N. Kenmochi and Y. Mizuta
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POTENTIAL THEORETIC PROPERTIES OF THE GRADIENT OF A CONVEX FUNCTION ON A FUNCTIONAL SPACE

NOBUYUKI KENMOCHI AND YOSHIHIRO MIZUTA

Introduction

In the previous paper [11], introducing the notions of potentials and of capacity associated with a convex function Φ given on a regular functional space $\mathfrak{X}(X;\xi)$, we discussed potential theoretic properties of the gradient $\nabla\Phi$ which were originally introduced and studied by Calvert [5] for a class of nonlinear monotone operators in Sobolev spaces. For example:

(i) The modulus contraction operates.

(ii) The principle of lower envelope holds.

(iii) The domination principle holds.

(iv) The contraction T_k onto the real interval $[0,k]$ $(k>0)$ operates.

(v) The strong principle of lower envelope holds.

(vi) The complete maximum principle holds.

In this paper we shall investigate relations among the properties mentioned above. For this purpose, we shall consider an operator A from a subset of $L^2(X;\xi)\cap\mathfrak{X}(X;\xi)$ into $L^2(X;\xi)$ associated with $\nabla\Phi$ and its resolvent $R_\lambda=(I+\lambda A)^{-1}$, $\lambda>0$.

One aim is to show (in Theorem A) that each of properties (i) ~ (iii) is equivalent to:

(vii) for any $\lambda>0$, R_λ is order-preserving in $L^2(X;\xi)$. Another aim is to show (in Theorem B) that if (i) is satisfied, then each of properties (iv) ~ (vi) holds if and only if

(viii) $R_\lambda(f+T_kg)\le R_\lambda f+k$ holds for any $\lambda>0$ and any $f,g\in L^2(X;\xi)$.

These assertions are nonlinear analogues of results in the Dirichlet space (cf. Deny [7; Théorèmes 1 et 2] and Itô [8; Theorems 3 and 5]). The crucial step in the proofs is to deduce both (i) and (iv) from (vii)

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and (viii). The key tool is the fact that if $f \in L^2(X;\xi)$ and if

$$\left\{\int_X \lambda^{-1}(f - R_\lambda f) f d\xi \,;\, \lambda > 0\right\}$$

is bounded, then $f \in \mathfrak{X}(X;\xi)$. This fact is a nonlinear extension of a result in [7]. At the end of this paper we shall consider the nonlinear contraction semigroup $S = \{S(t)\,;\, t > 0\}$ on $L^2(X;\xi)$ generated by $-A$ and show (in Theorem C) that each of properties (i) $\sim$ (iii) and (vii) is necessary and sufficient for S to be order-preserving in $L^2(X;\xi)$.

§ 1. Preliminaries

Let X be a locally compact Hausdorff space with a countable base and ξ be a positive (Radon) measure on X. Let $\mathfrak{X} = \mathfrak{X}(X;\xi)$ be a real reflexive Banach space whose elements are real-valued locally ξ-summable functions defined ξ-a.e. on X. We denote by $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, by $\|u\|$ (resp. $\|u^*\|$) the norm of $u \in \mathfrak{X}$ (resp. $u^* \in \mathfrak{X}^*$) and by $\langle u^*, u\rangle$ the value of $u^* \in \mathfrak{X}^*$ at $u \in \mathfrak{X}$. By "$\xrightarrow{s}$" (resp. "$\xrightarrow{w}$") we mean the strong (resp. weak) convergence. For functions $u, v \in L^1_{loc}(X;\xi)$, we write $u \vee v$, $u \wedge v$, u^+ and u^- for $\max(u, v)$, $\min(u, v)$, $\max(u, 0)$ and $-\min(u, 0)$, respectively, and for a ξ-measurable set S in X we write simply "$u \geq v$ (resp. $u = v$) on S" for "$u \geq v$ (resp. $u = v$) ξ-a.e. on S". Especially we write "$u \geq v$ (resp. $u = v$)" for "$u \geq v$ (resp. $u = v$) on X".

Throughout this paper, let $1 < p < \infty$ and Φ be a strictly convex function on $\mathfrak{X}$ such that

$$\text{(1.1)} \qquad \begin{cases} \Phi(0) = 0\,, \\ \Phi(u) \geq C\|u\|^p\,, \qquad \text{for any } u \in \mathfrak{X}, \end{cases}$$

where C is a positive constant. Suppose that Φ is bounded on each bounded subset of $\mathfrak{X}$ and is everywhere differentiable in the sense of Gâteaux, that is, there is an operator $G: \mathfrak{X} \to \mathfrak{X}^*$ such that for any u, $v \in \mathfrak{X}$,

$$\langle Gu, v\rangle = \lim_{t \downarrow 0} \frac{\Phi(u + tv) - \Phi(u)}{t}\,.$$

This operator G is called the gradient of Φ and denoted by $\nabla\Phi$. We shall use the following properties of Φ and $\nabla\Phi$ without proof:

(Φ_1) Φ is weakly sequentially lower semicontinuous in $\mathfrak{X}$.

(Φ_2) Let $u \in \mathfrak{X}$ and $u^* \in \mathfrak{X}^*$. Then $u^* = \nabla\Phi(u)$ if and only if

$$\langle u^*, v - u\rangle \le \Phi(v) - \Phi(u) \qquad \text{for any } v \in \mathfrak{X} .$$

(Φ_3) $\nabla\Phi$ is strictly monotone, i.e.,

$$\langle \nabla\Phi(u) - \nabla\Phi(v), u - v\rangle > 0 \qquad \text{for any } u, v \in \mathfrak{X}, u \neq v .$$

(Φ_4) $\nabla\Phi$ is bounded, i.e., it maps bounded sets in $\mathfrak{X}$ to bounded sets in $\mathfrak{X}^*$.

(Φ_5) $\nabla\Phi$ is demicontinuous, i.e., if $u_n \xrightarrow{s} u$ in $\mathfrak{X}$ as $n \to \infty$, then $\nabla\Phi(u_n) \xrightarrow{w} \nabla\Phi(u)$ in $\mathfrak{X}^*$ as $n \to \infty$.

