J. Noguchi
Nagoya Math. J.
Vol. 59, (1975), 97-106

A RELATION BETWEEN ORDER AND DEFECTS OF MEROMORPHIC MAPPINGS OF C^n INTO $P^N(C)$

JUNJIRO NOGUCHI

1. Introduction

Let f be a meromorphic mapping of the n-dimensional complex plane C^n into the N-dimensional complex projective space $P^N(C)$. We denote by $T(r, f)$ the characteristic function of f and by $N(r, f^*H)$ the counting function for a hyperplane $H \subset P^N(C)$.[1] The purpose of this paper is to establish the following results.

THEOREM 1. *Let $f: C^n \to P^N(C)$ be a meromorphic mapping of finite order ρ which is not a positive integer. Then for any $N+1$ hyperplanes $H_\mu \subset P^N(C)$, $\mu = 0, 1, \cdots, N$, in general position*

$$K(f) = \varliminf_{r\to\infty} \frac{\sum_{\mu=0}^{N} N(r, f^*H_\mu)}{T(r, f)} \geqq k(\rho)\ , \tag{1.1}$$

where $k(\rho)$ is a positive constant depending only on ρ and satisfies

$$k(\rho) \geqq \frac{2\Gamma^4(3/4)\,|\sin \pi\rho|}{\pi^2\rho + \Gamma^4(3/4)\,|\sin \pi\rho|}\ .\text{[2]} \tag{1.2}$$

In case $0 \leqq \rho < 1$, we shall also obtain

THEOREM 2. *The positive constant $k(\rho)$ in* (1.1) *satisfies*

$$k(\rho) \geqq 1 - \rho \quad \textit{for} \quad 0 \leqq \rho < 1\ . \tag{1.3}$$

Remark. When ρ takes values near 0, the evaluation (1.3) is better than (1.2). On the other hand (1.2) is better than (1.3) when ρ is close to 1.

From these theorems we have readily

Received December 7, 1974.

1) Throughout the present paper we only consider hyperplanes H such that f^*H do not contain the origin.

2) As usual, $\Gamma(\cdot)$ stands for the gamma-function.

COROLLARY. *If a meromorphic mapping $f: C^n \to P^N(C)$ admits $N+1$ hyperplanes in general position whose defects are equal to one, then the order of f is infinite or a positive integer.*

In case $n = N = 1$, the existence of the positive lower bound $k(\rho)$ in (1.1) was first proved by R. Nevanlinna [7, Chap. III] and he posed the problem to determine the best possible value of $k(\rho)$. In the same case Theorem 1 was proved by Edrei-Fuchs [1] and they determined the correct value of $k(\rho)$ for $0 \leqq \rho < 1$ in [2]. In case $n = 1$ and $N \geqq 1$, Toda [10] obtained the evalution (1.2) and moreover Ozawa [8] obtained the correct value of $k(\rho)$ for $\rho < 1$.

One notes that $k(\rho)$ may be determined independently of the dimension n.

The author is very thankful to Professors N. Toda and H. Fujimoto for their helpful advices and to Professors K. Niino and M. Ozawa for their valuable suggestions.

2. Notation

Let $(z_1, \cdots, z_n)$ be the natural coordinate system in C^n and set

$$\|z\|^2 = \sum_{\mu=1}^{n} z_\mu \bar{z}_\mu , \qquad B(r) = \{\|z\| < r\} ,$$
$$A(r) = A \cap B(r) \quad \text{for a subset} \quad A \subset C^n ,$$
$$d^c = \frac{i}{4\pi}(\bar{\partial} - \partial) ,$$
$$\chi = (dd^c \log \|z\|^2)^{n-1} , \qquad \eta = d^c \log \|z\|^2 \wedge \chi .$$

For a positive divisor D on C^n not containing the origin, set

$$n(t, D) = \int_{D(t)} \chi, \qquad N(r, D) = \int_0^r \frac{n(t, D)}{t} dt .$$

In case $n = 1$, $n(t, D)$ is the number of elements of D in $B(t)$ with counting multiplicities. Let L denote the hyperplane bundle over $P^n(C)$ and ω the positive definite curvature form of L arising from an hermitian metric h in L. For a meromorphic mapping $f: C^n \to P^N(C)$ which is holomorphic at the origin, the characteristic function is defined by

$$T(r, f) = \int_0^r \frac{dt}{t} \int_{B(t)} f^*\omega \wedge \chi .$$

It is noted that the pull-back form $f^*\omega$ is a differential form with coefficients belonging to L^1_{loc} which is closed and positive in the sense of currents (cf. Lelong [6]) and that $T(r, f)$ is independent of the curvature form ω of L, up to an $O(1)$-term (cf. Griffiths-King [3]).

