}(U_A)$ is torsion-free; this fact, when A is the local ring at the vertex of the affine cone associated to a projective variety, has precisely the geometric implications we were looking for.

In the second section, using results of [1], [2], [11], [12], we show that in our hypotheses not always the divisor class group of A is finitely generated, although it is so in the general case (see proposition and corollary).

The last paragraph contains the geometric theorem, which we have

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already spoken about and a consequence which is related to the well-known problem as to whether any curve of $\boldsymbol{P}^3$ is a set-theoretic complete intersection. More precisely we get that if in a good ambient space a curve C is a set-theoretic complete intersection but not a complete intersection, then there are two possibilities: C can be obtained as a set-theoretic complete intersection of either two normal surfaces having non factorial points on C, or two non normal surfaces having C as a multiple curve; in both cases the surfaces can be chosen to be smooth outside C.

In this paper all rings are supposed to be commutative, noetherian and with identity. If (R, m) is a local ring, $(U_R, \mathcal{O}_{U_R})$ denotes the punctured spectrum of R with its structure sheaf. Throughout this work we shall say that a ring R is "good" if every ring which is obtained from R by a finite succession of localizations at prime ideals and completions with respect to the corresponding maximal ideal is a UFD (Unique factorization domain) and a C-M (Cohen-Macaulay) ring; in the same way we shall say that a projective variety is "good" if the local ring at the vertex of the cone associated with it is "good". "Good" varieties are for instance projective spaces, Grassmannians and non singular complete intersections of dimension bigger than 2.

§1. The main result is the following

THEOREM. *Let (A, m, k) be a local ring which is a complete intersection in a "good" ring.*

i) *If* $\dim A \geqslant 4$, *then* $\operatorname{Pic}(U_A) = 0$ *(hence A is parafactorial).*

ii) *If* $\dim A = 3$ *and* $\operatorname{char} k = 0$, *then* $\operatorname{Pic}(U_A)$ *is torsion-free.*

iii) *If* $\dim A = 3$ *and* $\operatorname{char} k = p > 0$, *then* $\operatorname{Pic}(U_A)$ *is e-torsion-free for any e prime to p.*

The proof requires several steps.

Step 1. If $(\hat{A}, \hat{m}, k)$ denotes the completion of (A, m, k) with respect to its m-adic topology, the canonical map $\operatorname{Pic}(U_A) \to \operatorname{Pic}(U_{\hat{A}})$ is injective.

The proof is in [7] Lemma 1.4.

After step 1 we may assume A complete.

Now, if A is a quotient of a "good" ring by a regular sequence of length r, we say that $c(A) \leqslant r$; after remarking that $c(A) = 0$ means A "good", hence implies $\operatorname{Pic}(U_A) = 0$, we may assume that the theorem is true when $c(A) \leqslant r-1$ and prove it if $c(A) \leqslant r$. So let us assume that R is a "good" ring, $(t_1, \cdots, t_r)$ a regular R-sequence, and $A \simeq R/(t_1, \cdots, t_r)R$;

call B the ring $R/(t_1, \cdots, t_{r-1})R$, $\bar{t}_r$ the canonical image of t_r in B and m_B the maximal ideal of B.

Step 2. The pair $(U_B, V(\bar{t}_r) - \{m_B\})$ satisfies the effective Lefschetz condition (see [5] Exp X).

First of all we remark that B too may be assumed to be complete, hence a quotient of a regular ring; according to [5] Exp X, it is sufficient to prove that for any closed point $\mathfrak{P}$ of U_B, depth $B_{\mathfrak{P}} \geqslant 3$. But, since B is a C-M ring, it is catenarian, and since $\dim B \geqslant 4$, $\operatorname{ht}(\mathfrak{P}) \geqslant 3$ for any closed point $\mathfrak{P}$ of U_B, hence depth $B_{\mathfrak{P}} \geqslant 3$, being also $B_{\mathfrak{P}}$ a C-M ring. We complete U_B along its closed subscheme $V(\bar{t}_r) - \{m_B\}$, which is isomorphic to U_A, and we get a formal scheme denoted by $\hat{U}_B$.

Step 3. In case i) $\operatorname{Pic}(\hat{U}_B) \simeq \operatorname{Pic}(U_A)$

In case ii) $\operatorname{Tors}(\operatorname{Pic}(\hat{U}_B)) \simeq \operatorname{Tors}(\operatorname{Pic}(U_A))$

In case iii) e-$\operatorname{Tors}(\operatorname{Pic}(\hat{U}_B)) \simeq e$-$\operatorname{Tors}(\operatorname{Pic}(U_A))$ for any integer e prime to p.