(Φ_6) For each $u \in \mathfrak{X}$, $\langle \nabla\Phi(v), v - u\rangle/\|v\| \to \infty$ as $\|v\| \to \infty$.

(Φ_7) For any $u, v \in \mathfrak{X}$,

$$\Phi(u) - \Phi(v) = \int_0^1 \langle \nabla\Phi(v + t(u - v)), u - v\rangle dt .$$

Remark. $\nabla\Phi$ is one to one and onto. For a proof, see Browder [2; Theorem 3].

LEMMA 1.1. *Let $\{u_n\}$ be a sequence in $\mathfrak{X}$ which converges weakly to u in $\mathfrak{X}$. If $\limsup_{n\to\infty} \langle \nabla\Phi(u_n), u_n - u\rangle \le 0$, then $\nabla\Phi(u_n) \xrightarrow{w} \nabla\Phi(u)$ in $\mathfrak{X}^*$ and $\Phi(u_n) \to \Phi(u)$ as $n \to \infty$.*

Proof. From (Φ_2) and our assumption it follows that

$$0 \ge \limsup_{n\to\infty} \langle \nabla\Phi(u_n), u_n - u\rangle \ge \limsup_{n\to\infty} \Phi(u_n) - \Phi(u) .$$

On account of (Φ_1) we obtain

$$\lim_{n\to\infty} \Phi(u_n) = \Phi(u) . \tag{1.2}$$

Next, by (Φ_3), $\liminf_{n\to\infty} \langle \nabla\Phi(u_n), n_n - u\rangle \ge \lim_{n\to\infty} \langle \nabla\Phi(u), u_n - u\rangle = 0$. Hence

$$\lim_{n\to\infty} \langle \nabla\Phi(u_n), u_n - u\rangle = 0 . \tag{1.3}$$

The sequence $\{\nabla\Phi(u_n)\}$ is weakly relatively compact in $\mathfrak{X}^*$, since it is bounded in $\mathfrak{X}^*$ on account of (Φ_4). Now, let $\{u_{n_j}\}$ be any subsequence of $\{u_n\}$ such that $\nabla\Phi(u_{n_j}) \xrightarrow{w} u^*$ in $\mathfrak{X}^*$ as $j \to \infty$ for some $u^* \in \mathfrak{X}^*$. Then, using (Φ_2), (1.2) and (1.3), we see that for any $v \in \mathfrak{X}$

$$\begin{aligned}\langle u^*, v-u\rangle &= \lim_{j\to\infty} \langle \nabla\Phi(u_{n_j}), v-u_{n_j}\rangle \\ &\le \lim_{j\to\infty} \{\Phi(v)-\Phi(u_{n_j})\} \\ &= \Phi(v)-\Phi(u)\ ,\end{aligned}$$

which implies that $u^* = \nabla\Phi(u)$, and simultaneously that $\nabla\Phi(u_n) \xrightarrow{w} \nabla\Phi(u)$ in $\mathfrak{X}^*$ as $n\to\infty$. q.e.d,

DEFINITION 1.1. (cf. [7], [8]) $\mathfrak{X} = \mathfrak{X}(X;\xi)$ is called a functional space if the following axiom is satisfied:

AXIOM (a) For each compact subset K of X, there is a constant $M(K) > 0$ such that

$$\int_K |u|\, d\xi \le M(K)\, \|u\| \qquad \text{for all } u \in \mathfrak{X}\ .$$

Henceforth let $\mathfrak{X}$ be a functional space satisfying the following axiom:

AXIOM (b′) $L^2 \cap \mathfrak{X}$ is dense both in L^2 and in $\mathfrak{X}$, where $L^2 = L^2(X;\xi)$.

We consider an operator A from a subset of L^2 into L^2 associated with $\nabla\Phi$ which is defined as follows: If v is a function in $L^2\cap\mathfrak{X}$ and f is a function in L^2 such that

$$\text{(1.4)} \qquad \langle \nabla\Phi(v), w\rangle = \int_X wf\,d\xi \qquad \text{for any } w \in L^2\cap\mathfrak{X}\ ,$$

then we put $Av = f$. By Axiom (b′), such an f is uniquely determined by v. Thus we can define an operator $A: D_0 \to L^2$, where $D_0 = \{v \in L^2\cap\mathfrak{X};$ there is a function $f\in L^2$ such that (1.4) holds$\}$. By (Φ_3), A is strictly monotone, i.e.,

$$\text{(1.5)} \qquad (Au-Av, u-v) > 0 \qquad \text{for any } u, v \in D_0, u \ne v\ ,$$

where $(\cdot,\cdot)$ denotes the inner product in L^2, i.e., $(v,w) = \int_X vw\,d\xi$ for any $v, w \in L^2$.

§ 2. The resolvent of A

In order to show that the resolvent of A exists, we prove

LEMMA 2.1. *Given $\lambda > 0$ and $f\in L^2$, we find a unique function $u \in L^2\cap\mathfrak{X}$ such that*

$$(2.1) \qquad 2\lambda\Phi(u) + \|u - f\|_2^2 = \inf\{2\lambda\Phi(v) + \|v - f\|_2^2;\, v \in L^2 \cap \mathfrak{X}\}\,,$$

where by $\|v\|_2$ *we mean the norm of* v *in* L^2.

Proof. Set $F(v) = 2\lambda\Phi(v) + \|v - f\|_2^2$ for $v \in L^2 \cap \mathfrak{X}$ and $\alpha = \inf\{F(v);\, v \in L^2 \cap \mathfrak{X}\}$. Then, clearly, $0 \le \alpha < \infty$. We find a sequence $\{v_n\} \subset L^2 \cap \mathfrak{X}$ such that $F(v_n) \downarrow \alpha$ as $n \to \infty$. Since $\{v_n\}$ is bounded in $\mathfrak{X}$ by (1.1) as well as L^2, it is weakly relatively compact both in $\mathfrak{X}$ and in L^2 and hence, by Axiom (a), there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that $v_{n_j} \xrightarrow{w} u$ both in $\mathfrak{X}$ and in L^2 for some $u \in L^2 \cap \mathfrak{X}$. Noting ($\Phi_1$) and the weak lower semicontinuity of the functional: $v \to \|v - f\|_2^2$ relative to the topology of L^2, we have

$$\Phi(u) \le \liminf_{j\to\infty} \Phi(v_{n_j})$$

and

$$\|u - f\|_2^2 \le \liminf_{j\to\infty} \|v_{n_j} - f\|_2^2\,,$$

so that $F(u) \le \lim_{j\to\infty} F(v_{n_j}) = \alpha$. This implies that $F(u) = \alpha$. The uniqueness of such a u follows from the strict convexity of Φ. q.e.d.

