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ON A HYPOELLIPTIC BOUNDARY VALUE PROBLEM

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§ 1. Introduction.

This paper is devoted to the investigation of the hypoellipticity of the following first boundary value problem:

$$(1.1)\qquad \begin{aligned} Lu &= u_{tt} + (a(x,t)u_x)_x + g(x,t)u_{xt} + b(x,t)u_x + b^0(x,t)u_t + c(x,t)u \\ &= f(x,t) \qquad \text{in } \Omega\,, \end{aligned}$$

$$(1.2)\qquad u(x,t)|_{t=0} = 0\,, \qquad |x| < R\,,$$

where Ω is an open rectangular domain in (x,t)-plane:

$$\Omega = (-R < x < R) \times (0 < t < T) \qquad R > 0,\ T > 0\,.$$

We assume that the coefficients $a(x,t), b(x,t), b^0(x,t)$ and $c(x,t)$ are all C^∞ functions in $\bar{\Omega}$ satisfying the following conditions:

$$(1.3)\qquad \operatorname{Re} a(x,t) \geqq 0 \qquad \text{in } \bar{\Omega}\,,$$

(1.4) for all x with $|x| < R$, the function $t \mapsto \operatorname{Re} a(x,t)$ has only finite zeros of order less than or equal to ℓ ($\geqq 0$) in the interval $[0 \leqq t \leqq T]$

$$(1.5)\qquad |\operatorname{Im} a(x,t)| \leqq C^{1)} \operatorname{Re} a(x,t) \qquad \text{in } \bar{\Omega}\ (C > 0)\,,$$

$$(1.6)\qquad |\operatorname{Im} a_x(x,t)| \leqq C[\operatorname{Re} a(x,t)]^{1/2} \qquad \text{in } \bar{\Omega}\,,$$

$$(1.7)\qquad t\,|\operatorname{Im} b(x,t)|^2 \leqq C \operatorname{Re} a(x,t) \qquad \text{in } \bar{\Omega}\,,$$

$$(1.8)\qquad |g(x,t)| \leqq \frac{\varepsilon_1}{2}[\operatorname{Re} a(x,t)]^{1/2} \qquad \text{in } \bar{\Omega},\ 0 < \varepsilon_1 < 1\,,$$

$$(1.9)\qquad |g_t(x,t)| \leqq C[\operatorname{Re} a(x,t)]^{1/2} \qquad \text{in } \bar{\Omega}\,.$$

We set $\tilde{\Omega} = (-R < x < R) \times [0 \leqq t < T)$. The main result of this paper is to prove the following theorem.

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1) We use the symbols $C, C^1, \ldots$ to express the different positive constants throughout this paper.

THEOREM 1.1. *Suppose that the operator L given in* (1.1) *satisfies the condition* (1.3)$\sim$(1.9). *Then any distribution* $u \in \mathscr{D}'(\tilde{\Omega})$ *satisfying* (1.1) *and* (1.2)[2] *with* $f(x,t) \in C^\infty(\tilde{\Omega})$ *must be a* C^∞ *function in* $\tilde{\Omega}$.

We remark that if we consider the partial differential operator of first order

$$(1.10) \qquad L_1 = \frac{\partial}{\partial t} + ia(x,t)\frac{\partial}{\partial x} + c(x,t) \qquad \text{in } \Omega ,$$

a sufficient condition of Nirenberg and Treves (cf. [10], [11]) for the operator L_1 to be hypoelliptic is expressed by (1.3) and (1.4). This is a necessary and sufficient condition when $a(x,t)$ is analytic in $\bar{\Omega}$. Our problem is motivated by this fact (cf. [6]) and the proof of Theorem 1.1 will be obtained in the following paragraphs by a refinement of the method used in [2] and [4]. For the equations of the second order with real coefficients we refer to [2] and [9].