Let $S(r)$ be a real, non-negative and increasing function of $r \geqq 0$. Then $\overline{\lim}_{r\to\infty} \log S(r)/\log r$ is called the order of $S(r)$. In particular the order of $T(r, f)$ ($N(r, D)$ resp.) is called the order of f (D resp.). Let U be an open set in $\boldsymbol{P}^N(\boldsymbol{C})$ such that $L|_U \cong U \times \boldsymbol{C}$. Then the restriction $\sigma|_U$ of a global holomorphic section $\sigma \in H^0(\boldsymbol{P}^N(\boldsymbol{C}), L)$ is naturally regarded as a holomorphic function in U and similarly $h|_U$ as a positive smooth function in U. The length of σ is defined by

$$|\sigma| = \left(\frac{|\sigma|_U|^2}{h|_U}\right)^{1/2} \quad \text{in } U ,$$

which is independent of the local trivialization, $L|_U \cong U \times \boldsymbol{C}$. For a hyperplane H in $\boldsymbol{P}^N(\boldsymbol{C})$, choose always a global section $\sigma \in H^0(\boldsymbol{P}^N(\boldsymbol{C}), L)$ so that the divisor (σ) is equal to H and $|\sigma| \leqq 1$, and set

$$m(r, H) = \int_{\partial B(r)} \log \frac{1}{f^*|\sigma|}\eta .$$

Now the following is well-known (Nevanlinna's first main theorem):

$$T(r, f) = N(r, f^*H) + m(r, H) + \log f^*|\sigma|(0) \tag{2.1}$$

provided that $f^*H \not\ni 0$.

In case $N = 1$, f is a meromorphic function in $\boldsymbol{C}^n$. Let $(f)_0$ and $(f)_\infty$ denote respectively the divisors of zeros and poles of f and suppose that $(f)_0 \cup (f)_\infty \not\ni 0$. Then (2.1) yields that

$$\begin{aligned} T(r, f) &= N(r, (f)_\infty) + \int_{\partial B(r)} \log^+ |f|\eta + O(1) \\ &= N(r, (f)_0) + \int_{\partial B(r)} \log^+ \frac{1}{|f|}\eta + O(1), \end{aligned} \tag{2.2}$$

where $\log^+ s = \max\{0, \log s\}$ for $s \geqq 0$. We return to the general case, $N \geqq 1$. We set for a hyperplane H

$$\delta(H, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r, f^*H)}{T(r, f)}$$

which is called the defect of H.

3. An estimate for canonical functions

For an entire function F in C^n, we set

$$M(r,F)=\max_{\|z\|=r}|F(z)| .$$

LEMMA 1. *Let F be an entire function. Then for $r<R$*

$$T(r,F)+O(1)\leqq \log M(r,F)\leqq \frac{1-(r/R)^2}{(1-r/R)^{2n}}\{T(R,F)+O(1)\} . \tag{3.1}$$

Proof. The first inequality follows from (2.2). We prove the second. Let $Aut(B(R))$ denote the group of holomorphic automorphisms of $B(R)$. For $z_0\in B(R)$, there is an element $\gamma(\cdot,z_0)\in Aut(B(R))$ with $\gamma(z_0,z_0)=0$. We define

$$\begin{aligned}\phi(z,z_0)&=dd^c\log\|\gamma(z,z_0)\|^2 ,\\ \chi(z,z_0)&=\phi(z,z_0)^{n-1}\\ \eta(z,z_0)&=d^c\log\|\gamma(z,z_0)\|^2\wedge\chi(z,z_0) .\end{aligned}$$