For any $\lambda > 0$, we denote by R_λ the operator from L^2 into L^2 which assigns the function u satisfying (2.1) to each $f \in L^2$.

LEMMA 2.2. *Let* λ *be any positive number. Then:*

(i) $\Phi(R_\lambda f) \le \Phi(f)$ *for any* $f \in L^2 \cap \mathfrak{X}$.

(ii) $\langle \nabla\Phi(R_\lambda f), v\rangle = (f - R_\lambda f, v)/\lambda$ *for any* $f \in L^2$ *and* $v \in L^2 \cap \mathfrak{X}$.

(iii) $R_\lambda f \in D_0$ *and* $AR_\lambda f = (f - R_\lambda f)/\lambda$ *for any* $f \in L^2$.

(iv) $R(I + \lambda A)$ *(the range of* $I + \lambda A$*)* $= L^2$ *and* $R_\lambda = (I + \lambda A)^{-1}$.

(v) $\|R_\lambda f - R_\lambda g\|_2 \le \|f - g\|_2$ *for any* $f, g \in L^2$, *especially* $\|R_\lambda f\|_2 \le \|f\|_2$ *for any* $f \in L^2$.

Proof. (i) is clear. For any $f \in L^2$, $v \in L^2 \cap \mathfrak{X}$ and $t > 0$, setting $u = R_\lambda f$, we observe that

$$2\lambda\Phi(u + tv) + \|u + tv - f\|_2^2 \ge 2\lambda\Phi(u) + \|u - f\|_2^2\,,$$

that is

$$\frac{1}{t}\{\Phi(u + tv) - \Phi(u)\} \ge \frac{1}{2\lambda}\frac{1}{t}\{\|u - f\|_2^2 - \|u + tv - f\|_2^2\}\,.$$

Letting $t \downarrow 0$ in this inequality, we obtain

$$\langle \nabla\Phi(u), v\rangle \geq \frac{1}{\lambda}(f-u, v) .$$

Hence we have (ii) and at the same time (iii). Besides, we see that

$$f = R_\lambda f + \lambda A R_\lambda f = (I+\lambda A)R_\lambda f \qquad \text{for any } f \in L^2 .$$

From this we infer (iv), since $(I+\lambda A)$ is one to one by (1.5). Finally we shall show (v). Let f and g be any functions in L^2, and set $u = R_\lambda f$ and $v = R_\lambda g$. Then, as was seen above, $f = u + \lambda Au$ and $g = v + \lambda Av$. From (1.5) it follows that

$$\begin{aligned} \|f-g\|_2 \|u-v\|_2 &\geq (f-g, u-v) \\ &= (u+\lambda Au - v - \lambda Av, u-v) \\ &= \|u-v\|_2^2 + \lambda(Au-Av, u-v) \\ &\geq \|u-v\|_2^2 , \end{aligned}$$

so we have (v). q.e.d.

The fact (iv) of Lemma 2.2 says that R_λ is the (nonlinear) resolvent (at λ) of A for each $\lambda > 0$.

LEMMA 2.3 (cf. [9; Lemma 4.3]). *If* $v \in D_0$, *then* $\|AR_\lambda v\|_2 \leq \|Av\|_2$ *for any* $\lambda > 0$.

Proof. We have $v = R_\lambda(I+\lambda A)v = R_\lambda(v+\lambda Av)$, since $R_\lambda = (I+\lambda A)^{-1}$ by (iv) of Lemma 2.2. From (iii) and (v) of Lemma 2.2 it follows that

$$\|AR_\lambda v\|_2 = \frac{1}{\lambda}\|v - R_\lambda v\|_2 = \frac{1}{\lambda}\|R_\lambda(v+\lambda Av) - R_\lambda v\|_2 \leq \|Av\|_2 . \qquad \text{q.e.d.}$$

LEMMA 2.4. (1) *For any* $f \in L^2$, $R_\lambda f \xrightarrow{s} f$ *in* L^2 *as* $\lambda \downarrow 0$.

(2) *For any* $f \in L^2 \cap \mathfrak{X}$, $R_\lambda f \xrightarrow{w} f$ *in* $\mathfrak{X}$ *as* $\lambda \downarrow 0$.

Proof. First, let f be any function in $L^2 \cap \mathfrak{X}$. We observe from (1.1) and (i), (v) of Lemma 2.2 that

$$C\|R_\lambda f\|^p \leq \Phi(R_\lambda f) \leq \Phi(f) \qquad \text{for any } \lambda > 0$$

and

$$\|R_\lambda f\|_2 \leq \|f\|_2 \qquad \text{for any } \lambda > 0 . \tag{2.2}$$

Therefore $\{R_\lambda f; \lambda > 0\}$ is bounded in $\mathfrak{X}$ as well as in L^2. From (ii) of Lemma 2.2 and (Φ_4) we derive that for each $v \in L^2 \cap \mathfrak{X}$

$$(f - R_\lambda f, v) = \lambda \langle \nabla\Phi(R_\lambda f), v\rangle \to 0 \qquad \text{as } \lambda \downarrow 0 .$$

This fact and the boundedness of $\{R_\lambda f\,;\,\lambda > 0\}$ in L^2 imply that

$$R_\lambda f \xrightarrow{w} f \qquad \text{in } L^2 \text{ as } \lambda \downarrow 0 \tag{2.3}$$

and $\liminf_{\lambda\downarrow 0}\|R_\lambda f\|_2 \geq \|f\|_2$. On the other hand, we have by (2.2) $\limsup_{\lambda\downarrow 0}\|R_\lambda f\|_2 \leq \|f\|_2$. Hence, $\lim_{\lambda\downarrow 0}\|R_\lambda f\|_2 = \|f\|_2$. This together with (2.3) implies that

$$R_\lambda f \xrightarrow{s} f \qquad \text{in } L^2 \text{ as } \lambda \downarrow 0 . \tag{2.4}$$

Now, let $\{\lambda_j\}$ be any sequence tending to 0 such that $\{R_{\lambda_j} f\}$ is weakly convergent in $\mathfrak{X}$. Then, denoting the weak limit by g, we see from (2.4) and Axiom (a) that $g = f$. This shows that $R_\lambda f \xrightarrow{w} f$ in $\mathfrak{X}$ as $\lambda \downarrow 0$. Finally we can obtain (1) by using Axiom (b′) and (v) of Lemma 2.2 and noting the fact that $R_\lambda f \xrightarrow{s} f$ in L^2 as $\lambda \downarrow 0$ for $f \in L^2 \cap \mathfrak{X}$.
q.e.d.

