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BOUNDARY VALUE PROBLEMS OF BIHARMONIC FUNCTIONS

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1. Introduction

Let Ω be a bounded domain of n-dimensional Euclidean space $\boldsymbol{R}^n$ ($n \geq 2$). On Ω we consider the biharmonic equation

$$(1) \qquad \Delta^2 u = \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)^2 u = 0 .$$

A function u in $C^4(\Omega)$ is called biharmonic in Ω if it satisfies the equation (1). In this note we shall deal with the following boundary value problems. Find a biharmonic function u in Ω such that the following couples of functions have boundary values given on the boundary of Ω:

(a) $\dfrac{\partial u}{\partial n}, \quad \dfrac{\partial(\Delta u)}{\partial n};$

(b) $\Delta u, \quad \dfrac{\partial u}{\partial n};$

(c) $u, \quad \dfrac{\partial(\Delta u)}{\partial n}.$

J. L. Lions [4] treated these problems for the operator $\Delta^2 + I$ and gave solutions in case that Ω is a Nikodym domain. But in his method, the boundary of Ω or boundary functions are not referred to.

In this note we take as the boundary the Martin boundary M of Ω and define notations $\gamma_0(u)$ and $\gamma_1(u)$ for a function u on Ω as follows. If u has a fine boundary function f on M we denote f by $\gamma_0(u)$ and if u has φ, as generalized normal derivative of Doob [3] (in a slightly modified sense), we denote φ by $\gamma_1(u)$ (c.f. Definitions 1 and 2).

Now our boundary value problems are described as follows. Find a biharmonic function u in Ω such that the following couples of functions are equal to boundary functions given on the Martin boundary M:

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(a) $\gamma_1(u)$, $\gamma_1(\Delta u)$;
(b) $\gamma_0(\Delta u)$, $\gamma_1(u)$;
(c) $\gamma_0(u)$, $\gamma_1(\Delta u)$.

Let $K(x,\xi)$ be the Martin kernel and μ be the harmonic measure on M. Define new measures $\tilde{\mu}$ and $\tilde{\tilde{\mu}}$ on M by $d\tilde{\mu}(\xi) = k(\xi)d\mu(\xi)$ and $d\tilde{\tilde{\mu}}(\xi) = \frac{1}{k(\xi)}d\mu(\xi)$, where $k(\xi) = \int K(x,\xi)dx$.

Then we shall show that for any $\varphi \in L^2(\tilde{\tilde{\mu}})$ with $\int \varphi(\xi)d\mu(\xi) = 0$, there exists a square integrable harmonic function h on Ω with $D(h) < \infty$ such that $\gamma_1(h) = \varphi$ if and only if Ω is a Nikodym domain (Lemma 8). As an application of this fact we shall solve the above boundary value problems as follows.

Assume that Ω is a Nikodym domain, then

(a) for any φ and ψ in $L^2(\tilde{\tilde{\mu}})$ with $\int \psi(\xi)d\mu(\xi) = 0$ there exists a biharmonic function u such that $\gamma_1(u) = \varphi$ and $\gamma_1(\Delta u) = \psi$;

(b) for any $f \in L^2(\tilde{\mu})$ and $\varphi \in L^2(\tilde{\tilde{\mu}})$ with $\int \varphi(\xi)d\mu(\xi) = -\int H_f(x)dx$ there exists a biharmonic function u such that $\gamma_0(\Delta u) = f$ and $\gamma_1(u) = \varphi$;

(c) for any $f \in L^1(\mu)$ and $\varphi \in L^2(\tilde{\tilde{\mu}})$ with $\int \varphi(\xi)d\mu(\xi) = 0$ there exists a biharmonic function u such that $\gamma_0(u) = f$ and $\gamma_1(\Delta u) = \varphi$.

Moreover the uniqueness of the above solutions will be shown.

2. Preliminaries

Let Ω be an arbitrary bounded domain of the n-dimensional Euclidean space $R^n (n \geq 2)$ and $G(x,y)$ be it's Green function with respect to the equation $\Delta u = 0$, that is $(-\Delta_y)G(x,y) = \varepsilon_x$ in Ω.

We shall mention the definition of the Martin boundary of Ω.

We put

$$K(x,y) = \frac{G(x,y)}{G(x_0,y)}$$

on $\Omega \times \Omega$ if $y \neq x_0$ and $K(x,x_0) = 0$ if $x \neq x_0$ and $K(x_0,x_0) = 1$, where x_0 is a fixed reference point in Ω.

