GAUSSIAN RANDOM FIELDS WITH PROJECTIVE INVARIANCE

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§ 1.   Introduction

We shall consider the class of Gaussian random fields $X_\alpha = \{X(A)\,;\ A \in R^n\}$ such that $E\{X(A) - X(B)\} = 0$ and $E\{(X(A) - X(B))^2\} = |A - B|^\alpha$ $(0 < \alpha < 2)$, where $|A - B|$ denotes the Euclidean distance between two points A and B.

The correlation function of X_α under the condition $X(P) = 0$, which is denoted by $\rho_\alpha(A, B\,|\,P)$, is

$$\rho_\alpha(A, B\,|\,P) = (|A - P|^\alpha + |B - P|^\alpha - |A - B|^\alpha)/2\,|A - P|^{\alpha/2}\,|B - P|^{\alpha/2}\,.$$

Clearly, the equality

$(*)$ $$\rho_\alpha(TA, TB\,|\,TP) = \rho_\alpha(A, B\,|\,P)$$

holds for a translation or an orthogonal transformation T on R^n. This property corresponds to stationarity and isotropy of X_α (see [10]). Moreover, we have $(*)$ when T is a similar transformation or an inversion with center P. We shall call these properties projective invariance.

In § 2, we shall show that projective invariance characterizes the class of X_α's (Theorem 1) and that only the above mentioned transformations satisfy $(*)$ unless $\alpha = n = 1$ (Theorem 2).

In § 3, we shall apply projective invariance to the prediction problem and get the new formulas. In particular, the two important properties of X_α—non-deterministic property and Markov property—will be mentioned and be used in § 4.

In § 4, we shall consider the conditional correlation function of X_α. For a closed set e in R^n, we calculate the conditional expectation of $X(A)$ relative to the σ-field generated by $\{X(P)\,;\,P \in e\}$, which is denoted by $\mu(A\,|\,e)$. Then the conditional correlation function relative to e is defined as follows:

$$\rho_\alpha(A, B \,|\, e) = \frac{E\{(X(A) - \mu(A \,|\, e))(X(B) - \mu(B \,|\, e))\}}{[E\{(X(A) - \mu(A \,|\, e))^2\}E\{(X(B) - \mu(B \,|\, e))^2\}]^{1/2}} \,.$$

We are interested in the problem to determine the transformation T on R^n satisfying the following equality:

$$(**) \qquad\qquad \rho_\alpha(TA, TB \,|\, Te) = \rho_\alpha(A, B \,|\, e) \,.$$

It is easy to see that translations, orthogonal transformations, similar transformations and inversions with the center in e satisfy $(**)$ (Theorem 3). We are able to determine T satisfying $(**)$ in the case n $= 1$ (Theorem 4). There occurs a remarkable difference between the cases $\alpha = 1$ and $\alpha \neq 1$, and this is due to Markov property. The case $\alpha = 1$ in Theorem 4 corresponds to the principle of projective invariance of the Brownian motion ([5]) which was the starting point of our study.

§ 2. Gaussian random field X_α with projective invariance

Let $X = \{X(A) \,;\, A \in R^n\}$ be a Gaussian random field such that $X(A) - X(B)$ has mean 0 and variance $r(A, B)$, where $r(A, B)$ is a function on $R^n \times R^n$ satisfying the following conditions:

$$(1) \quad \begin{cases} r(A, B) = r(B, A), \ r(A, A) = 0, \ r(A, B) \geq 0, \ \text{and} \ r \ \text{is a function} \\ \text{of negative type, i.e.} \ \sum a_i a_j r(A_i, A_j) \leq 0 \ \text{for any} \ A_i \in R^n \ \text{and any} \\ a_i \in R^1 \ \text{such that} \ \sum a_i = 0 \ (i = 1, 2, \cdots, N). \end{cases}$$

We assume in the sequel that $r(A, B)$ is continuous and $r(A, B) \not\equiv 0$.

With the above assumptions we can speak of the probability law only for the system of the differences $\{X(A) - X(B) \,;\, A, B \in R^n\}$. Therefore, we need to impose an additional condition, say $X(P) = 0$, in order to determine $X(A)$ uniquely. (In this section, by P we denote an arbitrarily fixed point in R^n.)