Remark. By the above lemma, D_0 is dense in L^2. Also, we can prove that D_0 is weakly sequentially dense in $\mathfrak{X}$.

LEMMA 2.5. *If* $f \in L^2 \cap \mathfrak{X}$, *then* $\nabla\Phi(R_\lambda f) \xrightarrow{w} \nabla\Phi(f)$ *in* $\mathfrak{X}^*$ *and* $\Phi(R_\lambda f) \to \Phi(f)$ *as* $\lambda \downarrow 0$.

Proof. Let $f \in L^2 \cap \mathfrak{X}$. Then we see from (ii) of Lemma 2.2 that

$$\limsup_{\lambda\downarrow 0} \langle \nabla\Phi(R_\lambda f), R_\lambda f - f\rangle = \limsup_{\lambda\downarrow 0} -\frac{1}{\lambda}\|f - R_\lambda f\|_2^2 \leq 0 .$$

Applying Lemma 1.1, we obtain the lemma. q.e.d.

Following Deny (cf. [7; Théorème 2]), we define

$$H_\lambda(f) = \frac{1}{\lambda}(f - R_\lambda f, f) , \qquad f \in L^2 .$$

We note that the following relations hold:

$$H_\lambda(f) \geq \frac{1}{\lambda}(f - R_\lambda f, R_\lambda f) = \langle \nabla\Phi(R_\lambda f), R_\lambda f\rangle \geq \Phi(R_\lambda f) . \tag{2.5}$$

LEMMA 2.6. *If* $f \in L^2$ *and if* $\{H_\lambda(f)\,;\,\lambda > 0\}$ *is bounded, then* $f \in \mathfrak{X}$.

Proof. Let $f \in L^2$ and assume that $\{H_\lambda(f)\,;\,\lambda > 0\}$ is bounded. Then we see from (2.5) and (1.1) that $\{R_\lambda f; \lambda > 0\}$ is bounded in $\mathfrak{X}$. Hence there is a sequence $\{\lambda_j\}$ tending to 0 such that $R_{\lambda_j} f \xrightarrow{w} g$ in $\mathfrak{X}$ for some $g \in \mathfrak{X}$. By (1) of Lemma 2.4 and Axiom (a), we have $g = f$, so that $f \in \mathfrak{X}$. q.e.d.

§ 3. Potential theoretic properties and their equivalence

In this section, we state potential theoretic properties of Φ, A and R_λ and our main results about their equivalence.

DEFINITION 3.1 (cf. [5], [11]). Let k be a positive number or ∞. We say that the contraction T_k operates in $\mathfrak{X}$ with respect to Φ if the following two conditions are satisfied:

(C_k) $T_k v \equiv v^+ \wedge k \in \mathfrak{X}$ for all $v \in \mathfrak{X}$, where $T_k v = v^+$ if $k = \infty$.

(ΦC_k) For any $u, v \in \mathfrak{X}$,

$$\Phi(u + T_k(v-u)) + \Phi(v - T_k(v-u)) \leq \Phi(u) + \Phi(v) .$$

In particular, we say that the modulus contraction operates in $\mathfrak{X}$ with respect to Φ, when T_∞ operates in $\mathfrak{X}$ with respect to Φ.

Remark. It was shown in [11; Proposition 2.1] that under (C_k), condition (ΦC_k) is equivalent to the following:

(ΦC_k)' $\langle \nabla\Phi(u + T_k v) - \nabla\Phi(u), v - T_k v\rangle \geq 0$ for any $u, v \in \mathfrak{X}$.

In particular, (ΦC_∞)' is of the form:

$$\langle \nabla\Phi(u + v^+) - \nabla\Phi(u), v^-\rangle \leq 0 \qquad \text{for any } u, v \in \mathfrak{X} .$$

DEFINITION 3.2 (cf. [5], [11]). We say that the principle of lower envelope with respect to Φ holds if (C_∞) and the following are satisfied:

(ΦL) If u and v are functions in D_0, then

$$\langle \nabla\Phi(u \wedge v), w\rangle \geq (Au \wedge Av, w)$$

for any non-negative function $w \in L^2 \cap \mathfrak{X}$.

DEFINITION 3.3 (cf. [5], [11]). We say that the domination principle with respect to Φ holds, if the following is satisfied:

(ΦD) If u and v are functions in D_0 and if there is $f \in L^2$ such that $Au \geq f$, $Av \geq f$ and $(Au - f, (u-v)^+) = 0$, then $u \leq v$.

DEFINITION 3.4 (cf. [5], [11]). We say that R_λ, $\lambda > 0$, is order-preserving in L^2, if the following holds:

(ΦO) $R_\lambda f \leq R_\lambda g$ for any $f, g \in L^2$ such that $f \leq g$.

Now, we are in a position to state one of main theorems.

THEOREM A. *The following statements are equivalent to each other*:

(a_1) *The modulus contraction operates in* $\mathfrak{X}$ *with respect to* Φ.

(a_2) *The principle of lower envelope with respect to* Φ *holds.*

(a_3) *The domination principle with respect to* Φ *holds.*

(a_4) *For any* $\lambda > 0$, *the resolvent* R_λ *is order-preserving in* L^2.

Next, to state another main theorem we give some definitions. Throughout the remainder of this section, let k be a positive number.