Henceforth $\mathscr{T}$ denotes the sheaf induced on U_B by the ideal $(\bar{t}_r)$ of B and U_A^n the scheme $(U_A, \mathcal{O}_{U_B}/\mathscr{T}^n)$.

Since $\mathscr{T}$ is a sheaf of ideals of definition, we get $\operatorname{Pic}(\hat{U}_B) \simeq \varprojlim_n \operatorname{Pic}(U_A^n)$ (see [8] lemma 8.2).

Using the following exact sequences of sheaves (see [5] Exp XI)

$$0 \to \mathscr{T}^{n-1}/\mathscr{T}^n \to (\mathcal{O}_{U_B}/\mathscr{T}^n)^* \to (\mathcal{O}_{U_B}/\mathscr{T}^{n-1})^* \to 1$$

we get the following exact sequences of groups

$$H^1(U_A, \mathcal{O}_{U_A}) \to \operatorname{Pic}(U_A^n) \to \operatorname{Pic}(U_A^{n-1}) \to H^2(U_A, \mathcal{O}_{U_A}) \;.$$

By local cohomology (see [6]) $H^i(U_A, \mathcal{O}_{U_A}) \simeq H_m^{i+1}(A)$ and A is a Gorenstein ring; we get in case i) depth $A \geqslant 4$, hence $H_m^2(A) = H_m^3(A) = 0$, and so $\operatorname{Pic}(U_A) \simeq \operatorname{Pic}(U_A^2) \simeq \cdots \simeq \varprojlim_n \operatorname{Pic}(U_A^n) \simeq \operatorname{Pic}(\hat{U}_B)$; in case ii) depth $A = 3$, hence $H_m^2(A) = 0$, $H_m^3(A) \simeq D(A)$ the injective envelope of k (see [6] coroll. 3.10 and prop. 4.13) which is a torsion-free group, being char $k = 0$, and so

$$\begin{aligned}\operatorname{Tors}(\operatorname{Pic}(U_A)) &\simeq \operatorname{Tors}(\operatorname{Pic}(U_A^2)) \simeq \cdots \\ &\simeq \operatorname{Tors}(\varprojlim_n \operatorname{Pic}(U_A^n)) \simeq \operatorname{Tors}(\operatorname{Pic}(\hat{U}_B)) \;.\end{aligned}$$

In case iii) it is sufficient to note that $D(A)$ is e-torsion-free for any integer e prime to p and the conclusion follows as in case ii).

Step 4. LEMMA. *Let X be a locally noetherian scheme and $x \in X$ a closed point such that* depth $\mathcal{O}_{X,x} \geqslant 2$; *denote the open subscheme $X - \{x\}$ by U and the scheme* Spec $(\mathcal{O}_{X,x}) - \{x\}$ *by U_x.*

α) *If* Pic (X) *and* Pic (U_x) *are trivial,* Pic (U) *is trivial.*

β) *If* Pic (X) *and* Pic (U_x) *are torsion-free,* Pic (U) *is torsion-free.*

γ) *If* Pic (X) *and* Pic (U_x) *are e-torsion-free, then* Pic (U) *is e-torsion-free (e an integer).*

Proof. α) By hypothesis $\mathcal{O}_{X,x}$ is a parafactorial ring, hence any invertible sheaf on U can be extended to X, and the conclusion immediately follows. β) Let $\mathscr{L}$ be an invertible sheaf on U such that $\mathscr{L}^{\otimes n} \simeq \mathcal{O}_U$ and let $i: U_x \to U$ be the canonical morphism; then $i^*(\mathscr{L}^{\otimes n}) \simeq (i^*\mathscr{L})^{\otimes n} \simeq \mathcal{O}_{U_x}$, which implies $i^*\mathscr{L} \simeq \mathcal{O}_{U_x}$ since Pic (U_x) is torsion free. Hence $\mathscr{L}$ is trivial on the complement of x in an affine neighbourhood of x (see [4] 8.5.2 (ii) and 8.5.5) and therefore it can be extended to an invertible sheaf $\mathscr{E}$ on X. Let now $j: U \to X$ be the canonical immersion; we get $i^*(\mathscr{E}^{\otimes n}) \simeq (j^*\mathscr{E})^{\otimes n} \simeq \mathscr{L}^{\otimes n} \simeq \mathcal{O}_U$, but depth $\mathcal{O}_x \geqslant 2$, hence $\mathscr{E}^{\otimes n} \simeq \mathcal{O}_X$ (see [7] 21.13.3 and 21.13.4); since Pic (X) is torsion-free, $\mathscr{E} \simeq \mathcal{O}_X$ and $\mathscr{L} \simeq \mathcal{O}_U$. γ) Same proof as β.