We take a fixed exhaustion $\{\Omega_n\}$ of Ω such that $x_0 \in \Omega_1$, and put

$$d(x_1, x_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x \in \Omega_n} \left| \frac{K(x, x_1)}{1 + K(x, x_1)} - \frac{K(x, x_2)}{1 + K(x, x_2)} \right| .$$

Then d defines a metric on Ω. We denote by Ω^* the completion of Ω by this metric. For a point $\xi \in \Omega^* - \Omega$, we can find a sequence $\{y_n\}$ in Ω such that $d(\xi, y_n) \to 0$ and so we can define

$$K(x, \xi) = \lim_{n \to \infty} K(x, y_n) .$$

We say that Ω^* is the Martin compactification of Ω and the set $M = \Omega^* - \Omega$ is called the Martin boundary of Ω. The function $K(x, \xi)$ on $\Omega \times \Omega^*$ is called the Martin kernel. We denote by μ the harmonic measure on M with respect to the fixed reference point x_0.

Now let $G_1(x, y)$ be the Green function of Ω with respect to the equation $(\Delta - 1)u = 0$, that is $(-\Delta_y + 1)G_1(x, y) = \varepsilon_x$ in Ω. For $x \in \Omega$ and $\xi \in M$, we put

$$K_1(x, \xi) = K(x, \xi) - \int G_1(x, y) K(y, \xi) dy . \tag{2}$$

We set for $f \in \boldsymbol{L}^1(\mu)$,

$$H_f(x) = \int K(x, \xi) f(\xi) d\mu(\xi) \tag{3}$$

and

$$H_f^1(x) = \int K_1(x, \xi) f(\xi) d\mu(\xi) . \tag{4}$$

Denote by $D(u)$ the Dirichlet integral of u on Ω. For measurable functions f and g on M, we put

$$\boldsymbol{D}(f, g) = \frac{1}{2} \int_M \int_M (f(\xi) - f(\eta))(g(\xi) - g(\eta)) \theta(\xi, \eta) d\mu(\xi) d\mu(\eta) \tag{5}$$

and $\boldsymbol{D}(f) = \boldsymbol{D}(f, f)$, where $\theta(\xi, \eta)$ is the Naim kernel (c.f. [7]).

The following lemma is obtained by Doob [3].

LEMMA 1. *If u is a harmonic function with $D(u) < \infty$, then u has a fine boundary function u' and $\boldsymbol{D}(u') = D(u)$. Conversely if f is an arbitrary measurable function on M with $\boldsymbol{D}(f) < \infty$, then $f \in \boldsymbol{L}^2(\mu)$ and $D(H_f) = \boldsymbol{D}(f)$.*

Put $k(\xi)=\int K(x,\xi)dx$, and $k(\xi)$ is a strictly positive lower semi-continuous function on M and so $\inf_{\xi\in M} k(\xi)=c>0$. Since

$$\int k(\xi)d\mu(\xi)=\int\left(\int K(x,\xi)d\mu(\xi)\right)dx=|\Omega| \qquad (\text{area of } \Omega)\,,$$

we see that $k(\xi)\in \boldsymbol{L}^1(\mu)$.

Define new measures $\tilde{\mu}$ and $\tilde{\tilde{\mu}}$ on M by $d\tilde{\mu}(\xi)=k(\xi)d\mu(\xi)$ and $d\tilde{\tilde{\mu}}(\xi)=\frac{1}{k(\xi)}d\mu(\xi)$ respectively, and we have the following relations

$$\boldsymbol{B}(M)\subset \boldsymbol{L}^2(\tilde{\mu})\subset \boldsymbol{L}^2(\mu)\subset \boldsymbol{L}^2(\tilde{\tilde{\mu}})\subset \boldsymbol{L}^1(\mu)\,, \tag{6}$$

where $\boldsymbol{B}(M)$ is the space of all bounded measurable functions on M. We also see that

$$\|f\|_{L^2(\tilde{\tilde{\mu}})}\le\frac{1}{\sqrt{c}}\|f\|_{L^2(\mu)}\le\frac{1}{c}\|f\|_{L^2(\tilde{\mu})} \tag{7}$$

for any $f\in \boldsymbol{L}^2(\tilde{\mu})$.

By the Fubini theorem, $\int H_{f^2}(x)dx<\infty$ for any $f\in \boldsymbol{L}^2(\tilde{\mu})$. Hence we know

$$\begin{aligned}\int H_{|f|}(x)H^1_{|g|}(x)dx&\le\int H_{|f|}(x)H_{|g|}(x)dx\\&\le\left(\int(H_{|f|}(x))^2dx\cdot\int(H_{|g|}(x))^2dx\right)^{1/2}\\&\le\left(\int H_{f^2}(x)dx\cdot\int H_{g^2}(x)dx\right)^{1/2}<\infty\end{aligned}$$

for any f and g in $\boldsymbol{L}^2(\tilde{\mu})$.

LEMMA 2. *Let f and g be in $\boldsymbol{L}^2(\tilde{\mu})$. Then*

$$\int H_f(x)H^1_g(x)dx=\int H_g(x)H^1_f(x)dx \tag{8}$$

and

$$\int H_f(x)H^1_f(x)dx\le\int(H_f(x))^2dx\le c'\cdot\int H_f(x)H^1_f(x)dx \tag{9}$$

for some constant $c'\ge 1$.