DEFINITION 3.5 (cf. [5], [11]). We say that the strong principle of lower envelope with respect to Φ and k holds, if the following two conditions are satisfied:

$(C_k)^*$ $\quad u \wedge (v + k) \in \mathfrak{X} \qquad$ for any $u, v \in \mathfrak{X}$.

(ΦSL_k) If u and v are functions in D_0, then

$$\langle \nabla\Phi(u \wedge (v + k)), w\rangle \geq (Au \wedge Av, w)$$

for any non-negative function $w \in L^2 \cap \mathfrak{X}$.

It should be noticed that under (C_∞), conditions $(C_k)^*$ and (C_k) are equivalent.

DEFINITION 3.6 (cf. [5], [11]). We say that the complete maximum principle with respect to Φ and k holds, if the following is satisfied:

(ΦCM_k) If u and v are functions in D_0 and if there is $f \in L^2$ such that $Au \geq f$, $Av \geq f$ and $(Au - f, (u - v - k)^+) = 0$, then $u \leq v + k$.

The second main theorem is stated as follows:

THEOREM B. *Assume that the modulus contraction operates in* $\mathfrak{X}$ *with respect to* Φ. *Then the following statements are equivalent to each other*:

(b_1) *The contraction* T_k *operates in* $\mathfrak{X}$ *with respect to* Φ.

(b_2) *The strong principle of lower envelope with respect to* Φ *and* k *holds.*

(b_3) *The complete maximum principle with respect to* Φ *and* k *holds.*

(b_4) *For any* $\lambda > 0$ *and any* $f, g \in L^2$

$$R_\lambda(f + T_k g) \leq R_\lambda f + k .$$

§ 4. Proofs of Theorems A and B

Before proving the theorems, we recall an existence theorem for nonlinear variational inequalities. Let $\mathfrak{Y}$ be a real reflexive Banach space and P be a (nonlinear) semicontinuous monotone[1] operator from $\mathfrak{Y}$ into $\mathfrak{Y}^*$ (the dual space of $\mathfrak{Y}$). Let Ψ be a lower semicontinuous convex function on $\mathfrak{Y}$ with values in $(-\infty, \infty]$.

THEOREM (cf. [3; Theorem 3], [10; Theorem 4.1]). *Let $\mathfrak{K}$ be a nonempty closed convex subset of $\mathfrak{Y}$ and assume that for some $w \in \mathfrak{K}$ with $\Psi(w) < \infty$,*

$$\frac{\langle Pv, v - w\rangle_{\mathfrak{Y}} + \Psi(v)}{\|v\|_{\mathfrak{Y}}} \to \infty \qquad \text{as } \|v\|_{\mathfrak{Y}} \to \infty,\ v \in \mathfrak{K}\,,$$

where we denote by $\langle\cdot,\cdot\rangle_{\mathfrak{Y}}$ the natural pairing between $\mathfrak{Y}^*$ and $\mathfrak{Y}$ and by $\|\cdot\|_{\mathfrak{Y}}$ the norm in $\mathfrak{Y}$. Then, there is $u \in \mathfrak{K}$ such that

$$\langle Pu, u - v\rangle_{\mathfrak{Y}} \le \Psi(v) - \Psi(u) \qquad \text{for all } v \in \mathfrak{K}\,.$$

Moreover, if P is strictly monotone, then such a u is unique.

LEMMA 4.1. *The function Ψ on L^2 defined by*

$$\Psi(v) = \begin{cases} \Phi(v)\,, & \text{if } v \in L^2 \cap \mathfrak{X}\,, \\ \infty\,, & \text{otherwise,} \end{cases}$$

is lower semicontinuous on L^2.

Proof. Let $\{v_n\}$ be a sequence in L^2 which converges to v_0 in L^2, and assume that $\alpha \equiv \liminf_{n\to\infty} \Psi(v_n) < \infty$. Then, by (1.1) there is a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that $v_{n_j} \xrightarrow{w} v$ in $\mathfrak{X}$ for some $v \in \mathfrak{X}$ and $\Phi(v_{n_j}) \to \alpha$ as $j \to \infty$. By Axiom (a), $v_0 = v \in \mathfrak{X}$. Hence, from (Φ_1) it follows that $\alpha \ge \Phi(v_0) = \Psi(v_0)$. Thus Ψ is lower semicontinuous on L^2.
q.e.d.

Proof of $(a_1) \to (a_2)$: Let $u \in D_0$ and $v \in D_0$, and set $f = Au$ and $g = Av$. Define an operator $P: L^2 \to L^2$ by $Pw = w - u \wedge v - f \wedge g$ and let Ψ be the same function as in Lemma 4.1. Obviously, P is a demicontinuous monotone operator from L^2 into L^2 and Ψ is a convex function on L^2 with values in $[0, \infty]$ such that $\{(Pw, w - u) + \Psi(w)\}/\|w\|_2 \to \infty$ as $\|w\|_2 \to \infty$. By Lemma 4.1, Ψ is lower semicontinuous on L^2. Furthermore, setting $\mathfrak{K} = \{w \in L^2; w \ge u \wedge v\}$, we see that $\mathfrak{K}$ is closed and convex in L^2, $u \in \mathfrak{K}$ and $\Psi(u) = \Phi(u) < \infty$. By virtue of the existence theorem

1) An operator P from $\mathfrak{Y}$ into $\mathfrak{Y}^*$ is called monotone, if for any $u, v \in \mathfrak{Y}$, $\langle Pu - Pv, u - v\rangle_{\mathfrak{Y}} \ge 0$.

mentioned above, there is $u_0 \in \mathfrak{K}$ such that $(Pu_0, u_0 - w) \leq \Psi(w) - \Psi(u_0)$ for all $w \in \mathfrak{K}$. Clearly, $u_0 \in \mathfrak{K} \cap \mathfrak{X}$, so we have $(Pu_0, u_0 - w) \leq \Phi(w) - \Phi(u_0)$ for all $w \in \mathfrak{K} \cap \mathfrak{X}$. Since $u_0 + t(w - u_0) \in \mathfrak{K} \cap \mathfrak{X}$ for any $w \in \mathfrak{K} \cap \mathfrak{X}$ and $1 > t > 0$, we see that

$$(Pu_0, u_0 - w) \leq \frac{1}{t}\{\Phi(u_0 + t(w - u_0)) - \Phi(u_0)\} .$$

As $t \downarrow 0$, we have

$$(4.1) \qquad (Pu_0, u_0 - w) \leq \langle \nabla\Phi(u_0), w - u_0 \rangle \qquad \text{for any } w \in \mathfrak{K} \cap \mathfrak{X}.$$