The covariance function of X is

$$E[X(A)X(B) \,|\, X(P) = 0] = (r(A, P) + r(B, P) - r(A, B))/2 \,,$$

and the correlation function of X is defined by

$$(2) \quad \begin{aligned} \rho(A, B \,|\, P) &\equiv E\left[\frac{X(A)}{r(A, P)^{1/2}} \cdot \frac{X(B)}{r(B, P)^{1/2}} \,\middle|\, X(P) = 0\right] \\ &= \frac{r(A, P) + r(B, P) - r(A, B)}{2r(A, P)^{1/2}r(B, P)^{1/2}} \,. \end{aligned}$$

DEFINITION 1. We say that a transformation T on $\mathbf{R}^n$ leaves the correlation function of X invariant if the equality

$$(3) \qquad \rho(TA, TB \,|\, TP) = \rho(A, B \,|\, P)$$

holds for any A, B in $\mathbf{R}^n$. If all the translations, orthogonal transformations, similar transformations and inversions with center P leave the correlation function of X invariant, we say X has projective invariance.

THEOREM 1. *X has projective invariance if and only if*

$$(4) \qquad r(A, B) = |\text{const.}| \cdot |A - B|^\alpha \qquad (0 < \alpha \leq 2) .$$

The proof of the "if" part is clear. The "only if" part is divided into the following two lemmas.

LEMMA 1. *Both translations and orthogonal transformations leave the correlation function of X invariant if and only if $r(A, B)$ depends only on $|A - B|$, i.e. $r(A, B) = r(|A - B|)$.*

The proof of Lemma 1 is elementary, so is omitted.

LEMMA 2. *If all the translations, orthogonal transformations and similar transformations leave the correlation function of X invariant, then (4) holds.*

Proof. By Lemma 1, $r(A, B) = r(|A - B|)$. We can easily show that $r(x) > 0$ whenever $x > 0$. Without loss of generality, we can assume that $r(1) = 1$. Since similar transformations leave the correlation function invariant, we have $\rho(-b, ab \,|\, 0) = \rho(-1, a \,|\, 0)$ for any $a, b > 0$. By (2), we have

$$\rho(-b, ab \,|\, 0) = (r(b) + r(ab) - r((a + 1)b))/2r(b)^{1/2}r(ab)^{1/2} ,$$
$$\rho(-1, a \,|\, 0) = (1 + r(a) - r(a + 1))/2r(a)^{1/2} .$$

Hence,

$$\{(r(a)r(b))^{1/2} - r(ab)^{1/2}\}\{r(b)^{1/2} - (r(a)r(ab))^{1/2}\}$$
$$= r(a)^{1/2}r((a + 1)b) - (r(b)r(ab))^{1/2}r(a + 1) .$$

Therefore, $r(ab) = r(a)r(b)$ yields $r((a + 1)b) = r(a + 1)r(b)$. By induction, we see that $r(nb) = r(n)r(b)$ for any integer n. This implies that $r(qb) = r(q)r(b)$ for any rational number q. It follows from the continuity of

$r(x)$ that $r(ab) = r(a)r(b)$ for any $a, b > 0$. Thus, $r(x) = x^\alpha$ and the fact that the condition (1) implies $0 < \alpha \le 2$ is well known.   (Q.E.D.)

The random field X which corresponds to $r(A, B) = |A - B|^\alpha$ is denoted by X_α and the associated correlation function is denoted by ρ_α as in §1. Since X_2 is a trivial random field, we shall consider only the case $0 < \alpha < 2$ in the sequel.

We introduce a notation. $\mathscr{T}$ is the transformation group generated by translations, orthogonal transformations, similar transformations and inversions. $T|_D$ is the restriction of a transformation T to a subset D of R^n. For a transformation T such that $TA = \tilde{T}A$ on D for some $\tilde{T} \in \mathscr{T}$, we say $T|_D$ belongs to $\mathscr{T}|_D$.