Since the isotropy subgroup of $Aut(B(R))$ at the origin consists of unitary transformations of the coordinates, these differential forms are independent of the choice of $\gamma(\cdot,z_0)$. Note that $\chi(z,0)=\chi(z)$ and $\eta(z,0)=\eta(z)$. Since $\log|F\circ\gamma(\cdot,z_0)^{-1}|$ is plurisubharmonic in a neighborhood of $\overline{B(R)}$,

$$\begin{aligned}\log|F(z_0)|&=\log|F\circ\gamma(0,z_0)^{-1}|\\ &\leqq\int_{\partial B(R)}\log|F\circ\gamma(z,z_0)^{-1}|\,\eta(z)=\int_{\partial B(R)}\log|F(z)|\,\eta(z,z_0)\\ &\leqq\int_{\partial B(R)}\log^+|F(z)|\,\eta(z,z_0) .\end{aligned}\tag{3.2}$$

Let $\log M(r,F)=\log|F(z_0)|$ with $z_0\in\partial B(r)$. By a unitary transformation of the coordinates, we can carry z_0 to $(r,0,\cdots,0)$. Therefore we may assume that $z_0=(r,0,\cdots,0)$. Let us take $\gamma(z,z_0)$ as follows:

$$\gamma(z,z_0)=\frac{R}{R-(r/R)z_1}(z_1-r,\sqrt{1-(r/R)^2}\,z_2,\cdots,\sqrt{1-(r/R)^2}\,z_n) .$$

By an elementary calculation we have

$$\begin{aligned}\phi(z,z_0)&\leqq\frac{1}{(1-r/R)^2}\phi(z,0) ,\\ d^c\log\|\gamma(z,z_0)\|^2&=\frac{R^2-r^2}{|R-(r/R)z_1|^2}d^c\log\|\gamma(z,0)\|^2\end{aligned}$$

and so $\eta(z, z_0) \leqq \{1 - (r/R)^2\}\eta(z)/(1 - r/R)^{2n}$. Combining this with (3.2) and (2.2), we obtain the required inequality. Q.E.D.

Let ℓ be a complex line in $\boldsymbol{C}^n$ through the origin and $F_\ell(u)$ denote the restriction of F on ℓ. From Lemma 1 it follows that for every ℓ,

$$\text{order of } F_\ell(u) \leqq \text{order of } F(z) . \tag{3.3}$$

Let D be a positive divisor on $\boldsymbol{C}^n$ not containing the origin and suppose that for an integer q

$$\int^\infty \frac{1}{t^{q+1}} dn(t, D) < \infty . \tag{3.4}$$

Then according to Lelong [5, Theorem 5] (see also Stoll [9]), there exists an entire function F such that $(F) = D$, $F(0) = 1$, all the partial derivatives of $\log F$ of order $\leqq q$ vanish at the origin, the order of F is not greater than $\max\{q, \text{order of } D\}$ and

$$\begin{aligned} \log|F(z)| \leqq A(n, q)\Big\{\|z\|^q \int_0^{\|z\|} \frac{n(t, D)}{t^{q+1}} dt \\ + \|z\|^{q+1} \int_{\|z\|}^\infty \frac{n(t, D)}{t^{q+2}} dt\Big\} , \end{aligned} \tag{3.5}$$

where $A(n, q)$ is a constant depending only on n and q. Such a function F is called the canonical function of genus q associated with the divisor D.

Let D be a positive divisor on $\boldsymbol{C}^n$ not containing the origin, whose order is less than $q + 1$. Then (3.4) is satisfied. Let F be the canonical function of genus q associated with D, ℓ a complex line in $\boldsymbol{C}^n$ through the origin and suppose that $F_\ell(u)$ does not vanish for all $u \in \ell \cong \boldsymbol{C}$. Then by (3.3), $F_\ell(u) = e^{P(u)}$ where $P(u)$ is a polynomial of degree $\leqq q$. Since all the derivatives of $\log F$ of order $\leqq q$ vanish at the origin and $F(0) = 1$, $P(u) \equiv 0$ and then $F_\ell(u) \equiv 1$. Regarding ℓ as a point of $\boldsymbol{P}^{n-1}(\boldsymbol{C})$ in the natural manner, we see

LEMMA 2. *Let $E = \{\ell \in \boldsymbol{P}^{n-1}(\boldsymbol{C}) : \ell \cdot D = \phi\}$, ($\ell \cdot D$ = intersection of ℓ and D with counting multiplicities). Then E is an analytic subset and for $\ell \in E$, $F_\ell \equiv 1$ and for $\ell \notin E$, F_ℓ coincides with the Weierstrass product of genus q associated with $\ell \cdot D$.*

Remark. It follows from (3.3) that $\int^\infty dn(t, \ell \cdot D)/t^{q+1} < \infty$.