By (a_1), $u_0 \wedge u \in \mathfrak{K} \cap \mathfrak{X}$. Hence, taking $w = u_0 \wedge u$ in (4.1)

$$(4.2) \qquad (Pu_0, u_0 - u \wedge u_0) \leq \langle \nabla\Phi(u_0), u \wedge u_0 - u_0 \rangle .$$

Since $f + Pu = u - u \wedge v + f - f \wedge g \geq 0$, it follows from $(\Phi C_\infty)'$ that $\langle \nabla\Phi(u \wedge u_0), u_0 - u \wedge u_0 \rangle = \langle \nabla\Phi(u - (u - u_0)^+), (u - u_0)^- \rangle \geq \langle \nabla\Phi(u), (u - u_0)^- \rangle = (f, u_0 - u \wedge u_0) \geq (-Pu, u_0 - u \wedge u_0)$. Hence, by (4.2), $\langle \nabla\Phi(u_0) - \nabla\Phi(u \wedge u_0), u_0 - u \wedge u_0 \rangle \leq -(Pu_0 - Pu, (u_0 - u)^+) \leq 0$, so we have by (Φ_3) $u_0 = u_0 \wedge u$, that is, $u_0 \leq u$. In a similar manner, we have $u_0 \leq v$. Therefore $u_0 \leq u \wedge v$, while $u_0 \geq u \wedge v$ because $u_0 \in \mathfrak{K}$. Consequently, $u_0 = u \wedge v$ and hence (4.1) yields (ΦL). q.e.d.

Proof of $\{(a_1), (b_1)\} \to (b_2)$: Let u and v be any functions in D_0, and set $f = Au$ and $g = Av$. Define an operator P from L^2 into L^2 by $Pw = w - u \wedge (v + k) - f \wedge g$ and denote by $\mathfrak{K}$ the set of all $w \in L^2$ such that $w \geq u \wedge (v + k)$. Then, in the same manner as in the proof of $(a_1) \to (a_2)$, we can find $u_0 \in \mathfrak{K} \cap \mathfrak{X}$ such that $u_0 \leq u$ and

$$(4.3) \qquad (Pu_0, u_0 - w) \leq \langle \nabla\Phi(u_0), w - u_0 \rangle \qquad \text{for any } w \in \mathfrak{K} \cap \mathfrak{X} .$$

Moreover, $u_0 = u_0 \wedge (v + k)$ holds. In fact, by using $(\Phi C_k)'$ we observe that

$$\begin{aligned} &\langle \nabla\Phi(u_0 \wedge (v + k)), u_0 - u_0 \wedge (v + k) \rangle \\ &\quad = \langle \nabla\Phi(u_0 \wedge v + T_k((u_0 - v)^+)), (u_0 - v)^+ - T_k((u_0 - v)^+) \rangle \\ &\quad \geq \langle \nabla\Phi(u_0 \wedge v), (u_0 - v)^+ - T_k((u_0 - v)^+) \rangle \\ &\quad = \langle \nabla\Phi(v - (v - u_0)^+), (u_0 - v - k)^+ \rangle . \end{aligned}$$

Now, putting $z = (v - u_0)^+ - (u_0 - v - k)^+$, we see that $w^+ = (v - u_0)^+$ and $w^- = (u_0 - v - k)^+$, so by $(\Phi C_\infty)'$ the right hand side of the above inequalities is

$$\geq \langle \nabla\Phi(v - (v - u_0)^+ + w^+), w^- \rangle = (g, u_0 - u_0 \wedge (v + k)) .$$

From this and (4.3) it follows that

$$\langle \nabla\Phi(u_0) - \nabla\Phi(u_0 \wedge (v + k)), u_0 - u_0 \wedge (v + k) \rangle \\ \leq -(Pu_0 + g, u_0 - u_0 \wedge (v + k)) \leq 0 .$$

By (Φ_3), $u_0 = u_0 \wedge (v + k)$. Hence we see that $u_0 = u \wedge (v + k)$ and obtain (ΦSL_k) from (4.3). q.e.d.

Proof of $(b_2) \to (b_3)$: Let u and v be any functions in D_0 and f be a function in L^2 such that $Au \geq f$, $Av \geq f$ and $(Au - f, (u - v - k)^+) = 0$. Then it follows from (ΦSL_k) that

$$\langle \nabla\Phi(u \wedge (v + k)), w \rangle \geq (f, w) \qquad \text{for any non-negative } w \in L^2 \cap \mathfrak{X} .$$

Hence

$$\begin{aligned} &\langle \nabla\Phi(u) - \nabla\Phi(u \wedge (v + k)), u - u \wedge (v + k) \rangle \\ &\quad = \langle \nabla\Phi(u) - \nabla\Phi(u \wedge (v + k)), (u - v - k)^+ \rangle \\ &\quad \leq (Au - f, (u - v - k)^+) \\ &\quad = 0 . \end{aligned}$$

From this and (Φ_3) we obtain $u = u \wedge (v + k)$, that is, $u \leq v + k$.

Proof of $(a_2) \to (a_3)$: By taking 0 instead of k in the above proof, we have the proof of $(a_2) \to (a_3)$.

Proof of $(b_3) \to (b_4)$: Let λ be any positive number and f and g be any functions in L^2. Set $u = R_\lambda(f + T_k g)$ and $v = R_\lambda f$. Then we see by (iii) of Lemma 2.2 that $Au \geq \lambda^{-1}\{f - (u - T_k g) \vee v\}$ and $Av \geq \lambda^{-1}\{f - (u - T_k g) \vee v\}$. Moreover,

$$\left(Au - \frac{f - (u - T_k g) \vee v}{\lambda}, (u - v - k)^+\right) = 0 .$$

Hence, we obtain from (ΦCM_k) that $u \leq v + k$, i.e., $R_\lambda(f + T_k g) \leq R_\lambda f + k$. q.e.d.