Step 5 (conclusion). For concluding the proof it is sufficient to prove in case i) that Pic $(\hat{U}_B) = 0$, in case ii) that Pic $(\hat{U}_B)$ is torsion-free, in case iii) that Pic $(\hat{U}_B)$ is e-torsion-free for any e prime to p.

Let $\hat{\mathscr{L}}$ be an invertible sheaf on $\hat{U}_B$; by step 2 we know that $\hat{\mathscr{L}}$ is induced on U_B by an invertible sheaf $\mathscr{L}$ on a neighbourhood U of $V(\bar{t}_r) - \{m_B\}$ (see 5 Exp X). Hence it is sufficient to prove that, for any neighbourhood U of $V(\bar{t}_r) - \{m_B\}$ in U_B, Pic (U) is trivial in case i), torsion-free in case ii), e-torsion-free for any e prime to p in case iii). First of all we remark that B is a local ring which is a complete intersection in a "good" ring and such that $\dim B \geqslant 4$, $c(B) \leqslant r-1$; hence, by induction, Pic $(U_B) = 0$. Further, if U is a neighbourhood of $V(\bar{t}_r) - \{m_B\}$ in U_B, then $U_B - U = \{\mathfrak{P}_1, \cdots, \mathfrak{P}_s\}$ where ht $(\mathfrak{P}_i) \geqslant 4$ in case i, ht $(\mathfrak{P}_i) = 3$ in cases ii) and iii) (see [5] Exp X p. 112); we get the conclusion, applying the lemma of step 4, with $X = U_B$, $U_x = U_{B_{\mathfrak{P}_i}}$ for $i = 1, \cdots, s$, the local character of the property of being a "good" ring and the inductive hypothesis.

COROLLARY. *Let (A, m, k) be a three-dimensional local ring, which*

is a complete intersection in a "good" ring and equicharacteristic zero. Assume that U_A is locally factorial (for instance A has an isolated singularity) and A is almost factorial.
Then A is factorial.

Proof. Since U_A is locally factorial and $\dim A > 1$, $\mathrm{Pic}(U_A) \simeq \mathrm{Cl}(U_A) \simeq \mathrm{Cl}(A)$ which is a torsion group (see [3] or [13]); after the theorem, it is also torsion-free, hence null.

§ **2.** In the previous paragraph we proved that in certain cases $\mathrm{Pic}(U_A)$ is torsion-free; as a natural question, we can ask if under the same hypotheses $\mathrm{Pic}(U_A)$ or better $\mathrm{Cl}(A)$ is finitely generated. If (R, m) is a local normal ring, we shall denote the strict henselisation of A by ${}^{sh}A$ (see [4] 18.8). We shall restrict our attention to the case of characteristic 0 and use the following

PROPOSITION (Danilov). *Let (A, m) be a local normal excellent k-algebra with $\mathrm{char}\, k = 0$ and satisfying the property (S_3) of Serre. Assume that for any prime ideal $\mathfrak{P}$ of height $2, A_{\mathfrak{P}}$ is either regular or a rational singularity.*
Then $\mathrm{Cl}({}^{sh}A)$ is a finitely generated group.

Proof. After [2] theorem 1 and remark 4 § 6 it is sufficient to show that A is a ring with DCG (i.e. such that $\mathrm{Cl}(A) \simeq \mathrm{Cl}(A[[T]])$, T an indeterminate). But this follows from [1] theorem 2 and proposition 7.