THEOREM 2. *If a transformation T leaves the correlation function of X_α invariant, then T belongs to $\mathscr{T}$ unless $n = \alpha = 1$. For $n = \alpha = 1$, $T_{|(-\infty, P]}$ and $T_{|[P, +\infty)}$ belong to $\mathscr{T}_{|(-\infty, P]}$ and $\mathscr{T}_{|[P, +\infty)}$, respectively.*

For the proof of Theorem 2, we use the following properties of ρ_α. On R^2, fix $B = (1, 0)$ and $P = (0, 0)$ as the origin, and consider the variable $A = (r, \theta)$ with the polar coordinates. Then, we have

$$\rho_\alpha(A, B \,|\, P) = (r^{\alpha/2} + r^{-\alpha/2} - (r + r^{-1} - 2 \cos \theta)^{\alpha/2})/2 \ .$$

i)   When r is fixed, ρ_α is strictly increasing in $0 \le \theta \le \pi$.

ii) When $\theta (\le \pi/2)$ is fixed, ρ_α is strictly increasing in $0 \le r \le 1$.

iii) When $\theta = \pi$, ρ_α is strictly decreasing, identically 0 and strictly increasing in $0 \le r \le 1$ according as $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$.

iv) If $\alpha > 1$, ρ_α takes the minimal value $1 - 2^{\alpha-1}$ at $A = -B$. If $\alpha \le 1$, the minimal value of ρ_α is 0.

Proof of Theorem 2. For T in Theorem 2, take the translation T_0 such that $T_0(TP) = P$. Then, $T_0 \circ T$ leaves the correlation function invariant and fixes the point P. Therefore, we have only to prove Theorem 2 under the assumption $TP = P$.

The proof in the case $n = 1$ is simple. We show here only the case $n = 2$. For a general n we can prove in a similar manner. On R^2 we use the polar coordinates (r, θ) with origin P.

When $\alpha \ne 1$, we first claim that $T|_S \in \mathscr{T}|_S$ for a circle $S = \{r = a\}$. We may assume that $T(a, 0) = (a, 0)$. For T which does not fix the point $(a, 0)$, we find a transformation T_1 in $\mathscr{T}$ such that $T_1(T(a, 0)) = (a, 0)$ and

$T_1 P = P$, and consider $T_1 \circ T$ instead of T. Under this assumption we show that $T(a, \pi) = (a, \pi)$. When $\alpha > 1$, $\rho_\alpha(A, (a, 0) | P)(= \rho_\alpha(TA, (a, 0) | P))$ takes the minimal value only at (a, π), so we have $T(a, \pi) = (a, \pi)$. While, when $\alpha < 1$, we describe the contour $\{A ; \rho_\alpha(A, (a, 0) | P) = \rho_\alpha((a, \pi), (a, 0) | P)\}$ and see that it crosses itself just at (a, π). Hence we have $T(a, \pi) = (a, \pi)$ by the fact that T is a homeomorphism and leaves the contour invariant. Since $(a, 0)$ and (a, π) are invariant under T, we see that $T(a, \pi/2)$ is either $(a, \pi/2)$ or $(a, -\pi/2)$ by using the properties (i), (ii) of ρ_α. First assume that $T(a, \pi/2) = (a, \pi/2)$. Since $(a, 0)$ and $(a, \pi/2)$ are invariant under T, so is $(a, \pi/4)$. Thus all the points $(a, k\pi 2^{-n})$ are invariant under T $(-2^n < k \leq 2^n ; n = 0, 1, 2, \cdots)$. By the continuity of T, we have $T(a, \theta) = (a, \theta)$ for any point (a, θ) in S. On the other hand, if $T(a, \pi/2) = (a, -\pi/2)$, we can prove that $T(a, \theta) = (a, -\theta)$ by the similar argument. Namely, we have proved that $T|_S \in \mathcal{T}|_S$.