Proof. The first two assertions follow immediately from the above arguments. We show the last. Let $\Pi(u)$ denote the Weierstrass product of genus q associated with $\ell\cdot D$. Noting that the orders of $\Pi(u)$ and $F_\ell(u)$ are less than $q+1$, we have

$$F_\ell(u) = e^{P(u)}\Pi(u) ,$$

where $P(u)$ is a polynomial of degree $\leqq q$. For the same reason as above, $P(u) \equiv 0$. Q.E.D.

Let us set

$$\phi(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|te^{i\theta}-1|} .$$

Then by Edrei-Fuchs [1, p. 303] we have for $0<\beta<1$

$$\int_0^\infty \phi(t)t^{\beta-1}dt \leqq \frac{\pi^2}{\Gamma^4(3/4)\sin(\pi\beta)} . \tag{3.6}$$

LEMMA 3. *The above canonical function F satisfies*

$$\int_{\partial B(r)} \log^+|F|\eta \leqq \frac{1}{2}\int_0^r \frac{n(t,D)}{t}dt + \frac{r^q}{2}\int_0^\infty n(t,D)t^{-q-1}\phi\Big(\frac{t}{r}\Big)dt . \tag{3.7}$$

Furthermore in case $q=0$ we have

$$\int_{\partial B(r)} \log^+|F|\eta \leqq \int_0^r \frac{n(t,D)}{t}dt + r\int_r^\infty \frac{n(t,D)}{t^2}dt . \tag{3.8}$$

Proof. First we show (3.7). From Lemma 2 and Edrei-Fuchs [1, p. 302] we obtain for $u\in\ell\in \boldsymbol{P}^{n-1}(\boldsymbol{C})$ with $\|u\|=r$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log^+|F_\ell(ue^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|F_\ell(ue^{i\theta})|}d\theta \\ \leqq r^q\int_0^\infty \frac{n(t,\ell\cdot D)}{t^{q+1}}\phi\Big(\frac{t}{r}\Big)dt .\end{aligned}$$

From Nevanlinna's first main theorem and $F_\ell(0)=1$ it follows that

$$\frac{1}{\pi}\int_0^{2\pi}\log^+|F_\ell(ue^{i\theta})|\,d\theta \leqq N(r,\ell\cdot D) + r^q\int_0^\infty \frac{n(t,\ell\cdot D)}{t^{q+1}}\phi\Big(\frac{t}{r}\Big)dt . \tag{3.9}$$

Letting $\lambda(\ell)$ denote the standard volume form on $\boldsymbol{P}^{n-1}(\boldsymbol{C})$ defined by χ, we have

$$\int_{\partial B(r)} \log^+ |F| \eta = \int_{\ell \in P^{n-1}(C)} \lambda(\ell) \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F_\ell(ze^{i\theta})| \, d\theta \, , \tag{3.10}$$

where $z \in \ell$ and $\|z\| = r$. Since $n(t, D) = \int n(t, \ell \cdot D)\lambda(\ell)$ by definition, using Fubini's theorem we get (3.7) from Lemma 2, (3.9) and (3.10).

In case $q = 0$ we have by Lemma 2 and Hayman [4, p. 28]

$$\log |F_\ell(u)| \leqq \int_0^r \frac{n(t, \ell \cdot D)}{t} dt + r \int_r^\infty \frac{n(t, \ell \cdot D)}{t^2} dt$$

for $u \in \ell \in \boldsymbol{P}^{n-1}(\boldsymbol{C})$ with $\|u\| = r$. Then the rest of the proof is similar to the above. Q.E.D.