Proof. By the definition of $K_1(x,\xi)$ and the resolvent equation,

$$H_f^1(x) = H_f(x) - \int G_1(x, y)H_f(y)dy \tag{10}$$

and

$$H_f(x) = H_f^1(x) + \int G(x, y)H_f^1(y)dy \ . \tag{11}$$

Hence

$$\begin{aligned}\int H_g(x)H_f^1(x)dx &= \int H_g(x)\Big(H_f(x) - \int G_1(x, y)H_f(y)dy\Big)dx \\ &= \int H_g(x)H_f(x)dx - \int H_f(y)\Big(\int G_1(x, y)H_g(x)dx\Big)dy \\ &= \int H_g(x)H_f(x)dx - \int H_f(y)(H_g(y) - H_g^1(y))dy \\ &= \int H_f(x)H_g^1(x)dx\end{aligned}$$

and

$$\begin{aligned}\int (H_f(x))^2dx - \int H_f(x)H_f^1(x)dx &= \int H_f(x)(H_f(x) - H_f^1(x))dx \\ &= \int H_f(x)\Big(\int G_1(x, y)H_f(y)dy\Big)dx \\ &= \iint G_1(x, y)H_f(x)H_f(y)dxdy \geq 0 \ .\end{aligned} \tag{12}$$

By (11)

$$\int(H_f(x))^2dx - \int H_f(x)H_f^1(x)dx = \int H_f(x)\Big(\int G(x, y)H_f^1(y)dy\Big)dx$$

and hence

$$\begin{aligned}&\Big(\int (H_f(x))^2dx - \int H_f(x)H_f^1(x)dx\Big)^2 \\ &\quad \leq \int (H_f(x))^2dx\cdot\Big(\int\Big(\int G(x, y)dy\cdot\int G(x, y)(H_f^1(y))^2dy\Big)dx\Big) \\ &\quad \leq c_0^2\cdot\int (H_f(x))^2dx\cdot\int (H_f^1(x))^2dx\end{aligned}$$

where $c_0 = \sup_{x\in\Omega}\int G(x, y)dy$. Similarly to (12), we know

$$\int H_f(x)H_f^1(x)dx - \int (H_f^1(x))^2dx \geq 0 \ ,$$

and so we have an inequality

$$\int (H_f(x))^2 dx - \int H_f(x) H_f^1(x) dx$$
$$\leq c_0 \cdot \left(\int (H_f(x))^2 dx\right)^{1/2} \left(\int H_f(x) H_f^1(x) dx\right)^{1/2} .$$

Hence

$$\int (H_f(x))^2 dx \leq c' \cdot \int H_f(x) H_f^1(x) dx$$

for some constant $c' \geq 1$. This completes the proof.

Now we set

$$\tilde{\boldsymbol{H}}(M) = \{f\,;\, f \in \boldsymbol{L}^2(\tilde{\mu}) \text{ and } \boldsymbol{D}(f) < \infty\}\,, \tag{13}$$

and define two inner products on $\tilde{\boldsymbol{H}}(M)$ by

$$(f, g)_1 = \boldsymbol{D}(f, g) + \int H_f(x) H_g(x) dx \tag{14}$$

and

$$(f, g)_2 = \boldsymbol{D}(f, g) + \int H_f(x) H_g^1(x) dx \tag{15}$$

for functions f and g in $\tilde{\boldsymbol{H}}(M)$. By the above lemma, we know that $(\cdot, \cdot)_2$ is an inner product on $\tilde{\boldsymbol{H}}(M)$. We put $\|f\|_1^2 = (f, f)_1$ and $\|f\|_2^2 = (f, f)_2$ for $f \in \tilde{\boldsymbol{H}}(M)$. Then we have

LEMMA 3. *Norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent and* $\tilde{\boldsymbol{H}}(M)$ *is a Hilbert space with respect to these norms.*

Proof. By the above lemma,

$$\|f\|_2 \leq \|f\|_1 \leq (\max(1, c'))^{1/2} \|f\|_2\,, \tag{16}$$

and so these norms are equivalent. Let f be in $\tilde{\boldsymbol{H}}(M)$. Then by the Riesz decomposition of $-(H_f)^2$ we have

$$(H_f)^2 = H_{f^2} - \int G(\cdot, y) d\nu_f(y)\,.$$

Since $D(H_f) = \dfrac{1}{2}\displaystyle\int d\nu_f$, we have

$$
\begin{aligned}
(17) \qquad \|f\|^2_{L^2(\tilde{\mu})} &= \int H_{f^2}(x)dx \\
&= \int ((H_f(x))^2 + \int G(x, y)d\nu_f(y))dx \\
&\le \int (H_f(x))^2dx + c_0 \cdot \int d\nu_f \\
&\le \max(1, 2c_0)\Big(\int (H_f(x))^2dx + D(H_f)\Big) \\
&= \max(1, 2c_0)\Big(\int (H_f(x))^2dx + \boldsymbol{D}(f)\Big) \\
&= \max(1, 2c_0)\, \|f\|_1^2 .
\end{aligned}
$$

Hence we see that $\tilde{\boldsymbol{H}}(M)$ is a Hilbert space.