For any $r_1 \neq r_2$, we can write $T|_{\{r = r_i\}} (r_i, \theta) = (r'_i, g_i(\theta))$ with some r'_i $(i = 1, 2)$. It is easy to show that $g_1(\theta) \equiv g_2(\theta)$. Therefore, T is expressed in the form $T(r, \theta) = (f(r), g(\theta))$. Clearly, $f(r)$ is either cr or c/r, and $g(\theta)$ is either $\theta + \theta_0$ or $-\theta + \theta_0$ (c and θ_0 are constants.). Thus we have proved that $T \in \mathcal{T}$ when $\alpha \neq 1$.

When $\alpha = 1$, we see that $T_{|\ell} \in \mathcal{T}_{|\ell}$ for a half-line $\ell = \{\theta = \theta_0\}$ by using the fact that ℓ is the contour $\{A ; \rho_1(A, (1, \theta_0 + \pi) | P) = 0\}$. We can write $T|_{\{\theta = \theta_i\}}(r, \theta_i) = (f_i(r), \theta'_i)$, where we set $T\{\theta = \theta_i\} = \{\theta = \theta'_i\}$ $(i = 1, 2)$. It is easy to show that $f_1(r) = f_2(r)$ unless $|\theta_1 - \theta_2| = \pi$. Therefore, T is expressed in the form $T(r, \theta) = (f(r), g(\theta))$, so T must belong to $\mathcal{T}$.

(Q.E.D.)

§3. Prediction formulas and Markov property of X_α

For a closed set e in R^n, we set

$$\mu(A | e) = E[X(A) | X(P) ; P \in e] ,$$
$$\sigma^2(A | e) = E\{(X(A) - \mu(A | e))^2\} .$$

Let a point P_0 be chosen arbitrarily in e. Under the additional condition $X(P_0) = 0$, the conditional expectation $\mu(A | e)$ relative to the σ-field generated by $\{X(P) ; P \in e\}$ is calculated.

Let T_* be the inversion with respect to the unit sphere with center P_0. The simple application of projective invariance gives us the following lemma.

LEMMA 3. *If the conditional expectation $\mu(A \mid e)$ is expressed in the form*

$$\mu(A \mid e) = \int_e X(P) dF_A(P) ,$$

with a signed measure $dF_A(P)$ on e depending on A, then we have

$$\mu(T_* A \mid T_* e) = \int_e X(T_* P) |P - P_0|^\alpha |A - P_0|^{-\alpha} dF_A(P) ,$$

and

$$\sigma^2(T_* A \mid T_* e) = |A - P_0|^{-2\alpha} \sigma^2(A \mid e) .$$

By virtue of Lemma 3 we can obtain the new formulas for

$$\mu(T_* A \mid T_* e) \quad \text{and} \quad \sigma^2(T_* A \mid T_* e)$$

from the known formulas for $\mu(A \mid e)$ and $\sigma^2(A \mid e)$. For example, when e is a spherical surface S and A is its center, $T_* S$ becomes a hyperplane. So the simple formula

$$\mu(A \mid S) = \int_S X(P) d\mu(P)$$

$$(d\mu \text{ is the uniform probability measure on } S.)$$

gives us the prediction formulas relative to a hyperplane L.

Put $A = (1, 0, \cdots, 0)$, $L = \{(x_1, \cdots, x_n) ; x_1 = 0\}$ and $P = (0, x_2, \cdots, x_n)$. Then,

$$\mu(A \mid L) = \pi^{-(n-1)/2} \Gamma\left(n - 1 + \frac{\alpha}{2}\right) \Gamma\left(\frac{n - 1 + \alpha}{2}\right)^{-1}$$

$$\times \int_L X(P) |P - A|^{-2n+2-\alpha} dx_2 \cdots dx_n ,$$

$$\sigma^2(A \mid L) = \Gamma\left(n - 1 + \frac{\alpha}{2}\right) \Gamma\left(\frac{n - 1}{2}\right) \Big/ 2 \Gamma\left(\frac{n - 1 + \alpha}{2}\right) \Gamma(n - 1) .$$

When e is a half-space H, M. I. Fortus ([2]) gave the formula:

$$(5) \qquad \sigma^2(A \mid H) = 2^{\alpha-1} \Gamma\left(\frac{n}{2}\right) |A - H|^\alpha \Big/ \Gamma\left(\frac{2 - \alpha}{2}\right) \Gamma\left(\frac{n + \alpha}{2}\right) .$$