EXAMPLES. The following operators satisfy the condition (1.3)$\sim$(1.7) in a neighbourhood of the origin.

$$(1.11) \quad L_2 = \frac{\partial^2}{\partial t^2} + ta(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} + b^0(x,t)\frac{\partial}{\partial t} + c(x,t) ,$$

$\operatorname{Re} a(x,t) > 0$ in $\bar{\Omega}, b, b^0$ and c are arbitrary complex valued C^∞ functions in $\bar{\Omega}$,

$$(1.12) \quad \begin{aligned} L_3 = {} & \frac{\partial^2}{\partial t^2} + t^3[t-g(x)]^{2\ell}\frac{\partial^2}{\partial x^2} + (1+i)t[t-g(x)]^{\ell}\frac{\partial}{\partial x} \\ & + b^0(x,t)\frac{\partial}{\partial t} + c(x,t) , \end{aligned}$$

ℓ integer, $\geqq 0$; $g(x)$ is a real valued C^∞ function in $(-R \times x \times R), b^0, c$ are arbitrary C^∞ functions in $\bar{\Omega}$.

§ 2. Preliminaries for the proof of Theorem 1.1.

LEMMA 2.1. ([9], Lemma 1.7.1) *Let* $a(x,t)$ *be the function given in* § 1. *Then there exists a positive constant* C *such that*

$$(2.1) \qquad |a_x(x,t)|^2 \leqq C \operatorname{Re} a(x,t) \qquad (x,t) \in \bar{\Omega} .$$

Being suggested by [2] and [4], we now introduce the norm $|||\cdot|||$

2) By the partial hypo-ellipticity of L in t, condition (1.2) is meaningful in the sense of distributions (cf. [1], Ch. 4).

and its dual norm $|||\cdot|||'$ by

$$|||u|||^2 = ||u_t||^2 + ||\sqrt{\mathrm{Re}\, a}u_x||^2 + ||u||^2 ,$$
$$|||v|||' = \sup_{w\in C_0^\infty(\tilde{\Omega})} \frac{|\langle v, w\rangle|}{|||w|||} ,$$

where $||\cdot||$ is the usual L^2-norm on $\tilde{\Omega}$ and $\langle v, w\rangle$ is the value of $v \in \mathscr{D}'(\tilde{\Omega})$ evaluated at w.

LEMMA 2.2. *Let L be the operator given in* (1.1). *We have the following estimate with some positive constant C*

$$|||v||| \leqq C\,||v|| + |||Lv|||' , \qquad v \in C_0^\infty(\tilde{\Omega}),\ v(x,0) = 0 . \tag{2.2}$$

Proof. Obviously we have

$$|\langle Lv, \bar{v}\rangle| \leqq |||Lv|||'\,|||v||| , \qquad v \in C_0^\infty(\tilde{\Omega}),\ v(x,0) = 0 . \tag{2.3}$$

Next, integrating by parts, we have

$$-\mathrm{Re}\,\langle Lv, \bar{v}\rangle = ||v_t||^2 + ||\sqrt{\mathrm{Re}\,a}v_x||^2 + \mathrm{Re}\,\langle g_t v_x, \bar{v}\rangle + \mathrm{Re}\,\langle g v_x, \bar{v}_t\rangle$$
$$- \mathrm{Re}\,\langle b v_x, \bar{v}\rangle - \mathrm{Re}\,\langle b^0 v_t, \bar{v}\rangle - \mathrm{Re}\,\langle cv, \bar{v}\rangle .$$

For the term of the right hand side, we have for any positive number δ

$$|\langle \mathrm{Im}\, b v_x, \bar{v}\rangle| = |\langle t^{1/2}\,\mathrm{Im}\, b v_x, t^{-1/2}\bar{v}\rangle|$$
$$\leqq \delta\langle t\,|b|^2 v_x, \bar{v}_x\rangle + \frac{1}{\delta}\,||t^{-1/2}v||^2 .$$

On the other hand, for any $\varepsilon > 0$, we easily have

$$||t^{-1/2}v||^2 \leqq \varepsilon\,||v_t||^2 + C(\varepsilon)\,||v||^2 .$$

Thus, by virtue of the assumption (1.7), we are given the inequality

$$|\langle \mathrm{Im}\, b v_x, \bar{v}\rangle| \leqq \delta\,||\sqrt{\mathrm{Re}\,a}v_x||^2 + \frac{\varepsilon}{\delta}\,||v_t||^2 + \frac{C(\varepsilon)}{\delta}\,||v||^2 .$$