Proof of $(a_3) \to (a_4)$: In the above proof, replace k by 0 and $T_k g$ by $-g^+$. Then, by (ΦD) we have $R_\lambda(f - g^+) \leq R_\lambda f$. Since this inequality holds for any $f, g \in L^2$, R_λ is order-preserving in L^2.

Remark. The proofs of $(a_2) \to (a_3) \to (a_4)$ and of $(b_2) \to (b_3) \to (b_4)$ given above are essentially due to Calvert [5; §2].

In order to prove the assertions $(a_4) \to (a_1)$ and $\{(a_4), (b_4)\} \to (b_1)$, we prepare some lemmas. In the rest of this section, let k be a positive number.

LEMMA 4.2. (i) *If* (a_4) *and* (b_4) *are satisfied, then* $T_k f \in \mathfrak{X}$ *for any* $f \in D_0$.

(ii) *If* (a_4) *is satisfied, then* $f^+ \in \mathfrak{X}$ *for any* $f \in D_0$.

Proof. Assume (a_4) and (b_4). Let $f \in D_0$ and set $g = T_k f$. Then

$$H_\lambda(f - g) = \frac{1}{\lambda}((f - g) - R_\lambda(f - g), f - g)$$
$$= \frac{1}{\lambda}\sum_{i=1}^{3}\int_{X_i}\{(f - g) - R_\lambda(f - g)\}(f - g)d\xi ,$$

where $X_1 = \{x \in X; f(x) < 0\}$, $X_2 = \{x \in X; 0 \leq f(x) \leq k\}$ and $X_3 = \{x \in X; f(x) > k\}$. By (a_4) and (b_4) we have

$$R_\lambda(f - g) \leq R_\lambda f \leq R_\lambda(f - g) + k ,$$

so that $(f - g) - R_\lambda(f - g) \geq f - R_\lambda f$ on X_1 and $\leq f - R_\lambda f$ on X_3. Moreover, $f - g = f < 0$ on X_1, $f - g = 0$ on X_2 and $0 < f - g < f$ on X_3. Hence we have $H_\lambda(f - g) \leq \lambda^{-1}\|f - R_\lambda f\|_2\|f\|_2 \leq \|Af\|_2\|f\|_2$ because of (iii) of Lemma 2.2 and Lemma 2.3, so from Lemma 2.6 it follows that $f - g \in \mathfrak{X}$, i.e., $g \in \mathfrak{X}$. Thus (i) is obtained. (ii) is similarly proved. q.e.d.

LEMMA 4.3. (i) *If* (a_4) *and* (b_4) *are satisfied, then*

$$\langle \nabla\Phi(u + T_k v) - \nabla\Phi(u), v - T_k v\rangle \geq 0 \qquad \textit{for any } u \in \mathfrak{X} \textit{ and } v \in D_0 . \tag{4.4}$$

(ii) *If* (a_4) *is satisfied, then*

$$\langle \nabla\Phi(u + v^+) - \nabla\Phi(u), v^-\rangle \leq 0 \qquad \textit{for any } u \in \mathfrak{X} \textit{ and } v \in D_0 .$$

Proof. Assume (a_4) and (b_4). According to Axiom (b') and property (Φ_5), it is sufficient to show the inequality in case $u \in L^2 \cap \mathfrak{X}$ and $v \in D_0$. By Lemma 4.2 we see that $T_k v \in L^2 \cap \mathfrak{X}$. Furthermore we have

$$\langle \nabla\Phi(R_\lambda(u + T_k v)) - \nabla\Phi(R_\lambda u), v - T_k v\rangle$$
$$= \frac{1}{\lambda}((u + T_k v) - R_\lambda(u + T_k v) - (u - R_\lambda u), v - T_k v)$$
$$= \frac{1}{\lambda}(T_k v + R_\lambda u - R_\lambda(u + T_k v), v = T_k v) .$$

Now, $T_k v + R_\lambda u - R_\lambda(u + T_k v) \geq 0$ on $X_1 = \{x \in X; v(x) \geq k\}$ by (b_4) and ≤ 0 on $X_2 = \{x \in X; v(x) \leq 0\}$ by (a_4). Moreover, $v - T_k v \geq 0$ on X_1, ≤ 0 on X_2 and $= 0$ on $\{x \in X; 0 < v(x) < k\}$, so the right hand side of the above equalities is non-negative. Hence, by Lemma 2.5,

$$\begin{aligned}&\langle \nabla\Phi(u + T_k v) - \nabla\Phi(u), v - T_k v\rangle \\ &= \lim_{\lambda \downarrow 0} \langle \nabla\Phi(R_\lambda(u + T_k v)) - \nabla\Phi(R_\lambda u), u - T_k v\rangle \\ &\geq 0 .\end{aligned}$$

The assertion (ii) is similarly proved. q.e.d.

LEMMA 4.4. (i) *If* (a_4) *and* (b_4) *are satisfied, then*

$$\Phi(u) + \Phi(u + v) \geq \Phi(u + T_k v) + \Phi(u + v - T_k v) \tag{4.5}$$

for any $u \in \mathfrak{X}$ *and* $v \in D_0$.

(ii) *If* (a_4) *is satisfied, then*

$$\Phi(u) + \Phi(u + v) \geq \Phi(u + v^+) + \Phi(u - v^-)$$

for any $u \in \mathfrak{X}$ *and* $v \in D_0$.

Proof. We shall show (i). By using property (Φ_7) and (i) of Lemma 4.3 we have

$$\begin{aligned}&\Phi(u + v) - \Phi(u + T_k v) \\ &= \int_0^1 \langle \nabla\Phi(u + T_k v + t(v - T_k v)), v - T_k v\rangle dt \\ &\geq \int_0^1 \langle \nabla\Phi(u + t(v - T_k v)), v - T_k v\rangle dt \\ &= \Phi(u + v - T_k v) - \Phi(u)\end{aligned}$$

for any $u \in \mathfrak{X}$ and $v \in D_0$. Similarly from (Φ_7) and (ii) of Lemma 4.3 we obtain (ii). q.e.d.