Applying Lemma 3 to (5), we have

$$\sigma^2(A \mid \mathscr{S}^c) = \Gamma\left(\frac{n}{2}\right) R^{-\alpha} (R^2 - |A - O|^2)^\alpha \Big/ 2 \Gamma\left(\frac{2 - \alpha}{2}\right) \Gamma\left(\frac{n + \alpha}{2}\right) > 0$$

$$\text{for } A \in \mathscr{S} ,$$

where $\mathscr{S}$ is the sphere $\{P\,;\,|P - O| < R\}$. This implies the following proposition.

PROPOSITION 1. X_α *is non-deterministic. That is,* $\sigma(A\,|\,e) > 0$ *for any* e *and any* $A \notin e$.

We cite here the following two formulas from [2], [4] and [9] which will be used in the next section.

Let H be the half-space $\{(x_1, \cdots, x_n)\,;\,x_1 \le 0\}$ and $A = (1, 0, \cdots, 0)$. For $\alpha \ne 1$ or for an even n,

$$
(6) \quad
\begin{aligned}
\mu(A\,|\,H) &= \pi^{-(n/2)-1} \Gamma\!\left(\frac{n}{2}\right) \sin\!\left(\frac{\alpha + n}{2}\pi\right) \\
&\quad \times \int_0^\infty p^{-(\alpha+n)/2} dp \left\{ N(p) - \sum_{j=0}^{[(n-2+\alpha)/2]} \frac{p^j}{j!} N^{(j)}(0) \right\} ,
\end{aligned}
$$

where

$$
N(p) = \int_{R^{n-1}} X((-p, y))((p + 1)^2 + |y|^2)^{-n/2} dy ,
$$

and $N^{(j)}$ is the j-th derivative of $N(p)$.

Next, let $\mathscr{S}$ be the sphere $\{P\,;\,|P - O| < R\}$ and A be any point of $\mathscr{S}$. If $\alpha = 1$ and n is odd, we have

$$
(7) \quad
\begin{aligned}
\mu(A\,|\,\mathscr{S}^c) &= \frac{(R^2 - |A - O|^2)^{(n+1)/2}}{((n-1)/2)!} \\
&\quad \times \left(-\frac{1}{2r}\frac{d}{dr}\right)^{(n-1)/2} \left(\int_{S(r)} \frac{r^{n-2} X(P)}{|A - P|^n} d\mu(P)\right)\Bigg|_{r=R} ,
\end{aligned}
$$

where $S(r) = \{P\,;\,|P - O| = r\}$ $(r > R)$ and $d\mu$ is the uniform probability measure on $S(r)$.

From these formulas we can see the following important property of X_α.

PROPOSITION 2. X_α *has Markov property if and only if* $\alpha = 1$ *and* n *is odd.*

This proposition has been proved in [6] and [7]. Furthermore we may expect that X_α has anti-Markov property for $\alpha \ne 1$ or for an even n. For an open set U in R^n we define a sub-σ-field $F(U)$ by

$F(U) = $ the smallest σ-field for which all the $X(A)$ $(A \in U)$ is
measurable.

For each domain D in $\mathbf{R}^n$, we introduce the following four sub-σ-fields in order to explain anti-Markov property:

$F^-(D) = \bigcap_{\varepsilon>0} F(D_\varepsilon)$, where D_ε means an ε-neighborhood of D.

$F^+(D) = \bigcap_{\varepsilon>0} F((D^C)_\varepsilon)$.

$\partial F(D) = \bigcap_{\varepsilon>0} F((\partial D)_\varepsilon)$, where ∂D means the boundary of D.

$F^{+/-}(D) =$ the smallest splitting field of $F^+(D)$ upon $F^-(D)$ (see [6]).