For the remaining terms we have

$$|\langle g_t v_x, \bar{v}\rangle| \leqq \delta\,||\sqrt{\mathrm{Re}\,a}v_x||^2 + \frac{C(\delta)}{\delta}\,||v||^2 ,$$
$$|\langle g v_x, \bar{v}_t\rangle| \leqq \varepsilon_1(||\sqrt{\mathrm{Re}\,a}v_x||^2 + ||v_t||^2) ,$$
$$|\langle b^0 v_t, \bar{v}\rangle| \leqq \delta\,||v_t||^2 + \frac{C'}{\delta}\,||v||^2$$
$$|\langle cv, \bar{v}\rangle| \leqq C''\,||v||^2 .$$

Taking δ and ε sufficiently small, we have

$$-\mathrm{Re}\langle Lv, \bar{v}\rangle \geqq C_1 |||v|||^2 - C_2 ||v||^2$$
$$\geqq C_1 |||v|||^2 - C_2 ||v|| \cdot |||v||| .$$

This, combining with (2.3), gives the estimate (2.2). Q.E.D.

LEMMA 2.3. *Let $a(x, t)$ be as above, then we have*

$$|||a_x v_x|||' \leqq C ||v|| , \tag{2.4}$$

$$|||a v_x|||' \leqq C ||v|| , \tag{2.5}$$

$$|||\mathrm{Im}\, b v_x|||' \leqq C(||v_t|| + ||v||) \tag{2.6}$$

for all $v \in C_0^\infty(\tilde{\Omega})$, ($v(x, 0) = 0$ *for* (2.6)) *with some positive constant* C.

Proof. For any $w \in C_0^\infty(\tilde{\Omega})$, we have

$$\langle a_x v_x, w\rangle = \langle v_x, a_x w\rangle$$
$$= -\langle v, a_x w_x\rangle - \langle v, a_{xx} w\rangle .$$

Taking account of the assumption (1.6) and Lemma 2.1, we have

$$|\langle a_x v_x, w\rangle| \leqq C ||v|| \cdot |||w||| ,$$

from which follows the estimate (2.4). By the same way we have (2.5). As in the proof of Lemma 2.2, we have

$$\langle \mathrm{Im}\, b v_x, w\rangle = -\langle v, \mathrm{Im}\, b w_x\rangle - \langle v, \mathrm{Im}\, b_x w\rangle ,$$
$$|\langle \mathrm{Im}\, b v_x, w\rangle| \leqq |\langle t^{-1/2} v, t^{1/2}\, \mathrm{Im}\, b w_x\rangle| + C ||v|| \, ||w||$$
$$\leqq C' (||v_t|| + ||v||) \, |||w||| ,$$

which imply (2.6).

Now we introduce the norm $|| \cdot ||_{(s,k)}$, with s any real number and k non negative integer (cf. [1], § 2.6), defined by

$$||v||^2_{(s,k)} = (2\pi)^{-1} \int_0^\infty \int_{R_\xi} |\hat{v}(\xi, t)|^2 (1 + |\xi|^2)^s d\xi dt + \sum_{j=0}^{k} ||D_t^j v||^2_{L^2(R_+^2)} ,$$
$$\hat{v}(\xi, t) = \int e^{-ix\xi} v(x, t) dx , \qquad v \in C_0^\infty(\bar{R}_+^2) .$$

We denote by $H_{(s,k)}(\bar{R}_+^2)$ the completion of $C_0^\infty(\bar{R}_+^2)$ in the norm $|| \cdot ||_{(s,k)}$.