Proof of $\{(a_4), (b_4)\} \rightarrow (b_1)$: Let v be any function in $L^2 \cap \mathfrak{X}$. Then $R_\lambda v \in D_0$, $\lambda > 0$, because of (iii) of Lemma 2.2. Taking 0 and $R_\lambda v$ for u and v in (4.5) respectively and using (i) of Lemma 2.2, we have

$$\Phi(T_k(R_\lambda v)) \leq \Phi(R_\lambda v) \leq \Phi(v) ,$$

so by (1.1), $\{T_k(R_\lambda v); \lambda > 0\}$ is bounded in $\mathfrak{X}$ and hence it is weakly relatively compact in $\mathfrak{X}$. Now, let $\{T_k(R_{\lambda_n} v)\}$ be any sequence weakly convergent in $\mathfrak{X}$ such that $\lambda_n \downarrow 0$ as $n \rightarrow \infty$, and denote by g the weak limit.

Then, since $T_k(R_{\lambda_n}v) \xrightarrow{s} T_k v$ in L^2 as $n \to \infty$ by (i) of Lemma 2.4, it follows that $g = T_k v$. This shows that $T_k v \in \mathfrak{X}$. Thus we have seen that $T_k v \in \mathfrak{X}$ for any $v \in L^2 \cap \mathfrak{X}$. Moreover, just as Lemma 4.3, we can prove that

$$\langle \nabla\Phi(u + T_k v) - \nabla\Phi(u), v - T_k v\rangle \geq 0 \quad \text{for any } u \in \mathfrak{X} \text{ and } v \in L^2 \cap \mathfrak{X}\,.$$

From this, by the same calculation as in the proof of Lemma 4.4, we deduce that

$$\Phi(u + T_k v) + \Phi(u + v - T_k v) \leq \Phi(u) + \Phi(u + v) \qquad \text{for any } u \in \mathfrak{X} \text{ and } v \in L^2 \cap \mathfrak{X}\,. \tag{4.6}$$

Again by the limit process, we see from (4.6) that $T_k v \in \mathfrak{X}$ for any $v \in \mathfrak{X}$, i.e., (C_k) holds, and simultaneously that

$$\Phi(u + T_k v) + \Phi(u + v - T_k v) \leq \Phi(u) + \Phi(u + v)$$

for any $u, v \in \mathfrak{X}$. Clearly, this inequality is equivalent to (ΦC_k). q.e.d.

Proof of $(a_4) \to (a_1)$: We can prove the assertion just as $\{(a_4), (b_4)\} \to (b_1)$.

§ 6. The nonlinear contraction semigroup generated by $-A$

In this section we discuss the nonlinear contraction semigroup on L^2 generated by $-A$. In view of the generation theorem for contraction semigroups due to Kōmura [12; Theorem 4], there is a unique contraction semigroup $S = \{S(t)\,; t \geq 0\}$ on L^2 whose infinitesimal generator is $-A$. Here we mean by a contraction semigroup $S = \{S(t)\,; t \geq 0\}$ on L^2 a one-parameter family of operators $S(t)$, $t \geq 0$, from L^2 into L^2 with the following properties:

(s_1) $S(t)S(s) = S(t + s)$ for $t, s \geq 0$.

(s_2) $S(0) = I$.

(s_3) $\|S(t)v - S(t)w\|_2 \leq \|v - w\|_2$ for $t \geq 0$ and $v, w \in L^2$.

(s_4) For each $v \in L^2$, $t \to S(t)v$ is continuous on $[0, \infty)$.

The contraction semigroup $S = \{S(t)\,; t \geq 0\}$ on L^2 generated by $-A$ is in fact given by

$$S(t)v = \lim_{n\to\infty} R^n_{t/n} v \qquad \text{for } t \geq 0 \text{ and } v \in L^2$$

(see Crandall-Liggett [6; Theorem I]).

THEOREM C. *The following statements* $(c_1) \sim (c_3)$ *and* $(a_1) \sim (a_4)$ (*in Theorem A*) *are equivalent to each other*:

(c_1) *A is T-accretive* (*or* $-A$ *is dispersive*) (*cf.* [4], [13]), *i.e.*,

$$(Au - Av, (u-v)^+) \geq 0 \qquad \textit{for any } u, v \in D_0 .$$

(c_2) *For any* $t, s \in [0, \infty)$, $s \leq t$, *and any* $u, v \in L^2$,

$$\|(S(t)u - S(t)v)^+\|_2 \leq \|(S(s)u - S(s)v)^+\|_2 .$$

(c_3) *For any* $\lambda > 0$ *and any* $u, v \in L^2$,

$$\|(R_\lambda u - R_\lambda v)^+\|_2 \leq \|(u-v)^+\|_2 .$$

Proof. From (a_1) we see that

(c_1') Condition (C_∞) is satisfied and $\nabla\Phi$ is T-monotone (cf. [1]), i.e., for any $u, v \in \mathfrak{X}$,

$$\langle \nabla\Phi(u) - \nabla\Phi(v), (u-v)^+ \rangle \geq 0 .$$

In fact, from $(\Phi C_\infty)'$ and the monotonicity of $\nabla\Phi$ we derive that

$$\begin{aligned} &\langle \nabla\Phi(u) - \nabla\Phi(v), (u-v)^+ \rangle \\ &\quad = \langle \nabla\Phi(u) - \nabla\Phi(v), (v-u)^- \rangle \\ &\quad \geq \langle \nabla\Phi(u + (v-u)^+) - \nabla\Phi(v), (v-u)^- \rangle \\ &\quad \geq \langle \nabla\Phi(u \vee v) - \nabla\Phi(v), u \vee v - v \rangle \\ &\quad \geq 0 \end{aligned}$$

for any $u, v \in \mathfrak{X}$ (cf. [5; Proposition 1.2]). The assertions $(c_1') \to (c_1)$ and $(c_3) \to (a_4)$ are trivial, and $(c_1) \to (c_2)$ and $(c_2) \to (c_3)$ are known (cf. [13]).

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N. KENMOCHI
Department of Mathematics
Faculty of Education
Chiba University
Chiba, Japan

Y. MIZUTA
Department of Mathematics
Faculty of Science
Hiroshima University
Hiroshima, Japan