Following to Dym-McKean [1], we call the case $F^{+/-}(D) = F^-(D)$ anti-Markovian, while the case $F^{+/-}(D) = \partial F(D)$ is termed Markovian. For an even n, it is known that X_1 has anti-Markov property ([6]). For $n = 1$ and $\alpha \neq 1$, X_α has anti-Markov property· We can prove this fact by the use of the canonical representation of X_α due to Molchan-Golosov [8], and we shall use it in the proof of Theorem 4.

§4. Invariance for the conditional correlation function of X_α

For a closed set e in $\mathbf{R}^n$, we set

$$X(A) = \mu(A \,|\, e) + \sigma(A \,|\, e)\xi(A \,|\, e) .$$

The conditional correlation function of X_α relative to e has been defined in §1 by

$$\rho_\alpha(A, B \,|\, e) = E\{\xi(A \,|\, e)\xi(B \,|\, e)\} .$$

We can easily show the following facts 1) ~ 4). In particular, 2) and 3) follow from the non-deterministic property.

1) $\rho_\alpha(A, B \,|\, e) = 0$ for $A \in e$,
2) $\rho_\alpha(A, A \,|\, e) = 1$ for $A \notin e$,
3) $|\rho_\alpha(A, B \,|\, e)| < 1$ for $A \neq B$,
4) $\rho_\alpha(A, B \,|\, e)$ is continuous in A and B on e^C.

DEFINITION 2. We say a transformation T on $\mathbf{R}^n$ leaves the conditional correlation function of X_α relative to e invariant if the equality (∗∗) holds for any $A, B \in \mathbf{R}^n$.

(∗∗) $\rho_\alpha(TA, TB \,|\, Te) = \rho_\alpha(A, B \,|\, e) .$

We note that for T satisfying (∗∗), a transformation T' such that $T'e = Te$ and $T'|_{e^C} = T|_{e^C}$ also satisfies the same equality (∗∗).

THEOREM 3. *For each e, $\mathscr{T}$ leaves the conditional correlation function*

of X_α relative to e invariant, provided that as to inversions we consider only the inversions with the center in e.

Theorem 3 is an easy consequence of Theorem 1 and the following simple lemma.

LEMMA 4. *If a transformation T leaves the conditional correlation function relative to e invariant, then for each e' containing e T leaves the conditional correlation function relative to e' invariant.*

LEMMA 5. *For an open ball $\mathscr{S}$ in $\mathbf{R}^n$, if a transformation T leaves the conditional correlation function relative to $\mathscr{S}^c$ invariant and if $T\mathscr{S}$ is also an open ball, then $T|_{\mathscr{S}}$ belongs to $\mathscr{T}|_{\mathscr{S}}$.*

We note that in the case $n = 1$ the condition that $T\mathscr{S}$ is an open ball is always fulfilled.

Proof of Lemma 5. Let O be the center of $\mathscr{S}$ and R be the radius. Assuming that $TO = O$, $T\mathscr{S} = \mathscr{S}$ and T leaves the conditional correlation function relative to $\mathscr{S}^c$ invariant, we shall prove that T is an orthogonal transformation on $\mathscr{S}$. Once this is established, we can prove Lemma 5 as follows. For T in Lemma 5, we find T_0 in $\mathscr{T}$ such that $T_0(TO) = O$ and $T_0(T\mathscr{S}) = \mathscr{S}$. Then $T_0 \circ T$ satisfies the hypotheses of the above assertion, so $T_0 \circ T$ is an orthogonal transformation on $\mathscr{S}$. Hence $T|_{\mathscr{S}} \in \mathscr{T}|_{\mathscr{S}}$.

For the proof of the above assertion, we calculate $\rho_\alpha(A, O \mid \mathscr{S}^c) \equiv \tilde{\rho}_\alpha(|A - O|)$ explicitly by using the prediction formulas (5), (6) and (7) cited in §3. Then we can show that $(d/da)\tilde{\rho}_\alpha(a) < 0$ in $0 < a < \varepsilon$, hence $\tilde{\rho}_\alpha$ is strictly decreasing in $(0, \varepsilon)$, where ε is some sufficiently small positive number. Therefore, for any A in some sufficiently small neighborhood of O, $\tilde{\rho}_\alpha(|TA - O|) = \tilde{\rho}_\alpha(|A - O|)$ yields $|TA - O| = |A - O|$.