4. Representation of meromorphic mappings

In this section let us fix a homogeneous coordinate system $(w_0; \cdots; w_N)$ in $\boldsymbol{P}^N(\boldsymbol{C})$. Then we may take

$$\begin{aligned} h &= \sum_{\mu=0}^N |w_\mu|^2 / |w_\nu|^2 \qquad \text{if } w_\nu \neq 0 \, , \\ \omega &= dd^c \log \left(\sum_{\mu=0}^N |w_\mu|^2 \right) . \end{aligned} \tag{4.1}$$

A meromorphic mapping $f: \boldsymbol{C}^n \to \boldsymbol{P}^N(\boldsymbol{C})$ is represented as

$$f = (f_0; \cdots; f_N) \, , \tag{4.2}$$

where f_μ are entire functions and codim $\{f_0 = \cdots = f_N = 0\} \geqq 2$. If $f = ('f_0; \cdots; 'f_N)$ is another representation of f, then there is an entire function g such that $'f_\mu = e^g f_\mu$ for all μ. By (4.1) and (4.2) we have

$$T(r, f) = \int_{\partial B(r)} \log \left(\sum_{\mu=0}^N |f_\mu|^2 \right)^{1/2} \eta - \log \left(\sum_{\mu=0}^N |f_\mu(0)|^2 \right)^{1/2} \tag{4.3}$$

provided that $\sum_{\mu=0}^N |f_\mu(0)|^2 \neq 0$, i.e., f is holomorphic at the origin.

LEMMA 4. *Let $f: \boldsymbol{C}^n \to \boldsymbol{P}^N(\boldsymbol{C})$ be a meromorphic mapping of order $< q + 1$ and suppose that $f^*\{w_\mu = 0\}$, $\mu = 0, \cdots, N$ do not contain the origin. Then f is represented as*

$$f = (F_0; F_1 e^{P_1}; \cdots; F_N e^{P_N}) \, ,$$

where each F_μ is the canonical function of genus q associated with $f^*\{w_\mu = 0\}$ if $f^*\{w_\mu = 0\} \neq \phi$, or $\equiv 1$ if $f^*\{w_\mu = 0\} = \phi$ and P_μ are polynomials in $z_1, \cdots, z_n$ of degree $\leqq q$.

Proof. By the assumption and (2.1) the orders of $f^*\{w_\mu = 0\}$ are less than $q+1$. Thus we may take the canonical functions F_μ of genus q associated with $f^*\{w_\mu = 0\}$ (if $f^*\{w_\mu = 0\} = \phi$, we take $F_\mu \equiv 1$). f is represented as

$$f = (F_0; F_1 e^{g_1}; \cdots; F_N e^{g_N}) , \tag{4.4}$$

where g_μ are entire functions. Hence it suffices to show that the order of e^{g_μ}, say e^{g_1}, is less than $q+1$. From (4.4), (4.1) and (2.1) it follows that

$$\int_{\partial B(r)} \log^+ \left| \frac{F_1}{F_0} e^{g_1} \right| \eta \leqq T(r, f) + O(1) . \tag{4.5}$$

Noting that $\log^+ ab \leqq \log^+ a + \log^+ b$, we have

$$\int_{\partial B(r)} \log^+ |e^{g_1}| \eta \leqq \int_{\partial B(r)} \log^+ \left| \frac{F_1}{F_2} e^{g_1} \right| \eta + \int_{\partial B(r)} \log^+ |F_0| \eta + \int_{\partial B(r)} \log^+ \frac{1}{|F_1|} \eta .$$

From (2.2),

$$\int_{\partial B(r)} \log^+ \frac{1}{|F_1|} \eta \leqq \int_{\partial B(r)} \log^+ |F_1| \eta + O(1) .$$

So we see that

$$\int_{\partial B(r)} \log^+ |e^{g_1}| \eta \leqq T(r, f) + T(r, F_0) + T(r, F_1) + O(1) .$$

As the orders of f, F_0 and F_1 are less than $q+1$, so is that of e^{g_1}.

Q.E.D.