LEMMA 2.4. *There exists a positive constant* C *such that*

$$||v||_{(1/(\ell+1),1)} \leqq C |||v||| , \qquad v \in C_0^\infty(\tilde{\Omega}),\ v(x, 0) = 0 . \tag{2.7}$$

Proof. Since we have

$$\operatorname{Re} a(x,t) \leqq C_1\sqrt{\operatorname{Re} a(x,t)}\ , \qquad (x,t)\in\bar{\Omega}\ ,$$
$$\leqq C_1^2\,|||v|||^2\ , \qquad v\in C_0^\infty(\tilde{\Omega})\ .$$

If we consider the differential operator

$$L_4 = \frac{\partial}{\partial t} + i \operatorname{Re} a(x,t)\frac{\partial}{\partial x} \qquad \text{in } \tilde{\Omega}\ ,$$

we have

$$||L_4 v||^2 = ||v_t||^2 + ||\operatorname{Re} a v_x||^2\ .$$

On the other hand, as a particular case of Theorem I in [10], it follows that

$$||v||_{(1/(\ell+1),1)} \leqq C_2||L_4 v|| \qquad v\in C_0^\infty(\tilde{\Omega}),\ v(x,0)=0$$

with another constant $C_2 > 0$. Combining the above investigations, we have (2.7). Q.E.D.

By using Lemma 2.2 and Lemma 2.4 we now come to the main estimate:

LEMMA 2.5. *There exists a positive constant C such that*

$$(2.8)\quad ||v||_{(\varepsilon,1)} \leqq C(||v|| + |||Lv|||')\ , \quad v\in C_0^\infty(\tilde{\Omega}),\ v(x,0)=0,\ \left(\varepsilon = \frac{1}{\ell+1}\right).$$

LEMMA 2.6. *Every* $v\in H_{(0,2)}(R_+^2)\cap \mathscr{E}'(\tilde{\Omega})$ *such that* $v(x,0)=0$ *and* $|||Lv|||' < \infty$ *belongs to* $H_{(\varepsilon,2)}(R_+^2)$ *with* $\varepsilon = \dfrac{1}{\ell+1}$.

Proof. The inequality (2.8) is valid for all $v\in H_{(2,2)}(R_+^2)\cap\mathscr{E}'(\tilde{\Omega})$, $v(x,0)=0$. Indeed, we can find a sequence $v_j\in C_0^\infty(\tilde{\Omega})$ such that $v_j(x,0) = 0$, $D_x^\alpha D_t^\beta v_j - D_x^\alpha D_t^\beta v \to 0$, $j\to\infty$, when $\alpha+\beta\leqq 2$. Hence $||Lv_j - Lv|| \to 0$, which implies that $|||Lv_j - Lv|||' \to 0$. In particular,

$$\overline{\lim}\,|||Lv_j|||' \leqq |||Lv|||'\ .$$

So it follows from (2.8) applied to v_j that

$$\overline{\lim}\,||v_j||_{(\varepsilon,1)} \leqq C(||v|| + |||Lv|||')\ .$$

Next if v satisfies the required conditions, we choose $\chi\in D_0^\infty(\tilde{\Omega})$ so that $0\leqq\chi\leqq 1$ and $\chi=1$ in a neighbourhood ω of supp. v and we set

$$v_\delta = \chi(1 - \delta^2\Delta)^{-1}v \ .$$

Here $(1 - \delta^2\Delta)^{-1}v$ is defined as the inverse Fourier transform of $(1 + \delta^2|\xi|^2)^{-1}\hat{v}(\xi, t)$:

$$v_\delta = (2\pi)^{-1}\int_{R_\xi} e^{ix\xi}(1 + \delta^2\xi^2)^{-1}\hat{v}(\xi, t)d\xi \ .$$

It is clear that v_δ is then in $H_{(2,2)}(R^2_+) \cap \mathscr{E}'(\tilde{\Omega})$, and that $v_\delta \to v$ in L^2 norm as $\delta \to 0$. Hence we may apply (2.8) to v_δ to conclude that $\|v\|_{(\varepsilon,1)} < \infty$ provided that we can show that $|||Lv_\delta|||'$ remains bounded when $\delta \to 0$. To prove the last assertion we must prepare some remarks which correspond to $1^\circ \sim 4^\circ$ of [2].