We denote by $T_{(A)}$ the transformation in $\mathscr{T}$ such that $T_{(A)}A = O$, $T_{(A)}\mathscr{S} = \mathscr{S}$. The transformation $T_{(TA)} \circ T \circ T_{(A)}^{-1}$ has the same properties as T, so we have $|T_{(TA)}(TB) - O| = |T_{(A)}B - O|$ for small $|T_{(A)}B - O|$. On the other hand we can show that

$$|T_{(A)}B - O| = R^2 |A - B| \{|A - O|^2 |B - O|^2$$
$$+ R^4 - 2|A - O||B - O| R^2 \cos(\angle AOB)\}^{-1/2}.$$

From the equalities:

$$|TA - O| = |A - O|, \qquad |TB - O| = |B - O|$$

and

$$|T_{(TA)}(TB) - O| = |T_{(A)}B - O|\,,$$

we see that $\cos(\angle TA \cdot O \cdot TB) = \cos(\angle AOB)$. Hence T is an orthogonal transformation on a sufficiently small neighborhood of O. Repeating this argument, it can be seen that T is an orthogonal transformation on $\mathscr{S}$.   (Q.E.D.)

Let $n = 1$. For a closed set e in $\boldsymbol{R}^1$, put $e^C = \bigcup_i I_i$, where I_i's are the connected components of the open set e^C.

THEOREM 4. *If T leaves the conditional correlation function relative to e invariant, then in the case $\alpha \neq 1$, $T|_{e^C}$ belongs to $\mathscr{T}|_{e^C}$. While, in the case $\alpha = 1$, $T|_{I_i}$ belongs to $\mathscr{T}|_{I_i}$ for every i.*

Proof. For each i, T leaves the conditional correlation function relative to I_i^C invariant by Lemma 4. So we have $T|_{I_i} \in \mathscr{T}|_{I_i}$ by Lemma 5. In the case $\alpha = 1$, the proof is completed. But in the case $\alpha \neq 1$, we must show that $T|_{e^C} \in \mathscr{T}|_{e^C}$. For any intervals I_i and I_j, we shall show that if $T|_{I_i}$ is the identical transformation, so is $T|_{I_j}$. Then we see that $T|_{I_i \cup I_j} \in \mathscr{T}|_{I_i \cup I_j}$ by the fact that $T|_{I_i} \in \mathscr{T}|_{I_i}$. It follows from arbitrariness of I_i and I_j that $T|_{e^C} \in \mathscr{T}|_{e^C}$.

By considering $T_0 \in \mathscr{T}$ such that $T_0 I_i = (-1, 0)$ and $T_0 I_j = (a, \infty)$ $(a > 0)$, we may prove the above assertion for $I_i = (-1, 0)$ and $I_j = (a, \infty)$. Thus we assume that T leaves the conditional correlation function relative to $((-1, 0) \cup (a, \infty))^C$ invariant, that $T|_{(-1,0)}$ is the identity, and that $T|_{(a,\infty)}(\in \mathscr{T}|_{(a,\infty)})$ is not the identity. Then, we show that these assumptions contradict the anti-Markov property of X_α. Put $T(a, \infty) = (b, c)$. We show only the case $b > a$. All other cases are shown similarly.

Consider the transformation T' such that

$$T'x = \frac{b}{a}T|_{(-a/b,0) \cup (a,(a/b)c)}\left(\frac{v}{b}x\right) \qquad \text{for } x \in (-1, 0) \cup (b, c)\,.$$

This transformation T' leaves the conditional correlation function relative to $(-\infty, -1] \cup [0, b] \cup [c, \infty)$ invariant, T' belongs to $\mathscr{T}$ on (b, c) and T' is the identity on $(-1, 0)$. Put $T'(b, c) = (b_1, c_1)$. Then we see that $b_1/b \geq b/a$. Next consider T'' such that T'' leaves the conditional correlation function relative to $(-\infty, -1] \cup [0, b_1] \cup [c_1, \infty)$ invariant, T'' belongs to $\mathscr{T}$ on (b_1, c_1) and T'' is the identity on $(-1, 0)$. Then we see that b_2/b_1