3. Definitions of $\gamma_0(u)$ and $\gamma_1(u)$ for a function u on Ω

We shall define $\gamma_0(u)$ and $\gamma_1(u)$ for a function u on Ω as follows.

DEFINITION 1. If a function u on Ω has a fine boundary function f on M, we denote f by $\gamma_0(u)$.

The definition of $\gamma_1(u)$ is a slight modification of the definition of the generalized normal derivative of u (c.f. Doob [3]).

DEFINITION 2. Consider the function $u(x) = H_f(x) + u_p(x)$, where f is a measurable function on M with $\boldsymbol{D}(f) < \infty$ and u_p is a potential of a measure ν on Ω. We assume that for any $g \in \boldsymbol{H}(M), H_g$ is integrable on Ω with respect to the absolute variation of ν. If there exists a function φ on M such that $\int \varphi(\xi)g(\xi)d\mu(\xi) < +\infty$ and

$$
(18) \qquad \boldsymbol{D}(f, g) = -\int \varphi(\xi)g(\xi)d\mu(\xi) + \int H_g(x)d\nu(x)
$$

for any $g \in \tilde{\boldsymbol{H}}(M)$, we denote φ by $\gamma_1(u)$.

We shall show the following

LEMMA 4. *Let φ be in $\boldsymbol{L}^2(\tilde{\mu})$. Then there exists a unique function $f \in \tilde{\boldsymbol{H}}(M)$ such that $\gamma_1(u) = \varphi$, where*

$$
u(x) = H_f(x) - \int G(x, y)H_f(y)dy .
$$

Proof. In the Hilbert space $\tilde{\boldsymbol{H}}(M)$ with the norm $\|\cdot\|_1$, the mapping

$g \to -\int g(\xi)\varphi(\xi)d\mu(\xi)$ is a linear functional. By the Schwarz inequality and (17), we have

$$\begin{aligned}\left|-\int g(\xi)\varphi(\xi)d\mu(\xi)\right|^2 &\le \left(\int |g(\xi)|\, k(\xi)^{1/2}\frac{1}{k(\xi)^{1/2}}|\varphi(\xi)|\, d\mu(\xi)\right)^2 \\ &\le \|\varphi\|^2_{L^2(\tilde{\mu})}\cdot\|g\|^2_{L^2(\bar{\mu})} \\ &\le \max(1, 2c_0)\,\|\varphi\|^2_{L^2(\tilde{\mu})}\cdot\|g\|_1^2 .\end{aligned}$$

Hence the above mapping is bounded on $\tilde{H}(M)$. Therefore there exists a unique function $f \in \tilde{H}(M)$ such that $(f, g)_1 = -\int \varphi(\xi)g(\xi)d\mu(\xi)$, namely

$$D(f, g) = -\int \varphi(\xi)g(\xi)d\mu(\xi) + \int H_g(x)(-H_f(x))dx$$

for any $g \in \tilde{H}(M)$. If we put $u(x) = H_f(x) - \int G(x, y)H_f(y)dy$, then from the definition we have $\gamma_1(u) = \varphi$.

Similarly we have

LEMMA 5. *Let φ be in $L^2(\bar{\mu})$. Then there exists a unique function $f \in \tilde{H}(M)$ such that $\gamma_1(H_f^1) = \varphi$.*

Proof. By Lemma 3, the mapping $g \to -\int g(\xi)\varphi(\xi)d\mu(\xi)$ is a bounded linear functional on the Hilbert space $\tilde{H}(M)$ with the norm $\|\cdot\|_2$. Hence there exists a unique function $f \in \tilde{H}(M)$ such that

$$D(f, g) = -\int \varphi(\xi)g(\xi)d\mu(\xi) + \int H_g(x)(-H_f^1(x))dx$$

for any $g \in \tilde{H}(M)$. Since $H_f^1(x) = H_f(x) - \int G(x, y)H_f^1(y)dy$, we have $\gamma_1(H_f^1) = \varphi$.