In our cases we get the following

COROLLARY. *Let (A, m) be a normal three-dimensional local ring which is a complete intersection in a "good" ring and equicharacteristic zero. Assume that A is an excellent k-algebra such that for any prime ideal of height $2, A$ is either regular or a rational singularity. Then $\mathrm{Cl}({}^{sh}A)$ (hence $\mathrm{Pic}(U_A)$) is a finitely generated group.*

Proof. It is sufficient to remark that A is a C-M ring hence it satisfies the property (S_3), and $\mathrm{Pic}(U_A)$ can be embedded in $\mathrm{Cl}(U_A)$ which is the same as $\mathrm{Cl}(A)$ since A is normal and $\dim A > 1$; the conclusion follows since the canonical maps $\mathrm{Pic}(U_A) \to \mathrm{Cl}(A) \to \mathrm{Cl}({}^{sh}A)$ are injective (see [3] corollary 6.11. p. 35).

We want now to produce an example which shows that not only in the above proposition but also in the corollary, if we drop the hypothesis that for any prime ideal $\mathfrak{P}$ of height $2, A_{\mathfrak{P}}$ is at worst a rational singularity, we can get a divisor class groups which is not finitely generated.

EXAMPLE. Let $A = k[x, y, z, u]_{(x,y,z,u)} \simeq k[X, Y, Z, U]_{(X,Y,Z,U)}/(F)$, where $F = X^2 + Y^3 + UZ^6$ and k is a field of characteristic 0, and denote the strict henselisation of A by B. It is immediate to see that the singular locus of A is $V(\mathfrak{P})$ where $\mathfrak{P} = (x, y, z)$ and this implies the normality of A by Serre's criterion.

Further we have that A is excellent and equicharacteristic zero. Let us suppose for contradiction, that $\mathrm{Cl}(B)$ is finitely generated; then $\mathrm{Cl}(A) \simeq \mathrm{Cl}(A[[T]])$ by [2] theorem 1, hence $\mathrm{Cl}(A_{\mathfrak{P}}) \simeq \mathrm{Cl}(A_{\mathfrak{P}}[[T]])$ by [1] proposition 4. But $A_{\mathfrak{P}} \simeq k(U)[X, Y, Z]_{(X,Y,Z)}/(F)$ which is a UFD since so is its completion with respect to the maximal ideal (see [11] prop. 5). On the other hand, a general statement found in [12] (see also [3] p. 117) shows that $A_{\mathfrak{P}}[[T]]$ is not factorial. In conclusion we get that $\mathrm{Cl}(B)$ is not a finitely generated group.

§ **3.** We want now to deduce some "geometric" consequences from the theorem of § 1; so we fix a field k of characteristic 0 (not necessarily algebraically closed) and consider projective spaces over k; we get the following

THEOREM. *Let X be a normal projective surface which is a complete intersection in a "good" projective variety V and C an irreducible reduced curve on X which is a set-theoretic complete intersection in X and locally defined by a single equation in X. Then C is actually a complete intersection of X with a suitable hypersurface of V.*

Proof. The curve C can be seen as a Cartiér divisor on X, hence we can speak about the invertible sheaf $\mathcal{O}_X(C)$ associated with C. Since X is normal and a complete intersection in a "good" projective variety, it is projectively normal, therefore it is sufficient to show that $\mathcal{O}_X(C) \simeq \mathcal{O}_X(n)$ for a suitable positive integer n.

Following the proof of [10], the key point is to show that $\mathrm{Pic}(X)/([\mathcal{O}_X(1)])$ is torsion-free ($([\mathcal{O}_X(1)])$ is the subgroup of $\mathrm{Pic}(X)$ generated by the class of the invertible sheaf $\mathcal{O}_X(1)$).

Let (A, m) be the local ring at the vertex of the affine cone associated to X, and let us consider the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \boldsymbol{Z} & \longrightarrow & \mathrm{Cl}\,(X) & \longrightarrow & \mathrm{Cl}\,(A) & \longrightarrow 0 \\ & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & \\ 0 \longrightarrow & \boldsymbol{Z} & \longrightarrow & \mathrm{Pic}\,(X) & \longrightarrow & \mathrm{Pic}\,(U_A) & \end{array}$$

where $\boldsymbol{Z}$ is the subgroup generated by $[\mathcal{O}_X(1)]$, i is the identity map, α, β and the horizontal arrows are the canonical ones.

Since X is a normal variety α is injective and since X is projectively normal β is injective too; moreover the first sequence is exact by a well-known result of Samuel (see [3] prop. 10.2). Hence the second sequence is exact, but using ii) of the main theorem, we get that $\mathrm{Pic}\,(U_A)$ is torsion-free and this completes the proof.