5. Proof of Theorem 1

First we take a homogeneous coordinate system $(w_0; w_1; \cdots; w_N)$ in $P^N(C)$ so that $H_\mu = \{w_\mu = 0\}$. Let q denote the largest integer not exceeding ρ. By Lemma 4, f is represented as

$$f = (F_0; F_1 e^{P_1}; \cdots; F_N e^{P_N}) .$$

By (4.3) and Lemma 4 we see that

$$\begin{aligned} T(r, f) &\leqq \sum_{\mu=0}^{N} \int_{\partial B(r)} \log^+ |F_\mu| \eta + \sum_{\mu=1}^{N} \int_{\partial B(r)} \log^+ |e^{P_\mu}| \eta + O(1) \\ &\leqq \sum_{\mu=0}^{N} \int_{\partial B(r)} \log^+ |F_\mu| \eta + O(r^q) . \end{aligned}$$

Now we apply Lemma 3 to this. Setting $n(t)=\sum_{\mu=0}^{N} n(t, f^*H_\mu)$ and $N(r)=\int_0^r n(t)dt/t$, we get from (3.7)

$$2T(r, f) \leqq N(r) + r^q \int_0^\infty n(t)t^{-q-1}\phi\left(\frac{t}{r}\right)dt + O(r^q) .$$

Similarly to Edrei-Fuchs [1, § 4] this inequality yields

$$2 - K(f) \leqq K(f)\rho \int_0^\infty t^{\rho-q-1}\phi(t)dt .$$

From this and (3.6) we deduce that

$$K(f) \geqq \frac{2\Gamma^4(3/4)\,|\sin \pi\rho|}{\pi^2\rho + \Gamma^4(3/4)\,|\sin \pi\rho|} .$$

Hence we have (1.2). Q.E.D.

6. Proof of Theorem 2

As in the previous section, f may be represented as

$$f = (F_0\,;\, c_1F_1\,;\, \cdots\,;\, c_NF_N) ,$$

where c_μ are non-zero constants. By (4.3) we have

$$T(r, f) \leqq \sum_{\mu=0}^{N} \int_{\partial B(r)} \log^+ |F_\mu|\,\eta + O(1) .$$

Using the same notation $n(t)$ and $N(r)$ as in section 5, we have by Lemma 3

$$T(r, f) \leqq N(r) + r\int_r^\infty \frac{n(t)}{t^2}dt + O(1) .$$

In view of integration by parts this implies

$$T(r, f) \leqq r\int_r^\infty \frac{N(t)}{t^2}dt + O(1) . \tag{6.1}$$

Noting that the order of $N(r)$ is ρ, by Hayman [4, Lemma 4.7] we can take a sequence $'r \uparrow \infty$ for an arbitrarily small $\varepsilon > 0$ such that

$$N(t) \leqq \left(\frac{t}{'r}\right)^{\rho+\varepsilon} N('r) \qquad \text{for } t \geqq 'r . \tag{6.2}$$

From (6.1) and (6.2) we get

$$T(r, f) \leqq {}'r^{1-\rho-\varepsilon} N('r) \int_{'r}^{\infty} t^{\rho+\varepsilon-2} dt + O(1)$$
$$= \frac{N('r)}{1-\rho-\varepsilon} + O(1) .$$

Thus $K(f) \geqq 1 - \rho - \varepsilon$. Letting $\varepsilon \to 0$, we deduce (1.3). Q.E.D.

REFERENCES

[1] Edrei, A. and Fuchs, W. H. J., On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc. **93** (1959), 292–328.

[2] ——, The deficiencies of meromorphic functions of order less than one, Duke Math. J. **27** (1964), 233–249.

[3] Griffiths, P. and King, J., Nevanlinna theory and holomorphic mappings between algebraic varieties, Acta Math. **130** (1973), 145–220.

[4] Hayman, W. K., Meromorphic functions, Oxford University Press, London, 1964.

[5] Lelong, P., Fonctions entières (n variables) et fonctions plurisousharmoniques d'ordre fini dans C^n, J. Analyse Math. **12** (1964), 365–407.

[6] ——, Fonctions plurisousharmoniques et formes différentielles positives, Gordon and Breach, Paris, 1968.

[7] Nevanlinna, R., Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris, 1929.

[8] Ozawa, M., On the growth of algebroid functions with several deficiencies, II, Kōdai Math. Sem. Rep. **22** (1970), 129–137.

[9] Stoll, W., Ganze Funktionen endlicher Ordnung mit gegebenen Nullstellenflächen, Math. Z. **57** (1953), 211–237.

[10] Toda, N., Sur la croissante de fonctions algebroides à valeurs deficientes, Kōdai Math. Sem. Rep. **22** (1970), 324–337.

Department of Mathematics
Faculty of Science
Hiroshima University