We set

$$\widehat{H(M)} = \{f \in \tilde{H}(M)\,;\ \text{there exists } \gamma_1(H_f) \in L^2(\bar{\mu})\} .$$

Then we have similarly to Folgesatz 17.27 in [1] and Theorem 6 in [6] the following

LEMMA 6. *$\widehat{H(M)}$ is dense in $\tilde{H}(M)$.*

Proof. Let f_0 be in $\tilde{H}(M)$ and $(f_0, g)_1 = 0$ for any $g \in \tilde{H}(M)$. Then we have

$$D(f_0, g) + \int H_{f_0}(x) H_g(x) dx = 0 . \tag{19}$$

Since f_0 is in $L^2(\tilde{\mu})$, by Lemma 4 there exists $f_0' \in \tilde{H}(M)$ such that

$$\gamma_1\Big(H_{f_0'} - \int G(\cdot, y) H_{f_0'}(y) dy\Big) = f_0 . \tag{20}$$

On the other hand

$$\gamma_1\Big(\int G(\cdot, y) H_{f_0'}(y) dy\Big) = \int K(x, \cdot) H_{f_0'}(x) dx$$

and

$$\Big\| \int K(x, \cdot) H_{f_0'}(x) dx \Big\|_{L^2(\tilde{\mu})} \leq \| f_0' \|_{L^2(\tilde{\mu})} < \infty .$$

Hence $\gamma_1(H_{f_0'}) \in L^2(\tilde{\mu})$ and f_0' is in $\widehat{H(M)}$. By (19), we have

$$D(f_0, f_0') + \int H_{f_0}(x) H_{f_0'}(x) dx = 0$$

and by (20),

$$D(f_0, f_0') = -\int f_0^2(\xi) d\mu(\xi) - \int H_{f_0}(x) H_{f_0'}(x) dx$$

therefore we know that $f_0 = 0$. This completes the proof.

4. Nikodym domain

In this section we shall treat the problem whether we are able to find $f \in \tilde{H}(M)$ such that $\gamma_1(H_f) = \varphi$ for any $\varphi \in L^2(\tilde{\mu})$ with $\int \varphi(\xi) d\mu(\xi) = 0$.

DEFINITION 3. (Deny-Lions [2]) We shall say that Ω is a Nikodym domain if every distribution T with $\frac{\partial}{\partial x_i} T \in L^2(\Omega)$ $(1 \leq i \leq n)$ is in $L^2(\Omega)$.

We set $\mathscr{E}^1_{L^2}(\Omega) = \Big\{u ;\ u \in L^2(\Omega) \text{ and } \frac{\partial}{\partial x_i} u \in L^2(\Omega)\ (1 \leq i \leq n)\Big\}$.

A necessary and sufficient condition for Ω to be a Nikodym domain is given by the following inequality of Poincaré: there exists a constant $P(\Omega)$ such that

$$\int (u(x))^2 dx - \frac{1}{|\Omega|}\left|\int u(x)\,dx\right|^2 \leq P(\Omega)D(u)$$

for any $u \in \mathscr{E}^1_{L^2}(\Omega)$ (c.f. [2]).

Deny-Lions [2] gives another characterization of a Nikodym domain by setting

$$N = \left\{ \begin{matrix} u \in \mathscr{E}^1_{L^2}(\Omega);\ \Delta u \in L^2(\Omega) \text{ and } (-\Delta u, v)_{L^2(\Omega)} = D(u, v) \\ \text{for any } v \in \mathscr{E}^1_{L^2}(\Omega) \end{matrix} \right\}.$$

LEMMA 7. (Deny-Lions) *For any* $F \in L^2(\Omega)$ *with* $\int F(x)dx = 0$ *we can find* u *in* N *(unique up to an additive constant) such that* $-\Delta u = F$ *if and only if* Ω *is a Nikodym domain.*

The following lemma gives an answer to our above problem and it gives a characterization of a Nikodym domain.

LEMMA 8. *For any* $\varphi \in L^2(\tilde{\mu})$ *with* $\int \varphi(\xi)d\mu(\xi) = 0$ *we can find* f *in* $\tilde{H}(M)$ *(unique up to an additive constant) such that* $\gamma_1(H_f) = \varphi$ *if and only if* Ω *is a Nikodym domain.*

Proof. Assume that Ω is a Nikodym domain. Let φ be in $L^2(\tilde{\mu})$ with $\int \varphi(\xi)d\mu(\xi) = 0$. Then by Lemma 4 there exists a unique function $f_0 \in \tilde{H}(M)$ such that

$$\gamma_1\left(H_{f_0} - \int G(\cdot, y)H_{f_0}(y)dy\right) = \varphi\,.$$

Hence

$$D(f_0, g) = -\int \varphi(\xi)g(\xi)d\mu(\xi) + \int H_g(x)(-H_{f_0}(x))dx \tag{21}$$

for any $g \in \tilde{H}(M)$. We put $g = 1$ in (21), then $\int H_{f_0}(x)dx = 0$ from the condition $\int \varphi(\xi)d\mu(\xi) = 0$.