1°. We have

$$\frac{1}{2}e^{-|x|} = \mathscr{F}^{-1}[(1 + \xi^2)^{-1}] = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{ix\xi}\frac{d\xi}{1 + \xi^2} \ , \qquad -\infty < x < \infty \ .$$

Since

$$v_\delta = (1 - \delta^2\Delta)^{-1}v(x, t) = \delta^{-1}\int K\Big(\frac{x - y}{\delta}\Big)v(y, t)dy \ , \qquad K(x) = \frac{1}{2}e^{-|x|} \ ,$$

it follows that any derivative of $(1 - \delta^2\Delta)^{-1}v(x, t)$ decreases faster than any power of δ as $\delta \to 0$ if $(x, t) \notin \omega$.

2°. If Q is a differential operator of order $j \leqq 2$ $\left(\text{in } \dfrac{\partial}{\partial x}\right)$ with coefficients in $C^\infty(\bar{\Omega})$, it follows that

$$\|(1 - \delta^2\Delta)^{-1}Qu\| \leqq C\|u\| \ , \qquad u \in L^2(\tilde{\Omega}) \cap \varepsilon'(\tilde{\Omega}) \ . \tag{2.9}$$

3°. When $\chi \in C_0^\infty(\tilde{\Omega})$ we have

$$|||\chi(1 - \delta^2\Delta)^{-1}w||| \leqq C\,|||w||| \ , \qquad w \in C_0^\infty(\tilde{\Omega}) \ . \tag{2.10}$$

Indeed, we have

$$\|[\chi(1 - \delta^2\Delta)^{-1}w]_t\| \leqq C(\|w_t\| + \|w\|) \ , \qquad w \in C_0^\infty(\tilde{\Omega}) \tag{2.11}$$

and

$$\begin{aligned} &|\langle \mathrm{Re}\, a(x, t)(\chi(1 - \delta^2\Delta)^{-1}w)_x, \overline{(\chi(1 - \delta^2\Delta)^{-1}w)_x}\rangle| \\ &\quad \leqq C(\|w\|^2 + \|\sqrt{\mathrm{Re}\, a}(1 - \delta^2\Delta)^{-1}w_x\|^2) \ . \end{aligned}$$

For the second term of the right hand side, we have

$$\begin{aligned}&||\sqrt{\operatorname{Re} a}(1-\delta^2\Delta)^{-1}w_x||^2\\&\quad=||(1-\delta^2\Delta)^{-1}\sqrt{\operatorname{Re} a}w_x+[\sqrt{\operatorname{Re} a}D_x,(1-\delta^2\Delta)^{-1}]w||^2\\&\quad\leqq 2\,||\sqrt{\operatorname{Re} a}w_x||^2+2\,||[\sqrt{\operatorname{Re} a}D_x,(1-\delta^2\Delta)^{-1}]w||^2\,.\end{aligned}$$

By virtue of 1°, partial integration proves

$$\begin{aligned}&[\sqrt{\operatorname{Re} a}D_x,(1-\delta^2\Delta)^{-1}]w(x,t)\\&\quad=\frac{A}{\delta}\int\exp\Big(-\frac{|x-y|}{\delta}\Big)(\sqrt{\operatorname{Re} a(x,t)}-\sqrt{\operatorname{Re} a(y,t)})w_y dy\\&\quad=\frac{A}{\delta}\int\exp\Big(-\frac{|x-y|}{\delta}\Big)(\sqrt{\operatorname{Re} a(y,t)})_y w(y,t)dy\\&\qquad+\frac{A}{\delta^2}\int\exp\Big(-\frac{|x-y|}{\delta}\Big)(\sqrt{\operatorname{Re} a(x,t)}-\sqrt{\operatorname{Re} a(y,t)})w(y,t)dy\,.\end{aligned}$$

By Lemma 2.1, we can see that $\sqrt{\operatorname{Re} a(x,t)}$ is uniformly Lipschitz continuous in x and thus the L^2 norm of the last two terms is bounded above by $||w||^2$. This estimate combined with (2.11) gives (2.10).