Given a normal surface S, Nonfact(S) will denote the set of (singular) points whose local ring does not verify the UFD property. We get the following

COROLLARY 1. *Let X be a three-dimensional non singular projective variety which is a complete intersection in a "good" projective variety; let $C \subset X$ be an irreducible reduced curve, which is not a complete intersection in X and suppose that there exist two normal surfaces F, G in X such that $F \cap G = C$ (set-theoretically).*

Then $C \cap$ Nonfact(F) and $C \cap$ Nonfact(G) are nonempty.

Proof. If A denotes the local ring at the vertex of the affine cone over X, applying the theorem of § 1 we get that $\mathrm{Pic}\,(U_A) = 0$, which implies $\mathrm{Cl}\,(A) = 0$ since X is non singular.

Therefore X is projectively factorial, hence any divisor on X is cut out by a suitable hypersurface; this implies that F and G satisfy the hypotheses of the theorem of § 3.

If, for contradiction, $C \cap$ Nonfact (F) is empty, C is actually a Cartiér divisor on F, which implies that C is a complete intersection in X.

The same reasoning works for G.

COROLLARY 2. *Let X be as in corollary 1 and $C \subset X$ an irreducible reduced curve which is a set-theoretic complete intersection but not a complete intersection.*

Then, either there exist in X two normal surfaces F, G non singular outside C, having (isolated) non factorial points on C and such that $C = F \cap G$, or there exist in X two surfaces F', G', non singular outside

C, both having C as a multiple curve and such that $C = F \cap G$.

Proof. In the proof we shall use the same notation for a surface and for its equation.

First of all we may assume that $C = F \cap G$ with F, G reduced and irreducible surfaces.

As we pointed out in the proof of corollary 1, X is a projectively factorial variety, whose degree we denote by d. Since for any surface H on X, $\mathcal{O}_X(H) \simeq \mathcal{O}_X(h)$, h a positive integer, we get $h \cdot d$ as the degree of H. Assume that C is simple for F and multiple for G; in this case we may replace G with $G' = G^\nu + LF$ (L a suitable surface such that $\deg L = \nu \deg(G) - \deg(F)$) in such a way that G' is reduced irreducible and C is simple for G'. Assume now that C is simple for both F and G, but that G has singular points outside C; again replace G with $G' = G^\nu + LF$ and Bertini's second theorem assures that in such a way we get a suitable G' with no multiple points outside C. Finally, if C is multiple for both F and G we apply again Bertini's second theorem and we get C as a set-theoretic complete intersection of two surfaces F', G' having C as a multiple curve but non singular outside C. Combining with corollary 1 we get the goal.

References

[1] Danilov, V. I.: Rings with a discrete group of divisor classes. Math. USSR-Sb. **12** (1970), p. 368–386.

[2] Danilov, V. I.: On rings with a discrete divisor class group. Math. USSR-Sb. **17** (1972), p. 228–236.

[3] Fossum, R. M.: The divisor class group of a Krull domain. Ergebnisse der Mathematik und ihrer Grenzgebiete Bd. 74 Berlin-Heidelberg-New York Springer 1973.

[4] Grothendieck, A. and Dieudonné, J.: Eléments de Géométrie Algébrique (EGA IV) Publ. Math. I.H.E.S. (1964–1967).

[5] Grothendieck, A.: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2) North-Holland Publ. Amsterdam (1962).

[6] Grothendieck, A.: Local cohomology. Lecture Notes in Math. 41 Springer (1967).

[7] Hartshorne, R. and Ogus, A.: On the factoriality of local rings of small embedding codimension. Communications in Algebra **1** (1974).

[8] Hartshorne, R.: Cohomological dimension of algebraic varieties. Annals of Math. **88** (1968), p. 403–450.

[9] Hartshorne, R.: Topological conditions for smoothing algebraic singularities. Topology **13** (1974), p. 241–253.

[10] Robbiano, L.: A problem of complete intersections. Nagoya Math. J. **52** (1973), p. 129–132.

[11] Salmon, P.: Su un problema posto da P. Samuel. Rend. Accad. Naz. Lincei Sez VIII vol. XL, fasc. **5** (1966), p. 801–803.
[12] Samuel, P.: On unique factorization domains. Illinois J. Math. **5** (1961), p. 1–17.
[13] Storch, U.: Fastfaktorielle Ringe. Schriftenreihe des Math. Inst. (Münster) Heft **36** (1967).

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