Since f_0 is in $\tilde{H}(M)$, $H_{f_0} \in L^2(\Omega)$ and $D(H_{f_0}) = D(f_0) < \infty$. Therefore by Lemma 7, we can find u in N (unique up to an additive constant) such that $-\Delta u = H_{f_0}$. Hence we know that $\Delta^2 u = 0$, $u \in L^2(\Omega)$ and $D(u) < \infty$ and so by the uniqueness of the Royden decomposition of u, we have

$$u(x) = h(x) - \int G(x, y)\Delta u(y)dy$$
$$= h(x) + \int G(x, y)H_{f_0}(y)dy$$

for some harmonic function $h \in \boldsymbol{L}^2(\Omega)$ with $D(h) < \infty$. From (17), h has a fine boundary function h' in $\boldsymbol{L}^2(\tilde{\mu})$ and so $h = H_{h'}$ with $h' \in \tilde{\boldsymbol{H}}(M)$.

Since u is in $\boldsymbol{N}$ and $\{H_g;\ g \in \tilde{\boldsymbol{H}}(M)\} \subset \mathscr{E}^1_{L^2}(\Omega)$, we have

$$\int H_g(x)(-\Delta u(x))dx = D(u, H_g)$$

for any $g \in \tilde{\boldsymbol{H}}(M)$. Hence we have

$$\begin{aligned}\boldsymbol{D}(h', g) &- \int H_g(x)H_{f_0}(x)dx \\ &= D(h, H_g) - \int H_g(x)(-\Delta u(x))dx \\ &= D(h, H_g) - D(u, H_g) \\ &= D(h - u, H_g) \\ &= D\left(\int G(\cdot\,, y)\Delta u(y)dy, H_g\right) = 0\end{aligned}$$

for any $g \in \tilde{\boldsymbol{H}}(M)$ and so $\gamma_1(u) = 0$.

Now we put $f = f_0 + h'$, then f is determined (uniquely up to an additive constant) in $\tilde{\boldsymbol{H}}(M)$ and we have

$$\begin{aligned}\gamma_1(H_f) &= \gamma_1(H_{f_0} + h) \\ &= \gamma_1\left(H_{f_0} - \int G(\cdot\,, y)H_{f_0}(y)dy + u\right) \\ &= \varphi\ .\end{aligned}$$

Conversely assume that for any $\varphi \in \boldsymbol{L}^2(\tilde{\mu})$ with $\int \varphi(\xi)d\mu(\xi) = 0$ we can find f in $\tilde{\boldsymbol{H}}(M)$ such that $\gamma_1(H_f) = \varphi$. We shall show that for any $v \in \boldsymbol{L}^2(\Omega)$ with $\int v(x)dx = 0$, we can find u in $\boldsymbol{N}$ (unique up to an additive constant) such that $-\Delta u = v$. Then by Lemma 7 we conclude that Ω is a Nikodym domain. Let v be in $\boldsymbol{L}^2(\Omega)$ with $\int v(x)dx = 0$. Since

$$\int |v(x)|\cdot|H_g(x)|\, dx < \infty$$

for any $g \in \tilde{\boldsymbol{H}}(M)$, we know

$$\gamma_1\Big(-\int G(\cdot\,, y)v(y)dy\Big) = -\int K(x, \cdot)v(x)dx\ .$$

Put $\varphi_v = \gamma_1\Big(-\int G(\cdot\,, y)v(y)dy\Big)$, and we know

$$\begin{aligned}\int \varphi_v^2(\xi)d\tilde{\mu}(\xi) &= \int \frac{1}{k(\xi)}\varphi_v^2(\xi)d\mu(\xi)\\ &\leq \int \frac{1}{k(\xi)}\Big(\int K(x, \xi)dx\cdot\int K(x, \xi)v^2(x)dx\Big)d\mu(\xi)\\ &= \|v\|^2_{L^2(\Omega)} < \infty\end{aligned}$$

and

$$\int \varphi_v(\xi)d\mu(\xi) = \int\Big(-\int K(x, \xi)v(x)dx\Big)d\mu(\xi) = -\int v(x)dx = 0\ .$$

Hence we can find f in $\tilde{H}(M)$ (unique up to an additive constant) such that $\gamma_1(H_f) = \varphi_v$. We put

$$u(x) = H_f(x) + \int G(x, y)v(y)dy$$

thus u is determined (uniquely up to an additive constant) in $\mathscr{E}^1_{L^2}(\Omega)$, $-\Delta u = v$ and $\Delta u \in L^2(\Omega)$.

Now we shall show that u is in N, that is $D(u, w) = (-\Delta u, w)_{L^2(\Omega)}$ for any w in $\mathscr{E}^1_{L^2}(\Omega)$.