Completion of the proof of Lemma 2.6. We recall that with the notations introduced above it remains to prove that $|||Lv_\delta|||'$ is bounded as $\delta\to 0$. In the neighbourhood ω of supp. v we have $(1-\delta^2\Delta)v_\delta=v$ and

$$\begin{aligned}(1-\delta^2\Delta)Lv_\delta&=(1-\delta^2\Delta)(D_t^2v_\delta+(a(x,t)v_{\delta x})_x\\&\quad+(g(x,t)v_{\delta xt})+bv_{\delta xt}+b^0v_{\delta t}+cv_\delta\\&=v_{tt}+(a(x,t)v_x)_x+gv_{xt}+bv_x+b^0v_t+cv\\&\quad-2\delta^2(a_x(x,t)v_{\delta xx})_x-\delta^2(a_{xx}(x,t)v_{\delta x})_x\\&\quad-2\delta^2g_xv_{\delta xx}-\delta^2g_{xx}v_{\delta xt}\\&\quad-2\delta^2b_xv_{\delta xx}-\delta^2b_{xx}v_\delta x-2\delta^2b_x^0v_{\delta tx}\\&\quad-\delta^2b_{xx}^0v_{\delta t}-2\delta^2c_xv_{\delta x}-2\delta^2c_{xx}v_\delta\,.\end{aligned}$$

In view of 1° it follows that we have

$$(1-\delta^2\Delta)Lv=Lv+2\delta^2(a_x(x,t)v_{\delta xx})_x+\delta^2B_1v_\delta+\delta^2B_2v_{\delta t}+h_\delta\,,$$

where B_1 and B_2 are second order $\left(\text{in }\dfrac{\partial}{\partial x}\right)$ operators, and where h_δ is a function such that it vanishes in ω, supp. $h_\delta\subset$ supp. χ and $||h_\delta||\to 0$ as $\delta\to 0$. Hence

$$Lv_\delta = \chi_1\{(1-\delta^2\Delta)^{-1}Lv + 2\delta^2(1-\delta^2\Delta)^{-1}(a_x v_{\delta xx})_x + \delta^2(1-\delta^2\Delta)^{-1}B_1 v_\delta + \delta^2(1-\delta^2\Delta)^{-1}B^2 v_{\delta t} + (1-\delta^2\Delta)^{-1}h_\delta\,, \tag{2.12}$$

where χ_1 is a function in $C_0^\infty(\tilde\Omega)$ which is equal to 1 in supp. χ. We remark that from 3° we have

$$|||\chi_1(1-\delta^2\Delta)^{-1}f|||' \leqq C\,|||f|||'\,, \qquad f \in \mathscr{D}'(\tilde\Omega) \cap \mathscr{E}'(\tilde\Omega)\,. \tag{2.13}$$

Therefore, it follows that

$$|||\chi_1(1-\delta^2\Delta)^{-1}Lv|||' \leqq C\,|||Lv|||'\,.$$

The last three terms of (2.12) are bounded in L^2 norm in view of 2° and by the assumption $v \in {}_{(0,2)}(\tilde\Omega) \cap \mathscr{E}'(\tilde\Omega)$. For the second term, by (2.13), 2° and (2.4), we have

$$\begin{aligned} &|||\chi_1\delta^2(1-\delta^2\Delta)^{-1}(a_x v_{\delta xx})_x|||' \\ &\leqq C\,|||\delta^2(a_x v_{\delta xx})_x|||' \\ &\leqq 2C\,(|||\delta^2 a_{xx} v_{\delta xx}|||' + |||\delta^2 a_x D_x v_{\delta xx}|||') \\ &\leqq C'\,\|v_\delta\| \leqq C''\,\|v\|\,. \end{aligned}$$

This completes the proof of Lemma 2.6.

§ 3. Proof of Theorem 1.1.