$\geq b/a$ if we put $T''(b_1, c_1) = (b_2, c_2)$. Repeating this procedure, we have the following equalities: for any $A \in (-1, 0)$ and any $B \in (a, \infty)$,

$$
\begin{aligned}
\rho_\alpha(A, B \,|\, (-\infty, -1] \cup [0, a]) &= \rho_\alpha(A, TB \,|\, (-\infty, -1] \cup [0, b] \cup [c, \infty)) \\
&= \rho_\alpha(A, T'TB \,|\, (-\infty, -1] \cup [0, b_1] \cup [c_1, \infty)) \\
&= \rho_\alpha(A, T''T'TB \,|\, (-\infty, -1] \cup [0, b_2] \cup [c_2, \infty)) \\
&= \cdots \\
&= \rho_\alpha(A, T^{(n)} \cdots T''T'TB \,|\, (-\infty, -1] \cup [0, b_n] \cup [c_n, \infty)) \\
&= \cdots .
\end{aligned}
$$

From the fact that

$$
\rho_\alpha^2(A, B \,|\, e) = 1 - \sigma^2(A \,|\, e \cup \{B\})\sigma^{-2}(A \,|\, e) \qquad \text{for } A, B \notin e ,
$$

the $n + 2$nd member of the above equalities is dominated by

$$
\left(1 - \frac{\sigma^2(A \,|\, (-\infty, -1] \cup [0, b_n] \cup [c_n, \infty) \cup \{B_n\})}{\sigma^2(A \,|\, (-\infty, -1] \cup [0, b_n] \cup [c_n, \infty))}\right)^{1/2}
$$
$$
\leqq \left(1 - \frac{\sigma^2(A \,|\, (-\infty, -1] \cup [0, \infty))}{\sigma^2(A \,|\, (-\infty, -1] \cup [0, b_n])}\right)^{1/2},
$$

where $B_n = T^{(n)} \cdots T''T'TB$. Letting $b_n \to \infty$, we prove

$$
\sigma^2(A \,|\, (-\infty, -1] \cup [0, b_n]) \longrightarrow \sigma^2(A \,|\, (-\infty, -1] \cup [0, \infty)) .
$$

Consequently, we obtain $\rho_\alpha(A, B \,|\, (-\infty, -1] \cup [0, a]) = 0$ for any $A \in (-1, 0)$ and any $B \in (a, \infty)$, and this fact contradicts the anti-Markov property of X_α. (Q.E.D.)

REFERENCES

[1] H. Dym and H. P. McKean, Jr.: Application of de Branges spaces of integral functions to the prediction of stationary Gaussian processes, Ill. J. Math. **14** (1970), 299–343.

[2] M. I. Fortus: Formulas for extrapolation of random fields, Theory Prob. Appl. **7** (1962), 101–108. (English translation)

[3] R. Gangolli: Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters, Ann. Inst. Henri Poincaré section B vol. III n. **2** (1967), 121–225.

[4] I. Kubo: Some topics on random fields, Seminar on Probability vol. **26** (1967). (In Japanese)

[5] P. Lévy: Processus stochastiques et mouvement Brownien, Gauthier-Villars, Paris (1965).

[6] H. P. McKean, Jr.: Brownian motion with a several-dimensional time, Theory Prob. Appl. **8** (1963), 335–354.

[7] G. M. Molchan: Characterization of Gaussian fields with Markov property, Dokl. Akad. Nauk. SSSR **197** (1971), 784–787.

[8] G. M. Molchan and Ju. I. Golosov: Gaussian stationary processes with asymptotic power spectrum, Dokl. Akad. Nauk. SSSR **184** (1969), 546–549.

[9] A. M. Yaglom: Correlation theory of processes with random stationary n-th increments, Amer. Math. Soc. Transl. Ser **2** Vol. **8** (1958), 87–141.

[10] A. M. Yaglom: Some classes of random fields in n-dimensional space related to stationary random processes, Theory Prob. Appl. **2** (1957), 273–320. (English translation)

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