We have the following decomposition of $\mathscr{E}^1_{L^2}(\Omega)$:

$$\mathscr{E}^1_{L^2}(\Omega) = \{H^1_g;\ g \in \tilde{H}(M)\} \oplus L^2D_0(\Omega)\ ,$$

where $L^2D_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $D(\cdot)$ $+ \|\cdot\|_{L^2(\Omega)}$. In case $w = H^1_g$ for some $g \in \tilde{H}(M)$, we have

$$\begin{aligned}D(u, w) &= D(u, H^1_g)\\ &= D(H_f, H_g) - D\Big(\int G(\cdot\,, y)v(y)dy, \int G(\cdot\,, y)H^1_g(y)dy\Big)\\ &= D(f, g) - \int v(x)\Big(\int G(x, y)H^1_g(y)dy\Big)dx\ .\end{aligned}$$

Since $\gamma_1(u) = \gamma_1(H_f) + \int K(x, \cdot)v(x)dx = \varphi_v - \varphi_v = 0$, we know

$$D(f, g) = \int v(x)H_g(x)dx$$

for any $g \in \tilde{H}(M)$. Hence we have

$$D(u, H_g^1) = \int v(x)\Big(H_g(x) - \int G(x, y)H_g^1(y)dy\Big)dx$$
$$= -\int \Delta u(x)H_g^1(x)dx .$$

In case w is in $C_0^\infty(\Omega)$ we know that

$$w(x) = \int G(x, y)(-\Delta w(y))dy .$$

Hence

$$D(u, w) = D\Big(\int G(\cdot, y)v(y)dy, \int G(\cdot, y)(-\Delta w(y))dy\Big)$$
$$= \int v(x)\Big(\int G(x, y)(-\Delta w(y))dy\Big)dx$$
$$= -\int \Delta u(x)w(x)dx .$$

For any w in $L^2D_0(\Omega)$, we can find a sequence $\{w_n\}$ in $C_0^\infty(\Omega)$ such that $w_n \to w$ in $L^2D_0(\Omega)$. Since $D(u, w_n) = -\int \Delta u(x)w_n(x)dx$, letting $n \to \infty$, we have $D(u, w) = -\int \Delta u(x)w(x)dx$. Therefore we know

$$D(u, w) = (-\Delta u, w)_{L^2(\Omega)}$$

for any $w \in \mathscr{E}^1_{L^2}(\Omega)$ and so u is in N. This completes the proof.

5. Boundary value problems

In this section we shall solve the boundary value problems described in section 1 as an application of Lemma 8. We put

$$\mathscr{S}_1 = \{u \in C^4(\Omega)\,;\ u \text{ and } \Delta u \text{ are in } \mathscr{E}^1_{L^2}(\Omega)\} ,$$
$$\mathscr{S}_2 = \{u \in C^4(\Omega)\,;\ u \text{ is in } \mathscr{E}^1_{L^2}(\Omega) \text{ and } \Delta u \text{ is in } L^2(\Omega)\}$$

and

$$\mathscr{S}_3 = \{u \in C^4(\Omega)\,;\ \Delta u \text{ is in } \mathscr{E}^1_{L^2}(\Omega)\} .$$

Then we shall show

THEOREM. *Assume that Ω is a Nikodym domain, then*

(a) *for any φ and ψ in $L^2(\tilde{\mu})$ with $\int \psi(\xi)d\mu(\xi) = 0$, there exists u in*

$\mathscr{S}_1$ *unique up to an additive constant such that* $\Delta^2 u = 0$, $\gamma_1(u) = \varphi$ *and* $\gamma_1(\Delta u) = \psi$;

(b) *for any* f *in* $L^2(\tilde{\mu})$ *and* φ *in* $L^2(\tilde{\tilde{\mu}})$ *with*

$$\int \varphi(\xi)d\mu(\xi) = -\int H_f(x)dx \,, \tag{22}$$

there exists u *in* $\mathscr{S}_2$ *unique up to an additive constant such that* $\Delta^2 u = 0$, $\gamma_0(\Delta u) = f$ *and* $\gamma_1(u) = \varphi$;

(c) *for any* f *in* $L^1(\mu)$ *and* φ *in* $L^2(\tilde{\tilde{\mu}})$ *with* $\int \varphi(\xi)d\mu(\xi) = 0$, *there exists* u *in* $\mathscr{S}_3$ *such that* $\Delta^2 u = 0$, $\gamma_0(u) = f$ *and* $\gamma_1(\Delta u) = \varphi$.

Proof. (a) For any φ and ψ in $L^2(\tilde{\tilde{\mu}})$ with $\int \psi(\xi)d\mu(\xi) = 0$, by Lemma 8 there exists f in $\tilde{H}(M)$ such that $\gamma_1(H_f) = \psi$ and

$$\int \left(\varphi(\xi) + \int K(x, \xi)H_f(x)dx\right)d\mu(\xi) = 0 \,. \tag{23}$$

Since $\varphi + \int K(x, \cdot)H_f(x)dx$ is in $L^2(\tilde{\tilde{\mu}})$ and (23), there exists f_0 in $\tilde{H}(M)$ such that $\gamma_1(H_{f_0}) = \varphi + \int K(x, \cdot)H_f(x)dx$.

We put

$$u(x) = H_{f_0}(x) - \int G(x, y)H_f(y)dy \,.$$

Then we know that u is in $\mathscr{S}_1$, $\Delta^2 u = 0$, $\gamma_1(u) = \varphi$ and $\gamma_1(\Delta u) = \psi$.