Given a function $\psi(x,t) \in C_0^\infty(\tilde\Omega)$ and an integer $k \geqq 2$, we may assume, by the partial hypoellipticity of L in t (cf. [1], § 4.3), that $\psi u \in H_{(s,k)}(\bar R_+^2) \cap \mathscr{E}'(\tilde\Omega)$ for some real number s. For the proof of Theorem 1.1 it suffices to show that s can be replaced by $s+\varepsilon, \varepsilon = \dfrac{1}{\ell+1}$. Indeed, it follows that $u \in H_{(s,k)}^{\mathrm{loc}}(\tilde\Omega)$ for any s and k, which means that $u \in C^\infty(\tilde\Omega)$.

Let E be a pseudo-differential operator (in x) with symbol $e(\xi) = (1+\xi^2)^{s/2}$ (cf. [3]), and set $v = \chi E\psi u$ where $\chi \in C_0^\infty(\tilde\Omega)$. If we can show that $v \in H_{(\varepsilon,k)}$ for every χ and ψ we will have $E\psi u \in H_{(\varepsilon,2)}^{\mathrm{loc}}(\tilde\Omega)$, hence $u \in H_{(s+\varepsilon,2)}^{\mathrm{loc}}$ since E is elliptic. It is clear that $v \in H_{(0,2)}(\bar R_+^2) \cap \mathscr{E}'(\tilde\Omega)$, $v(x,0)=0$, so in view of Lemma 2.6 it remains only to show that $|||Lv|||' < \infty$. We note that $E' = \chi E\psi$ is a compactly supported pseudo-differential operator (in x) of order s with parameter $t \geqq 0$, (cf. [3]) and $Lv = LE'u$. Taking account of $E'Lu = E'f \in L^2(\tilde\Omega)$ and $|||E'f||| < \infty$, it now sufficies to show that

$$|||LE|u - E'f||| < \infty\,.$$

We have

$$LE'u - E'f = 2E'_t u_t + E'_{tt} u + [aD_x^2, E']u + [gD_x D_t, E']u$$
$$+ [bD_x, E']u + [b^0 D_t, E']u + [c, E']u$$
$$= [aD_x^2, E']u + E''u + E'''u_t ,$$

where E'' and E''' are compactly supported pseudo-differential operators (in x) of order $\leqq s$ with parameter $t \geqq 0$. Obviously $||E''u|| < \infty$ and $||E'''u_t|| < \infty$ and $|||gD_x E'_t u|||' < \infty$ by (1.8) and (2.5), so we shall analyse the first term in the right hand side. We have

$$[aD_x^2, E'] = [aD_x, E']D_x + aD_x E'_x ,$$

and

$$|||[aD_x^2, E']u|||' \leqq |||[aD_x, E']u_x|||' + |||aD_x E'_x u|||' .$$

By (2.5) the last term is estimated by $||E'_x u|| < \infty$. For any $w \in C_0^\infty(\tilde{\Omega})$ we have

$$|\langle [aD_x, E']u_x, w\rangle|$$
$$\leqq |\langle [aD_x, E']_x u, w\rangle| + |\langle [aD_x, E']u, w_x\rangle| .$$

Since the order of $[aD_x, E']_x$ is $\leqq s$ the first term is estimated by $C\,||w||$ with another constant C. Let $\sigma[aD_x, E']$ be a symbol of $[aD_x, E']$. A simple calculation (cf. [3]) proves the equality

$$\sigma[aD_x, E'] = a \cdot \sigma(E_1) + a_x \sigma(E_2) + E_3 ,$$

where E_1, E_2 and E_3 are compactly supported pseudo-differential operators (in x) of order $\leqq s$ and $\leqq s-1$, respectively. This equality leads us, by partial integration and by use of (2.4), to the following estimate

$$|\langle [aD_x, E']u, w_x| \leqq C\,|||w||| .$$

The above investigation implies that

$$|||[aD_x^2, E']u|||' < \infty .$$

Thus we have $|||Lv|||' < \infty$ and this completes the proof of Theorem 1.1.

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Added in proof. An investigation for the many variable cases will be given in a future publication.