Next we shall show the uniqueness of the solution. Let w be in $\mathscr{S}_1$ such that $\Delta^2 w = 0$, $\gamma_1(w) = 0$ and $\gamma_1(\Delta w) = 0$. By the uniqueness of the Royden decomposition of w, there exists f_w and g_w in $\tilde{H}(M)$ such that

$$w = H_{f_w} - \int G(\cdot, y)\Delta w(y)dy$$

and $\Delta w = H_{g_w}$. Since $\gamma_1(w) = 0$, we have

$$D(H_{f_w}, H_g) + \int \Delta w(x)H_g(x)dx = 0 \tag{24}$$

for any g in $\tilde{H}(M)$. Hence

$$
\begin{aligned}
(25)\qquad D(w,w) &= D(H_{f_w}, H_{f_w}) + \iint G(x,y)\Delta w(x)\Delta w(x)dxdy \\
&= -\int \Delta w(x)H_{f_w}(x)dx + \int \Delta w(x)\Big(\int G(x,y)\Delta w(y)dy\Big)dx \\
&= -\int \Delta w(x)w(x)dx\ .
\end{aligned}
$$

Since $\gamma_1(\Delta w) = 0$, we have

$$
(26)\qquad D(\Delta w, H_g) = 0
$$

for any g in $\tilde{\boldsymbol{H}}(M)$. We put $g = g_w$ in (24) and $g = f_w$ in (26), then we know that $\Delta w = 0$ and so $w =$ constant by (25).

(b) First we shall remark that the condition (22) is necessary for the existence of the solution. Let u be a solution, then

$$
u(x) = H_{f_u}(x) - \int G(x,y)\Delta u(y)dy
$$

for some $f_u \in \tilde{\boldsymbol{H}}(M)$. Since $\gamma_0(\Delta u) = f$ and $\gamma_1(u) = \varphi$, we know $\Delta u = H_f$ and

$$
(27)\qquad D(H_{f_u}, H_g) = -\int \varphi(\xi)g(\xi)d\mu(\xi) + \int H_g(x)(-\Delta u(x))dx
$$

for any $g \in \tilde{\boldsymbol{H}}(M)$. Put $g = 1$ in (27) and we have (22).

For any f in $\boldsymbol{L}^2(\tilde{\mu})$ and φ in $\boldsymbol{L}^2(\tilde{\tilde{\mu}})$ we know that $\int K(x,\cdot)H_f(x)dx$ is in $\boldsymbol{L}^2(\tilde{\tilde{\mu}})$ and by (22)

$$
\int \Big(\varphi(\xi) + \int K(x,\xi)H_f(x)dx\Big)d\mu(\xi) = 0\ .
$$

Hence there exists f_0 in $\tilde{\boldsymbol{H}}(M)$ such that

$$
\gamma_1(H_{f_0}) = \varphi + \int K(x,\cdot)H_f(x)dx\ .
$$

We put

$$
u(x) = H_{f_0}(x) - \int G(x,y)H_f(y)dy\ .
$$

Then u is in $\mathscr{S}_2$, $\Delta^2 u = 0$, $\gamma_0(\Delta u) = f$ and $\gamma_1(u) = \varphi$.

The uniqueness of the solution is shown in a similar manner to (a). Let w be in $\mathscr{S}_2$ such that $\Delta^2 w = 0$, $\gamma_0(\Delta w) = 0$ and $\gamma_1(w) = 0$, then we have

$$D(w, w) + \int \Delta w(x)w(x)dx = 0 .$$

Since Δw is harmonic and $\gamma_0(\Delta w) = 0$, we know $\Delta w = 0$ and so $w =$ constant.

(c) Put

$$u(x) = H_f(x) - \int G(x, y)H_{f_0}(y)dy ,$$

where f_0 is in $\tilde{H}(M)$ such that $\gamma_1(H_{f_0}) = \varphi$, and u is the desired solution. This completes the proof.

Remark 1. In the case of (c) the uniqueness of the solution is interpreted as follows. If u_0 is a solution of (c), then every solution is given by $u_0 + a\int G(\cdot, y)dy$, where a is some constant.

In fact if w is in $\mathscr{S}_3$, $\Delta^2 w = 0$, $\gamma_0(w) = 0$ and $\gamma_1(\Delta w) = 0$, then $h(x) = w(x) + \int G(x, y)\Delta w(y)dy$ is harmonic and $\gamma_0(h) = 0$. Hence we have $w(x) = -\int G(x, y)\Delta w(y)dy$. Since $\gamma_1(\Delta w) = 0$, we know $w(x) = a\int G(x, y)dy$ for some constant a.

Remark 2. Lemma 8 asserts that if one of the above boundary value problems has always a solution, then Ω is necessarily a Nikodym domain. Hence the above problems are solved if and only if Ω is a Nikodym domain.

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